Crux Mathematicorum

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Editorial Board

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IN MEMORIAM

The Canadian Mathematical Society (CMS) mourns the passing of Dr. Robert Woodrow, a dedicated scholar, mentor, and leader whose decades of service helped shape and strengthen the mathematical community of Canada.

Dr. Woodrow was a long standing member of the University of Calgary's Department of Mathematics and Statistics, where he combined his deep expertise with an unwavering commitment to students and colleagues alike. Within the CMS, he served in countless roles over the years, bringing thoughtful leadership and generous spirit.

Dr. Woodrow's most significant contributions were within the CMS Competitions program. He held multiple terms as Chair of the Canadian Open Mathematics Challenge (COMC) Sub-Committee and played a pivotal role in the Mathematical Competitions Committee. He was a true leader for the COMC, helping to shape and strengthen it into what it is today – a flagship competition of the CMS, recognized around the world and inspiring thousands of students across Canada each year.

Beyond competitions, Dr. Woodrow also contributed to editorial work with ***Crux*** and served as Faculty Advisor for the CMS Student Committee for many years, mentoring generations of young mathematicians and helping their voices be heard within the Society. In addition, Dr. Woodrow served on the CMS Board of Directors and contributed to the work of award selection and other committees. His recognition as a CMS Fellow Emeritus in 2018 reflects the high regard in which he was held by peers and the lasting impact of his dedication.

What stands out most about Robert's legacy is not only the breadth of his service, but the spirit with which he offered it – with kindness, patience, and a genuine commitment to community. He gave countless hours of his time and energy to nurturing mathematical talent across Canada and to strengthening the CMS as a whole. His generosity, guidance, and steady support left a lasting impression on the Society and on the many students and colleagues who were fortunate to work alongside him.

The CMS is profoundly grateful for Dr. Woodrow's many contributions and for the example he set as a mathematician, mentor, and friend. His presence will be deeply missed, but his impact will continue to be felt in the generations of students and colleagues he inspired.

Remembering Robert Woodrow, *CMS Notes*, September 2025

IN MEMORIAM

I first joined CMS as an ad-hoc member of the Student Committee. As one of the main organizers of the Canadian Undergraduate Math Conference to be held at SFU in 2007, I came to the CMS Winter meeting in December 2006 to present conference plans, discuss progress and hopefully get answers to some of our questions. For the meeting that was scheduled to span most of the day, I walked into the room of peers and Robert. In retrospect, what struck me most was the juxtaposition of Robert's undivided attention to Student Committee discussions and his ability to let us work through issues without intervention. First of all, he was fully listening and observing – no phone, computer or a math paper in sight. And secondly, he didn't interject unless prompted or absolutely necessary: he would prevent disasters, but would otherwise let us make our novice mistakes, teaching us the invaluable skill of knowing when to ask for guidance. You felt both supported and capable. He was the perfect safety net that you didn't know was there until it saved you. A busy faculty member with a bazillion tasks, he managed the patience and the commitment as we were going through the growing pains of building up our presence in the Canadian mathematical landscape.

Robert was the first CMS "adult" I got to know well. We saw each other every 6 months for well over a decade and always picked up where we left off: the conversation was engaging, easy, one to always look forward to. Always a twinkle in his eye, he was sarcastic and mischievous in a kind way, optimistic despite being admittedly jaded. We bonded over drinks at banquets, sticking around well after everyone had left to talk about all things CMS, academia, ***Crux***. Speaking of ***Crux*** – anyone that knows me will tell you that no CMS conversation I have goes by without my mentioning ***Crux***, as being its Editor-in-Chief is my long-standing pet project. With Robert, he has been around for such a long time that I often forget how involved he was in many of the initiatives I hold dear within ***Crux*** : editor of the Olympiad Corner from 1987 to 2011 and Editor-in-Chief alongside Bill Sands from 1992 to 1996; creator and editor of the Skoliad from 1995 to 2001. We shared the vision for the journal and the understanding of its importance. But for nearly a decade, every CMS meeting would include a discussion on discontinuation of ***Crux***. It was exhausting to have to make the same arguments over and over again to the Board voting on ***Crux***'s "to be or not to be", knowing full well that the same vicious cycle would repeat in 6 months. In many ways Robert's semi-annual support and exchange of experiences kept me going in the EIC role. ***Crux*** now has ongoing funding. It was the work of a village, including Robert, that got us here.

The last time I saw Robert was at the CMS Winter meeting in Montreal. As per tradition, we found each other as the banquet room was clearing out. He introduced me to the lovely Edgar Goodaire (who said Robert gave me a very favourable review before I came up to their table, but I've yet to find out the details of that), we all grabbed the last of the wine and chatted, about math and life. In hindsight, what a fitting last memory of Robert.

I raise a glass to you, Robert – my CMS meetings will never be the same.

Kseniya Garaschuk

Remembering Robert Woodrow, *CMS Notes*, September 2025

IN MEMORIAM

What to say about a guy I knew and worked closely with for over 40 years?

I think I first met Robert when he joined the University of Calgary in 1980, about a year after I did. We must have been thrown together shortly after, both of us seconded by older members of the Department to help out with the Calgary Junior Math Contest and the Alberta High School Math Competition. Both contests involved proposing problems and putting the exam together. For the Calgary Junior, there were also the usual logistics of staging the contest, marking, ordering prizes, selecting and notifying the winners, and scheduling an awards event. Robert's administrative skillset soon showed up, including an invaluable talent for finding funding. And it wasn't very long before he was in charge of the entire contest, a role he occupied right up to his passing.

