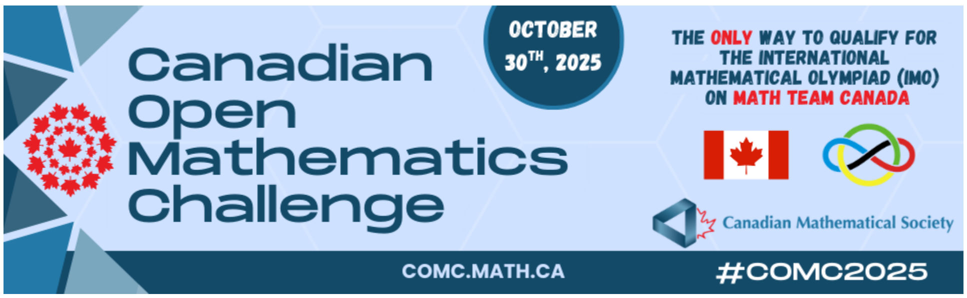
2025 Canadian Open Mathematics Challenge

Official Solutions



A competition of the Canadian Mathematical Society.

The COMC has three sections:

A. Short answer questions worth 4 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

B. Short answer questions worth 6 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

C. Multi-part full solution questions worth 10 marks each. Solutions must be complete and clearly presented to receive full marks.

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Section A

A1 What is 200% of 2% of a third of 2025?

Solution: We want to compute

$$200\% \cdot 2\% \cdot \frac{1}{3} \cdot 2025.$$

We can replace the percentages with fractions, obtaining

$$\frac{200}{100} \cdot \frac{2}{100} \cdot \frac{1}{3} \cdot 2025.$$

The first two terms simplify to $\frac{200}{100} \cdot \frac{2}{100} = \frac{4}{100} = \frac{1}{25}$. We notice that 25 divides 2025; namely, $2025 = 25 \cdot 81$, so our expression becomes

$$\frac{1}{25} \cdot \frac{1}{3} \cdot 81 \cdot 25 = \frac{81}{3} = 27.$$

Answer: $\boxed{27}$

A2 How many integers between 10 and 500, inclusive, have their digits in strictly decreasing order? For example, 41 and 320 are such integers, but 441 and 230 are not.

Solution 1: We first count the integers with two digits, then the integers with three digits, and then add these counts to obtain our final count.

If our integer has two digits, then there are $\binom{10}{2} = \frac{10 \cdot 9}{2!} = 45$ ways to choose the two digits to put in descending order. If our integer has three digits, then we note that we can only use the digits $0, 1, 2, 3,$ and 4, since our number must not exceed 500. There are $\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3!} = 10$ ways to choose the three digits that we put in decreasing order. The total count is 55.

Solution 2: We do a similar style of casework as above, splitting into cases based on whether our integer has two or three digits.

In the case that our number has two digits, let's assume the tens digit is k. Then, the units digit could be any of $0, 1, \ldots,$ up to $k-1$, for a total of k possibilities. For example, if the tens digit is 5, then the five possibilities for the units digit are 0, 1, 2, 3, and 4.

Since k can be any digit between 1 and 9, inclusive, we see that the total number of possibilities with two digits is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$.

In the case that our number has three digits, let's assume that the middle digit is k. Then, the units digit could be any of $0, 1, \ldots, k-1$, for a total of k possibilities, and the hundreds digit could be any of $k+1, k+2, \ldots, 4$, for a total of $4-k$ possibilities. k can be either 1, 2, or 3, so there are a total of $1 \cdot (4-1) + 2 \cdot (4-2) + 3 \cdot (4-3) = 3 + 4 + 3 = 10$ possibilities. As in Solution 1, these add up to 55.

Solution 3: List explicitly and systematically all the numbers.

The two digits numbers are:

10,

20, 21,

30, 31, 32,

40, 41, 42, 43,

etc

90, 91, 92, 93, 94, 95, 96, 97, 98.

So we have $1+2+3+4+5+6+7+8+9 = 45$ such integers that are less than 100.

Then we have additional $1+3+6 = 10$ integers between 100 and 500:

210,

310, 320, 321,

410, 420, 421, 430, 431, 432.

The total is 55.

Answer: $\boxed{55}$.

A3 An integer, $n > 1$, is called *doubly squared* if n is a perfect square and the number of positive divisors of n (including 1 and itself) is a perfect square.

Determine the smallest doubly squared integer greater than 1.

Solution 1: Let $n = k^2$. If k is a prime number, then n has three positive divisors: 1, k, and k^2, so n is not doubly squared. We now test composite numbers k: if $k = 4$, then n has five positive divisors: 1, 2, 4, 8, and 16. The next composite number is $k = 6$, in which case n has nine positive divisors: 1, 2, 3, 4, 6, 9, 12, 18, and 36. Thus, 36 is the smallest doubly squared integer greater than 1.

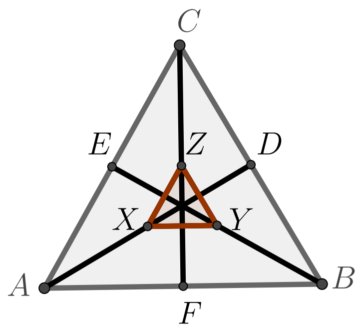
One can summarize this solution in the following table.

square number	divisors	number of divisors
4	1, 2, 4	3
9	1, 3, 9	3
16	1, 2, 4, 8, 16	5
25	1, 5, 25	3
36	1, 2, 3, 4, 6, 9, 12, 18, 36	9

Solution 2: We can be a little bit more systematic in our search. If we let $n = p_1^{2a_1} \cdot p_2^{2a_2} \cdot \cdots \cdot p_k^{2a_k}$, then we know that n has $(2a_1 + 1)(2a_2 + 1) \cdots (2a_k + 1)$ divisors. This is an odd number, so if n is doubly squared, the smallest number this could be is 9. We have that 9 can be represented as a product of odd numbers in two ways: $9 = 2 \cdot 4 + 1$ and $9 = (2 \cdot 1 + 1)(2 \cdot 1 + 1)$, which leads to the prime factorizations $n = p_1^8$ and $n = p_1^2 p_2^2$. In the first case, the smallest possible value for n is $2^8 = 256$, and in the second case, the smallest possible value for n is $2^2 \cdot 3^2 = 36$. The answer is thus 36.

