J1. Let a, b, c be *distinct* positive integers with $a + b + c = 29$. What is the smallest possible value of $a^2 + b^2 + c^2$?

Solution

We claim that the answer is $8^2 + 10^2 + 11^2 = \boxed{285}$.

First, let x, y be positive integers with $x \geq y + 3$, and note that

$$x^2 + y^2 - ((x-1)^2 + (y+1)^2) = 2x - 2y - 2 \geq 4 > 0.$$

Therefore decreasing x by 1 and increasing y by 1 keeps their sum constant, decreases their sum of squares, and keeps them distinct (as $x - 1 > y + 1$).

Pick the minimizing triple (a, b, c), where we can assume that $a > b > c$ without loss of generality. We claim that $a = 11$. If $a \leq 10$, then the maximum sum is $10 + 9 + 8 = 27 < 29$ (as they are distinct), contradiction. Next, assume that $a \geq 12$. If $b \leq a - 3$, then we can replace (a, b, c) by $(a - 1, b + 1, c)$ and get a new triple of distinct integers with smaller sum of squares (by the claim in the last paragraph). Thus we must have $b = a - 2$ or $b = a - 1$. In any case, $b \geq a - 2 \geq 10$, so $a + b \geq 22$, and $c = 29 - a - b \leq 7$. But now we can replace (a, b, c) by $(a, b - 1, c + 1)$, which decreases the sum of squares and keeps the numbers distinct as $b \geq 10 \geq c + 3$. This is a contradiction, hence $a = 11$.

We have $b + c = 29 - a = 18$ and $11 > b > c$, which uniquely gives $(b, c) = (10, 8)$. Therefore this triple attains the minimum of $11^2 + 10^2 + 8^2 = 285$, as claimed.

J2. Consider a trapezoid $ABCD$, where sides AB, CD are parallel. Points W, X, Y, Z are given such that $BCWX$ and $ADYZ$ form rhombi, and the interiors of the three quadrilaterals $ABCD$, $BCWX$, and $ADYZ$ are disjoint. Show that the distance between the centers of $BCWX$ and $ADYZ$ is at most half the perimeter of $ABCD$.

Remark: The center of a rhombus is the intersection of its diagonals.

Solution

Let P be the center of $BCWX$ and Q the center of $ADYZ$. Let M be the midpoint of BC and N the midpoint of AD. Then we have that $\angle BPC = \angle AQD = 90°$ since $BCWX$ and $ADYZ$ are rhombi. Thus we have

$$PQ \leq PM + MN + NQ = \frac{BC}{2} + \frac{AB + CD}{2} + \frac{AD}{2}.$$

The right hand side is precisely half the perimeter of $ABCD$, so we are done.

J3. Let $a, b \geq 2$ be relatively prime integers. Let S consist of the points in the plane with integer coordinates lying strictly inside the triangle with vertices $(0,0), (a,0), (0,b)$. Determine, with proof,

$$\sum_{(x,y)\in S} (a-2x)(b-2y)$$

in terms of a and b.

Remark. Here the summation denotes that we sum the value $(a-2x)(b-2y)$ over all points (x,y) in S.

Solution 1

For simplicity let $f(x,y) = (a-2x)(b-2y)$. Let R be the region in the plane consisting of lattice points (x,y) with $1 \leq x \leq a-1$ and $1 \leq y \leq b-1$. Then S consists of the lattice points in R that lie below the line $x/a + y/b = 1$. Let T consist of the lattice points in R that are not in the aforementioned right angled triangle and lie strictly above the line $x/a + y/b = 1$.

Since a and b are coprime the line $x/a + y/b = 1$, which is the line $bx + ay = ab$, has no lattice points with $1 \leq x \leq a-1$ and $1 \leq y \leq b-1$. Indeed $bx + ay = ab$ implies $b|ay$, so $b|y$ by coprimality, which contradicts $1 \leq y \leq b-1$. Therefore R is the disjoint union of S and T.