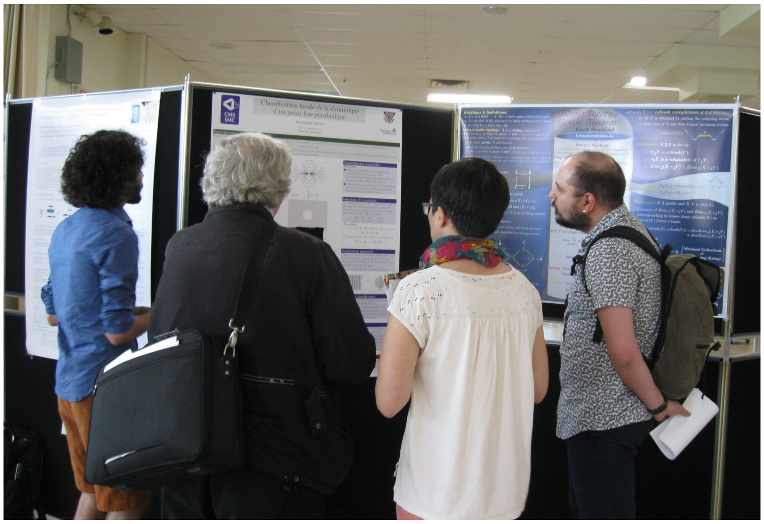
As for the Alberta contest, it was and is headquartered in Edmonton. So one day every fall, Robert and I (and sometimes a third person) would fly to Edmonton from Calgary to help make up that year's contest. It would be a day spent in selecting the contest questions from those collected from all members of the committee, with a break at mid-day for lunch. We were in the pleasant company of our hosts from the U of A, including Andy Liu, Murray Klamkin, and Alvin Baragar, among others. All are gone now. And the former in-person get-togethers have been replaced by zoom meetings. But the fond memories live on.

Robert and I also jointly inherited the task of organizing the department's weekly enrichment sessions for school students, which were called “Math Nights”. Thirty years, in concert with various other colleagues, of meeting and inspiring (or trying to inspire) youngsters with the beauties and oddities of math. And usually, at eight o'clock with the evening's work behind us, we would retire to the campus pub for dinner and unwinding. A more agreeable table companion would be hard to find. Sometimes it was impossible to dissuade Robert from picking up the tab for our evening's refreshment. On rare occasions he let me pay instead, but I knew that I was running up a debt to him I would never succeed in discharging, which indeed is now the case. Thank you, Robert.

Of course, Robert and I were also editors for *Crux* for several years, Robert inheriting the Olympiad Corner upon Murray Klamkin's retirement, and later starting up the Skoliad Corner, which has since morphed into MathemAttic.

As well, we collaborated (often with others) on a half-dozen mathematical research papers, one book, and a couple of problem session write-ups for BIRS conferences at the Banff Centre.

But maybe Robert was his most characteristic self when in a large group, such as a mathematics meeting. For me and all of Robert's mathematical friends, all future math conferences will be a little less memorable. No more that familiar rumpled figure, sandal-shod whatever the weather, shirt pocket stuffed with pens, heavy bag on shoulder.



None of us know when the game will be over. In Robert's case, it came immediately upon his return from attending a CMS conference. Immersed for a few days in a mathematics atmosphere, meeting old friends, socializing. Doing what he loved. Maybe not such a bad way to go, dare I say?

Thank you for everything, Robert. You are missed. Here's to you.

Bill Sands

MathemAttic

No. 71

The problems in this section are intended for students at the secondary school level.

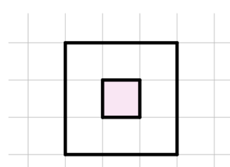
Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 15, 2026**.

MA351. The pairwise distances between nine cities are distinct. From each city, a traveller departs to visit the nearest city. Prove that some city is visited by at least two travellers.

MA352. A circle is divided by 27 points into 27 equal arcs. Each point is black or white. No two black points are adjacent or separated by one white point. Prove that there are 3 white points that are the vertices of an equilateral triangle.

MA353. Min and Max each have a 4×4 grid of 16 unit squares. Each of them removes three of the unit squares in their grid, and then computes the perimeter of their resulting shape. What is the maximum possible difference in their answers? Note: The perimeter of a shape is the sum of lengths of all the line segments that border the shape. For example, the following 3×3 square with the middle 1×1 square missing has the perimeter 16.



MA354. Show that, given five points in the plane in general position (that is, no three points are collinear), the number of convex quadrilaterals formed by these points is odd.

MA355.

(a) How many ways are there to pair up the elements of $1, 2, \ldots, 14$ into seven pairs so that each pair has sum at least 15?

(b) How many ways are there to pair up the elements of $1, 2, \ldots, 14$ into seven pairs so that each pair has sum at least 13?

(c) How many ways are there to pair up the elements of $1, 2, \ldots, 2024$ into 1012 pairs so that each pair has sum at least 2001?

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mars 2026**.

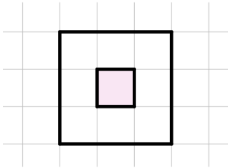
MA351. On considère neuf villes telles que toutes les distances entre paires de villes soient différentes. À partir de chaque ville, un voyageur part et se rend vers la ville qui est la plus proche de la sienne. Montrez qu'il existe nécessairement au moins une ville qui reçoit la visite d'au moins deux voyageurs.

MA352. Un cercle est divisé en 27 arcs égaux par 27 points. Chaque point est colorié soit en noir, soit en blanc. Aucun couple de points noirs n'est adjacent ni séparé par un seul point blanc.

Prouvez qu'il existe trois points blancs qui sont les sommets d'un triangle équilatéral.

MA353. Min et Max disposent chacun d'une grille 4×4 composée de 16 carrés unitaires. Chacun retire trois carrés unitaires de sa grille, puis calcule le périmètre de la figure obtenue. Quelle est la différence maximale possible entre leurs réponses ?

Remarque : Le périmètre d'une figure est la somme des longueurs de tous les segments de droite qui en constituent le contour. Par exemple, le carré 3×3 suivant, dont le carré central 1×1 a été retiré, a un périmètre égal á 16.



MA354. Montrez que, étant donnés cinq points du plan en position générale (c'est-à-dire qu'aucun trois points ne sont alignés), le nombre de quadrilatères convexes déterminés par ces points est impair.

MA355.

(a) De combien de façons peut-on regrouper les éléments de $1, 2, \ldots, 14$ en sept paires de sorte que la somme des nombres dans chaque paire soit au moins 15 ?

(b) De combien de façons peut-on regrouper les éléments de $1, 2, \ldots, 14$ en sept paires de sorte que la somme des nombres dans chaque paire soit au moins 13 ?

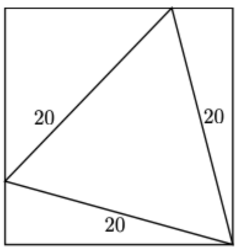
(c) De combien de façons peut-on regrouper les éléments de $1, 2, \ldots, 2024$ en 1012 paires de sorte que la somme des nombres dans chaque paire soit au moins 2001 ?