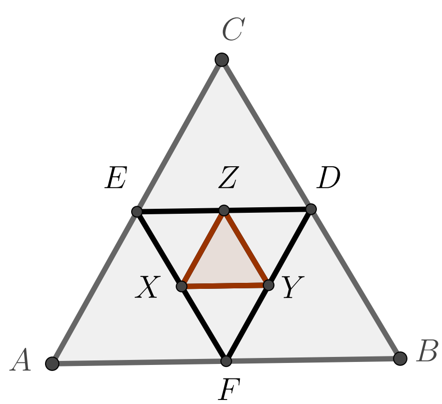
Answer: $\boxed{36}$.

A4 Let ABC be an equilateral triangle with area 80. Let D, E, and F be the midpoints of BC, CA, and AB, respectively. Then, let X, Y, and Z be the midpoints of AD, BE, and CF, respectively. Determine the area of triangle XYZ.



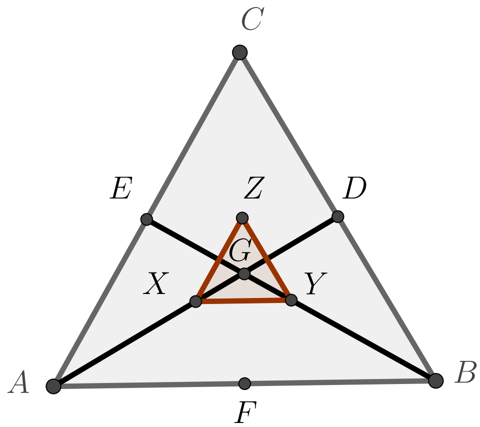
Solution 1: We first claim that X is the midpoint of EF. Note that triangles CED and CAB are similar by SAS similarity, since they share the common angle $\angle ACB$, and $CA/CE = CB/CD = 2$. Consequently, lines ED and AB are parallel. Similarly, lines DF and CA are parallel, so quadrilateral $AEDF$ is a parallelorgram. This implies that its diagonals AD and EF share a midpoint, which is X, as desired.

Now, we see from the similar triangles CED and CAB that $AB = 2 \cdot ED$. Similarly, from similar triangles FXY and FED, we have $ED = 2 \cdot XY$, so we have $AB = 4 \cdot XY$. Similarly, $AC = 4 \cdot XZ$ and $BC = 4 \cdot YZ$, so triangles ABC and XYZ are similar by SSS similarity. The ratio of their side lengths is 4, so the ratio of their areas is 16. Thus, the area of triangle XYZ is $80/16 = 5$.



Solution 2: Let $AD = x$, and let G be the centroid of triangle ABC. Then, we know that $AG = 2x/3$ and $AX = x/2$, so $XG = 2x/3 - x/2 = x/6$. This means that $AG = 4 \cdot XG$. Similarly, $BG = 4 \cdot YG$, so triangles ABG and XYG are similar with similarity ratio 4. Again, this implies that $AB = 4 \cdot XY$, so we can proceed as in

Solution 1.



Solution 3: We can use coordinates. Assume that the side length of triangle ABC is x, and we set $C = (0, \frac{\sqrt{3}}{2}x)$, $A = (-\frac{1}{2}x, 0)$, and $B = (\frac{1}{2}x, 0)$. We may then compute $F = (0,0), D = (\frac{1}{4}x, \frac{\sqrt{3}}{4}x), E = (-\frac{1}{4}x, \frac{\sqrt{3}}{4}x)$, and then $Z = (0, \frac{\sqrt{3}}{4}x)$, $X = (-\frac{1}{8}x, \frac{\sqrt{3}}{8}x)$, and $Y = (\frac{1}{8}x, \frac{\sqrt{3}}{8}x)$. To compute the area of triangle XYZ, we can use the Shoelace formula: it is equal to

$$\frac{1}{2}\left|\left(-\tfrac{1}{8}x\right)\left(\frac{\sqrt{3}}{8}x\right) - \left(\frac{\sqrt{3}}{8}x\right)\left(\tfrac{1}{8}x\right) + \left(\tfrac{1}{8}x\right)\left(\frac{\sqrt{3}}{4}x\right) - \left(\frac{\sqrt{3}}{4}x\right)\left(-\tfrac{1}{8}x\right)\right| = \frac{\sqrt{3}}{64}x^2.$$

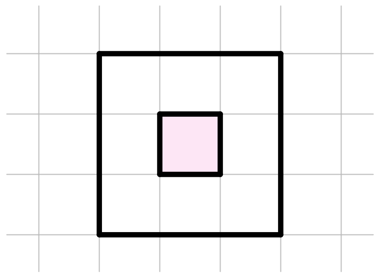
Note that the area of triangle ABC is $\frac{\sqrt{3}}{4}x^2 = 80$, so the area of triangle XYZ is $80/16 = 5$, as desired.

Answer: $\boxed{5}$

Section B

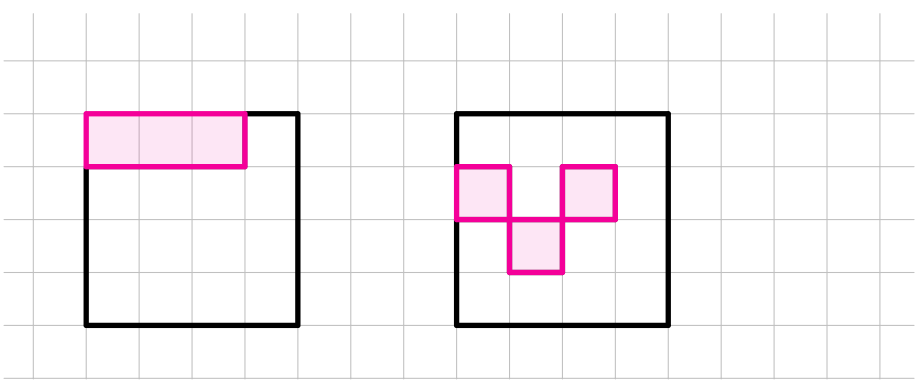
B1 Min and Max each have a 4×4 grid of 16 unit squares. Each of them removes three of the unit squares in their grid, and then computes the perimeter of their resulting shape. What is the maximum possible difference in their answers?

