Crux Mathematicorum

Volume/tome 52, issue/numéro 2
February/février 2026

Canadian Mathematical Society
Société mathématique du Canada

Crux Mathematicorum is a problem-solving journal at the secondary and university undergraduate levels, published online by the Canadian Mathematical Society. Its aim is primarily educational; it is not a research journal. Online submission:

`https://publications.cms.math.ca/cruxbox/`

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire publiée par la Société mathématique du Canada. Principalement de nature éducative, le Crux n'est pas une revue scientifique. Soumission en ligne:

`https://publications.cms.math.ca/cruxbox/`

ISSN 1496-4309 (Online)

ISSN 1496-4309 (électronique)

Editorial Board

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MathemAttic

No. 72

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2026**.

MA356. Let a, b, c, d be distinct positive numbers. What is the total number of different values for the expression $xy + zw$ if the variables x, y, z, w are assigned the values of a, b, c, d in all possible orders?

MA357. Forty-one rooks are placed on a 10×10 checkerboard. Prove that you can choose 5 of them that do not attack one another. (We say that one rook "attacks" another if they are in the same row or column of the checkerboard.)

MA358. *Proposed by Victor Manuel Mesa Solano.*

Show that if a number has exactly four proper factors (that is, four factors different from one or the number itself), then the sum of their reciprocals cannot be 1.

MA359. *Proposed by Michael Friday.*

Let R be the circumradius of triangle ABC with $\angle A = 15°$, $\angle B = 105°$, and T the foot of the bisector of $\angle C$. Prove that $R^2 = BC \times CA$ and $CT^2 = AT \times BT$.

MA360. The geometry group 4G has a number (more than one) of committees, each of which contains 4 members of the club. Membership has been established to guarantee that for every two members of 4G there is exactly one committee to which both belong, and every two committees have at least one member in common.

a) Show that each person belongs to exactly 4 committees.

b) How many persons belong to 4G?

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2026**.

MA356. Soient a, b, c et d des nombres positifs distincts. Quel est le nombre total de valeurs distinctes prises par l'expression $xy + zw$ lorsque les variables x, y, z et w reçoivent les valeurs a, b, c et d dans tous les ordres possibles ?

MA357. On place quarante et une tours sur un échiquier 10×10. Montrez qu'on peut en choisir 5 qui ne s'attaquent pas. (On dit qu'une tour "attaque" une autre si elles sont sur la même ligne ou la même colonne de l'échiquier.)

MA358. *Soumis par Victor Manuel Mesa Solano.*

Montrez que si un entier possède exactement quatre diviseurs propres (c'est-à-dire quatre diviseurs distincts de 1 et de lui-même), alors la somme des inverses de ces diviseurs ne peut pas être égale à 1.

MA359. *Soumis par Michael Friday.*

Soit R le rayon du cercle circonscrit au triangle ABC avec $\angle A = 15°$, $\angle B = 105°$, et soit T le pied de la bissectrice de l'angle $\angle C$. Montrez que $R^2 = BC \times CA$ et $CT^2 = AT \times BT$.

MA360. Le groupe de géométrie $4G$ possède un certain nombre (strictement supérieur à 1) de comités, chacun formé de 4 membres du club. La composition est telle que, pour toute paire de membres de $4G$, il existe exactement un comité auquel ils appartiennent tous les deux, et que deux comités quelconques ont au moins un membre en commun.

a) Montrez que chaque personne appartient exactement à 4 comités;

b) Combien de personnes appartiennent à $4G$?

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(7), p. 306–308.

MA331. *Proposed by Ivan Hadinata.*

Let M be the number of ordered pairs of natural numbers (a, b) satisfying the equation

$$a^b = (20!)^{24!}.$$

Find the last three digits of M.

We received 4 submissions, of which 2 were correct and complete. We present the solution by Konstantine Zelator.

We want to find the number of ordered pairs (a, b) such that $a^b = (20!)^{24!}$ and a, b are positive integers. We start by finding the prime factorization of 20! and 24!. The prime factorization of 20! is

$$\begin{aligned} 20! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20 \\ &= 1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot 2 \cdot 3 \cdot 7 \cdot 2^3 \cdot 3^2 \cdot 2 \cdot 5 \cdot 11 \cdot 2^2 \cdot 3 \cdot 13 \cdot 2 \cdot 7 \cdot 3 \cdot 5 \cdot 2^4 \cdot 17 \cdot 2 \cdot 3^2 \cdot 19 \cdot 2^2 \cdot 5 \\ &= 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19. \end{aligned}$$

The prime factorization of 24! is

$$\begin{aligned} 24! &= 20! \cdot 21 \cdot 22 \cdot 22 \cdot 23 \cdot 24 \\ &= 20! \cdot 3 \cdot 7 \cdot 2 \cdot 11 \cdot 23 \cdot 2^3 \cdot 3 \\ &= 2^{22} \cdot 3^{10} \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23. \end{aligned}$$

Since we want $a^b = (20!)^{24!}$, then a must have the same prime factors as 20!. Therefore, we have that $a = 2^{e_1}3^{e_2}5^{e_3}7^{e_4}11^{e_5}13^{e_6}17^{e_7}19^{e_8}$, where $e_i \in \mathbb{Z}^+$ for $i = 1, 2, \ldots, 8$. It follows that

$$\begin{aligned} (20!)^{24!} = a^b &= 2^{e_1 b}3^{e_2 b}5^{e_3 b}7^{e_4 b}11^{e_5 b}13^{e_6 b}17^{e_7 b}19^{e_8 b} \\ &= 2^{(18)(24!)}3^{(8)(24!)}5^{(4)(24!)}7^{(2)(24!)}11^{24!}13^{24!}17^{24!}19^{24!} \end{aligned}$$

From the above equalities, we see that we must have

$$e_1 = \frac{(18)(24!)}{b},\ e_2 = \frac{(8)(24!)}{b},\ e_3 = \frac{(4)(24!)}{b},\ e_4 = \frac{(2)(24!)}{b},$$
$$e_5, e_6, e_7, e_8 = \frac{(24!)}{b},$$

with $b \in \mathbb{Z}^+, e_i \in \mathbb{Z}^+$ for all $i = 1, \dots, 8$. Thus, b must be a positive divisor of 24!. Therefore, the number of ordered pairs is equal to the number of positive integer divisors of 24!. We state the following number theory result before applying it.

Let $n \geq 4$ be a positive integer and $n = p_1^{e_1} \dots p_k^{e_k}$ be its prime factorization for distinct primes p_i, and positive integer exponents $e_1, \dots, e_k$. Then, the number of positive integer divisors of n is $\Pi_{i=1}^k (e_i + 1)$.

Applying this theorem, the number of positive integer divisors of 24! is

$$(22+1)(10+1)(4+1)(3+1)(2+1)(1+1)(1+1)(1+1)(1+1) = 242880.$$

We conclude that the number of ordered pairs (a, b) is 242880 and so the last three digits are 880.

MA332. *Proposed by Michael Friday.*

In any triangle ABC, let H, O be the orthocenter and circumcenter, and let M_a, M_b, M_c be the midpoints of sides BC, CA, AB respectively. Prove that

$$OH^2 = (HM_a^2 - OM_a^2) + (HM_b^2 - OM_b^2) + (HM_c^2 - OM_c^2)$$

We received 4 submissions all of which were correct. We present the solution by Manescu-Avram Corneliu.

We will use complex numbers. Consider the circumcircle of the triangle ABC as the unit circle and denote the coordinates of the points by the small letters. Then

$$h = a + b + c, \quad o = 0, \quad m_a = \frac{b+c}{2}, \quad m_b = \frac{c+a}{2}, \quad m_c = \frac{a+b}{2} \tag{1}$$

where $a, b, c, \in \mathbb{C}$, $|a| = |b| = |c| = 1$. The proposed identity becomes

$$|a+b+c| = \left(\left| a+b+c - \frac{b+c}{2} \right|^2 - \left| \frac{b+c}{2} \right|^2 \right) + \left(\left| a+b+c - \frac{c+a}{2} \right|^2 - \left| \frac{c+a}{2} \right|^2 \right) + \left(\left| a+b+c - \frac{a+b}{2} \right|^2 - \left| \frac{c+a}{2} \right|^2 \right)$$

or equivalently

$$4|h|^2 = |h+a|^2 + |h+b|^2 + |h+c|^2 - (|h-a|^2 + ||h-b|^2 + |h-c|^2)$$

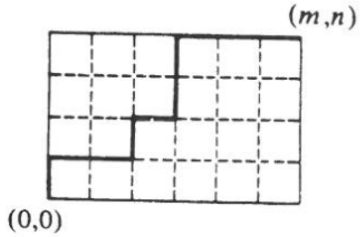
Now we calculate

$$\begin{aligned} &(h+a)\overline{(h+a)} + (h+b)\overline{(h+b)} + (h+c)\overline{(h+c)} \\ &- \left[(h-a)\overline{(h-a)} + (h-b)\overline{(h-b)} + (h-c)\overline{(h-c)} \right] \\ &= 2h(\overline{a} + \overline{b} + \overline{c}) + 2\overline{h}(a+b+c) = 2h\overline{h} + 2\overline{h}h = 4h\overline{h}. \end{aligned}$$

as desired.

MA333.

a) An $m \times n$ rectangle is divided into mn squares. A path is to be traced starting at $(0,0)$ and concluding at (m,n) by moving only in a positive sense along the ruled lines.



Show that the number of distinct paths is $\binom{m+n}{n}$

b) An $n \times n \times n$ cube has each of its faces ruled into n^2 squares. A path defined in part a), moving always in a positive sense on its faces, is to start at $(0,0,0)$ and reach the point (n,n,n). Determine the number of distinct paths.

Originally from the Descartes Contest 1986, problem 10, part b.

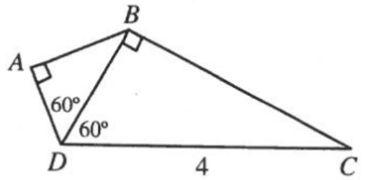
We received 6 submissions, of which 4 were correct and complete. We present the solution by Catherine Jian.

a) An $m \times n$ rectangle is divided into mn equal squares. We trace a path starting at point $(0,0)$ and ending at point (m,n) by moving only in a positive sense along the ruled lines. Therefore, the path has to move m units along the x-axis in a positive direction, and n units along the y-axis in a positive direction. The total number of possible paths is equal to the number of ways these units can be arranged. Therefore, there are $\binom{m+n}{n}$ different paths.

b) An $n \times n \times n$ cube has each of its faces ruled out into n^2 equal squares. We trace a path starting at point $(0,0,0)$ and ending at point (n,n,n) by moving in a positive sense on its faces. To reach the point (n,n,n), the path will need to move n units along the x-axis, n units along the y-axis, and n units along the z-axis, for a total of $3n$ units. There are n units to choose from among the starting $3n$ units along the x-axis, and then there are n units to choose from among the remaining $2n$ units left. The total number of possible paths is equal to the number of ways these units can be arranged, which is

$$\binom{3n}{n}\binom{2n}{n} = \frac{(3n)!}{n!n!n!}$$

MA334. In the quadrilateral $ABCD$, angles DBC and DAB are right angles. Also, angles ADB and BDC have measure of 60 degrees. If DC is 4 units, determine which one is greater $DA + AC$ or $DB + BC$.



Originally from the Descartes Contest 1998, problem B4.

We received 12 solutions, of which 9 were correct and complete. We present two solutions.

Solution 1, by multiple solvers.

Using the known ratios for the sides of a $30° - 60° - 90°$ triangle, we know that $BD = 2, BC = 2\sqrt{3}, AD = 1, AB = \sqrt{3}$. We can use the Law of Cosines on $\triangle ADC$ to find AC:

$$AC^2 = AD^2 + DC^2 - 2(AD)(DC)\cos 120° = 17 + 4 \implies AC = \sqrt{21}.$$

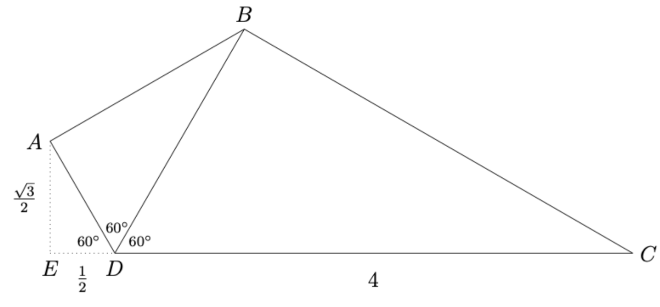
To compare $DA + AC$ and $DB + BC$, we know $21 > 81/4 \implies \sqrt{21} > 9/2$ and $3 < 49/16 \implies \sqrt{3} < 7/4$. Then

$$DA + AC > 11/2 > DB + BC,$$

so $DA + AC$ is the larger quantity.