Now on R we see that

$$\begin{aligned}\sum_{(x,y)\in R} f(x,y) &= \left(\sum_{x=1}^{a-1}(a-2x)\right)\left(\sum_{y=1}^{b-1}(b-2y)\right) \\ &= \left(a(a-1) - 2\cdot\frac{a(a-1)}{2}\right)\left(b(b-1) - 2\cdot\frac{b(b-1)}{2}\right) \\ &= 0 \cdot 0 = 0.\end{aligned}$$

However, the map $F(x,y) = (a-x, b-y)$ is a bijection from S to T. This is because T consists of lattice points in R above the line $x/a + y/b = 1$, S consists of lattice points in R above that line, and $x/a + y/b < 1$ if and only if $(a-x)/a + (b-y)/b = 2 - (x/a + y/b) > 1$. Moreover,

$$f(F(x,y)) = (a-2(a-x))(b-2(b-y)) = (2x-a)(2y-b) = (a-2x)(b-2y) = f(x,y).$$

Altogether then

$$\sum_{(x,y)\in S} f(x,y) = \sum_{(x,y)\in T} f(x,y).$$

Together with

$$0 = \sum_{(x,y)\in R} f(x,y) = \sum_{(x,y)\in S} f(x,y) + \sum_{(x,y)\in T} f(x,y)$$

this implies

$$\sum_{(x,y)\in S} f(x,y) = 0$$

as desired.

Solution 2

Let

$$T = \sum_{(x,y)\in S} (a-2x)(b-2y).$$

We prove that $T = 0$ by splitting the big triangle into two smaller regions using the median from $(0,0)$ to the midpoint $\left(\frac{a}{2}, \frac{b}{2}\right)$ of the hypotenuse. This median is the line

$$y = \frac{b}{a}x.$$

Since a and b are relatively prime and at least 2, there are no lattice points on the open line segment joining $(0,0)$ to $\left(\frac{a}{2}, \frac{b}{2}\right)$: indeed, if an integer point (x,y) lay on that segment, then $ay = bx$, and since $\gcd(a,b) = 1$, this would force $a \mid x$ and $b \mid y$, which is impossible for a point strictly between the endpoints. Thus every lattice point of S lies strictly on one side or the other of this median.

Write $S = S_1 \sqcup S_2$, where S_1 is the set of lattice points of S below the median and S_2 is the set of lattice points of S above the median. We first consider S_1. If $(x,y) \in S_1$, then $0 < y < \frac{b}{a}x$, and because the point also lies inside the original triangle we have $x > 0$, $y > 0$, and $x < a - ay/b$. Now reflect (x,y) horizontally across the median of the horizontal slice of the triangle by sending it to $(a-x, y)$. Since

$$\frac{x}{a} + \frac{y}{b} < 1 \quad\Longleftrightarrow\quad \frac{a-x}{a} + \frac{y}{b} > \frac{y}{b},$$

and since (x,y) lies below the median, one checks that $(a-x,y)$ lies in the upper part of the triangle determined by the same horizontal level, still strictly inside the original triangle. More importantly, the two corresponding summands cancel:

$$(a-2(a-x))(b-2y) = (-a+2x)(b-2y) = -(a-2x)(b-2y).$$

Thus points paired by $(x,y) \leftrightarrow (a-x,y)$ contribute zero in total.

Likewise, for the points in S_2, we pair vertically by sending (x,y) to $(x, b-y)$. This sends points in the left-hand region to corresponding points in the complementary region on the same vertical line, and again the summands cancel because

$$(a-2x)(b-2(b-y)) = (a-2x)(-b+2y) = -(a-2x)(b-2y).$$

The only points in S which cannot be paired are those on the lines $x = a/2$ or $y = b/2$, as they reflect to themselves. But $x = a/2$ and $y = b/2$ both imply