Note: The *perimeter* of a shape is the sum of length of all the line segments that border the shape. For example, the following 3×3 square with the middle 1×1 square missing has the perimeter 16.



Solution: Min cannot reduce the perimeter from the original, since there must be at least one cell in each row/column remaining. So her perimeter is 16. Max can increase the perimeter by 4 every time he removes a square that shares no side with the initial 4×4 configuration, and by 2 every time he removes a square that shares one side with the initial configuration. There are only 4 candidate squares that share no side, and he cannot choose 3 of those 4 that also do not share any sides with each other, so the best he can do is to increase the perimeter by 10.

This can be achieved for example as shown on the following diagram, where the purple squares are removed (Min's figure is on the left and Max's figure is on the right).



Thus, the answer is 10.

Answer: $\boxed{10}$

B2 Elizabeth keeps her photos in three boxes. There is some nonnegative integer k for which the first box contains $\frac{k}{5}$ of the total number of her photos, the second box contains 3/11 of the total number of her photos, and the third box contains 558 photos. How many photos does Elizabeth have in her collection?

Solution 1: Let N represent the number of photos in Elizabeth's collection, then

$$\frac{k}{5}N + \frac{3}{11}N + 558 = N,$$

where $0 < k < 5$ is a positive integer. Then

$$\begin{aligned} 55\left(\frac{k}{5}N + \frac{3}{11}N + 558\right) &= 55N \\ 55 \cdot 558 &= (40 - 11k)N \\ 2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 31 &= (40 - 11k)N \end{aligned}$$

where

k	$40 - 11k$
1	29
2	$18 = 2 \cdot 3^2$
3	7
4	-4

However, $40 - 11k$ must be a divisor of $2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 31 (= 30690)$, which is only possible if $k = 2$ which yields

$$\begin{aligned} (40 - 11 \cdot 2)N &= 2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 31 \\ 18N &= 2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 31 \\ N &= 5 \cdot 11 \cdot 31 = 1705. \end{aligned}$$

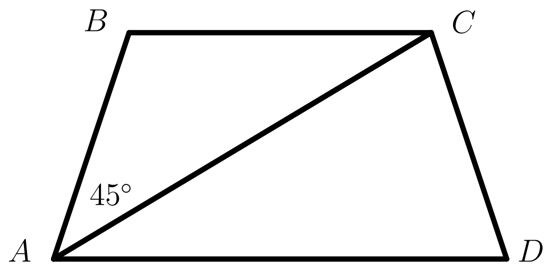
Solution 2: First, we note that the number of photos Elizabeth has must be divisible by 55. Assume that she has $55n$ photos. Then, we know that

$$11kn + 15n + 558 = 55n,$$

so we have that $558 = (40 - 11k)n$; that is, both n and $40 - 11k$ must be factors of 558. Note that the prime factorization of 558 is $558 = 2 \cdot 3^2 \cdot 31$. We plug in $k = 0, 1, 2,$ and 3; then, $40 - 11k = 40, 29, 18,$ and 7. Since 40, 29 and 7 are not factors of 558, we must have $k = 2$, in which case $40 - 11k = 18$ and so $n = 31$. We conclude that she has $55n = 55 \cdot 31 = 1705$ photos.

Answer: $\boxed{1705}$

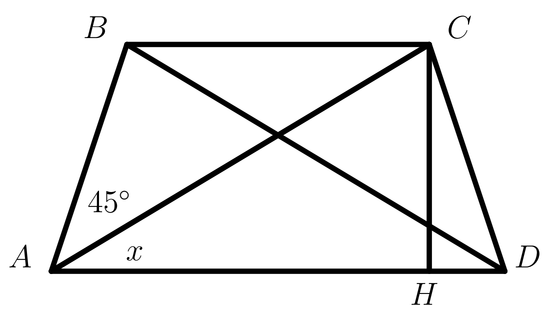
B3 Let $ABCD$ be an isosceles trapezoid with sides $AB = CD$ and parallel sides $AD > BC$. It is given that $AC = AD = 10$ and $\angle BAC = 45°$. Determine the area of the trapezoid.



Solution 1: We let $\angle CAD = x$ and aim to write some of the other angles in the diagram in terms of x. Because the trapezoid is isosceles, $\angle CDA = \angle BAD = 45° + x$. Because triangle ACD is isosceles, $\angle ACD = \angle CDA$ is also $45° + x$. Now, we see that the three angles in triangle ACD must sum to $180°$, so we have $x + 2(x + 45°) = 180°$, whence $x = 30°$.

Let H be the foot of the height from C to AD. Since $x = 30°$, we have that ACH is a 30-60-90 triangle, whence $CH = 5$ and $AH = 5\sqrt{3}$. Thus, $HD = 10-5\sqrt{3}$ units. Then, because the trapezoid is isosceles $BC = AD - 2HD = 10 - 2(10 - 5\sqrt{3}) = 10(\sqrt{3} - 1)$ units.

Finally, the area is $\frac{AD+BC}{2} \times CH = 25\sqrt{3}$.



A variation: If we let the foot of the height from A to BC be I, then we see that triangles ABI and CDH are congruent, so the area of trapezoid $ABCD$ is equal to that of rectangle $ICHA$, which is $25\sqrt{3}$.

Solution 2: Define x as in Solution 1 and find that $x = 30°$. We then know that the smallest angle between lines AC and BD is equal to 60 degrees, so the area of quadrilateral $ABCD$ is

$$\frac{1}{2}AC \cdot BD \cdot \sin 60^\circ = \frac{1}{2} \cdot 10 \cdot 10 \cdot \frac{\sqrt{3}}{2} = 25\sqrt{3}.$$

Answer: $\boxed{25\sqrt{3}}$

B4 Let $a_0, a_1, \ldots, a_{100}$ be fixed, pairwise distinct, and non-zero real numbers ($a_i \neq a_j$, $a_i \neq 0$, for all $0 \leq i \neq j \leq 100$). Consider the polynomial $p(x) = a_{100}x^{100} + a_{99}x^{99} + \cdots + a_1 x + a_0$. A polynomial $q(x)$ is obtained by choosing two distinct numbers among $a_0, a_1, \ldots, a_{100}$ uniformly at random, and swapping them in the expression of $p(x)$. The polynomials $p(x)$ and $q(x)$ are then graphed in the plane as functions of x. Determine the *expected* number of points of intersection in the graph of the two polynomials.