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(6), p. 260–262.

MA326. An equilateral triangle, 20 cm on a side, is inscribed in a square as shown in the diagram. Find the length of the side of the square.

Originally from the BC Secondary School Math Contest 2016, Junior Final Round, Part B, Q5.



We received 22 solutions from 21 solvers. Of these, 21 solutions were correct and complete. We present the solution by Harouach Madani, modified by the editor.

Call A the vertex that is shared by the square and the equilateral triangle. The angle of the square at A is $90°$, and the angle of the equilateral triangle at this same vertex is $60°$. The remaining part of the square's angle is split into two smaller angles, which we can call θ_1 and θ_2. Hence $\theta_1 + 60° + \theta_2 = 90°$, which simplifies to $\theta_1 + \theta_2 = 30°$.

Two right-angled triangles are formed in the corners of the square that are adjacent to vertex A. In both of these triangles, the hypotenuse is a side of the equilateral triangle, which has length 20 cm, and one leg is a side of the square, which we can call s. Applying the cosine function to these two right-angled triangles yields

$$\cos\theta_1 = \frac{s}{20} \qquad \text{and} \qquad \cos\theta_2 = \frac{s}{20}.$$

Since $\cos\theta_1 = \cos\theta_2$ and both angles are acute, we can conclude that $\theta_1 = \theta_2$. Substituting this back into our equation for the sum of the angles, we get $2\theta_1 = 30°$, so $\theta_1 = \theta_2 = 15°$.

The value of the side s of the square is therefore $s = 20\cos(15°)$ cm. An expression in terms of radicals can be computed by finding an exact value for $\cos(15°)$. Using the angle subtraction formula,

$$\cos(15°) = \cos(45° - 30°) = \cos(45°)\cos(30°) + \sin(45°)\sin(30°) = \frac{\sqrt{2}}{4}\big(\sqrt{3} + 1\big).$$

Substituting this value back into our equation for s yields

$$s = 20\cos(15^\circ) = 5\sqrt{2}\big(\sqrt{3}+1\big).$$

MA327. Find all triples (p, q, r) where p, q, r are positive integers of which at least two are prime for which

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Originally from the BC Secondary School Math Contest 2016, Senior Final Round, Part B, Q5.

We received 15 submissions, 5 of which are correct and complete. We present the solution by Harouach Madani.

The given equation is $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. We can rearrange it algebraically.

$$\frac{p+q}{pq} = \frac{1}{r} \implies r(p+q) = pq$$

Since p, q, r are positive integers, we can deduce from the original equation that $\frac{1}{r} > \frac{1}{p}$ and $\frac{1}{r} > \frac{1}{q}$, which implies $r < p$ and $r < q$.

From $r(p+q) = pq$, we can write $pq - rp - rq = 0$. Adding r^2 to both sides allows us to factor:

$$pq - rp - rq + r^2 = r^2 \implies (p-r)(q-r) = r^2$$

Let's consider the cases based on which two numbers are prime.

Case 1: p and q are prime.

Without loss of generality by well ordering principle assume $p \leq q$. From $r(p+q) = pq$, it's clear that p must divide $r(p+q)$. Since p divides rp, it must divide rq. As p is prime, either $p|r$ or $p|q$. We know $r < p$, so p cannot divide r. Thus, p must divide q. Since both p and q are prime, $p|q$ implies $p = q$.

Substituting $p = q$ into the original equation gives $\frac{2}{p} = \frac{1}{r}$, which means $p = 2r$. Since p is a prime number, its only positive divisors are 1 and p. For $p = 2r$ to be prime, we must have $r = 1$ (since r is a positive integer). This gives $p = 2$. So, we have the solution $(p, q, r) = (2, 2, 1)$

Case 2: p and r are prime.

From $(p-r)(q-r) = r^2$, we see that $p - r$ must be a divisor of r^2. Since r is prime, its divisors are 1 and r. So the divisors of r^2 are $1, r, r^2$. Also, $p > r$ so $p - r$ is a positive integer.

- If $p - r = 1$: Since p and r are prime, the only pair of consecutive primes is $(2, 3)$. So $r = 2$ and $p = 3$. Substituting into $(p-r)(q-r) = r^2$: $1 \cdot (q-2) = 2^2 \implies q - 2 = 4 \implies q = 6$.

- If $p - r = r$: Then $p = 2r$. As shown in Case 1, this means $r = 1$ if p is prime. But r must be prime in this case, so this is not possible.
- If $p - r = r^2$: Then $p = r^2 + r = r(r + 1)$. Since $r \geq 2$, p is a product of two integers greater than 1, so p cannot be prime.

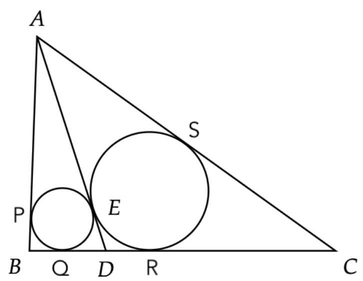
So, this case only yields the solution $(3, 6, 2)$.

Case 3: q and r are prime.

By symmetry with Case 2, we find the solution $(6, 3, 2)$.

Therefore, the set of all triples is $\{(2, 2, 1), (3, 6, 2), (6, 3, 2)\}$.

MA328. In the diagram, triangle ABC has sides of length $AB = 7$, $AC = 12$, $BC = 10$. There is a point D on BC such that the circles inscribed in triangles ABD and ACD are both tangent to the line AD at a common point E. Find the length of the line segment BD.



Originally from the BC Secondary School Math Contest 2016, Senior Final Round, Part B, Q5.

We received 17 submissions, of which 16 were correct and complete. We present the solution by Corneliu Manescu-Avram.

Denote by P, Q the tangency points of the incircle of triangle ABD with the sides AB, BD respectively; denote R, S the tangency points of the incircle of triangle ACD with the sides DC, AC respectively.