$(a-2x)(b-2y)=0$, so these terms do not contribute to the sum. Therefore the entire sum cancels under the above pairing:

$$\sum_{(x,y)\in S}(a-2x)(b-2y)=0.$$

Solution 3

Let

$$T=\sum_{(x,y)\in S}(a-2x)(b-2y).$$

We compute T by summing over the lattice points column by column. For each integer x with $1\le x\le a-1$, the points of S having first coordinate x are exactly $(x,1),(x,2),\ldots,(x,m_x)$, where m_x is the largest integer such that $\frac{x}{a}+\frac{y}{b}<1$. Since the hypotenuse of the triangle is the line $y=b-\frac{b}{a}x$, we have

$$m_x=\left\lfloor b-\frac{b}{a}x\right\rfloor.$$

Because $\gcd(a,b)=1$, the number $\frac{bx}{a}$ is not an integer for any $x=1,\ldots,a-1$, so

$$m_x=b-1-\left\lfloor\frac{bx}{a}\right\rfloor.$$

If we write $n_x=\left\lfloor\frac{bx}{a}\right\rfloor$, then $m_x=b-1-n_x$. The contribution from the x-th column to the entire sum is therefore

$$C_x=\sum_{y=1}^{m_x}(a-2x)(b-2y)=(a-2x)\sum_{y=1}^{m_x}(b-2y).$$

Now

$$\sum_{y=1}^{m_x}(b-2y)=m_xb-2\cdot\frac{m_x(m_x+1)}{2}=m_x(b-m_x-1),$$

and substituting $m_x=b-1-n_x$ yields

$$\sum_{y=1}^{m_x}(b-2y)=(b-1-n_x)n_x.$$

Thus

$$C_x=(a-2x)(b-1-n_x)n_x.$$

We now compare the x-th and $(a-x)$-th columns. Since

$$n_{a-x}=\left\lfloor\frac{b(a-x)}{a}\right\rfloor=\left\lfloor b-\frac{bx}{a}\right\rfloor=b-1-\left\lfloor\frac{bx}{a}\right\rfloor=b-1-n_x,$$

it follows that

$$(b-1-n_{a-x})n_{a-x} = n_x(b-1-n_x).$$

On the other hand,

$$a-2(a-x) = -(a-2x).$$

Hence

$$C_{a-x} = (a-2(a-x))(b-1-n_{a-x})n_{a-x} = -(a-2x)(b-1-n_x)n_x = -C_x.$$

So the contribution from column x cancels exactly with the contribution from column $a-x$. Therefore all columns cancel in pairs. If a is even, the middle column $x = \frac{a}{2}$ contributes 0 anyway, since then $a-2x=0$. It follows that

$$\sum_{(x,y)\in S} (a-2x)(b-2y) = 0.$$

J4. There are n types of coins in Wario's gold mine. Each coin of the ith type is worth d_i cents, where $d_1, \ldots, d_n$ are distinct positive integers. A positive integer D is denoted *lucky* if the following holds: For each positive integer k, any collection of coins (containing any number of coins of each type) with a total value of exactly kD cents can be split into k groups, each worth D cents. Does a lucky number necessarily exist?

Solution 1

The answer is yes. We apply induction on k, repeatedly extracting a group of value D from the remaining coins; the case $k = 1$ is immediate. Thus the problem is equivalent to the following inductive step:

Given positive integers $d_1, \ldots, d_n$, does there exist a positive integer D such that whenever nonnegative n_i satisfy $\sum_i n_i d_i = mD$ for a positive integer $m \geq 2$, there exist $0 \leq m_i \leq n_i$ satisfying $\sum_i m_i d_i = D$?

Let d be any multiple of all of $d_1, \ldots, d_n$ (e.g. $d = d_1 \ldots d_n$ or $d = \operatorname{lcm}(d_1, \ldots, d_n)$). We claim that $D = Md$ works for a suitable choice of M. Indeed, note that any d/d_i copies of d_i can be grouped together to a collection of numbers summing to d. There exist at least

$$\sum_i \left\lfloor \frac{n_i}{d/d_i} \right\rfloor$$

such groups, so it suffices to show that the quantity above is at least M. However we have that

$$\begin{aligned}
\sum_i \left\lfloor \frac{n_i}{d/d_i} \right\rfloor &\geq \left(\sum_i \frac{n_i}{d/d_i} \right) - n \\
&= \left(\frac{1}{d} \sum_i n_i d_i \right) - n \\
&= mM - n \\
&\geq M + (M - n).
\end{aligned}$$

Hence choosing $M = n$ works as desired.