Solution 2, composed by the editor using ideas from various solvers.

As above, we know $BD = 2, AD = 1$. Denote E as the perpendicular dropped from A to $\overline{DC}$. Note that $\angle EDA$ is $60°$, so $AE = \frac{\sqrt{3}}{2}$ and $ED = \frac{1}{2}$.



Since $EC = ED + DC = 9/2$, by the Pythagorean theorem on $\triangle AEC$, we find

$$AC = \sqrt{AE^2 + EC^2} = \sqrt{\frac{3}{4} + \frac{81}{4}} = \sqrt{21}.$$

The rest of the solution follows as in the previous solution to compare $DA + AC$ and $DB + BC$.

MA335.

a) Find all geometric series such that the sum of the first two terms is 2 and the sum of the first three terms is 3.

b) For each of the sequences determined in part a), calculate the sum of all terms having value less than 1.

Originally from the Descartes Contest 1986, problem 4.

We received 14 submissions of which 12 were correct and complete. We present the solution by Nicholas Fleece.

(a) We shall determine the sequences underlying these series. Let $A = \{ar^n\}_{n=0}^{\infty}$ with $a, r \in \mathbb{C}$ and $r \neq 0$ satisfying the above. We then have the system

$$\begin{aligned} a + ar &= 2, \\ a + ar + ar^2 &= 3, \end{aligned}$$

which through elimination yields $ar^2 = 1$ and $a = \frac{1}{r^2}$, giving us

$$\frac{1}{r^2} + \frac{1}{r} = 2 \implies \frac{1+r}{r^2} = 2 \implies 2r^2 - r - 1 = 0$$

yielding our two solutions $(a, r) = (1, 1)$ or $(a, r) = (4, -\frac{1}{2})$. So, our two geometric sequences are $A_1 = \{1\}_{n=0}^{\infty}$ and $A_2 = \left\{4\left(-\frac{1}{2}\right)^n\right\}_{n=0}^{\infty}$.

(b) Let $f(A)$ be the sum of all terms in the sequence A that are less than 1. It is clear that $f(A_1) = 0$. For $f(A_2)$ we look at

$$A_2 = \{4, -2, 1, -\tfrac{1}{2}, \tfrac{1}{4}, \ldots\}$$

and see that, given that $|r| < 1$ these are decreasing in magnitude so that everything beyond the second term is less than one, as well as the first term (we are using zero-indexing). Hence, only the zeroth and second term are not less than one, and they sum to 5. Thus

$$\begin{aligned} f(A_2) &= 4\sum_{n=0}^{\infty}\left(\left(-\tfrac{1}{2}\right)^n\right) - 5 \\ &= 4\left(\frac{1}{1 - \left(-\frac{1}{2}\right)}\right) - 5 \\ &= 4\left(\frac{2}{3}\right) - 5 \\ &= -\frac{7}{3} \end{aligned}$$

Competition Highlights

The Canada Jay Mathematical Competition

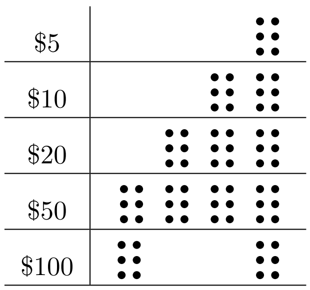
by Nicolae Strungaru

The Canada Jay Mathematical Competition is an yearly math contest aimed at grades 5-8. This year's contest took place on November 20th, 2025. In total, 3694 students wrote it and 178 students got a perfect score. Congratulations to all winners.

We continue last year's tradition of covering two questions from this year's contest, with their solutions. With start by presenting Problem B05 with its solution.

Problem B05

Canadian banknotes have Braille markings in the corner so people can identify the value of a bill by touch alone. A group of 6 dots is called a Braille block. The $5 bill has one Braille block, $10 has two, $20 has three, $50 has four, and $100 has two blocks that are far apart, as shown.



Brooke has some amount of bills and says, "the total number of Braille blocks on all the bills I'm holding is 6." Which of the following cannot be the total value of all the bills that Brooke is holding?

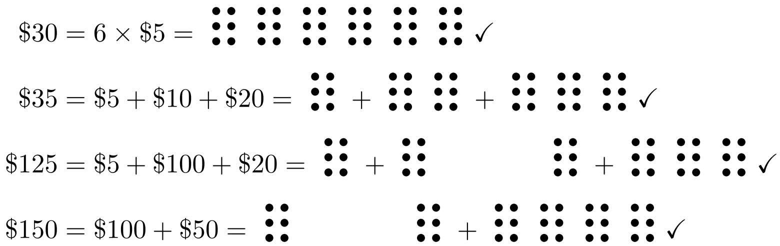
(A) $30 (B) $35 (C) $55 (D) $125 (E) $150

Solution

$ 55 cannot be the value of the bills Brooke holds.

Indeed, if Brooke holds a $50 bill, then she has at least $60 in total. Otherwise, she can have at most $20 for each 3 groups of Braille blocks and hence at most $40.

The other answers are possible:



Answer: (C) $55

Problem C04

The number 236 is an example of a three-digit number for which one of the digits is the product of the other two digits.

How many numbers between 100 and 999 inclusive have this property?

(A) 50 (B) 51 (C) $52 (D) 53 (E) 61

Solution

We split the problem into three cases.

Case 1: One of the digits is zero. Then, a second digit must be zero. The non-zero digit must be the first one.

There are 9 numbers in this case.

Case 2: Zero is not a digit and one of the digits is 1. Then, the other two digits must be equal. Therefore, the digits are $1, x, x$ for some $1 \leq x \leq 9$.

When $x = 1$, the number must be 111.

When $2 \leq x \leq 9$ then, there are 8 possibilities for choosing x and three possibilities of choosing the position of the digit 1 in the three digit number.

There are

$$1 + 8 \times 3 = 25$$

numbers in this case.

Case 3: No digit is 0 or 1. Therefore, all digits are between 2 and 9, inclusive, and the product of two of them is the third one. It follows that the digits must be one of the following groups

$$\{2, 2, 4\}; \{2, 3, 6\}; \{2, 4, 8\}, \{3, 3, 9\}.$$

There are 3 numbers which can be formed with the first group, 6 numbers which can be formed with the each of the second and third group, and 3 with the last group.

Therefore, in this case there are

$$3 + 6 + 6 + 3 = 18$$

numbers.

In total there are

$$9 + 25 + 18 = 52$$

numbers in total.

We can list them all out:

$$\begin{aligned}
&100, 111, 122, 133, 144, 155, 166, 177, 188, 199,\\
&200, 212, 221, 224, 236, 242, 248, 263, 284,\\
&300, 313, 326, 331, 339, 362, 393,\\
&400, 414, 422, 428, 441, 482,\\
&500, 515, 551,\\
&600, 616, 623, 632, 661,\\
&700, 717, 771,\\
&800, 818, 824, 842, 881,\\
&900, 919, 933, 991.
\end{aligned}$$

So, there are 52.

Answer: (C) 52

OLYMPIAD CORNER

No. 440

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **April 15, 2026**.

OC766. Suppose A is a point exterior to the unit sphere $\mathcal{S}$ in $\mathbb{R}^3$. The tangents from A to $\mathcal{S}$ form a right circular cone $\mathcal{K}$. (For our purposes this is a finite cone ending at the points of tangency. Note that the usual base of the cone is not included here.)

At what distance should A be from the centre of the sphere so that $\mathcal{S}$ and $\mathcal{K}$ have equal surface areas?

OC767. Sam goes into the gym and puts a marble of radius 1cm in the corner, touching two walls and the floor. (Assume all surfaces are flat and the angles are right.) A second student comes along and puts a ball in the corner so that it just touches both walls, the floor, and the marble. Several more students do the same, each ball touching the walls, the floor, and the previous ball.

What is the first ball radius that is greater than or equal to 1m? Give your answer in the form

$$\frac{a+b\sqrt{c}}{d},$$

where a, b, c, d are integers.

OC768. Late in the year 1 CE, the villagers of Marzipan invented a winter solstice tradition: each household sent a fruitcake to one other household. None of these fruitcakes were ever eaten: instead, each year afterward, each household passed their fruitcake on to the same household that they had given one to in the previous year. This tradition has lasted over the centuries, and last year (2025 CE), for the first time, every household got its original fruitcake back.

What is the smallest possible number of households in the village of Marzipan?

OC769. For each natural number n, define an $n \times n$ matrix $P(n)$ with $(P(n))_{ij} = \binom{n+i-2}{j-1}$ for $i, j = 1, 2, \ldots, n$. Find, with proof, $\det(P(n))$.

OC770. You are given positive integers p, q, r, s satisfying

$$qr - ps = 1.$$

Suppose x, y are positive integers with

$$\frac{p}{q} < \frac{x}{y} < \frac{r}{s}.$$

Find the smallest value of y satisfying this last condition and determine all x that go with it.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2026**.

OC766. Supposons que A soit un point extérieur à la sphère $\mathcal{S}$ de rayon 1 dans $\mathbb{R}^3$. Les tangentes de A à $\mathcal{S}$ forment un cône circulaire droit $\mathcal{K}$. (Pour ce problème, il s'agit d'un cône fini se terminant aux points de tangence. Notez que la base habituelle du cône n'est pas incluse ici.)

À quelle distance le point A doit-il se trouver du centre de la sphére pour que $\mathcal{S}$ et $\mathcal{K}$ aient des surfaces égales ?

OC767. Samuel va au gymnase et place une bille d'un rayon de 1 cm dans un coin, touchant deux murs et le sol. (Supposons que toutes les surfaces sont planes et que les angles sont droits.) Un deuxiéme élève arrive et place une balle dans le coin de façon à ce qu'elle touche juste les deux murs, le sol et la bille. Plusieurs autres éléves font de même, chaque balle touchant les murs, le sol et la balle précédente.

Quel est le rayon de la première balle qui a un rayon supérieur ou égal à 1 m ? Donnez votre réponse sous la forme

$$\frac{a + b\sqrt{c}}{d},$$

où a, b, c, d sont des nombres entiers.

OC768. À la fin de l'année 1 de notre ère, les villageois de Massepain ont inventé une tradition du solstice d'hiver : chaque foyer donne un gâteau aux fruits à un autre foyer. Aucun de ces gâteaux aux fruits n'est mangé : au contraire, chaque année, chaque foyer donne son gâteau aux fruits au même foyer auquel il avait offert un gâteau l'année précédente. Cette tradition s'est perpétuée au fil des siécles, et dernière année (2025 de notre ére), pour la première fois, chaque foyer retrouvera son gâteau aux fruits original.

Quel est le plus petit nombre possible de foyers dans le village de Massepain ?

OC769. Pour chaque nombre naturel n, définissez une matrice $P(n)$ de taille $n \times n$ avec $(P(n))_{ij} = \binom{n+i-2}{j-1}$ pour $i, j = 1, 2, \ldots, n$. Trouvez (avec preuve) la valeur de $\det(P(n))$.

OC770. Les nombres entiers positifs p, q, r, s sont tels qu'ils satisfont l'équation

$$qr - ps = 1.$$

Supposons que x et y soient des nombres entiers positifs tels que

$$\frac{p}{q} < \frac{x}{y} < \frac{r}{s}.$$

Trouvez la plus petite valeur de y permise, et pour cette valeur de y déterminez toutes les valeurs de x pour lesquelles cette dernière équation est satisfaite.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2026: 51(7), p. 324–325.

OC741. A triple of positive real numbers (a, b, c) is called *mysterious* if

$$\sqrt{a^2+\frac{1}{a^2c^2}+2ab}+\sqrt{b^2+\frac{1}{b^2a^2}+2bc}+\sqrt{c^2+\frac{1}{c^2b^2}+2ca}=2(a+b+c).$$

Prove that if the triple (a, b, c) is mysterious, then the triple (c, b, a) is also mysterious.

Originally from the All Russian Mathematical Olympiad 2024 - Final Round, Grade 11, Problem 2.

We received 11 correct solutions. We present the solution by Michel Bataille.