Here each intersection point is counted as one point regardless of its multiplicity.

The *expected value* of a random variable is the weighted average of the possible values the variable takes, weighted by their respective probabilities. For example, the expected number obtained from rolling a fair six-sided die is

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2}.$$

Solution 1: Assume that the coefficients a_i and a_j are swapped in $p(x)$ to form $q(x)$, where $i < j$. The set of points (x, y) where the two polynomials intersect is defined by the solutions of the equation $p(x) = q(x)$. This equation can be written as

$$a_{100}x^{100} + a_{99}x^{99} + \cdots + a_1 x + a_0 =$$
$$a_{100}x^{100} + \cdots + a_{j+1}x^{j+1} + a_i x^j + a_{j-1}x^{j-1} + \cdots + a_{i+1}x^{i+1} + a_j x^i + a_{i-1}x^{i-1} + \cdots + a_0.$$

We see that all terms other than the x^i and x^j cancel, so the equation can be rewritten as

$$a_j x^j + a_i x^i = a_i x^j + a_j x^i.$$

We now factor this expression as

$$(a_j - a_i)(x^{j-i} - 1)x^i = 0.$$

Since $a_i \neq a_j$, the solutions to this equation are

$$\begin{cases} x = 0 & \text{when } i \neq 0, \\ x = 1 & \text{always, and} \\ x = -1 & \text{when } j - i \text{ is even.} \end{cases}$$

Note that there are $\binom{101}{2} = 5050$ pairs of coefficients that could be swapped.

If the only intersection point is at $(1, p(1))$, then we must have that $i = 0$ and that j is odd. There are 50 possibilities for j, so the probability that this happens is $\frac{50}{\binom{101}{2}} = \frac{1}{101}$.

If all of $(-1, p(-1)), (0, p(0))$, and $(1, p(1))$ are intersection points, we must have that $i \neq 0$ and that $j - i$ is even. If i and j are odd, then there are $\binom{50}{2}$ possibilities for choosing i and j. Similarly, if i and j are even, there are also $\binom{50}{2}$ possibilities for choosing i and j. Thus, the probability that there are three intersection points is $\frac{2 \cdot \binom{50}{2}}{\binom{101}{2}} = \frac{49}{101}$.

We then establish that the probability that there are exactly two intersection points is $1 - \frac{1}{101} - \frac{49}{101} = \frac{51}{101}$.

The expected value of the number of intersection points is thus

$$1 \cdot \frac{1}{101} + 2 \cdot \frac{51}{101} + 3 \cdot \frac{49}{101} = \frac{250}{101}.$$

Solution 2: We can instead compute the expected number of points of intersection by linearity of expectation. As in Solution 1, we obtain the classification

$$\begin{cases} x = 0 & \text{when } i \neq 0, \\ x = 1 & \text{always, and} \\ x = -1 & \text{when } j - i \text{ is even.} \end{cases}$$

The expected number of points of intersection is then $P_{-1} + P_0 + P_1$, where P_k is the probability that $(x, y) = (k, p(k))$ is an intersection point of the two curves.

We see that $P_1 = 1$, and $P_0 = \frac{\binom{100}{2}}{\binom{101}{2}} = \frac{99}{101}$, since $x = 0$ is a intersection point whenever neither i and j are 0; there are 100 possibilities for i and j in this case, so there are $\binom{100}{2}$ ways to choose the pair (i, j).

Finally, in order to compute P_{-1}, we again do casework on the parity: the probability that both i and j are odd is $\frac{\binom{50}{2}}{\binom{101}{2}}$, and the probability that both are even is $\frac{\binom{51}{2}}{\binom{101}{2}}$. They add to $\frac{50 \cdot 50}{50 \cdot 101} = \frac{50}{101}$.

The expected value is thus $\frac{50}{101} + 1 + \frac{99}{101} = \frac{250}{101}$.

Answer: $\boxed{\frac{250}{101}}$

Section C

C1 For every integer $n \geq 2$, there is a *number system* with base n, where the integers are written using the digits $0, 1, \ldots, n-1$. For example, in the number system with base 16, also known as the *hexadecimal* system, the digits used are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E$, and F, where the letters A, B, C, D, E, and F represent the numbers $10, 11, 12, 13, 14$, and 15, respectively.

A string of digits $\overline{abc}$ in the system with base n represents the positive integer $an^2 + bn^1 + cn^0$. (We will not write the bar over numbers written in the commonly used system with base 10.) For example, the number 314 in base 10 can be written as $\overline{13A}$ in the hexadecimal system, because $1 \cdot 16^2 + 3 \cdot 16^1 + 10 \cdot 16^0 = 314$. In general, a string of digits $\overline{a_k a_{k-1} \ldots a_0}$ in base n represents the positive integer $a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n^1 + a_0 n^0$.

For this problem, you may assume that every positive integer can be represented uniquely in base n for each $n \geq 2$.

(a) Write the base 10 number 2025 in base 5.

(b) Solve the equation $x^2 - \overline{20}x + \overline{AF} = 0$ written in the hexadecimal system. The answer should be given in the hexadecimal system.

(c) In what systems with base n is it true that $\overline{24}$ divides $\overline{2000}$?

Solutions :

(C1-a) **Solution 1:**

In order to write 2025 in base 5, we first need to represent 2025 as a sum of powers of 5. We see that $5^4 = 625$ and $5^5 = 3125$. We see that 625 goes into 2025 three times, and $2025 = 625 \cdot 3 + 150$. Similarly, we see that $150 = 125 + 25$. Our full representation is

$$2025 = 3 \cdot 625 + 125 + 25,$$

so we see that $2025 = \overline{31100}_5$; that is, the base 5 representation of 2025 is $\overline{31100}$.

Solution 2:

Employ a method of repeated division and record the remainders.

division	quotient	remainder
$2025 \div 5$	405	0
$405 \div 5$	81	0
$81 \div 5$	16	1
$16 \div 5$	3	1
$3 \div 5$	0	3

Then read the remainders backward to obtain the answer $2025 = \overline{31100}_5$.