If $BQ = x$, $QD = y$, then

$$\begin{aligned}
&BP = BQ = x,\\
&AP = AB - BP = 7 - x,\\
&AS = AE = AP = 7 - x,\\
&RC = CS = AC - AS = 12 - (7 - x) = x + 5,\\
&DR = DE = DQ = y
\end{aligned}$$

as tangents to a circle from an exterior point.

We have $BC = BQ + QD + DR + RC = 10$, or $x + y + y + x + 5 = 10$, so $x + y = \frac{5}{2}$ or $BD = \frac{5}{2}$.

MA329. Nine people attend a dinner where there are three choices for the type of meal. Three people order combo A, three order combo B and three order combo C. The server distributes the nine meals in random order. In how many different ways can exactly one person receive the correct meal?

Originally from the BC Secondary School Math Contest 2016, Senior Preliminary, Q10.

We received 9 submissions, 7 of which were correct and complete. We present the solution by Meryem Bourget.

Since 9 people attend the dinner, there are 9 possibilities to select one person to receive the correct meal. Without loss of generality, we may assume this person to receive meal A. The remaining 8 people must receive meals different from those they originally ordered. We consider the different possibilities for the remaining two people who ordered meal A:

Case 1: One gets meal B and the other gets meal C.

This can happen in $\binom{2}{1}\binom{1}{1} = 2$ ways. The remaining six people who ordered meals B and C must be divided in the following way. Out of the three people who ordered meal B, one of them gets meal A and the other two get meal C. This can happen in $\binom{3}{1}\binom{2}{2} = 3$ ways. Similarly, out of the three people who ordered meal C, one of them gets meal A and the other two get meal B. This also happens in $\binom{3}{1}\binom{2}{2} = 3$ ways. Therefore, Case 1 leads to $2 \times 3 \times 3 = 18$ possibilities.

Case 2: They both get meal B or they both get meal C.

Suppose they both get meal B. Out of the three people who ordered meal C, two of them must receive meal A and the remaining one must get meal B. This can happen in $\binom{3}{2}\binom{1}{1} = 3$ ways. The three people who ordered meal B must all receive meal C. This happens in $\binom{3}{3} = 1$ way. Thus, this situation may happen in 3 ways. Similarly, there are 3 ways for both of the people who ordered meal A to get meal C. Therefore, Case 2 leads to $3 + 3 = 6$ possibilities.

Consequently, the total number of possibilities for exactly one person to receive the correct meal is

$$9 \times (18 + 6) = 216.$$

MA330. The x-coordinates of the vertices of a square in the plane are 1, 3, 8 and 10. Determine the area of the square.

Originally from the BC Secondary School Math Contest 2017, Junior Final Part B, Q4.

There were 15 submissions, 13 of them were complete and correct. We present the solution by The Ring Lords.

Since none of the square's vertices share the same x-coordinate, the square is tilted. To analyze it, we enclose the square inside a larger, axis-aligned square (whose sides are parallel to the x- and y-axes). Each vertex of the tilted square touches a side of this larger square, creating four right triangles around it.

Each of these four triangles has a right angle, coming from the sides of the enclosing square. Pick one of the non-right angles in one triangle and call it α. The other acute angle in that triangle must then be $90^\circ - \alpha$, since the sum of the angles in a triangle is 180°. This angle shares a vertex with another triangle, and because of the geometry of the enclosing square, the corresponding angle in that neighboring triangle must also equal α. Repeating this argument for all four triangles shows that all four are congruent.

Now consider the vertex of the tilted square with x-coordinate 3. Its horizontal neighbors have x-coordinates 1 and 10. The horizontal distance between 3 and 1 is 2, and the distance between 3 and 10 is 7. Thus, the two legs (catheti) of each of the congruent right triangles have lengths 2 and 7. The hypotenuse of each triangle equals the side length of the tilted square. By the Pythagorean theorem, the square's side length satisfies

$$(\text{side length})^2 = 2^2 + 7^2 = 4 + 49 = 53.$$

Therefore, the area of the tilted square is 53.

TEACHING PROBLEMS

No. 31

John Grant McLoughlin

The Game of *Sim*

> Six dots are drawn on a piece of paper to form the vertices of a hexagon. Two players are each assigned a colour. The players take turns joining any two of the dots with a line segment, using their assigned colours. The loser is the player who completes a triangle with three of the original six dots as its vertices and with all three edges the same colour.

The game of *Sim* was invented by Gustavus Simmons in 1969. My introduction to the game came back in the 1980's and it has been featured in many of my teaching contexts over the years. The game is played as described above. *Maths Week Scotland* profiled the game in 2024 on its site in a manner that would make it school or teacher friendly for use in classrooms. The link here may be helpful for those unfamiliar with the game: `https://mathsweek.scot/news/how-to-play-sim`

So why is the game being shared here in *Teaching Problems*? The answer is Robert Woodrow. When Kseniya Garaschuk extended an invitation to share something pertinent in this special issue, the game of *Sim* came to mind. This game bridges recreational mathematics and outreach in my experience, while serving as the foundation of a discussion and demonstration of proof in mathematics. Players of the game will find that a tie is impossible. The mathematics underpinning this fact is based on *Ramsey Theory*, an area of mathematics dear to the heart and work of Robert Woodrow. This one is for you, Robert.

Experiences and Observations around the Game of *Sim*

Most recently, I played this game this term in a class of prospective teachers of mathematics, mainly at the secondary level. Students enjoy many of the attractive features of this game. Its material simplicity is appealing. Likewise, the game does not take long to play. Yet, there appears to be a fairness with an understanding that both players alternate going first (hence, neither player has a disadvantage). The perception of fairness is not a mathematical one but rather an acknowledgment that devising a quick winning strategy is not going to happen. (Deeper analysis of the game may change that for interested readers.) Playing an easily accessible game with a sense that either person could win is a good starting point for engagement.

After students have had some chance to play, it is verified that indeed none of the games in the class resulted in a tie. This takes us to the question as to why.

> Prove that there must always be a winner (loser).