Let a, b, c be positive real numbers and

$$L=\sqrt{a^2+\frac{1}{a^2c^2}+2ab}+\sqrt{b^2+\frac{1}{b^2a^2}+2bc}+\sqrt{c^2+\frac{1}{c^2b^2}+2ca}.$$

We can write L as follows:

$$L=\sqrt{(a+b)^2+\frac{1-a^2b^2c^2}{a^2c^2}}+\sqrt{(b+c)^2+\frac{1-a^2b^2c^2}{b^2a^2}}+\sqrt{(c+a)^2+\frac{1-a^2b^2c^2}{c^2b^2}}$$

and therefore we have three options.

If $abc > 1$, then

$$L<\sqrt{(a+b)^2}+\sqrt{(b+c)^2}+\sqrt{(c+a)^2}=a+b+b+c+c+a=2(a+b+c).$$

If $abc < 1$, then (similarly) $L > 2(a+b+c)$. Finally, if $abc = 1$, then $L = 2(a+b+c)$.

We deduce that (a, b, c) is mysterious if and only if $abc = 1$.The required result obviously follows.

OC742. Let $A \in \mathcal{M}_n(\mathbb{R})$ be an invertible matrix.

(a) Show that the matrix AA^T has real and positive eigenvalues.

(b) Suppose that there exist distinct positive integers p and q so that $(AA^T)^p = (A^TA)^q$. Prove that $A^T = A^{-1}$.

Originally from the Romanian Mathematical Olympiad 2024 - Final Round, Grade 11, Problem 2.

We received 10 complete solutions. We present 2 of them.

Solution 1, by Niccolò Bucciantini.

(a) Let $B = AA^T$. Note that B is real symmetric since $(AA^T)^T = AA^T$, hence all eigenvalues of B are real. Let x be an eigenvector of B with eigenvalue λ. Then

$$\lambda\|x\|^2 = x^T Bx = x^T AA^T x = (A^T x)^T (A^T x) = \|(A^T x)\|^2 \geq 0.$$

If $\lambda = 0$, then $\|(A^T x)\|^2 = 0$, so $A^T x = 0$. Since A is invertible, A^T is invertible. Hence $x = 0$, contradiction. Therefore $\lambda > 0$.

(b) Any real matrix can be decomposed in the singular value decomposition

$$A = U\Sigma V^T.$$

with $U^T U = I$, $V^T V = I$ and Σ diagonal, say $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$. Since A is invertible, all singular values satisfy $\sigma_i > 0$. Now AA^T and $A^T A$ are real symmetric matrices and as such admit a set of orthonormal eigenvectors. Observe that

$$\begin{aligned} AA^T &= (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma U^T = U\Sigma^2 U^T \\ A^T A &= (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T. \end{aligned}$$

Assume $(AA^T)^p = (A^T A)^q$. Using the expressions above,

$$\begin{aligned} (AA^T)^p &= (U\Sigma^2 U^T)^p = U\Sigma^{2p} U^T \\ (A^T A)^q &= (V\Sigma^2 V^T)^q = V\Sigma^{2q} V^T. \end{aligned}$$

Therefore

$$U\Sigma^{2p} U^T = V\Sigma^{2q} V^T.$$

Equal matrices have the same eigenvalues (with multiplicity). The eigenvalues of $U\Sigma^{2p}U^T$ are σ_i^{2p}, and the eigenvalues of $V\Sigma^{2q}V^T$ are σ_i^{2q}. Hence, after matching multiplicities, $\sigma_i^{2p} = \sigma_i^{2q}$ for all i. Since $\sigma_i > 0$ and $p \neq q$, this implies $\sigma_i^{2(p-q)} = 1$, so $\sigma_i = 1$ for all i. Thus $\Sigma = I$ and

$$AA^T = U\Sigma^2 U^T = I.$$

Multiplying on the left by A^{-1} we get $A^T = A^{-1}$.

Solution 2, by Oliver Geupel.

(a) Since A is invertible, so is A^T. Hence, for any non-vanishing vector $x \in \mathbb{R}^{n\times 1}$, the associated $y = A^T x$ is also non-vanishing, so that

$$x^T AA^T x = y^T y = \|y\|^2 > 0.$$

Therefore, the symmetric matrix AA^T is positive definite, which implies that all its eigenvalues are positive reals by a standard result; see, e.g., theorem 6B at page 318 in Strang, G. (2006). *Linear Algebra and Its Applications*, 4th ed.

(b) By the spectral theorem (theorem 5O at page 285 in Strang as cited above), the symmetric matrix $A^T A$ can be factored into

$$A^T A = Q\Lambda Q^{-1}$$

with its eigenvalues in the diagonal of the $n \times n$ diagonal matrix Λ and with orthonormal eigenvectors in the columns of the real $n \times n$ matrix Q.

The matrices AA^T and $A^T A$ have the same set of positive eigenvalues, say,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

The eigenvalues of $\left(AA^T\right)^p$ and of $\left(A^T A\right)^q$ are then

$$0 < \lambda_1^p \leq \lambda_2^p \leq \cdots \leq \lambda_n^p$$

and

$$0 < \lambda_1^q \leq \lambda_2^q \leq \cdots \leq \lambda_n^q,$$

respectively, by theorem 5E at page 248 in Strang.

By the hypothesis of the problem, it follows for $1 \leq k \leq n$ that $\lambda_k^p = \lambda_k^q$; whence

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1.$$

Thus Λ and $A^T A$ are equal to the identity matrix, implying that $A^T = A^{-1}$.

OC743.

Let $(K, +, \cdot)$ be a division ring such that $x^2 y = yx^2$ for all $x, y \in K$. Prove that $(K, +, \cdot)$ is a field.

Originally from the Romanian Mathematical Olympiad 2024 - Final Round, Grade 12, Problem 2.

We received 10 solutions. We present the solution by Corneliu Manescu-Avram.

Let

$$Z(K) = \{c \in K \mid cx = xc,\ \forall x \in K\}$$

be the center of K (a subfield of K). The hypothesis says that for every $x \in K$ and every $y \in K$,

$$x^2 y = yx^2,$$

hence $x^2 \in Z(K)$ for all $x \in K$.

Case 1: $\mathrm{char}(K) \neq 2$. For any $a \in K$ we have

$$2a = (a+1)^2 - a^2 - 1 \in Z(K),$$

since $(a+1)^2, a^2, 1 \in Z(K)$. As $\text{char}(K) \neq 2$, the element $2 \cdot 1 \neq 0$ is invertible in the division ring K, and it lies in $Z(K)$, so $(2 \cdot 1)^{-1} \in Z(K)$ as well. Therefore,

$$a = (2 \cdot 1)^{-1}(2a) \in Z(K), \qquad \forall a \in K.$$

Hence $K = Z(K)$ and K is commutative.

Case 2: $\text{char}(K) = 2$**.** For any $a, b \in K$ set

$$w := ab + ba = (a+b)^2 - a^2 - b^2 \in Z(K),$$

because $(a+b)^2, a^2, b^2 \in Z(K)$. Consider $w \cdot ab$:

$$w \cdot ab = (ab+ba)ab = (ab)^2 + (ba)(ab) = (ab)^2 + baab = (ab)^2 + ba^2b.$$

Since $a^2 \in Z(K)$, we have $ba^2b = a^2b^2$, and since $b^2 \in Z(K)$, also $a^2b^2 \in Z(K)$. Moreover, $(ab)^2 \in Z(K)$ because it is a square. Thus

$$w \cdot ab = (ab)^2 + a^2b^2 \in Z(K).$$

If $w \neq 0$, then $w \in Z(K)^\star$, so multiplying by $w^{-1} \in Z(K)$ gives

$$ab = w^{-1}\big((ab)^2 + a^2b^2\big) \in Z(K).$$

In particular, ab commutes with a, hence

$$a(ab) = (ab)a.$$

But $a(ab) = a^2b$ and $(ab)a = aba$, so $a^2b = aba$. Since $a \neq 0$ (because $w \neq 0$ forces $ab \neq 0$, hence $a, b \neq 0$ in a division ring), we may cancel a on the left to obtain $ab = ba$.

If $w = 0$, then $ab = -ba$, and since $\text{char}(K) = 2$ we have $-ba = ba$, so again $ab = ba$.

In all cases, $ab = ba$ for all $a, b \in K$, so K is commutative, i.e. K is a field.

Editor's Comments. The conclusion remains valid if K is a domain (i.e., a ring without zero-divisors). Irving Kaplansky proved that if K is a division ring such that for every $x \in K$ there exists a positive integer $n(x)$ with

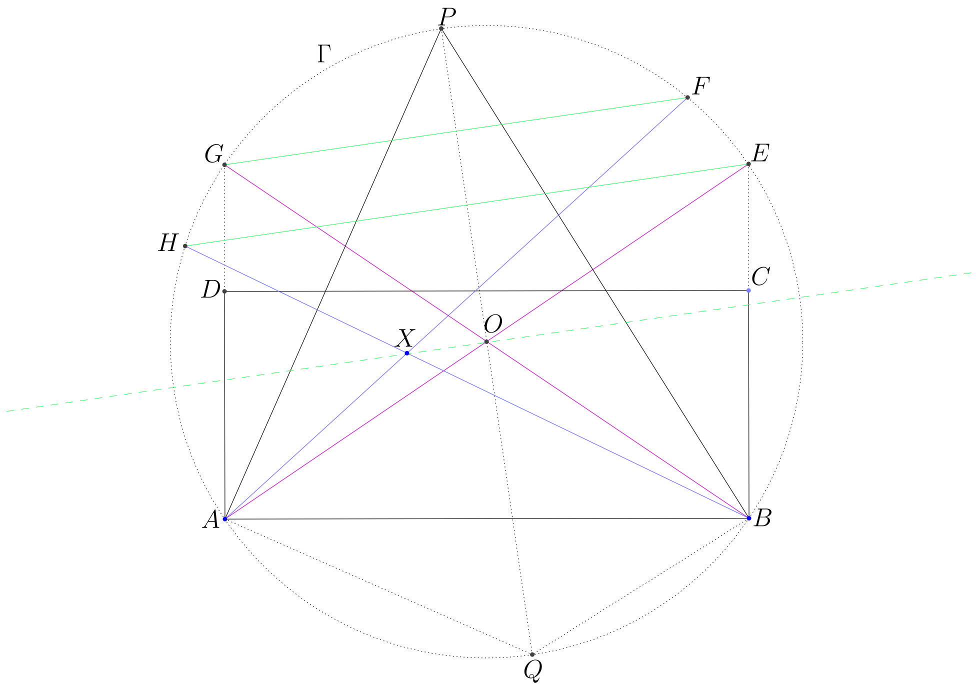
$$x^{n(x)}y = yx^{n(x)} \qquad \text{for all } y \in K,$$

then K is commutative.

OC744. Given a rectangle $ABCD$ and a point X lying inside it. The bisectors of angles DAX and CBX intersect at point P. Point Q satisfies the equalities $\angle QAP = \angle QBP = 90^\circ$. Prove that $PX = QX$.

Originally from Polish Mathematical Olympiad 2024 - Final Round, Problem 1.

We received 8 solutions. We present the solution by Oliver Geupel.



By the converse of Thales's theorem, the points A, B, P and Q lie on a common circle Γ whose center is the midpoint O of the line segment PQ. Let E, F, G, and H be the second points of intersection of Γ with the lines AO, AX, BO, and BX, respectively. By Thales's theorem, we have

$$\angle ABE = 90^\circ = \angle ABC.$$

Hence, the points B, C and E are collinear. Since, by hypothesis of the problem, it holds

$$\angle EBP = \angle CBP = \angle PBX = \angle PBH,$$

it follows that the arcs EP and PH of Γ are congruent. Thus, $EH \perp PQ$. Similarly, $FG \perp PQ$. Consider the hexagon $AEHBGF$ with vertices on Γ. Its pairs of opposite sides $\{AE, BG\}$ and $\{AF, BH\}$ intersect at the points O and X, respectively, which lie on its Pascal line. Since the opposite sides EH and FG are perpendicular to PQ, so is its Pascal line OX. Consequently, $PX = QX$.

OC745. Let n be a positive integer. Bolek draws $2n$ points on the plane, no two of which define a vertical or horizontal line. Then for each of these $2n$ points, Lolek draws two rays starting at that point, one of which is vertical and the other horizontal. Lolek wants to do this in such a way that the rays drawn divide the plane into as many areas as possible. Determine the largest integer k such that Lolek can obtain at least k areas regardless of the position of the points chosen by Bolek.

Originally from Polish Mathematical Olympiad 2024 - Final Round, Problem 2.

We received 8 solutions. We present the solution by Oliver Geupel.