(C1-b) **Solution 1:**

We complete the square. We have

$$x^2 - \overline{20}x + \overline{100} = \overline{100} - \overline{AF}.$$

In order to take the square root of the right hand side, we convert everything to base 10 for simplicity. We see that $\overline{100}$ represents 256 and $\overline{AF}$ represents $10 \cdot 16 + 15 = 175$, so the right hand side is equal to $256 - 175 = 81$. We thus have

$$(x - \overline{10})^2 = 81 \quad \text{so} \quad x = \overline{10} \pm 9 = 16 \pm 9, \quad \text{or} \quad x = \overline{19}, \quad \& \quad x = \overline{7}.$$

Solution 2:

$\overline{20}$ (base 16)$= 2 \cdot 16 = 32$. $\overline{AF}$ (base 16)$= A \cdot 16 + F = 10 \cdot 16 + 15 = 175$.

So we need to solve $x^2 - 32x + 175 = 0$, which gives $x = 7$ and $x = 25$.

Converting to the hexadecimal system we obtain $x = 7 = \overline{7}$(base 16) and $x = 25 = 1 \cdot 16 + 9 = \overline{19}$ (base 16).

(C1-c) **Solution 1:**

The requirement is equivalent to $n \geq 5$ (so that $\overline{24}$ and $\overline{2000}$ are defined), and that $2n + 4$ divides $2n^3$. This is equivalent to $n + 2$ dividing n^3. By doing polynomial long division, we see that $n^3 = (n + 2)(n^2 - 2n + 4) - 8$, so we require that $n + 2$ divides -8. The possibilities are $n + 2 = 1, 2, 4, 8$, since $n + 2$ must be positive; the only case where $n \geq 5$ is then $n = 6$.

Solution 2:

The statement becomes $2n+4$ divides $2n^3$ if n is the base. Noting that if $p|(n+2)$ then p does not divide n for all $p > 2$, we must have that the only primes dividing $2n + 4$ must be 2. Setting $n = 2m$, we obtain $2n + 4 = 4(m + 1)$ and $2n^3 = 16m^3$. We see that we need $m + 1$ dividing $4m^3$, then we can have $m = 1$ or 3, so $n = 2$ or 6. Since the system with base 2 uses only digits 0 and 1, the statement "24 divides 2000" makes sense in the system with base 6.

We can verify this by converting the numbers to decimal system. Indeed, $2 \cdot 6 + 4 = 16 = 2^4$ divides $2 \cdot 6^3 = 2^4 \cdot 3^3$.

C2 In mathematics, an anagram of a word is a rearrangement of its letters, including those that result in nonsensical words. A word is always an anagram of itself. For example, the word EAT has six anagrams: AET, ATE, EAT, ETA, TAE, and TEA.

(a) How many anagrams of the word $TENET$ start and end with T?

(b) How many anagrams of the word $YOOHOO$ start and end with O?

(c) Let N be the number of anagrams of the word $ABRACADABRA$ which start and end with different letters. Find, with proof, the largest prime number that divides N.

Solutions

(C2-a) There are only $3 = \frac{3!}{2!}$: $TENET, TEENT, TNEET$.

(C2-b) The repeating letter at the start and end must be O. This leaves the letters $YOOH$ to rearrange in the middle: there are $\frac{4!}{2!} = \binom{4}{2} \times \binom{2}{1} = 12$ ways of doing this.

(C2-c) Label all letters in the word $A_1B_1R_1A_2CA_3DA_4B_2R_2A_5$. There are 11! rearrangements of these specific letters, many of which form the identical words. For example, permuting B_1 and B_2 reads as the same word but is counted as distinct. We divide by the overcounting: there are $\frac{11!}{2!\cdot 2!\cdot 5!}$ distinct anagrams of $ABRACADABRA$. (This is the same as $\binom{11}{5}\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1}$, taken by choosing which of the 11 letters in an anagram gets to be A, which is B, which is R, and so on.)

How many of these arrangements use the same letter at the front and at the back? The repeated letter may be A, B, or R. There are 9 letters that are not first or last in the anagram. There are thus $\frac{9!}{5!2!}$ anagrams of the form $B\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_B$,
$\frac{9!}{5!2!}$ anagrams of the form $R\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_R$,
and $\frac{9!}{3!2!2!}$ anagrams of the form $A\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_A$.

Therefore, there are

$$\begin{aligned}N = \frac{11!}{5!2!2!} - \frac{9!}{5!2!} - \frac{9!}{5!2!} - \frac{9!}{3!2!2!} &= \frac{1}{5!2!2!}(11 \cdot 10 \cdot 9! - 9!(2! + 2! + 5 \cdot 4)) \\ &= \frac{9!}{5! \cdot 2 \cdot 2} \cdot (110 - 24) \\ &= 9 \cdot 2 \cdot 7 \cdot 6 \cdot 86 \\ &= 2^3 \cdot 3^3 \cdot 7 \cdot 43 \\ &= 65016 \text{ anagrams of } ABRACADABRA\end{aligned}$$

that start and end with different letters.

The largest prime that divides N is 43.

C3 Call a positive integer N *shifty* if there exists an integer $d > 1$ such that the product $d \cdot N$ is made by removing some of the first digits from N and shifting them to the end (in the same order). For example, $N = 157894736842105263$ is a shifty number since

$$5 \cdot \mathbf{15}7894736842105263 = 78947368421052631\mathbf{5}.$$

(a) Prove that all shifty numbers are divisible by 3.

(b) Let N be a shifty number. Prove that there are infinitely many shifty numbers divisible by N.

(c) Find, with proof, all shifty numbers N less than 10^{15} such that shifting only the first digit to the end produces a multiple of N not equal to N itself.

Solutions

(C3-a) **Solution 1:**
A shifty number N is of the form

$$N = a \cdot 10^{\alpha} + A$$

where

$$10^{\beta-1} \le a < 10^{\beta}$$
$$10^{\alpha-1} \le A < 10^{\alpha}$$
$$10^{\alpha+\beta-1} \le N < 10^{\alpha+\beta}$$

and $1 < d < 10$ or else $d \cdot N$ would have more digits than the original.

In our example, $N = 157894736842105263 = 15 \cdot 10^{16} + 7894736842105263$, $a = 15$, $A = 7894736842105263$, $\alpha = 16$, $\beta = 2$ and $d = 5$.