As mentioned, Ramsey Theory is related to the fact that a tie is impossible. Many sources exist for information on this topic. Those unfamiliar with Ramsey

Theory may find a helpful introduction in a collection of notes prepared by Veselin Jungic (`https://www.sfu.ca/~vjungic/Ramsey/RamseyNotes.pdf`) Those more formally versed in graph theory and Ramsey Theory may wish to consider research publications by Robert Woodrow amongst others. In our specific example, it is a geometric analogy to the fact that among 6 people there must be a group of 3 people that are mutual strangers or 3 people that are mutual acquaintances. This corresponds to the colouring of the edges of the various triangles, thus ensuring that there must be a triangle using the vertices in the hexagonal arrangement with all three of its edges having the same colour. Such a triangle is called *monochromatic*. Readers may wish to show that if there were only five dots as vertices, in a convex pentagonal arrangement, that it would be possible to alternate joining vertices with two different coloured edges without necessarily obtaining a triangle having three same-coloured edges. That is, a variation of *Sim* on a pentagonal board could end in a tie.

Proof by Contradiction

An outline of the proof is provided here. A keen ***Crux*** reader may note that the game of *Sim* along with the subsequent proof were shared in 2006 through a feature called *Polya's Paragon*. These appeared in successive issues under the titles *Playing Games with Mathematics (Part I)* and *Playing Games with Mathematics (Part II)*. Few geometric proofs employ contradiction as a method. Proving that a tie is impossible in a game of *Sim* offers a neat application of this technique.

Assume that it is possible to have a tie. Let the original dots be labelled A, B, C, D, E, and F. Consider any one of these dots. Suppose we choose A. Note that there are five segments that can be drawn from A. Select two colours, say blue and red. At least three of the segments from A must be one of these colours, say blue. Again, it does not matter which three segments are selected. Suppose that AB, AC, and AE are blue. It follows that none of BC, BE, or CE are blue because otherwise a blue triangle would be formed, thus, creating a loser. Aha! That makes triangle BCE a red triangle. Therefore, we have a proof by contradiction that a tie is impossible.

Closing Comments

The game of *Sim* is a rich example of a game that crosses ages. The game has been played in many elementary classrooms without a focus on any proof, aside from seemingly observing that a tie is impossible. The experience of playing such a game brings one closer to the spirit of doing mathematics. Messing around, playfully learning, and the like lead to conjectures and exploration. Then at appropriate stages the boundaries can be shifted to engage with variations, proofs, or strategies.

Robert Woodrow enjoyed playing with mathematics and reached out as a mentor to many students. May his spirit and energy be carried forth by many others in the mathematical community.

PROBLEM SOLVING VIGNETTES

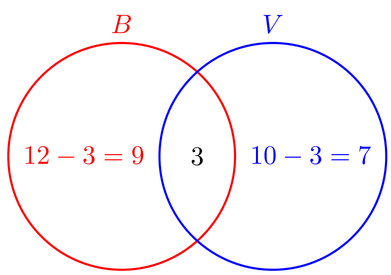
No. 40

Shawn Godin

Overcounting

Counting problems are a staple of math contests. In many cases, these simple sounding problems have snares waiting to trap the unwary contestant. One of the common places where people take the wrong path is by *overcounting*.

Overcounting occurs when some objects that we are counting get counted multiple times. As a simple example, imagine there was a party that was attended by the basketball team, with 12 members, and the volleyball team, with 10 members. How many people are at the party? One may be tempted to answer $12 + 10 = 22$, duh! However, it might not be that simple if anybody played on *both* teams. So if it turned out that 3 people are on both teams, then they are counted in the 12 members of the basketball team and again in the 10 members of the volleyball team. As such, our total of 22 has these members counted twice. If we draw a Venn diagram of the situation, where B and V represent the sets of members of the basketball and volleyball teams, respectively, then the total assembled members would be $9 + 7 + 3 = 19$.

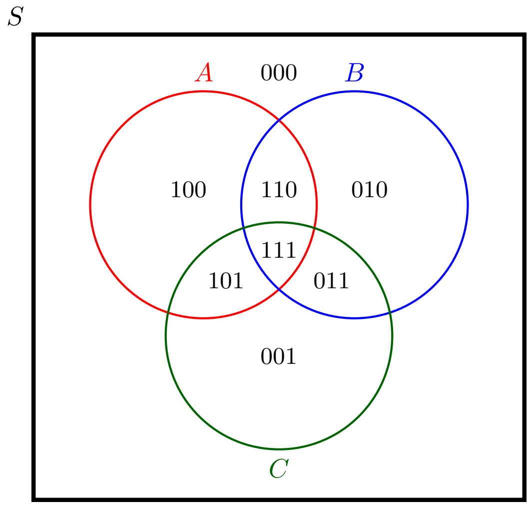


Using the notation $n(X)$ to represent the number of elements in set X, we can write

$$\begin{aligned} n(B \cup V) &= (n(B) - n(B \cap V)) + (n(V) - n(B \cap V)) + n(B \cap V) \\ &= n(B) + n(V) - n(B \cap V) \end{aligned}$$

where $B \cap V$ is the *intersection* of sets B and V — the set of shared elements — and $B \cup V$ is the *union* of sets B and V — the set created when the two sets are combined. Many readers will recognize the equation above as the *principle of inclusion and exclusion* for two sets. It says, to find the number of elements when two sets are combined, add the numbers of elements in each set and subtract those elements that are shared in both.

It seems that we should be able to generalize this to any combination of sets that we like. This can be done, but we must be careful. If we look at three sets, A, B, and C, right off the bat we should note that A can share elements with B, C, or both. We might be tempted to add the numbers in each set $n(A)$, $n(B)$, and $n(C)$ and then subtract the overlaps between two sets, that is, $n(A \cap B)$, $n(A \cap C)$, and $n(B \cap C)$. However, one must think about the elements shared between all three sets, that is the set $A \cap B \cap C$. How will this all go together? Below is the general Venn diagram for three sets. Each part of the diagram has been labelled with a three-digit binary number. The first digit is 1 if the element is in set A and 0 if it isn't. Similarly, the second and third digits give us an element's status in sets B and C, respectively. We have also included the *universal set*, S — the set of all elements that we are considering. Note that the region outside of sets A, B, and C is therefore labelled 000.