We show that the largest k is $k = n^2 + (n+1)^2$. Let us call any collection of a vertical and a horizontal ray starting at a common point an *elbow* and the common starting point its *corner*. A set of corners is said to be *suitable* if no two of them lie on a common vertical or horizontal line. A set of elbows is called suitable if its corners are suitable.

We claim that, if any E suitable elbows have in total V points of intersection and split the plane into F areas, then it holds $F = E+V+1$. Our proof is by induction on $E \geq 0$. The base case $E = 0$ is immediate from $F = 1$ and $V = 0$. Next, assume that the formula holds for E elbows and add another one, with corner C, in such a way that the $E+1$ elbows are suitable. The new elbow intersects the other elbows at points $P_1, \ldots, P_s$ and $Q_1, \ldots, Q_t$ on its vertical and horizontal ray, respectively, where $s \geq 0$, $t \geq 0$, $P_0 = Q_0 = C$ and each of the two lists of points is sorted by increasing distance from C. Then each of the segments (P_1, P_2), (P_2, P_3), $\ldots$, (P_{s-1}, P_s), (P_s, ∞), (Q_1, Q_2), (Q_2, Q_3), $\ldots$, (Q_{t-1}, Q_t), (Q_t, ∞) and (P_1, C, Q_1) divides an area. The numbers of intersection points and of areas are increased by $s+t$ and $s+t+1$, respectively, which completes the induction.

Having seen this, it is enough to prove that the largest number V of intersections in a suitable set of $E = 2n$ elbows is $V = k - E - 1 = 2n^2$.

First, we show that $V \leq 2n^2$ if the corners in Cartesian coordinates are $(-n, -n)$, $\ldots$, $(-2, -2)$, $(-1, -1)$, $(1, 1)$, $(2, 2)$, $\ldots$, (n, n). Our proof is by induction on $n \geq 0$. The base case $n = 0$ is trivial. For the induction step, assume that the relation holds for some $n \geq 0$, and add elbows e_- and e_+ at corners $(-n-1, -n-1)$ and $(n+1, n+1)$, respectively. Each of the $4n$ rays of the $2n$ inner elbows meets $e_- \cup e_+$ at most once, whereas e_- and e_+ have at most two common points. So the total number of intersection points is increased by not more than $4n + 2$, which completes the induction.

It remains to prove that $V \geq 2n^2$ holds for every suitable set of $2n$ corners. Since the corners are suitable, we may draw a horizontal line such that exactly n corners lie above that line. Similarly, we may draw a vertical line such that exactly n corners lie to its left. The two lines constitute four quadrants. If c corners lie in the NE quadrant, then the numbers of corners in the NW, SW and SE quadrant are $n-c$, c and $n-c$. We say that an elbow is *NE*, if its rays point to the right and upward. Similarly, we speak of *NW*, *SW* and *SE* elbows. Let us draw an SW, SE, NE and NW elbow to each corner in the NE, NW, SW and SE quadrant, respectively. Then any two elbows in neighboring quadrants have one point of intersection, while any two elbows in opposite quadrants have two points of intersection, so that

$$V \geq 4c(n-c) + 2c^2 + 2(n-c)^2 = 2n^2.$$

"And that is why you don't divide by zero."

Cartoon artist Brooke Bourgeois

Evaluating Limits of Integrals Using The Mean Value Theorem

Neculai Stanciu

1 Introduction

Consider the statement of the Mean Value Theorem (MVT) for integrals:
Theorem 1. *If a function f is continuous on the closed interval $[a, b]$, then there exists at least one number $c \in (a, b)$ such that*

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

In this article, we will present a way to use MVT to compute the limits of sequences defined using the Riemann integral.

Let $I = (a, b) \subset \mathbb{R}$, and define the sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ to satisfy:

(i) $a_n \leq b_n$, $c_n > 0$, for any $n \in \mathbb{N}^*$;

(ii) $a_n, b_n, c_n \in I$ for all $n \in \mathbb{N}^*$;

(iii) $a_n c_n$ and $b_n c_n$ as sequences for $n \geq 1$ have finite limits and $\lim_{n\to\infty} a_n c_n = \lim_{n\to\infty} b_n c_n = \ell \in I$;

(iv) there exists $\alpha \in \mathbb{R}$ such that $\lim_{n\to\infty} n^\alpha (b_n - a_n) = L \in \mathbb{R}$.

If the function $f : I \to \mathbb{R}$ is continuous on I, then

$$\lim_{n\to\infty} n^\alpha \int_{a_n}^{b_n} f(c_n x)\,dx = L \cdot f(\ell).$$

Below we present examples of application of this theorem on problems from math journals from all over the world.

2 Applications

Example 1. Find $\displaystyle\lim_{n\to\infty} \int_{e^{x_n}}^{e^{x_{n+1}}} f\left(\frac{x}{n}\right) dx$, where $f : (0, \infty) \to (0, \infty)$ is continuous on $(0, \infty)$ and $x_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$. (Problem 261, *Revista Escolar de la Olimpiada Iberoamericana de Matematica*, No. 52, January – June, 2015).

Solution. First note that

$$e^{x_{n+1}} - e^{x_n} = e^{x_n}\left(e^{\frac{1}{n+1}} - 1\right) = e^{x_n - \ln n} \cdot n\left(e^{\frac{1}{n+1}} - 1\right) = e^{\gamma_n} \cdot \frac{e^{\frac{1}{n+1}} - 1}{\frac{1}{n+1}} \cdot \frac{n}{n+1},$$

where $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma$. Then from the above simplifications, we get

$$\lim_{n\to\infty} (e^{x_{n+1}} - e^{x_n}) = e^{\gamma} \cdot 1 \cdot 1 = e^{\gamma}. \tag{1}$$

By MVT, there exists $\xi_n \in \lfloor e^{x_n}, e^{x_{n+1}} \rfloor$ such that

$$\int_{e^{x_n}}^{e^{x_{n+1}}} f\left(\frac{x}{n}\right) dx = (e^{x_{n+1}} - e^{x_n})\, f\left(\frac{\xi_n}{n}\right). \tag{2}$$

Since $e^{x_n} \leq \xi_n \leq e^{x_{n+1}}$ is equivalent to $\dfrac{e^{x_n}}{n} \leq \dfrac{\xi_n}{n} \leq \dfrac{e^{x_{n+1}}}{n+1} \cdot \dfrac{n+1}{n}$, we obtain

$$\lim_{n\to\infty} \frac{\xi_n}{n} = \lim_{n\to\infty} \frac{e^{x_n}}{n} = \lim_{n\to\infty} e^{x_n - \ln n} = \lim_{n\to\infty} e_n^{\gamma} = e^{\gamma}. \tag{3}$$

Hence from (1), (2) and (3), we get

$$\lim_{n\to\infty} \int_{e^{x_n}}^{e^{x_{n+1}}} f\left(\frac{x}{n}\right) dx = e^{\gamma} f\left(\lim_{n\to\infty} \frac{\xi_n}{n}\right) = e^{\gamma} f(e^{\gamma}).$$

Example 2. Calculate $\displaystyle\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right) dx$, where $f : (0,\infty) \to (0,\infty)$ is continuous on $(0,\infty)$.

(Problem 4127, *Crux Mathematicorum*, Vol. 42, No. 3, March 2016).

Solution. Using

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e},$$

and

$$\lim_{n\to\infty} \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} = \frac{1}{e}, \tag{4}$$

by MVT, there exists $\xi_n \in (\sqrt[n]{n!}, \sqrt[n+1]{(n+1)!})$ such that

$$\int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right) dx = (\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}) \cdot f\left(\frac{\xi_n}{n}\right). \tag{5}$$

Since $\sqrt[n]{n!} \leq \xi_n \leq \sqrt[n+1]{(n+1)!}$ is equivalent to $\dfrac{\sqrt[n]{n!}}{n} \leq \dfrac{\xi_n}{n} \leq \dfrac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \dfrac{n+1}{n}$, we obtain

$$\frac{1}{e} \leq \lim_{n\to\infty} \frac{\xi_n}{n} \leq \frac{1}{e} \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{\xi_n}{n} = \frac{1}{e}. \tag{6}$$

From (4), (5) and (6), we obtain

$$\lim_{n\to\infty}\int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right)\,dx = \lim_{n\to\infty}(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!})\cdot\lim_{n\to\infty} f\left(\frac{\xi_n}{n}\right)$$
$$= \frac{1}{e}\cdot f\left(\lim_{n\to\infty}\frac{\xi_n}{n}\right) = \frac{1}{e}f\left(\frac{1}{e}\right).$$

Example 3. Calculate $\lim\limits_{n\to\infty}\sqrt[n]{(2n-1)!!}\int_{\gamma}^{\gamma_n} f(x)\,dx$, where $\gamma_n = -\ln n+\sum_{k=1}^{n}\frac{1}{k}$ with $\lim_{n\to\infty}\gamma_n=\gamma$ and $f:(0,\infty)\to(0,\infty)$ is continuous.

(Problem 100, *MathProblems*, Vol. 4, No. 2, 2014).

Solution.

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n}} = \lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}$$
$$= \lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2}{e}. \qquad (7)$$

By MVT, there exists $\xi_n\in(\gamma,\gamma_n)$ such that $\int_\gamma^{\gamma_n} f(x)\,dx = f(\xi_n)(\gamma-\gamma_n)$ for all $n\in\mathbb{N}*$. Then

$$\sqrt[n]{(2n-1)!!}\int_\gamma^{\gamma_n} f(x)\,dx = \sqrt[n]{(2n-1)!!}(\gamma_n-\gamma)f(\xi_n) = \frac{\sqrt[n]{(2n-1)!!}}{n}\cdot n(\gamma_n-\gamma)f(\xi_n). \qquad (8)$$

From (7) and (8), we obtain

$$\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\int_\gamma^{\gamma_n} f(x)\,dx = \frac{2}{e}f(\lim_{n\to\infty}\xi_n)\lim_{n\to\infty} n(\gamma_n-\gamma) = \frac{2}{e}f(\gamma)\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}}.$$

Next, by Cèsaro-Stolz since the limit is of the form $(\frac{0}{0})$, we obtain that the above expression is equal to

$$\frac{2f(\gamma)}{e}\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1}-\frac{1}{n}} = \frac{2f(\gamma)}{e}\lim_{n\to\infty}\frac{-\gamma_{n+1}+\gamma_n}{\frac{1}{n(n+1)}} = \frac{2f(\gamma)}{e}\lim_{n\to\infty}\frac{\ln\frac{n+1}{n}-\frac{1}{n+1}}{\frac{1}{n^2}}$$

Set $\frac{1}{n}=x$ to get

$$\frac{2f(\gamma)}{e}\lim_{x\to\infty,x>0}\frac{\ln(1+x)-\frac{x}{1+x}}{x^2} = \frac{2f(\gamma)}{e}\lim_{x\to\infty,x>0}\frac{(1+x)\ln(1+x)-x}{x^2(1+x)}$$
$$= \frac{2f(\gamma)}{e}\lim_{x\to\infty,x>0}\frac{(1+x)\ln(1+x)-x}{x^2} = \frac{2f(\gamma)}{e}\lim_{x\to\infty,x>0}\frac{(1+x)+1-1}{2x}$$
$$= \frac{2f(\gamma)}{2e}\lim_{x\to\infty,x>0}\ln(1+x)^{\frac{1}{x}} = \frac{f(\gamma)}{e}\ln e = \frac{f(\gamma)}{e}.$$

Example 4. Calculate $\lim_{n\to\infty} n^2 \int_{\frac{1}{\sqrt[n+1]{(n+1)!}}}^{\frac{1}{\sqrt[n]{n!}}} \Gamma(nx)\,dx$, where Γ is the gamma function.

(Problem 11808, *The American Mathematical Monthly*, Vol. 121, No. 10, December 2014).