Hence,

$$N \cdot 10^{\beta} = A \cdot 10^{\beta} + a \cdot 10^{\alpha+\beta}.$$

The conditions of the problem give

$$d \cdot N = A \cdot 10^{\beta} + a.$$

Therefore, by subtracting the two equations we obtain

$$(10^{\beta} - d) \cdot N = (10^{\alpha+\beta} - 1) \cdot a.$$

As $1 < d < 10$, $9 \nmid (10^\beta - d)$, however it may happen that $3 \mid (10^\beta - d)$ (if $d = 4$ or $d = 7$). Since $9 \mid (10^{\alpha+\beta} - 1)$, we conclude that $3 \mid N$.

Solution 2:
Any integer $A = \sum_{n=0}^{M} d_n 10^n$ is congruent to the sum of its digits $\sum_{n=0}^{M} d_n$ modulo 9 because $10^n \equiv 1 \mod 9$ for all $n \geq 1$.

Since the digits of N and dN are the same, we have $dN \equiv N(\mod 9)$, which implies $(d-1)N \equiv 0(\mod 9)$.

Since $1 < d < 10$, we know that $d - 1$ is not $0(\mod 9)$, so N must be divisible by 3 (at least).

(C3-b) **Solution:**
Suppose $10^{k-1} < N < 10^k$, where N is a shifty number such that

$$d \cdot N = N'$$

satisfies the condition. Then if we let

$$n = N \cdot \sum_{i=1}^{m} 10^{(i-1)k},$$

the number obtained by appending m copies of N, then

$$d \cdot n = N' \cdot \sum_{i=1}^{m} 10^{(i-1)k}$$

the number obtained by appending m copies of N', showing that the same digits that shifted when we did $d \cdot N$, shift when we do $d \cdot n$.

For example, if $d \cdot aA = Aa$ then $d \cdot aAaAaA = AaAaAa$. (Here expression XY represents two numbers X and Y appended side by side.)

This is an infinite family.

(C3-c) **Solution 1:**
In this case we have $1 \leq a < 10$ and $\alpha < 15$.

$$N = a \cdot 10^\alpha + A$$

so

$$d \cdot N = 10 \cdot A + a$$

which leads to

$$(10 - d) \cdot N = (10^{\alpha+1} - 1) \cdot a.$$

Observe that $d \cdot a < 10$.

If $d = 7$ and $a = 1$, we have $3N = (10^{\alpha+1} - 1)$, so $N = 3...3$, which is not a shifty number.

If $d = 4$ and $a = 2$, we have $6N = 2(10^{\alpha+1} - 1)$, so $N = 3...3$ once again.

If $d = 4$ and $a = 1$, we have $6N = (10^{\alpha+1} - 1)$, which has no integer solution by the parity of numbers.

Thus, we can assume that $d \neq 4, 7$ and so $9 \mid N$. We can write $N = 9n$, and our equation can be rewritten as

$$(10 - d)n = \overbrace{111 \cdots 1}^{\alpha+1 \text{ 1s}} \cdot a.$$

Observe that a number of the form

$$\overbrace{111 \cdots 1}^{m \text{ 1s}} \text{ is divisible by } \begin{cases} 3, & \text{for } m = 3, 6, 9, 12, 15, \\ 7, & \text{for } m = 6, 12, \\ 9, & \text{for } m = 9. \end{cases}$$

(We are only interested in $\alpha < 15$, so $m = \alpha + 1 < 16$.)

$$\text{At the same time, } 10 - d = \begin{cases} 3, & \text{for } d = 7, \\ 7, & \text{for } d = 3, \\ 9, & \text{for } d = 1. \end{cases}$$

Since we ruled out $d = 7$ and $d = 1$, we are left with the case $d = 3$ corresponding to $m = 6, 12$ when both sides of the equation can be divided by 7.

For $m = 6$ we have

$$(10 - d) \cdot n = 111\,111 \cdot a.$$

If $d = 3$ and $a = 1$, we have $n = 15\,873$, so $N = 142\,857$.

If $d = 3$ and $a = 2$, we have $n = 31\,746$, so $N = 285\,714$.

If $d = 3$ and $a = 3$, we have $n = 47\,619$, so $N = 428\,571$.

However, the first digit of the last number $N = 428\,571$ is $4 \neq 3$, so this number does not satisfy our requirements.

For $m = 12$ we have

$$(10 - d) \cdot n = 111\,111\,111\,111 \cdot a$$

If $d = 3$ and $a = 1$, we have $n = 15\,873\,015\,873$, so $N = 142\,857\,142\,857$.

If $d = 3$ and $a = 2$, we have $6n = 31\,746\,031\,746$, so $N = 285\,714\,285\,714$.

If $d = 3$ and $a = 3$, we have $n = 47\,619\,047\,619$, so $N = 428\,571\,428\,571$.

However, the first digit of the last number $N = 428\,571\,428\,571$ is $4 \neq 3$, so this number does not satisfy our requirements.
In total we have four numbers in the given range. Two smallest numbers are:

$$N_1 = 142\,857 \text{ with } 3 \cdot 142\,857 = 428\,571,$$
$$N_2 = 285\,714 \text{ with } 3 \cdot 285\,714 = 857\,142.$$

The other two numbers could be alternatively obtained with the result from (b):

$$N_3 = 142\,857\,142\,857,$$
$$N_4 = 285\,714\,285\,714.$$

P.S. Note, the numbers $142\,857,\ 285\,714,$ and $428\,571$ obtained above are cyclic permutations of $142\,857$. They are shifty numbers: all products $d \cdot N$ (as long as $d \cdot N < 10^6 - 1$) shift some digits. For example,
$142\,857 \cdot 2 = 285\,714$, $142\,857 \cdot 3 = 428\,571$, $142\,857 \cdot 4 = 571\,428$,
$142\,857 \cdot 5 = 714\,285$, $142\,857 \cdot 6 = 857\,142$, while $142\,857 \cdot 7 = 999\,999$.