Thus, set A is made up of the regions labelled 100, 110, 101, and 111. We can rewrite this as $1XX$ where the X's can be either 0 or 1. Thus, if we add $n(A) + n(B) + n(C)$ and look at all the regions, we have accounted for 100, 010, and 001 once each; 110, 101, and 011 twice each; and 111 three times. Thus the sum $n(A) + n(B) + n(C)$ overcounts regions 110, 101, 011, and 111. However, when we take off the overlaps by subtracting $n(A \cap B) + n(A \cap C) + n(B \cap C)$, we remove one copy of each of 110, 101, and 011, but three copies of 111. Hence $n(A) + n(B) + n(C) - (n(A \cap B) + n(A \cap C) + n(B \cap C))$ counts each region once *except* 111, which is now unaccounted for. Therefore we can deduce the principle of inclusion and exclusion for three sets to be

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C).$$

This idea generalizes to any number of sets. To find the number of elements in the union of a number of sets, add the numbers in each set, then subtract the number of elements in the intersection of any two sets, then add the number of elements in the intersection of any three sets, then subtract intersections of four sets, etc.

Thus for four sets we get

$$
\begin{aligned}
&n(A \cup B \cup C \cup D) \\
&= n(A) + n(B) + n(C) + n(D) \\
&- n(A \cap B) - n(A \cap C) - n(A \cap D) - n(B \cap C) - n(B \cap D) - n(C \cap D) \\
&+ n(A \cap B \cap C) + n(A \cap B \cap D) + n(A \cap C \cap D) + n(B \cap C \cap D) \\
&- n(A \cap B \cap C \cap D).
\end{aligned}
$$

The interested reader may enjoy using the binary strategy to convince themselves of the formula above.

However, not all problems are given in terms of sets. Consider the problem below.

> *How many three-letter "words" can be formed using the letters A, B, and C that must contain at least one A? Letters may be used multiple times.*

A common incorrect solution that I would see as a teacher would be something like the following. We can pick a spot to put a letter A in $\binom{3}{1}$ ways. The other two positions can be filled by any letter, so the total is

$$\binom{3}{1} \times 3^2 = 27$$

which is actually the total number of possible words, so it must be wrong. But *how* is it wrong? Let's look at a couple of valid solutions and see if we can spot the problem.

Usually we handle this by breaking it down into cases: exactly one A, exactly two A's and exactly three A's — keeping in mind that the non-A spots must be filled with either B or C — which gives us

$$\binom{3}{1} \times 2^2 + \binom{3}{2} \times 2 + \binom{3}{3} = 3 \times 4 + 3 \times 2 + 1 = 19$$

The other common method is the so-called *indirect method* where we calculate the ones we *don't want* and subtract it from the total. Since we want at least one A, we don't want less than one A, that is no A's. Hence we get

$$3^3 - 2^3 = 27 - 8 = 19$$

So what went wrong with the first, incorrect, solution? If we list all possible three-letter words, we get a clue.

$$\begin{array}{ccc} AAA & BAA & CAA \\ AAB & BAB & CAB \\ AAC & BAC & CAC \\ \\ ABA & BBA & CBA \\ ABB & BBB & CBB \\ ABC & BBC & CBC \\ \\ ACA & BCA & CCA \\ ACB & BCB & CCB \\ ACC & BCC & CCC \end{array}$$

If we look at an entry like ACA, using the incorrect $\binom{3}{1} \times 3^2$, we can reason that we choose the first position for the A, then choose a C in the second position and A in the final position. However, it will also count if we pick the last position for the A and fill the first two positions with A and C, respectively. Hence, $\binom{3}{1} \times 3^2$ counts the entry ACA *twice*. Similarly, it will double-count all words with exactly two A's and will triple-count AAA. Thus, if we compare it to the solution by cases which calculated

$$12 + 6 + 1 = 19$$

for the cases with one, two, and three A's respectively, the incorrect $\binom{3}{1} \times 3^2$ actually counts it as

$$12 + 6 \times 2 + 1 \times 3 = 27$$

So overcounting is a bad thing that we should be careful to avoid . . . or is it?

Let's look at problem A6 from the 2025 Canadian Senior Mathematics Contest, run by the Centre for Education in Mathematics and Computing at the University of Waterloo.

> *All of the 8-digit positive integers are listed, and then every digit equal to* 0 *is erased. This causes some integers in the list to be replaced by another integer or by several other integers. For example, the integer* 89 160 000 *is replaced by* 8 916*, the integer* 34 041 034 *is replaced by the three integers* 34*,* 41*, and* 34*, and the integer* 49 671 349 *does not change. How many integers are in the list after the* 0*s have been erased?*

Let's attack this problem with recursion, like we discussed in a previous column [2022: 48(7), p. 388-392]. Define two functions $N(p, d)$ and $Z(p, d)$ which count the number of d-digit numbers that will break up into p pieces when the zeros are removed. The function $N(p, d)$ counts such numbers that end with a non-zero digit, while the function $Z(p, d)$ counts such numbers that end with a zero.

For example, if we look at three-digit numbers, they can be of four forms: XXX, $XX0$, $X0X$, and $X00$, where X represents a non-zero digit. Only $X0X$ will become two pieces when the zero is removed, all others will be remain a single piece. Thus

$$N(1,3) = \#(XXX) = 9^3 = 729$$
$$Z(1,3) = \#(XX0) + \#(X00) = 9^2 + 9 = 90$$
$$N(2,3) = \#(X0X) = 9^2 = 81$$

Note that $729 + 90 + 81 = 900 = 9 \times 10^2$, the number of three-digit numbers (note the first digit cannot be a zero). Note also that $N(p,3) = 0$, for $p > 2$ and $Z(p,3) = 0$ for $p > 1$. Let's define these functions recursively.

Clearly $N(1,1) = 9$ and $Z(1,1) = 0$, since we are dealing with positive integers. Also $N(p,1) = Z(p,1) = 0$ for $p > 1$. We will construct the $(d+1)$-digit numbers from the d-digit numbers by adding every possible digit to the end of each d-digit number. For each d-digit number that breaks into p pieces we have four cases.

- The original number ends in a non-zero digit and if we add a non-zero digit to the end, it still breaks into p pieces.
- The original number ends in a non-zero digit and if we add a zero to the end, it still breaks into p pieces.
- The original number ends in a zero and if we add a non-zero digit to the end, it now breaks into $p+1$ pieces.
- The original number ends in a zero and if we add a zero to the end, it still breaks into p pieces.