Solution. Recall that $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$. Denoting $x_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{(n)!}$, we have $\lim_{n\to\infty} x_n = \frac{1}{e}$. By MVT, there exists ξ_n in the integration interval such that

$$\int_{\frac{1}{\sqrt[n+1]{(n+1)!}}}^{\frac{1}{\sqrt[n]{n!}}} \Gamma(nx)\,dx = \left(\frac{1}{\sqrt[n]{n!} - \sqrt[n+1]{(n+1)!}}\right)\Gamma(n\xi_n) = \frac{x_n}{\sqrt[n]{n!}\,\sqrt[n+1]{(n+1)!}}\Gamma(n\xi_n).$$

So

$$\begin{aligned}
&\lim_{n\to\infty} n^2 \int_{\frac{1}{\sqrt[n+1]{(n+1)!}}}^{\frac{1}{\sqrt[n]{n!}}} \Gamma(nx)\,dx \\
&= \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{n!}} \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!}} \cdot \frac{n}{n+1}\right) \cdot \lim_{n\to\infty} \Gamma(n\xi_n) \\
&= \frac{1}{e} \cdot e \cdot e \cdot 1 \cdot \Gamma(\lim_{n\to\infty} n\xi_n).
\end{aligned}$$

Since

$$\frac{1}{\sqrt[n+1]{(n+1)!}} \le \xi_n \le \frac{1}{\sqrt[n]{(n)!}} \qquad \Longrightarrow \qquad \frac{n+1}{\sqrt[n+1]{(n+1)!}} \cdot \frac{n}{n+1} \le n\xi_n \le \frac{n}{\sqrt[n]{(n)!}}$$

yields that $\lim_{n\to\infty} n\xi_n = e$, hence the integral in question is equal to $e\Gamma(e)$.

3 Exercises

We invite the reader to practice MVT to compute the limits of integrals in the following exercises.

1. $\lim\limits_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} \Gamma\left(\frac{x}{n}\sqrt[n]{L_n^m}\right) dx$, where $m > 0$ and L_n for $n \ge 0$ is a Lucas sequence. (Problem 121, *MathProblems*, Vol. 5, No. 1, 2015 and Problem H-771, *The Fibonacci Quarterly*, Vol. 53, No. 2, 2015.)

2. $\lim\limits_{n\to\infty} x_{n+1} \int_{a^{E_n}}^{a^{E_{n+1}}} f(x)\,dx$, where $f : (0,\infty) \to (0,\infty)$ is a continuous function, $a \in (0,1) \cup (1,\infty)$, E_n is a sequence defined by $E_n = \sum_{k=0}^{n} \frac{1}{k!}$ and x_n is a sequence of positive terms such that $\lim_{n\to\infty} \frac{x_n}{n!} > 0$.

   (Problem 1, Romanian Mathematical Contest *Sperante Ramnicene*, Grade XII, 2014.)

3. $\lim\limits_{n\to\infty} \int_{s_n}^{\frac{\pi^2}{6}} f(x)\,dx$, where $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$ for $n \geq 1$ and $f : (0,\infty) \to (0,\infty)$ is a continuous function.

   (Problem Q123, Romanian Math Journal – *Sclipirea Mintii*, No. 37, 2026 and *Romanian Mathematical Magazine*, December 2025.)

4. $\lim\limits_{n\to\infty} \sqrt[n]{n!} \int_{\gamma}^{\gamma_n} f(x)\,dx$, where $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma$ and $f : (0,\infty) \to (0,\infty)$ is a continuous function.

   (Problem Q122, Romanian Math Journal – *Sclipirea Mintii*, No. 37, 2026 and *Romanian Mathematical Magazine*, December 2025.)

5. $\lim\limits_{n\to\infty} \sqrt[n]{\sqrt{3!!}\,\sqrt[3]{5!!}\,\sqrt[n]{(2n-1)!}} \int_{\gamma}^{\gamma_n} f(x)\,dx$, where $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma$ and $f : (0,\infty) \to (0,\infty)$ is a continuous function.

   (Problem Q121, Romanian Math Journal – *Sclipirea Mintii*, No. 37, 2026 and *Romanian Mathematical Magazine*, December 2025.)

Answers:

1. $\frac{\Gamma(\alpha^m e^{-1})}{e}$
2. $b a^e f(a^e)$
3. $\frac{f\left(\frac{\pi^2}{6}\right)}{e}$
4. $\frac{f(\gamma)}{2e}$
5. $\frac{f(\gamma)}{e^2}$

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2026**.

5111. *Proposed by Michel Bataille.*

Let ABC be an acute-angled triangle, Γ its circumcircle, B' and C' the feet of its altitudes from B and C. Let P and Q be distinct points on the arc BC of Γ not containing A and let AP and AQ intersect BB' and CC' at M and N, respectively. Prove that M, N, P, Q are concyclic if and only if MN is parallel to $B'C'$.

5112. *Proposed by Vasile Cîrtoaje.*

What is the smallest positive value of k such that

$$\frac{1}{a^2+b^2+k}+\frac{1}{b^2+c^2+k}+\frac{1}{c^2+a^2+k}\leq\frac{3}{2+k}$$

for any triangle with side lengths satisfying $a+b+c=3$?

5113. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Let u, v be complex numbers such that $|u-v|\geq\max(|u|,|v|)$ and let w be another complex number. Show that

$$|u+v+w|\leq|u-v+w|+|v-u+w|.$$

5114. *Proposed by Michael Friday, modified by the Editorial Board.*

Let ABC be a triangle in which $B-A=90^\circ$. If D and E are the feet of the altitudes from A and B, prove that the midpoint of DE is the foot of the symmedian from vertex C to the side AB.

5115. *Proposed by Tran Nhat Quang.*

Let n be a positive integer. Find all pairs of functions $f, g : (0,+\infty)\to(0,+\infty)$ such that:

$$f\big(g(x)+f(x)f(y)\big)=x^n+yf(x)$$

for al $x, y>0$.

5116. *Proposed by Wanlong Han.*

For $x_0 > 0$, we define

$$x_{n+1} = \sqrt{\frac{\sum_{k=0}^{n}(\arctan x_k)^2}{n+1}}.$$

for any nonnegative integer n. Find $\lim_{n\to\infty} x_n\sqrt{\ln n}$.

5117. *Proposed by Nguyen Minh Ha.*

Let $OXYZ$ be a tetrahedron and let k be a given positive constant. Points M, N, P lie respectively on the edges YZ, ZX, XY. Points A, B, C move respectively on the segments OX, OY, OZ such that

$$V[AXNP] + V[BYPM] + V[CZMN] = k,$$

where V represents volume. Prove that the center of the circumscribed sphere of the tetrahedron $OABC$ moves on a fixed plane.

5118. *Proposed by Tatsunori Irie.*

Let a, b, c, d be positive real numbers. Prove that

$$\left(\frac{a+b+c+d}{4}\right)^{\frac{a+b+c+d}{4}} \geq (a^b b^c c^d d^a)^{\frac{1}{4}}.$$

5119. *Proposed by Nazar Kirgizbaev, modified by the Editorial Board.*

For a triangle ABC with circumcenter O and a point X on the circumcircle, let D, E, and F be the points where the cevians AX, BX, and CX meet the opposite sides (or their extensions). Prove that if the circles with centers D, E, F and radii DX, EX, FX have a common tangent line, then that line passes through O.

5120. *Proposed by Michel Bataille.*

Let m, n be integers such that $0 \leq m < n$. Prove that

$$\left(\sum_{k=0}^{m}\binom{m}{k}\frac{(-2)^k}{n-k}\right)\left(\sum_{k=0}^{m}\binom{n}{k}(-2)^k\right) = \left(\sum_{k=0}^{m}\binom{m}{k}\frac{1}{n-k}\right)\left(\sum_{k=0}^{m}\binom{n}{k}\right).$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2026**.

5111. *Soumis par Michel Bataille.*

Soit ABC un triangle acutangle et soit Γ son cercle circonscrit. On note B' et C' les pieds des hauteurs issues de B et de C. Soient P et Q deux points distincts de l'arc BC de Γ ne contenant pas A, et soient M et N les points d'intersection de AP et AQ avec BB' et CC', respectivement. Montrez que M, N, P, Q sont concycliques si et seulement si MN est parallèle à $B'C'$.

5112. *Soumis par Vasile Cîrtoaje.*

Quelle est la plus petite valeur positive de k telle que

$$\frac{1}{a^2+b^2+k}+\frac{1}{b^2+c^2+k}+\frac{1}{c^2+a^2+k} \leq \frac{3}{2+k}$$

pour toutes les triangle avec les longueurs de côtés vérifiant $a+b+c=3$?

5113. *Soumis par Mihaela Berindeanu, modifié par le comité de rédaction.*

Soient u et v deux nombres complexes tels que

$$|u-v| \geq \max\big(|u|,|v|\big),$$

et soit w un autre nombre complexe. Montrez que

$$|u+v+w| \leq |u-v+w| + |v-u+w|.$$

5114. *Soumis par Michael Friday, modifié par le comité de rédaction.*

Soit ABC un triangle tel que $B-A=90°$. Si D et E sont les pieds des hauteurs issues de A et de B, montrez que le milieu de DE est le pied de la symédiane issue du sommet C sur le côté AB.

5115. *Soumis par Tran Nhat Quang.*

Soit n un entier strictement positif. Déterminez toutes les paires de fonctions $f, g : (0,+\infty) \to (0,+\infty)$ telles que

$$f\big(g(x)+f(x)f(y)\big) = x^n + yf(x)$$

pour tous $x, y > 0$.

5116. *Soumis par Wanlong Han.*

Pour $x_0 > 0$, on définit

$$x_{n+1} = \sqrt{\frac{\sum_{k=0}^{n}(\arctan x_k)^2}{n+1}}.$$

pour tout entier non négatif n. Trouvez $\lim_{n\to\infty} x_n\sqrt{\ln n}$.

5117. *Soumis par Nguyen Minh Ha.*

Soit $OXYZ$ un tétraèdre et soit k une constante strictement positive donnée. Les points M, N et P appartiennent respectivement aux arêtes YZ, ZX et XY. Les points A, B et C se déplacent respectivement sur les segments OX, OY et OZ de telle sorte que

$$V[AXNP] + V[BYPM] + V[CZMN] = k,$$

où V représente le volume. Montrez que le centre de la sphère circonscrite au tétraèdre $OABC$ se déplace sur un plan fixe.

5118. *Soumis par Tatsunori Irie.*

Soient a, b, c et d quatre nombres réels strictement positifs. Montrez que

$$\left(\frac{a+b+c+d}{4}\right)^{\frac{a+b+c+d}{4}} \geq (a^b b^c c^d d^a)^{\frac{1}{4}}.$$

5119. *Soumis par Nazar Kirgizbaev, modifié par le comité de rédaction.*

Soit ABC un triangle dont le centre du cercle circonscrit est noté O et soit X un point du cercle circonscrit. On note D, E et F les points où les céviennes AX, BX et CX rencontrent les côtés opposés (ou leurs prolongements). Montrez que, si les cercles de centres D, E, F et de rayons DX, EX, FX admettent une droite tangente commune, alors cette droite passe par O.

5120. *Soumis par Michel Bataille.*

Soient m et n deux entiers tels que $0 \leq m < n$. Montrez que

$$\left(\sum_{k=0}^{m}\binom{m}{k}\frac{(-2)^k}{n-k}\right)\left(\sum_{k=0}^{m}\binom{n}{k}(-2)^k\right) = \left(\sum_{k=0}^{m}\binom{m}{k}\frac{1}{n-k}\right)\left(\sum_{k=0}^{m}\binom{n}{k}\right).$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2025: 51(7), p. 342–346.

5061. *Proposed by Nguyen Van Huyen.*

Consider the polynomial $f(x) = x^4 - ax^3 + 6x^2 - bx + c$. Suppose that $f(x)$ has four distinct real roots. Prove that

$$a^3b + 4b^2 + 256 \leqslant 12a(a + 2b).$$

We received 17 *solutions to the problem. We present the one by Michal Adamaszek, lightly edited.*

If f has four distinct real roots then it has an extremum between each two consecutive roots, i.e. the derivative

$$f'(x) = 4x^3 - 3ax^2 + 12x - b$$

has three real roots. If a polynomial

$$g(x) = Ax^3 + Bx^2 + Cx + D$$

has three real roots then its discriminant $\Delta(g) \geq 0$, where

$$\Delta(g) = B^2C^2 + 18ABCD - 27A^2D^2 - 4AC^3 - 4B^3D \geq 0.$$

Thus we have

$$\begin{aligned} 0 &\leq \Delta(f') \\ &= (-3a)^2 \cdot 12^2 + 18 \cdot 4 \cdot (-3a) \cdot 12 \cdot (-b) - 27 \cdot 4^2 \cdot (-b)^2 \\ &\qquad - 4 \cdot 4 \cdot 12^3 - 4 \cdot (-3a)^3 \cdot (-b) \\ &= 108(12a^2 + 24ab - 4b^2 - 256 - a^3b) \end{aligned}$$

Dividing by 108 and rearranging yields

$$a^3b + 4b^2 + 256 \leq 12a(a + 2b)$$

as required.