Solution 1

We present a modified solution, based on the submission of Perry Dai, which proves $D = d_1 \ldots d_n$ is lucky. As in Solution 1, we use induction on k where the base case $k = 1$ is trivial. Thus, suppose that $k \geq 2$ and $a_1 d_1 + \cdots + a_n d_n = kD$. WLOG let $a_1 d_1 \geq a_2 d_2 \geq \ldots \geq a_n d_n$. Then $a_1 d_1 \geq \frac{kD}{n} \geq \frac{2d_1 \ldots d_n}{n}$, so $a_1 \geq \frac{2d_2 \ldots d_n}{n}$.

Claim. We have $a_1 \geq d_i$ for all $2 \leq i \leq n$, except for a case where $n = 3$ and $d_i = 1$ for some $i \geq 2$.

Proof of claim. If $n \le 2$ or $n \ge 4$, then

$$a_1 \ge \frac{2d_2 \dots d_n}{n} = d_i \cdot \frac{2}{n} \cdot \prod_{2 \le j \le n, j \ne i} d_j \ge d_i \cdot \frac{2}{n} \cdot (n-2)! \ge d_i.$$

If $n = 3$ and no d_i is equal to 1, then $a_1 \ge \frac{2d_2d_3}{3} \ge \frac{4}{3}d_i \ge d_i$. □

Let us ignore the $n = 3$ and some d_i equal to 1 case for now and continue with $a_1 \ge d_i$ for all $2 \le i \le n$.

Consider a sequence of nonnegative integers $x_1, \dots, x_n$ such that $x_1d_1 + \cdots + x_nd_n = D$ and $x_i \le a_i$ for all $2 \le i \le n$, where we choose x_1 to be minimal under these constraints. Note that $\frac{D}{d_1}, 0, \dots, 0$ satisfies the aforementioned constraints, so at least one such sequence exists. If $x_1 \le a_1$ as well, then $x_1, \dots, x_n$ corresponds to a group of coins worth D cents. Suppose this is not the case, so $x_1 > a_1$. For any $2 \le i \le n$, note that replacing (x_1, x_i) with $(x_1 - d_i, x_i + d_1)$ does not change $x_1d_1 + \cdots + x_nd_n$, keeps x_1 nonnegative because $x_1 > a_1 \ge d_i$ as per the claim above, and makes x_1 smaller. By the minimality of x_1, we must have $x_i + d_1 > a_i$ for all $2 \le i \le n$, so $x_i > a_i - d_1$. Then,

$$\begin{aligned} D = \sum_{i=1}^{n} x_id_i = x_1d_1 + \sum_{i=2}^{n} x_id_i &\ge (a_1+1)d_1 + \sum_{i=2}^{n}(a_i - d_1 + 1)d_i \\ &= \sum_{i=1}^{n} a_id_i + \sum_{i=1}^{n} d_i - d_1\sum_{i=2}^{n} d_i \ge 2D + \sum_{i=1}^{n} d_i - d_1\sum_{i=2}^{n} d_i \\ \implies d_1\sum_{i=2}^{n} d_i &\ge D + \sum_{i=1}^{n} d_i > d_1\left(1 + \prod_{i=2}^{n} d_i\right). \end{aligned}$$

We claim that $1 + \prod_{i=2}^{n} d_i \ge \sum_{i=2}^{n} d_i$ for any distinct positive integers $d_2, \dots, d_n$, which would imply a contradiction. Without loss of generality, rearrange them so that $d_2 < d_3 < \cdots < d_n$, so $2 \le d_i$ for all $3 \le i \le n$. Then $(d_2 - 1)(d_3 \dots d_n - 1) \ge 0$ and $d_3 \dots d_n \ge d_3 + \cdots + d_n$, so

$$1 + \prod_{i=2}^{n} d_i \ge d_2 + \prod_{i=3}^{n} d_i \ge d_2 + \sum_{i=3}^{n} d_i$$

as required. This completes the proof except for the case where $n = 3$ and some d_i is equal to 1.

