# 2026 CMO Qualifying Repêchage

## Problems

1. **[10 points]** Do there exist four lines on the same plane, no three concurrent and no two parallel, so that four particles can travel at (possibly different) constant non-zero speeds along these lines and such that at every point in time, the positions of the four particles are concyclic?

2. **[10 points]** Do there exist positive real numbers $a, b, c$, not all equal, such that $a^b = b^c = c^a$?

3. **[10 points]** Let point $P$ be outside circle $\Gamma$. The tangents from $P$ to $\Gamma$ hit $\Gamma$ at $A$ and $B$. A third line through $P$ hits $\Gamma$ at $C$ and $D$, such that $C$ is between $P$ and $D$. Point $Q$ is on chord $CD$ such that $\angle DAQ = \angle PBC$. Prove that $\angle DBQ = \angle PAC$.

4. **[15 points]** Fix a prime $p$ and natural numbers $k \geq n$. Consider choosing $k$ vectors $v_1, \ldots, v_k$ uniformly at random (possibly with repetition) from the set $\{(x_1, \ldots, x_n) | x_i \in \{0, 1, \ldots, p-1\}\}$. Also, let $v_{k+i}$ be the vector with value $p$ in the $i$-th coordinate and 0 elsewhere for $1 \leq i \leq n$.

   These $k+n$ vectors are said to *generate* $\mathbb{Z}^n$ if for any vector of integers $v = (y_1, \ldots, y_n)$, there are some integers $c_1, \ldots, c_{k+n}$ such that $v = \sum_{i=1}^{k+n} c_i v_i$. Let $P(p, n, k)$ be the probability that these $k+n$ vectors generate $\mathbb{Z}^n$.

   Then, there is some constant $c$, depending on $p$ and $n$, such that $p^k(1 - P(p, n, k)) - c$ approaches 0 as $k \to \infty$. Determine $c$ in terms of $p$ and $n$.

5. **[15 points]** Determine whether there exists a surjective function $f : (0, \infty) \to \mathbb{N}$, such that for any positive numbers $a < b$, the image of the open interval $(a, b)$ under $f$ is a set of the form $\{1, 2, \ldots, N\}$ for some finite positive integer $N$.

   Recall that a function $f : (0, \infty) \to \mathbb{N}$ is *surjective*, if for every natural number $M$, there exists some positive real number $r$ such that $f(r) = M$.

6. **[20 points]** A set of positive integers $T$ is given. For a positive integer $n$, Alice and Bob play a game, where there are initially $n$ stones in a pile, they take turns taking $m$ stones from the pile, where $m \in T, m \leq n$, and whoever takes the last stone wins. If at some point the number of stones in the pile is less than $\min T$, then the game is a draw. Call a number $n$ *good* if Alice can guarantee a win, *bad* if Bob can guarantee a win, and *neutral* if neither Alice nor Bob can guarantee a win.

   For example, if $T$ is the set of prime numbers, then $n = 4$ is neutral and $n = 5$ is good; in the first game, Alice can guarantee a draw by taking 3 stones on her first move, and cannot do better, and in the second game, she can take all the stones on her first move to win.

   (a) Prove that if $T$ is finite, then the set of good numbers is eventually periodic: that is, there exists some positive integers $N$ and $p$ such that for all positive integers $n > N$, we have that $n$ is good if and only if $n + p$ is good.

(b) Construct an infinite set $T$ such that the set of good numbers is not eventually periodic: that is, for every pair of positive integers $N$ and $p$, there exists some $n > N$ such that exactly one of $n$ and $n + p$ is good.

*[note]: the word 'eventually' was originally missing from this question part b but we added it and notified students during the exam.*

7. **[20 points]** Determine the least positive integer that is not a perfect square that can be written in the form $\frac{a^2+b^2}{ab+1}$ for some rational numbers $a$ and $b$ such that $ab \neq -1$.