Editor's Comments. As noted by several readers the inequality in the question is in fact strict since $\Delta(f') > 0$, as f' has three distinct real roots.

5062. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with $AC > BC$. The midpoint of AB is M, the orthocenter of ΔABC is H, and the feet of the altitudes from A, B, C are D, E, F, respectively. Let X be the point of intersection of AB and ED. If O is the circumcenter of ΔCMX, then prove that

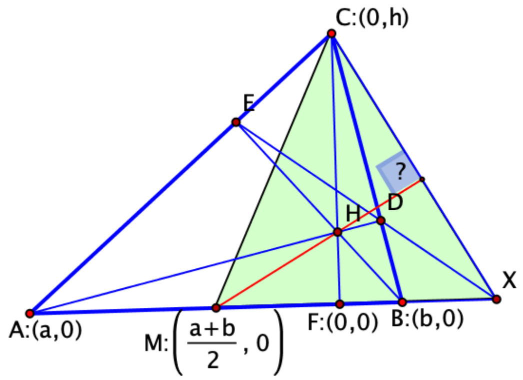
$$\overrightarrow{OH} = \frac{2(\overrightarrow{OC} + \overrightarrow{OX}) + \overrightarrow{OA} + \overrightarrow{OB}}{2}.$$

All of the 11 submissions that we received were correct; we feature the solution by Francisco Javier García Capitán.

Since $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$, the proposed formula can be written as

$$\overrightarrow{OH} = \overrightarrow{OC} + \overrightarrow{OX} + \overrightarrow{OM}.$$

Because O is the circumcenter of CXM, it follows that the problem reduces to proving that H is not only the orthocenter of ABC, but also the orthocenter of CXM. For this, it is sufficient to prove that $MH \perp CX$ (because we already have $CH \perp XM$).



Taking as origin the foot $F = (0, 0)$ of the C-altitude, we can consider the coordinates

$$C = (0, h), \quad A = (a, 0), \quad B = (b, 0), \quad M = \left(\frac{a+b}{2}, \ 0\right).$$

Since X is the harmonic conjugate of F with respect to A and B, we have

$$X = \left(\frac{2ab}{a+b}, 0\right).$$

We can easily find

$$H = \left(0, -\frac{ab}{h}\right).$$

The slopes of MH and CX are, respectively,

$$\frac{ab/h}{(a+b)/2} = \frac{2ab}{h(a+b)} \quad \text{and} \quad -\frac{h(a+b)}{2ab},$$

so that $MH \perp CX$, as desired.

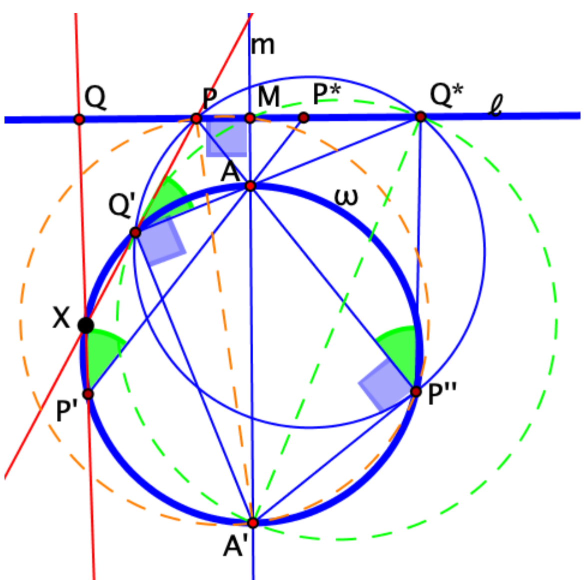
Editor's comments. Theo Koupelis used Brocard's theorem to prove that H is the orthocenter of CXM: Because of the right angles at D and E, $ABDE$ is a quadrilateral inscribed in the circle whose center is M (the midpoint of the circle's diameter AB); furthermore, HXC is its diagonal triangle. Brocard's theorem states that the points M, H, X, C form an orthocentric system; that is, each of the four points is the orthocenter of the triangle formed by the other three.

5063. *Proposed by Bing Jian.*

Given a circle ω with center O and a line ℓ not tangent to ω, let m be the line passing through O and perpendicular to ℓ, and denote by A one of the points where m intersects the circle. For points P and Q on ℓ, let P^* and Q^* be their respective reflections in the mirror m, and let P' and Q' be the second points of intersection of the lines P^*A and Q^*A with ω. Prove that the cross-joins PQ' and QP' intersect on ω.

All of the 11 submissions were correct, and we will sample four of them.

Solution 1, by Antoine Mhanna.



Denote by X the intersection of the cross-joins PQ', QP', while M is the intersection of ℓ and m, and P'' the reflection of P' in the mirror m. Our goal is to prove that $X \in \omega$.

Because of the right angles at M and P'', M and P'' are on the circle whose diameter is $A'P$; similarly Q^*, M, A', Q' are concyclic. It follows that

$$AP \times AP'' = AM \times AA' = AQ^* \times AQ'.$$

Consequently P, P'', Q^*, Q' are concyclic. Let us use directed angles so that the inscribed angles subtended by the arc Q^*P are equal (the third equality in the following display), and by the symmetry of P'' and P' (the second equality) we get

$$\angle AP'X = \angle P^*P'Q = \angle Q^*P''P = \angle Q^*Q'P = \angle AQ'X,$$

which implies that X is on the circle $P'AQ'$, as desired.

Solution 2, by Oliver Geupel.

We consider the problem in the plane of complex numbers where the affixes of points $A, P, Q, \ldots$ are denoted by the respective lower-case $a, p, q, \ldots$. Suppose that ω is the unit circle, m is the real axis, and $a = 1$. Note that $p^* = \bar{p}$ and $q^* = \bar{q}$. It is a basic fact that a point Z belongs to a chord XY of the unit circle if and only if $z + xy\bar{z} = x + y$. Since P^* belongs to the chord AP', we have

$$\bar{p} + p'p = 1 + p',$$

so that

$$p' = \frac{1-\bar{p}}{p-1}.$$

Similarly,

$$q' = \frac{1-\bar{q}}{q-1}.$$

Let S and T be the respective second points of intersection of the lines PQ' and QP' with ω. Since P belongs to the chord $Q'S$, we have

$$p + q's\bar{p} = q' + s.$$

Hence,

$$s = \frac{q'-p}{\bar{p}q'-1} = \frac{\frac{1-\bar{q}}{q-1} - p}{\bar{p}\frac{1-\bar{q}}{q-1} - 1} = \frac{p - \bar{q} + 1 - pq}{\bar{p} - q + 1 - \bar{p}\bar{q}}$$

Analogously,

$$t = \frac{q - \bar{p} + 1 - pq}{\bar{q} - p + 1 - \bar{p}\bar{q}}.$$

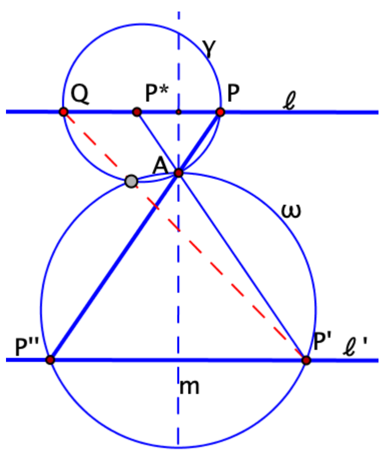
Since P and Q lie on a line parallel to the imaginary axis, we have $p + \bar{p} = q + \bar{q}$, so that $p - \bar{q} = q - \bar{p}$ and $\bar{p} - q = \bar{q} - p$. Thus, $s = t$; that is, $S = T$. Consequently, the lines PQ' and QP' meet at the point $S = T$ on ω.

Solution 3 by Michal Adamaszek for readers familiar with Miquel's theorem; the Wikipedia article on the theorem provides all the background that is required.

Let γ be the circumcircle of APQ. It is possible for γ to be tangent to ω; that happens only if P, Q are symmetric with respect to m, in which case $P^* = Q$, $Q^* = P$ and the lines PQ', $P'Q$ both pass through A.

Otherwise $P'Q$ and PQ' do not pass through A. We aim to show that $P'Q$, γ and ω have a common point. Since the roles of P and Q in this claim are symmetric, the same will be true for PQ', which completes the proof since that common point will in both cases be the intersection point of γ and ω other than A.

Let ℓ' be the line through P' parallel to ℓ. Let $P'' = \ell' \cap \omega$, which is the symmetric image of P' with respect to m. The points P'', P, A are colinear.

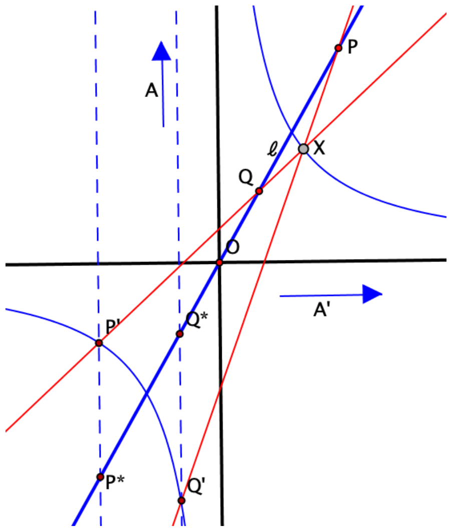


Consider the "triangle" with vertices P'', P, and the point ∞ at infinity whose sides are $P''P$ and the parallel lines ℓ and ℓ'. The point P' lies on the line ℓ', the point A lies on the line $P''P$, and the point Q lies on the line ℓ. By Miquel's theorem the circumcircle of $P''PA$ (namely ω), the circumcircle of PAQ (namely γ), and the circumcircle of $\infty P'Q$ (that is the line $P'Q$) intersect in a single point, as claimed.

Solution 4 provides a generalization by J. Chris Fisher.

The result is, in fact, a theorem of projective geometry: m is the axis of an involution that fixes an arbitrary conic ω and a line ℓ (different from m). Specifically,

> Given an involution $\mathcal{I}$ of the real projective plane having an axis m (of fixed points) and a center $O \notin m$ (of fixed lines), let ω be a conic that intersects m in two points, A, A' and is fixed by $\mathcal{I}$. Let P, Q be any two points different from O on one of the fixed lines different from OA, and denote by P^*, Q^* their images under $\mathcal{I}$. Define P' and Q' to be the second points of intersection of the lines P^*A and Q^*A with ω. Then the cross-joins PQ' and QP' intersect on ω.



The generalized version of the theorem can always be reduced to the special case of problem 5063 because a projective collineation is defined by its effect on four points, no three collinear: the collineation that takes the points A, A', O, X to the corresponding points of the original problem necessarily takes the conic ω to a circle and $\mathcal{I}$ to a reflection. In other words, to prove the general theorem it suffices to prove the special case. Alternatively, if we take m to be the line at infinity of the Euclidean plane, O to be the origin, A to be the point at infinity of the y-axis, and ω to be the hyperbola $xy = 1$, then $\mathcal{I}$ is the reflection in the origin, taking (x, y) to $(-x, -y)$. We let ℓ be the line $y = rx,\ r \neq 0$, and take $P = (p, rp), Q = (q, rq)$ (different from $O = (0, 0)$), whence

$$P^* = (-p, -rp),\ Q^* = (-q, -rq),\ P' = \left(-p, -\frac{1}{p}\right),\ Q' = \left(-q, -\frac{1}{q}\right).$$

The equations of PQ' and $P'Q$ are respectively

$$q(p+q)y = (pqr+1)x + (pq^2r - p)$$

and

$$p(p+q)y = (pqr+1)x + (p^2qr - q).$$

These lines intersect in the point

$$\left(\frac{p+q}{pqr+1}, \frac{pqr+1}{p+q}\right),$$

which satisfies $xy = 1$ and, therefore, lies on ω as claimed.

5064. *Proposed by Michel Bataille.*

Let the sequence $(a_n)_{n\geq 1}$ be defined by $a_1 = 0$ and $a_{n+1} = a_n + \ln(2^n e^{a_n} - 1)$ for all $n \geq 1$. Evaluate

$$\ell = \lim_{n\to\infty} \frac{a_n}{2^n} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{a_n - 2^n \ell}{n}.$$

We received 7 solutions, only 3 of which were completely correct. We present the solution by Michał Adamaszek.