In this remaining case, suppose the coins have values $d_1, d_2, 1$. If we have at least d_2 coins of value d_1, or at least d_1 coins of value d_2, then we can extract a group worth $D = d_1d_2$. Otherwise, the total value of the coins of value d_1 and d_2 is $\le d_1(d_2 - 1) + d_2(d_1 - 1) \le kD - d_1 - d_2$, so there are at least $d_1 + d_2$ coins of value 1. We can form a group of D by taking coins of value d_1 or d_2 until we exhausted all such coins or adding another one would make the total exceed D, then fill the remainder with coins of value 1.

J5. Turbo the snail plays a game on a board with $2n$ rows and $2n$ columns. There are $2n^2$ monsters who first choose to occupy $2n^2$ distinct cells, with Turbo's knowledge. After this, Turbo chooses any cell and labels it 1. Starting from this cell, Turbo then walks through all other $4n^2 - 1$ cells exactly once, labelling them in order with $2, 3, \ldots, 4n^2$. Turbo only moves between cells which share an edge, and never returns to a cell. The final score is the sum of the labels of the cells with monsters. The monsters are trying to place themselves to maximize the score, while Turbo is trying to minimize the score based on the monsters' positions. Find, in terms of n, the largest score which the monsters can guarantee.

Solution 1

The largest score which the monsters can achieve is $4n^4$. This is obtained by placing themselves in a checkerboard pattern. Call a cell marked if a monster occupies it and empty otherwise. Clearly, Turbo's path will alternate between marked and empty. By starting at a marked cell, the score is

$$1 + 3 + \cdots + (4n^2 - 1) = 4n^4.$$

Now we will show that Turbo can always achieve a score at most $4n^4$, regardless of the monsters' positions. Note that for a $2n \times 2n$ board, there exists a Hamiltonian cycle through the cells. Consider an arbitrary monster and say that it is on cell 1. Now consider the two paths starting at this cell and going along the Hamiltonian cycle in each direction. For any monster other than the one at cell 1, its index in one path will be i and in the other will be $4n^2 + 2 - i$. Thus, the sum of the scores of these two paths is

$$\begin{aligned} 2 + \sum_{\text{monster at } i} \left(i + 4n^2 + 2 - i\right) &= 2 + (2n^2 - 1)(4n^2 + 2) \\ &= 8n^4. \end{aligned}$$

Thus, one of these two paths has a score at most $4n^4$.

Solution 2

As in the previous solution, the monsters can arrange themselves in a checkboard fashion to attain a lower bound of $4n^4$. Now we will show that any other configuration will allow Turbo to get a score at most $4n^4 - n^2$, which is slightly sharper than the previous solution.

As in the previous solution, there exists some Hamiltonian cycle. If the monsters are not in a checkboard pattern, there must be two adjacent cells in this cycle where both are monsters. Call these cells A and B. We will consider two paths going along the cycle. The first path begins at A, goes to B next, and continues along the cycle in that direction until ending in a cell adjacent to A. Likewise, the second path begins at B, goes to A next, and continues along until ending next to B.

We claim that the scores of the paths, denoted as $S_{A\to B}$ and $S_{B\to A}$, sum to $8n^4 - 2n^2$. Consider any of the $2n^2 - 2$ monsters not on A or B and say it is at index i of the first path. Then on the second path, this monster will be reached at index $4n^2 + 3 - i$. So we have

$$\begin{aligned} S_{A\to B} + S_{B\to A} &= 6 + \sum_{\text{monster at } i} \big(i + 4n^2 + 3 - i\big) \\ &= 6 + (2n^2 - 2)(4n^2 + 3) \\ &= 8n^4 - 2n^2. \end{aligned}$$

Thus, $\min(S_{A\to B}, S_{B\to A}) \le 4n^4 - n^2$ and so there exists a path which Turbo can take to achieve a score $\le 4n^4 - n^2$.