This suggests the following recursion relationships:

$$N(p, d+1) = 9N(p,d) + 9Z(p-1,d)$$
$$Z(p, d+1) = N(p,d) + Z(p,d)$$

Since we are dealing with 8-digit numbers and we need one or more zeros between non-zero digits to create extra pieces, the configuration $X0X0X0X0$ suggests that, at most, there will be four pieces when we remove the zeros. Hence we can use our recursion relationships to calculate $N(p,8)$ and $Z(p,8)$ for $p \in \{1,2,3,4\}$, as shown in the Table 1.

A quick calculation, using the entries from the last row of Table 1, shows that

$$\sum_{p=1}^{4} (N(p,8) + Z(p,8)) = 90\ 000\ 000 = 9 \times 10^7$$

which is the total number of 8-digit numbers, which checks out. Thus, our desired result is going to be

$$\sum_{p=1}^{4} p\,(N(p,8) + Z(p,8)) = 138\ 600\ 000$$

d	$N(1,d)$	$Z(1,d)$	$N(2,d)$	$Z(2,d)$
1	9	0	0	0
2	81	9	0	0
3	729	90	81	0
4	6 561	819	1 539	81
5	59 049	7 380	21 222	1 620
6	531 441	66 429	257 418	22 842
7	4 782 969	597 870	2 914 623	280 260
8	43 046 721	5 380 839	31 612 437	3 194 883
d	$N(3,d)$	$Z(3,d)$	$N(4,d)$	$Z(4,d)$
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	729	0	0	0
6	21 141	729	0	0
7	395 847	21 870	6 561	0
8	6 084 963	417 717	255 879	6 561

Table 1: The functions $N(p,d)$ and $Z(p,d)$ for $p,d \in \mathbb{Z}$, $1 \le p \le 4$, and $1 \le d \le 8$.

since each number counted by $N(p,8)$ and $Z(p,8)$ will yield p numbers when the zeros are erased.

OK, so what does that have to do with overcounting? The brilliant official solution to the problem uses overcounting as an advantage. Let's take a quick look at this solution.

First we recall that there are 9×10^7 8-digit numbers. Each of these numbers contributes at least one number to the final list. Those that contribute more than one number to the list must have a zero followed by a non-zero digit somewhere in the list. This zero can occur in any of six places, that is we could have

$X\,0\,X$ __ __ __ __ __	X __ $0\,X$ __ __ __ __	X __ __ $0\,X$ __ __ __
X __ __ __ $0\,X$ __ __	X __ __ __ __ $0\,X$ __	X __ __ __ __ __ $0\,X$

where the blanks could be any of the ten digits and the X's are non-zero digits. There are therefore $6 \times 9^2 \times 10^5$ numbers of this form.

However, as we saw in our first example with the three-letter words, numbers with only one occurrence of a zero followed by a non-zero digit will be counted once, numbers with two zeros — each followed by a non-zero digit — will be counted twice, numbers with three zeros — each followed by a non-zero digit — will be counted three times. It is impossible to get an 8-digit number with four zeros, each followed by a non-zero digit as shown by the extreme case $X0X0X0X0$. The numbers with one zero followed by non-zero digits — such as 83 0519 274 or 20 003 999 or 15 006 000 — will break into 2 pieces, the one piece that every

8-digit number contributes plus one more. Similarly, if it has two zeros followed by non-zeros the number will be broken into three parts, but double counted by our method above. Hence

$$9 \times 10^7 + 6 \times 9^2 \times 10^5 = 138\ 600\ 000$$

counts all the pieces formed when zeroes are removed, agreeing with our earlier calculation. What a beautiful way to use overcounting to our advantage! As always, one must be ready to adapt known methods to new scenarios.

. . . .

Here are a few counting problems for your amusement. Careful not to overcount, unless overcounting one thing helps you to determine something else. Have fun!

1. How many integers between 10 and 500, inclusive, have their digits in strictly decreasing order? For example, 41 and 320 are such integers, but 441 and 230 are not. (2025 Canadian Open Mathematics Challenge, Question A2)

2. Alice has a lock whose combination consists of three integers a, b, c which need to be entered in that order. The three integers satisfy the following:

   - each of a, b, and c is between 1 and 40, inclusive;
   - a, b, and c are all different;
   - b is less than a, and b is less than c; and
   - one of the integers is 20 and another of the integers is 30.

   How many possible combinations satisfy these conditions? (2025 Euclid Contest, Question 5 a)

3. In mathematics, an anagram of a word is a rearrangement of its letters, including those that result in nonsensical words. A word is always an anagram of itself. For example, the word EAT has six anagrams: AET, ATE, EAT, ETA, TAE, and TEA.

   Let N be the number of anagrams of the word $ABRACADABRA$ which start and end with different letters. Find, with proof, the largest prime number that divides N. (2025 Canadian Open Mathematics Challenge, Question C2(c))

4. 20 different students are lined up in a row. 8 students wear a red hat and 12 students wear a green hat. There are 20! ways to arrange the students. Call a pair of students *complementary* if they are next to each other and have different coloured hats. For example, an arrangement of the students resulting in the following order of hat colours

   $$GRRGGRRGGGRGGRGGGGRGGGGR$$

   has 11 complementary pairs. Among the 20! ways to arrange the students, what is the average number of complementary pairs of students? (2025 Canadian Team Mathematics Contest, Individual problem 10)

5. Consider all 9-letter strings, where each letter is an A or a B. We say that a string is *diverse* if it does not contain two consecutive As nor three consecutive Bs. For example, $ABABABABA$ and $BBABABBAB$ are diverse, but $AAABABBBB$ and $BBABBAABB$ are not.

   How many 9-letter strings are diverse? (2025 Canada Lynx Mathematical Competition, Question C4)

Competition Highlights

The Canadian Open Mathematics Challenge

Margo Kondratieva

The Canadian Open Mathematics Challenge (COMC) is an annual, individual competition designed for the secondary school students and available to any interested participant who enjoys solving mathematical problems.

Two years ago, we wrote an overview of this contest (***Crux***, Vol. 50(6), June 2024) together with Robert Woodrow, who was the Chair of the COMC committee. This year we were asked to write another article related to the contest. Unfortunately, our plan to work collaboratively on that project was interrupted by the sudden death of my dear colleague. I would like to dedicate this short article to his memory.