Let $z_n := 2^{n+1}e^{a_n} - 1$. Then

$$\begin{aligned} z_{n+1} &= 2^{n+2}e^{a_{n+1}} - 1 \\ &= 2^{n+2}e^{a_n + \ln(2^n e^{a_n} - 1)} - 1 \\ &= 2^{n+2}e^{a_n}(2^n e^{a_n} - 1) - 1 \\ &= 2^{2n+2}e^{2a_n} - 2^{n+2}e^{a_n} - 1 \\ &= (2^{n+1}e^{a_n})^2 - 2 \cdot 2^{n+1}e^{a_n} + 1 - 2 \\ &= (2^{n+1}e^{a_n} - 1)^2 - 2 \\ &= z_n^2 - 2. \end{aligned}$$

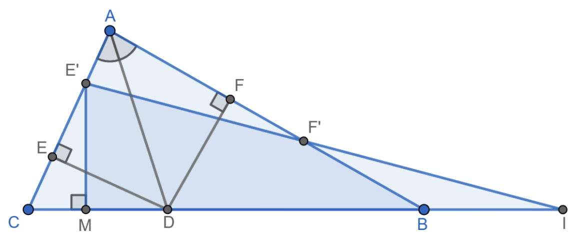
This recurrence is satisfied by $z_n = x^{2^n} + x^{-2^n}$ for any x, and by comparing with $z_1 = 2^2 e^0 - 1 = 3$ we get $x = \varphi = \frac{1+\sqrt{5}}{2}$. Therefore

$$a_n = \ln \frac{1 + \varphi^{2^n} + \varphi^{-2^n}}{2^{n+1}} = \ln \frac{\varphi^{2^n}(1 + \varphi^{-2^n} + \varphi^{-4^n})}{2^{n+1}} = 2^n \ln \varphi - (n+1)\ln 2 + o(1)$$

so the limits in question are $\ln \varphi$ and $-\ln 2$.

5065. *Proposed by Yagub Aliyev.*

Let ABC be a triangle with acute angles at the vertices B and C, such that $\angle B < \angle C$. The angle bisector AD of the triangle ABC is drawn. Let DE and DF be perpendiculars to the sides AC and AB, respectively. Let E' and F' be points on the sides AC and AB, respectively, such that $AE = CE'$ and $AF = BF'$. Let $E'M$ be perpendicular to the side BC. Prove that $E'M + E'I > AB + AC$.



We received 5 submissions, none of which were complete and correct. All the submissions skipped heavily computational steps, making them difficult to verify. We present an overview of the submission by the proposer.

Denote the sides of $\triangle ABC$ by a, b and c and the angles by α, β and γ in the usual way. We start by calculating some of the relevant lengths.

By the angle bisector theorem, $CD = \dfrac{ab}{b+c}$ and $BD = \dfrac{ac}{b+c}$. From $\triangle CED$,

$$CE = CD \cdot \cos(\gamma) = \frac{ab}{b+c} \cdot \frac{a^2+b^2-c^2}{2ab} = \frac{a^2+b^2-c^2}{2(b+c)},$$

where for the second equality we used the cosine law in $\triangle ABC$ to calculate $\cos(\gamma)$ in terms of a, b and c. Similarly, from $\triangle DFB$ we calculate that

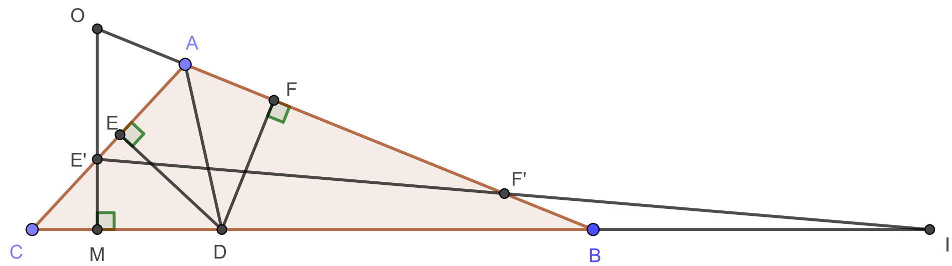
$$BF = \frac{a^2+c^2-b^2}{2(b+c)}.$$

We also have

$$AE = b - CE = b - \frac{a^2+b^2-c^2}{2(b+c)} = \frac{(b+c)^2-a^2}{2(b+c)}.$$

Since AD is the angle bisector, $\triangle AED \cong \triangle AFD$; it follows that $AF = AE$. From the problem setup we know that $AE' = CE$, $AE = CE'$ and also that $BF = AF'$, $BF' = AF$; we will use the above calculations for these lengths as well, as appropriate.

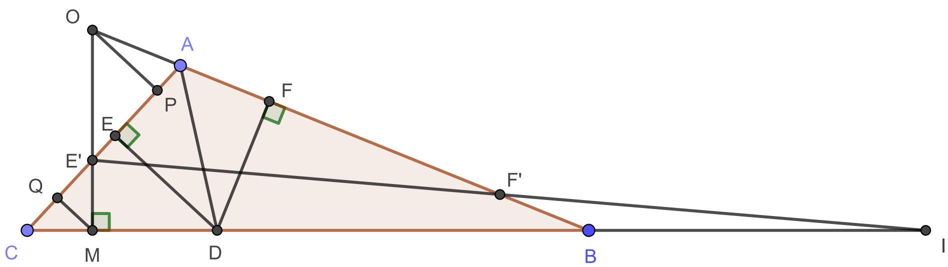
Extend $E'M$ and AB and denote their intersection by O:



The proof then consists of proving the two inequalities:

1. $OM + OB > AB + AC$, and
2. $E'M + E'I > OM + OB$.

First prove that $OM + OB > AB + AC$. Let P and Q, respectively, be the feet of the perpendiculars from O and M to AC:



In the interest of brevity, assume that P is interior to AC (i.e. $\alpha > 90°$, as in the diagram); the proof when it is not proceeds similarly. Note that $OE' > PE'$ and $E'M > QE'$, so we only have left to prove that

$$OA > AP + QC.$$

In $\triangle OAE'$, $\angle OAE' = 180° - \alpha$ and $\angle OE'A = \angle CE'M = 90° - \gamma$, so $\angle AOE' = 90° - \beta$. Applying the sine law we get

$$OA = \frac{AE'}{\sin(90° - \beta)} \sin(90° - \gamma) = \frac{AE'}{\cos(\beta)} \cos(\gamma).$$

From $\triangle OPA$ we get

$$AP = OA\cos(180° - \alpha) = -OA\cos(\alpha) = -\frac{AE'}{\cos(\beta)} \cos(\gamma)\cos(\alpha).$$

Finally, from $\triangle QCM$ and $\triangle CE'M$ we get

$$QC = CM\cos(\gamma) = (CE'\cos(\gamma))\cos(\gamma) = AE\cos^2(\gamma).$$

Therefore we need to show that

$$\frac{AE'}{\cos(\beta)} \cos(\gamma) > -\frac{AE'}{\cos(\beta)} \cos(\gamma)\cos(\alpha) + AE\cos^2(\gamma).$$

Since $\angle B$ and $\angle C$ are acute, we have $\cos(\beta) > 0$ and $\cos(\gamma) > 0$ so we can multiply through by $\frac{\cos(\beta)}{\cos(\gamma)}$ without changing the inequality. Use

$$AE' = CE = \frac{a^2 + b^2 - c^2}{2(b+c)} \quad \text{and} \quad AE = \frac{(b+c)^2 - a^2}{2(b+c)}$$

and the formulas for $\cos(\alpha)$, $\cos(\beta)$, $\cos(\gamma)$ obtained by applying the cosine law in $\triangle ABC$ to rewrite the inequality we want to prove in terms of a, b and c only:

$$\frac{a^2+b^2-c^2}{2(b+c)} >$$
$$-\frac{a^2+b^2-c^2}{2(b+c)} \cdot \frac{b^2+c^2-a^2}{2bc} + \frac{(b+c)^2-a^2}{2(b+c)} \cdot \frac{a^2+c^2-b^2}{2ac} \cdot \frac{a^2+b^2-c^2}{2ab};$$

this is equivalent to

$$\frac{a^2+b^2-c^2}{2(b+c)}\left[1+\frac{b^2+c^2-a^2}{2bc}\right] > \frac{(b+c)^2-a^2}{2(b+c)}\cdot\frac{a^2+c^2-b^2}{2ac}\cdot\frac{a^2+b^2-c^2}{2ab}$$
$$\iff \frac{a^2+b^2-c^2}{2(b+c)}\cdot\frac{(b+c)^2-a^2}{2bc} > \frac{(b+c)^2-a^2}{2(b+c)}\cdot\frac{a^2+c^2-b^2}{2ac}\cdot\frac{a^2+b^2-c^2}{2ab}.$$

From the triangle inequality, $b+c>a$, so $(b+c)^2-a^2>0$. Divide both sides by $((b+c)^2-a^2)$ and multiply by $4(b+c)bc$ to obtain

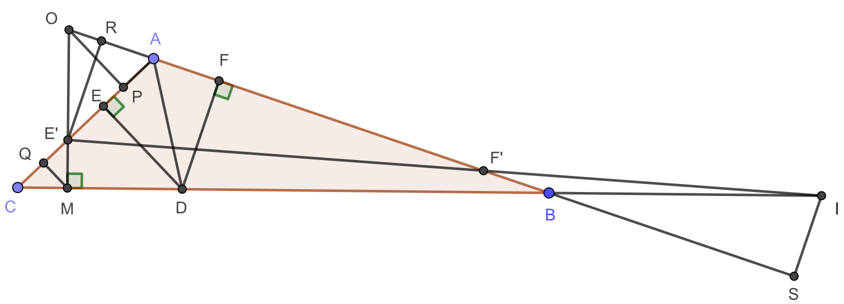
$$(a^2+b^2-c^2) > \frac{(a^2+c^2-b^2)(a^2+b^2-c^2)}{2a^2}.$$

Finally, multiplying through by $2a^2$, multiplying out and moving all the terms to one side, we need to prove that

$$a^4+2a^2b^2-2a^2c^2-2b^2c^2+b^4+c^4>0,$$

which is clearly true because the left hand side is equal to $(a^2+b^2-c^2)^2$ (and not equal to zero because $\triangle ABC$ is not a right triangle).

The proof that $E'M+E'I>OM+OB$ is similar; we start by projecting E' and I onto AB, as in the diagram below. Again, one has to consider cases based on whether the feet of the perpendiculars are interior to AB or not.



Using $E'I>RS$ and cancelling terms which show up on both sides of the inequality, in the particular case shown in the diagram we then have to show that

$$BS>OE'+OR.$$

This part of the proof is left as a (calculation intensive) exercise to the interested reader. It will be helpful to use the theorem of Menelaus to calculate the length of the segment BI: note that $\frac{AE'}{E'C}\cdot\frac{CI}{BI}\cdot\frac{BF'}{F'A}=1$, that $CI=a+BI$, and that we have already calculated formulas for the remaining lengths.

5066. *Proposed by Tatsunori Irie.*

Let n be a positive integer. Initially, n stones – each coloured either white or black – are arranged in a single row. The game is played by repeatedly performing the following operation:

- Randomly select two white stones that are not adjacent (i.e. if two stones appear consecutively, they cannot be selected as a pair).
- Reverse the colour (i.e. switch from white to black or black to white) of every stone located between the two selected white stones.
- Finally, change the colours of the two chosen white stones to black.

The game terminates when no pair of white stones satisfying the above condition (that is, non-adjacent) can be selected.

Prove that, regardless of the initial configuration of the stones and irrespective of the order and combination in which the valid pairs of white stones are chosen, the game always terminates.

We received 5 submissions and they were all complete and correct. We feature the following solution by Michal Adamaszek, Oliver Geupel, and the proposer (independently).

We treat the sequence of stones as an n-digit binary number with white stones representing 1 and black stones representing 0. Note that each operation strictly reduces the value of the number. Indeed, among all the modified bits, the most significant one changes from 1 to 0. Since the number is at most $2^n - 1$, the game always terminates.

5067. *Proposed by Paul Bracken.*

Prove that

$$\sum_{n=1}^{\infty} \frac{4^n}{(2n-1)^2(4n+1)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{16}{9} \cdot \sqrt{2} - \frac{20}{9}.$$

We received 11 submissions, all of which were correct and complete. We present a solution by Ezerskis Liveta that uses a method different from all the others.

Idea

The series is *hypergeometric* because the ratio a_{n+1}/a_n is a rational function of n. Gosper's algorithm looks for a sequence S_n such that $a_n = S_{n+1} - S_n$, so that the sum is telescopic:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (S_{n+1} - S_n) = -\, S_1 + \lim_{n\to\infty} S_n$$

(provided the limit exists).