Solution 2:
Obtain $(10 - d)N = (10^{\alpha+1} - 1)a$, as previously.
Observe that we must have $d \geq 3$, since if $d = 2$, then $8|(10^{\alpha+1} - 1)a$, so $a = 8$ and $N = 10^{\alpha+1} - 1$, which is a contradiction.
Now that we know $d \geq 3$, we have that $N < (10^{\alpha+1} - 1)/3$, since $3N$ must have α digits and cannot be $10^{\alpha+1} - 1$, since this would imply that N has all 3s as its digits. Plugging in this inequality, we get $10 - d > 3a$.
This reduces to a finite case check: we have that the possibilities for (d, a) are $(3, 1), (3, 2), (4, 1), (5, 1), (6, 1)$. We can eliminate the last three cases by looking at divisibility by 2 and 5.
For the cases $(d, a) = (3, 1)$ and $(3, 2)$, we require that $10^{\alpha+1} - 1$ is divisible by 7. This happens precisely when 6 divides $\alpha + 1$, so the possibilities are $\alpha = 5$ and 11, since $\alpha \leq 14$ is assumed. We can now solve for N in all of these cases and obtain our desired four answers.

Solution 3:
After obtaining $N = a \cdot 10^\alpha + A$ and $dN = 10A + a$, solve for A to get

$$A = \frac{a}{10 - d}(d \cdot 10^\alpha - 1) = \frac{ad}{10 - d} \cdot \frac{d \cdot 10^\alpha - 1}{d}.$$

Now, since $d > 1$, we have

$$A \leq 10^\alpha - 1 < 10^\alpha - \frac{1}{d} = \frac{d \cdot 10^\alpha - 1}{d}.$$

Therefore, $\frac{ad}{10-d} < 1$, or

$$ad < 10 - d \iff (a+1)d < 10.$$

Since $a \geq 1$ this gives $d \leq 4$.
Case 1: $d = 2$. Then

$$A = \frac{a}{8}(2 \cdot 10^\alpha - 1)$$

and hence $8|a$ which gives $a = 8$. But this contradicts $(a+1)d < 10$.
Case 2: $d = 4$. Then

$$A = \frac{a}{6}(4 \cdot 10^\alpha - 1)$$

and hence $6|a$ which gives $a = 6$. But again this contradicts $(a+1)d < 10$.
Case 3: $d = 3$. Then, $(a+1)d < 10$ gives $a = 1, 2$.

$$A = \frac{a}{7}(3 \cdot 10^\alpha - 1)$$

and hence

$$3 \cdot 10^\alpha = 1 \pmod 7 \implies \alpha = 5 \pmod 6.$$

This gives $\alpha = 5, 11$ and combined with $a = 1, 2$ gives the 4 solutions.

Solution 4:
We write N in the form $N = a10^\alpha + A$, where $a \in \{1, 2, \ldots, 9\}$ and $\alpha < 15$. For N to be a shifty number, we must have

$$dN = 10A + a. \tag{1}$$

We also note that

$$da < 10, \tag{2}$$

as otherwise dN will have more digits than N. Equation (1) implies that if d is even, then a must be even. Also, the case $d = 5$ is impossible (as then a must be divisible by 5, which contradicts (2) and the condition $a \geq 1$). Thus we need to check the following **eight** combinations:

- $d = 2$ and $a \in \{2, 4\}$
- $d = 3$ and $a \in \{1, 2, 3\}$
- $d = 4$ and $a = 2$
- $d \in \{7, 9\}$ and $a = 1$

We will start with the case when $d = 3$ and $a = 1$. The last digit of $3N$ is $a = 1$, thus we obtain the following long multiplication setup

$$\begin{array}{r} ???????w \\ \times 3 \\ \hline ???????w1 \end{array}$$

It is clear that $w = 7$ is the only digit that satisfies $w \times 3 = 1(\mod 10)$, thus the last digit of N must be 7, and we go to the next step of this "backtracking" algorithm:

$$\begin{array}{r} ???????u7 \\ \times 3 \\ -\;-\;-\;-\;- \\ ??????u71 \end{array}$$

Since $7 \times 3 = 21$, $7 - 2 = 5$, and $5 \times 3 = 15$, the next digit must be $u = 5$:

$$\begin{array}{r} ??????v57 \\ \times 3 \\ -\;-\;-\;-\;- \\ ?????v571 \end{array}$$

We continue in the same way: $5 \times 3 = 15$, $5 - 1 = 4$, and $8 \times 3 = 24$, thus the next digit must be $v = 8$:

$$\begin{array}{r} ??????857 \\ \times 3 \\ -\;-\;-\;-\;- \\ ?????8571 \end{array}$$

Repeating this procedure two more times, we arrive at

$$\begin{array}{r} ???142857 \\ \times 3 \\ -\;-\;-\;-\;- \\ ???428571 \end{array}$$

At this stage we found one shifty number that satisfies the conditions of C3(c): 142857. We can stop here, or we can continue with the same backtracking procedure for another 6 steps, when we will arrive at

$$\begin{array}{r} ???142857142857 \\ \times 3 \\ -\;-\;-\;-\;- \\ ???428571428571 \end{array}$$

This gives us another shifty number: 142857142857. Alternatively, we could obtain it using the results of C3(b).

Consider next the case $d = 3$ and $a = 2$. We start with long multiplication in the form

$$\begin{array}{r} ???????w \\ \times 3 \\ \hline ??????w2 \end{array}$$

Thus the next digit is $w = 4$ and we have

$$\begin{array}{r} ??????u4 \\ \times 3 \\ \hline ?????u42 \end{array}$$

The next digit is $u = 1$

$$\begin{array}{r} ??????14 \\ \times 3 \\ \hline ?????142 \end{array}$$

and proceeding in this way we recover two more shifty numbers: 285714 and 285714285714.

We claim that all remaining combinations of a and d do not produce shifty numbers. Consider the case when $d = a = 3$:

$$\begin{array}{r} ???????w \\ \times 3 \\ \hline ??????w3 \end{array}$$

Then the last digit of N must be $w = 1$

$$\begin{array}{r} ??????u1 \\ \times 3 \\ \hline ?????u13 \end{array}$$

The next digit is $u = 7$

$$\begin{array}{r} ??????71 \\ \times 3 \\ \hline ?????713 \end{array}$$

Continuing in this way we would eventually arrive at

$$\begin{array}{r} ??1428571 \\ \times 3 \\ \hline ?14285713 \end{array}$$

Again we arrived at a digit 1, so now the sequence of digits becomes periodic with period 6:

$$\begin{array}{r} ??1428571428571 \\ \times 3 \\ \hline ?14285714285713 \end{array}$$

It is clear wherever we decide to stop in this backtracking algorithm, we will never obtain a shifty number, since the digit 3 does not appear in N (while it appears as the last digit in $3N$).

Next we consider the case $d = 7$ and $a = 1$. We have

$$\begin{array}{r} ????????? \\ \times 7 \\ \hline ????????1 \end{array}$$

which leads to

$$\begin{array}{r} ????????3 \\ \times 7 \\ \hline ???????31 \end{array}$$

which leads to

$$\begin{array}{r} ???????33 \\ \times 7 \\ \hline ??????331 \end{array}$$

and this algorithm can only produce N of the form $333\ldots3$ (the digit 3 repeated k times), whereas $3N$ has the last digit $a = 1$. Thus the combination $d = 7$ and $a = 1$ does not produce a shifty number.

Exactly same situation occurs when $d = 9$ and $a = 1$, when we'll have

$$\begin{array}{r} ??????999 \\ \times 9 \\ \hline ?????9991 \end{array}$$

The case $d = 4$ and $a = 2$ gives the same result

$$\begin{array}{r} ??????333 \\ \times 4 \\ \hline ?????3332 \end{array}$$

When $d = 2$ and $a = 2$ and we will arrive at

$$\begin{array}{r} ???????w1 \\ \times 2 \\ \hline ?????w12 \end{array}$$

Now we can not continue the backtracking algorithm, since $2w \neq 1 \pmod{10}$. No shifty numbers are produced by this combination.

When $d = 2$ and $a = 4$ we arrive at the same situation as in the previous case after one more step:

$$\begin{array}{r} ??????w12 \\ \times 2 \\ \hline ????w124 \end{array}$$

Again, this combination can not produce a shifty number.

Thus the only shifty numbers satisfying conditions of C3(c) are the four numbers that we obtained above: 142857, 142857142857, 285714, and 285714285714.

C4 Veronica plays a game against her 2025 students. First, each of the students points to exactly one other student in the room. Multiple people could point to the same student, and some students could have no one point to them. Veronica sees who pointed to whom.

Second, Veronica chooses N students to leave the room.

Third, Veronica assigns a number to each of the remaining $2025 - N$ students, under the condition that if one student is pointing to another, then they must receive the same number. If she assigns k distinct numbers to the students, then Veronica scores k points.

The students are trying to minimize the number of points Veronica scores, and she is trying to maximize the number of points she scores. Under optimal play from both Veronica and her students, what score does she receive when

(a) $N = 1$?

(b) $N = 100$?

(c) $N = 1000$?

Solutions:

For part (a), we claim that the answer is 1. Veronica can guarantee saying at least one number by default, by giving everyone remaining in the room the same number. Thus, it remains to devise a way that the students can point at each other.

Arrange the students in a circle, and have each student point to the student directly to their left. Then, after Veronica removes any one student, the remaining students form a chain, where each student points to the next, with the last student in the chain pointing to the departed student. Veronica must then assign all of them the same number.

We now give the answer for general N. Under optimal play, Veronica's score is equal to

$$\begin{cases} N & \text{if } N \in [1, 674] \\ 675 & \text{if } N \in [675, 1349] \\ 2025 - N & \text{if } N \in [1350, 2025] \end{cases}$$

Thus, the answer to part (b) is 100, and the answer to part (c) is 675.

In order to show that this is optimal for both Veronica and her students, it will be convenient to rephrase the problem in terms of graph theory. The students will form a graph with 2025 vertices and 2025 edges, with the property that there is a bijection between the vertices and edges such that every edge maps to one of its vertices. Then, Veronica will remove N of the vertices, and her score is the number of connected components of the remaining graph.

Under this terminology, we first present the students' strategy. Label the students $1, 2, \ldots, 2025$.

When we have $N \in [1, 674]$, the students point to each other in a circle, like in part (a), say, with person N pointing at $N+1$. Then, the resulting graph is a cycle. Say Veronica removes students $a_1, a_2, \ldots, a_N$. Then, the remaining graph has at most N connected components, since each of the students $a_i + 1, a_i + 2, \ldots, a_{i+1} - 1$ are in the same component for $i = 1, 2, \ldots, N-1$, and the students $a_N+1, a_N+2, \ldots, a_{2025}, a_1, \ldots, a_i-1$ are also in the same component. Thus, the students can guarantee that Veronica scores at most N points.

When we have $N \in [675, 1349]$, the students form groups of 3 and point to each other in a cycle there. Then, after Veronica removes students, each group must only contribute at most 1 connected component, so Veronica can only score at most 675 points.

When we have $N \geq 1350$, the students can do whatever they want; there will be $2025 - N$ students remaining, so Veronica can score at most $2025 - N$ points.

Now, we present Veronica's strategy. She will apply the greedy algorithm and remove a vertex of maximal degree iteratively, until she has removed N students.

If Veronica ever removes a vertex of degree 0, then the graph she ends with will have no edges, so she will score $2025 - N$. This is greater than or equal to the score we claimed above, so we may assume that Veronica never removes a vertex of degree 0 in this process.

Assume that Veronica removes k vertices of degree at least 2, and $N - k$ vertices of degree 1. Assume that among the first k vertices removed, Veronica removed e edges. Then, we make the following observations:

- $e \geq 2k$ (and so $e - k \geq k$).
- Veronica removes exactly $(N - k) + e$ edges, so the resulting graph has $2025 - N$ vertices and $2025 - N + k - e$ edges. Since a graph with a vertices and $b < a$ edges has at least $a - b$ components, this graph must have at least $e - k$ connected components.
- When $k = N$, then we see that $e - k \geq k = N$.
- If $k \neq N$, then after removing the first k vertices, the resulting graph must consist of single points and pairs of points connected by one edge. In any of such graph, $|V| \geq 2|E|$, where $|V|$ is the number of vertices and $|E|$ is the number of edges. In our case the graph has $2025 - k$ vertices and $2025 - e$ edges, so we must have $2025 - k \geq 2(2025 - e)$, which simplifies to $2e \geq 2025 + k$. Then, since $e \geq 2k$, we see that $3e \geq 2k + (2025 + k)$, so $e - k \geq 675$.
- In either case, Veronica scores at least $\min(N, 675)$ points, which is again greater than or equal to the score we claimed for Veronica above.

Remark: In Veronica's strategy, there is no need for the graph to be a bijection from edges to vertices; Veronica can guarantee her score from any graph with 2025 vertices and 2025 edges.