Computing the ratio $R(n) = \frac{a_{n+1}}{a_n}$

Starting from the definition,

$$a_n = \frac{4^n}{(2n-1)^2(4n+1)} \cdot \frac{\binom{2n}{n}}{\binom{4n}{2n}},$$

we get

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}}{4^n} \cdot \frac{(2n-1)^2(4n+1)}{(2n+1)^2(4n+5)} \cdot \frac{\binom{2n+2}{n+1}}{\binom{2n}{n}} \cdot \frac{\binom{4n}{2n}}{\binom{4n+4}{2n+2}}.$$

We use the standard binomial ratios:

$$\frac{\binom{2n+2}{n+1}}{\binom{2n}{n}} = \frac{(2n+2)(2n+1)}{(n+1)^2},$$

and

$$\frac{\binom{4n}{2n}}{\binom{4n+4}{2n+2}} = \frac{(2n+2)(2n+1)}{(4n+4)(4n+3)(4n+2)(4n+1)}.$$

Substituting and simplifying (many factors cancel), we obtain

$$\frac{a_{n+1}}{a_n} = \frac{4(2n-1)^2}{(4n+5)(4n+3)}.$$

It is convenient to rewrite this as a ratio of polynomials:

$$\frac{a_{n+1}}{a_n} = \frac{16n^2-16n+4}{16n^2+32n+15} = \frac{p(n)}{q(n)},$$

where

$$p(n) = 16n^2 - 16n + 4, \qquad q(n) = 16n^2 + 32n + 15.$$

Setting up Gosper's equation

We want to find S_n such that $a_n = S_{n+1} - S_n$. A natural choice (and exactly the one suggested by Gosper's method) is to look for S_n of the form

$$S_n = q(n-1)\,x(n)\,a_n,$$

where $x(n)$ is an unknown function (typically a polynomial) to be determined.

Why this form?

$$S_{n+1} - S_n = q(n)\,x(n+1)\,a_{n+1} - q(n-1)\,x(n)\,a_n.$$

Since $a_{n+1} = a_n \frac{p(n)}{q(n)}$, we obtain

$$S_{n+1} - S_n = q(n)\,x(n+1)\,a_n \frac{p(n)}{q(n)} - q(n-1)\,x(n)\,a_n = a_n\Big(p(n)x(n+1) - q(n-1)x(n)\Big).$$

Therefore the condition $S_{n+1} - S_n = a_n$ is equivalent to requiring

$$\boxed{p(n)x(n+1) - q(n-1)x(n) = 1.} \qquad (\star)$$

Note also that

$$q(n-1) = 16(n-1)^2 + 32(n-1) + 15 = 16n^2 - 1.$$

Solving $(\star)$ by seeking a linear $x(n)$

Gosper's method often tries $x(n)$ as a low-degree polynomial. Try

$$x(n) = un + v.$$

Then

$$x(n+1) = u(n+1) + v = un + (u+v).$$

Substituting into $(\star)$ with $p(n) = 16n^2 - 16n + 4$ and $q(n-1) = 16n^2 - 1$:

$$(16n^2 - 16n + 4)\big(un + (u+v)\big) - (16n^2 - 1)(un + v) = 1.$$

Expanding and collecting terms, the resulting polynomial simplifies to

$$(-11u - 16v)\, n + (4u + 5v) = 1.$$

For this to hold for every n, we must have

$$\begin{cases} -11u - 16v = 0, \\ 4u + 5v = 1. \end{cases}$$

Solving gives

$$u = \frac{16}{9}, \qquad v = -\frac{11}{9}.$$

Hence

$$x(n) = \frac{16}{9}n - \frac{11}{9}.$$

Explicit telescoping and the summation formula

Define

$$S_n := q(n-1)\, x(n)\, a_n = (16n^2 - 1)\Big(\frac{16}{9}n - \frac{11}{9}\Big)a_n.$$

By construction, using $(\star)$,

$$S_{n+1} - S_n = a_n.$$

Therefore, for every N,

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (S_{n+1} - S_n) = S_{N+1} - S_1.$$

If $\lim_{n\to\infty} S_n$ exists, letting $N \to \infty$ yields

$$\sum_{n=1}^{\infty} a_n = -\,S_1 + \lim_{n\to\infty} S_n.$$

First compute a_1:

$$a_1 = \frac{4^1}{(1)^2(5)} \cdot \frac{\binom{2}{1}}{\binom{4}{2}} = \frac{4}{5} \cdot \frac{2}{6} = \frac{4}{15}.$$

Then

$$S_1 = (16 \cdot 1^2 - 1)\Big(\frac{16}{9} \cdot 1 - \frac{11}{9}\Big) a_1 = 15 \cdot \frac{5}{9} \cdot \frac{4}{15} = \frac{20}{9}.$$

Rewriting S_n in a form suitable for taking limits

Substitute a_n into S_n and simplify in a convenient way:

$$S_n = (16n^2 - 1)\Big(\frac{16}{9}n - \frac{11}{9}\Big) \cdot \frac{4^n}{(2n-1)^2(4n+1)} \cdot \frac{\binom{2n}{n}}{\binom{4n}{2n}}.$$

Note that

$$16n^2 - 1 = (4n-1)(4n+1), \qquad \frac{16}{9}n - \frac{11}{9} = \frac{16}{9}\Big(n - \frac{11}{16}\Big).$$

Thus the factor $(4n+1)$ cancels, and we obtain

$$S_n = \frac{16}{9}\Big(n - \frac{11}{16}\Big)\frac{(4n-1)}{(2n-1)^2} \cdot 4^n \cdot \frac{\binom{2n}{n}}{\binom{4n}{2n}}.$$

Now split $(2n-1)^2$ into two equal factors to isolate ratios with simple limits:

$$S_n = \frac{16}{9}\ \underbrace{\frac{n - \frac{11}{16}}{2n-1}}_{\longrightarrow 1/2}\ \underbrace{\frac{4n-1}{2n-1}}_{\longrightarrow 2}\ \underbrace{\left(4^n \frac{\binom{2n}{n}}{\binom{4n}{2n}}\right)}_{(*)}.$$

The limit of the factor $(*)$

We show that

$$4^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \xrightarrow[n\to\infty]{} \sqrt{2}.$$

Use the classical asymptotic for the central binomial coefficient (from Stirling's formula):

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}.$$

Applying it also to $\binom{4n}{2n} = \binom{2(2n)}{2n}$:

$$\binom{4n}{2n} \sim \frac{4^{2n}}{\sqrt{\pi(2n)}} = \frac{16^n}{\sqrt{2\pi n}}.$$

Hence

$$4^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \sim 4^n \cdot \frac{\frac{4^n}{\sqrt{\pi n}}}{\frac{16^n}{\sqrt{2\pi n}}} = 4^n \cdot \frac{4^n}{16^n} \cdot \sqrt{2} = \sqrt{2}.$$

Therefore the limit of the factor $(*)$ is $\sqrt{2}$.

Therefore, from the boxed form of S_n we obtain

$$\lim_{n\to\infty} S_n = \frac{16}{9} \cdot \frac{1}{2} \cdot 2 \cdot \sqrt{2} = \frac{16}{9}\sqrt{2}.$$

Finally,

$$\sum_{n=1}^{\infty} a_n = -\,S_1 + \lim_{n\to\infty} S_n = -\frac{20}{9} + \frac{16}{9}\sqrt{2} = \frac{16}{9}\sqrt{2} - \frac{20}{9}.$$

This completes the proof.

Editor's Comments/ This problem is very similar to Problem 2208 from the December 2024 issue of Mathematics Magazine. Most solutions followed the same approach, relying on integration and properties of the Gamma function. The featured solution, however, used a distinct method and was well explained.

5068. *Proposed by Nguyen Viet Hung.*

Prove that in any triangle ABC,

$$\frac{4R}{r} \geq \left(\frac{1}{r_a} + \frac{1}{r_b}\right)(\sqrt{r_a} + \sqrt{r_b})^2.$$

When does the equality happen?

We received 16 correct solutions. The following is the solution by Marius Stanean.

The inequality can be written as

$$\frac{abc}{(s-a)(s-b)(s-c)} \geq \left(\frac{s-a}{S} + \frac{s-b}{S}\right)\left(\sqrt{\frac{S}{s-a}} + \sqrt{\frac{S}{s-b}}\right)^2,$$

or

$$ab \geq (s-c)\left(\sqrt{s-a} + \sqrt{s-b}\right)^2,$$

where s and S are the semiperimeter and area of triangle ABC, respectively.

By the Cauchy-Schwarz Inequality, we have

$$ab = (s-b+s-c)(s-c+s-a) \geq \left(\sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)}\right)^2$$
$$= (s-c)\left(\sqrt{s-a} + \sqrt{s-b}\right)^2,$$

as desired. The equality holds when $(s-c)^2 = (s-a)(s-b)$, which means $c = \frac{a^2+b^2}{a+b}$.

5069. *Proposed by Michael Friday.*

Let ABC be a triangle in which $B - A = 90^\circ$. Let M and S be the feet of the median and symmedian, respectively, from vertex C. Prove that triangles ABC and CMS have the same orthocenter, and the circumcircle of CMS is internally tangent to the circumcircle of ABC at C.

We received 12 solutions of which all but one were correct and complete. We present a short synthetic solution by Michał Adamaszek, slightly edited.

Let us call a triangle XYZ *good* if $\angle Y = \angle X + 90^\circ$.

Lemma. If XYZ is a good triangle with orthocenter H and circumcenter O then

(1) H is the reflection of Z in the line XY;

(2) $ZO||XY$.

Proof. By simple angle chasing, $\angle ZHY = \angle X = \angle YZH$, which proves the first claim, and

$$\angle OZX = \frac{1}{2}(180^\circ - \angle ZOX) = \frac{1}{2}(180^\circ - (360^\circ - 2\angle Y)) = \angle X,$$

which proves the second claim.

Solution. Since $CA > CB$, the points B, S, M, A lie on BA in that order. Let $\angle A = \alpha$ and $\angle ACM = \angle SCB = \beta$. Then we compute

$$\angle CMS = \angle A + \angle ACM = \alpha + \beta,$$
$$\angle CSM = \angle B + \angle SCB = 90^\circ + \alpha + \beta.$$

It follows that the triangle MSC is good. By p. (1) of the lemma, the orthocenters of MSC and ABC coincide, and by p. (2), the circumcenters of ABC and MSC both lie on the line through C parallel to AB, from which the second claim follows.

Editor's Comments. Chikara Tsugawa pointed out that M and S could be any points on the line AB such that $\measuredangle ACM = \measuredangle SCB$ (directed angles). The solution above does not also use that M is the midpoint of AB.

5070. *Proposed by Vasile Cîrtoaje.*

Prove that 2 is the largest positive value of the constant k such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}-4\geq k(a+b+c+d-4)$$

for any positive real numbers a,b,c,d with at most one of them less than 1 and $ab+bc+cd+da=4$.

We received 6 solutions, all correct and complete. We present the solution by the proposer.

For $a\in(0,1), b=d=\dfrac{2}{a+1}>1$, and $c=1$, the constraints are satisfied and the inequality takes the form

$$\frac{1}{a}+\frac{2}{b}-3\geq k(a+2b-3),\qquad \frac{1}{a}+a-2\geq k\left(a+\frac{4}{a+1}-3\right),$$

$$\frac{(a-1)^2}{a}\geq\frac{k(a-1)^2}{a+1},\qquad \frac{1}{a}\geq\frac{k}{a+1}.$$

For $a\to 1$ we get $k\leq 2$. To prove that 2 is the largest positive value of k, we need to show that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+4\geq 2(a+b+c+d).$$

Assume that $a\leq 1$, $b,c,d\geq 1$ and denote

$$x=\frac{a+c}{2},\qquad y=\frac{b+d}{2}.$$

Since $y\geq 1$, we have $x=\dfrac{1}{y}\leq 1$. In addition, from $(a-1)(c-1)\leq 0$ we get $0<ac\leq 2x-1$. Since

$$\frac{1}{a}+\frac{1}{c}=\frac{2x}{ac}\geq\frac{2x}{2x-1}$$

and

$$\frac{1}{b}+\frac{1}{d}\geq\frac{4}{b+d}=\frac{2}{y}=2x,$$

it suffices to show that

$$\frac{2x}{2x-1}+2x+4\geq 4\left(x+\frac{1}{x}\right),$$

which is equivalent to an obvious inequality

$$(1-x)^3\geq 0.$$

For $k=2$, the equality occurs when $a=b=c=d=1$.