Through the work related to mathematics competitions over the years we looked together at a number of problems from various areas of mathematics and enjoyed discussing interesting solutions, especially those submitted by the contest participants. Somehow, I never knew what area of school mathematics was the most exciting to Robert. It is quite possible that there were many. However, considering that his mathematical research was related to Ramsey theory, I can only guess that he could very likely favour problems involving graphs. Ramsey theory in combinatorics was developed as a generalization of a simple but powerful derivation that is presented below as a solution to Problem 1. Then I discuss a combinatorics problem from the most recent COMC contest that allows a graph-based solution.

A graph is a mathematical object that consists of dots (called vertices), some of which are connected by lines (called edges). Graphs can be convenient for describing real life situations. Consider the following example.

Problem. Show that *in any group of 6 people there are always either 3 people who know each other or three people who are all strangers to each other.*

To model this statement with a graph, one can schematically draw 6 people standing in a circle and connect each pair of people who know each other by a red solid line. Then, connect any pair of people who do not know each other by a grey dashed line. Any three people connected by three lines will form a triangle. A red solid triangle, if it appears, will indicate three persons who know each other. A grey dashed triangle, if it appears, will indicate three persons who are strangers to each other.

Note that people could be shown as dots, so vertices of the graph represent the six people. The graph in Figure 1 shows that among six people called Ann, Bob, Cora, Don, Ed, and Fred, four pairs of people know each other: Ann and Don, Bob and Cora, Bob and Ed, Cora and Fred. However, there is no red solid triangle connecting any three of them. The edges connecting Ann, Bob and Fred form a grey dashed triangle, meaning that these three individuals do not know each other. Similarly, Ann, Cora and Ed are strangers to each other.

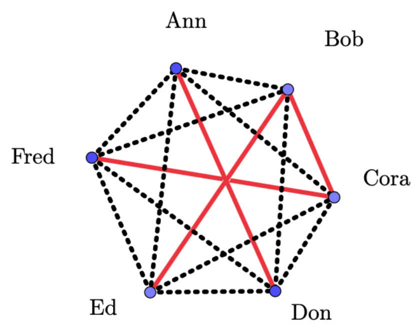


Figure 1: A graph representing six people who either know each other or not

In the graph shown in Figure 1 each pair of vertices is connected by an edge. A graph having this property is called complete.

Then the statement above formulated in mathematical terms becomes: *in a complete graph with six vertices and edges of two types there always will be a triangle formed of edges of one of these types.*

There is a very elegant proof of this statement. Consider five edges adjacent to one of the vertices, say A. Among them there will be at least three edges of the same type. This follows from the Pigeonhole principle as there are two types and 5 edges. Let AB, AC and AF be three edges of type 1. Then if it happens that all edges connecting vertices B, C, and F are of type 2, then we have a type 2 triangle BCF. Otherwise, if one of the edges connecting vertices B, C, and F is type 1, say CF, then edges connecting A, C, and F form a type 1 triangle ACF.

Observe that this derivation does not work for a group of 5 people and indeed Figure 2 gives an example where neither 3 strangers nor 3 acquaintances are present in a group of 5 people.

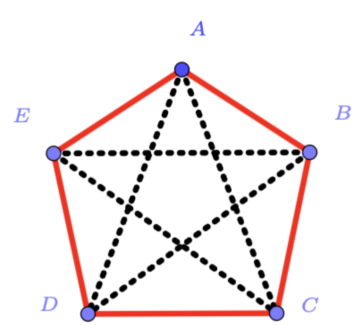


Figure 2: Example with no monochrome triangles in a group of five people

Now let us consider a problem from COMC 2025. Being the last problem in section C, this problem is one of the most challenging in the contest.

Problem (C4, COMC 2025). *Veronica plays a game against her* 2025 *students. First, each of the students points to exactly one other student in the room. Multiple people could point to the same student, and some students could have no one point to them. Veronica sees who pointed to whom. Second, Veronica chooses N students to leave the room. Third, Veronica assigns a number to each of the remaining* $2025 - N$ *students, under the condition that if one student is pointing to another, then they both must receive the same number. If she assigns k distinct numbers to the students, then Veronica scores k points. The students are trying to minimize the number of points Veronica scores, and she is trying to maximize the number of points she scores. Under optimal play from both Veronica and her students, what score does she receive when*

(a) $N = 1$?

(b) $N = 100$?

(c) $N = 1000$?

Solution.

We can start by exploring a simpler case with fewer than 2025 students. We expect that considering a simplified version of part (a) will possibly provide insights into the solution of further parts. Let there be 3 students. As previously, we can schematically represent people by dots (vertices) and draw a directed line each time one student is pointing to another. We can have the following situations shown in Figure 3: (i) first student points to the second, second to the third and third to the first; (ii) two students are pointing to the third and the third student is pointing to one of them.

In case (ii) Veronica can remove the third student and receive 2 points. In case (i) she scores 1 point regardless of whom she will remove. Thus, the optimal strategy for students is to form a circle so that Veronica will get only one point. Increasing the number of students to 4, 5, 6, and even 2025, we see that the students' strategy of pointing to each other in a circular fashion always leaves Veronica with one point. Indeed, after Veronica removes any one student, the remaining students form a chain, where each student points to the next, with the last student in the chain pointing to the departed student. Veronica must then assign all of them the same number. So, the answer in (a) is 1.

Before attempting (b) and (c) we again can consider smaller models. Suppose, for example, that Veronica is playing against nine students and she can remove $1 < N < 9$ students. If $N = 2$ and the nine students follow the strategy of pointing in a circular fashion (Figure 4, left), Veronica can take any two students among whom neither is pointing to the other one. Then the remaining students form two chains, where each student points to the next and the last student in each chain is the one who has been pointing to the departed student (note that a 'chain' may

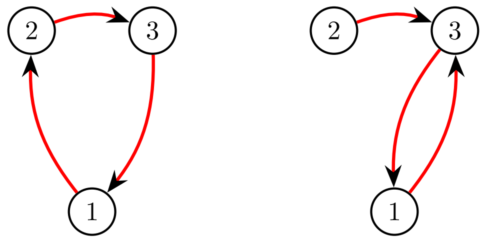


Figure 3: Two possible scenarios for three students

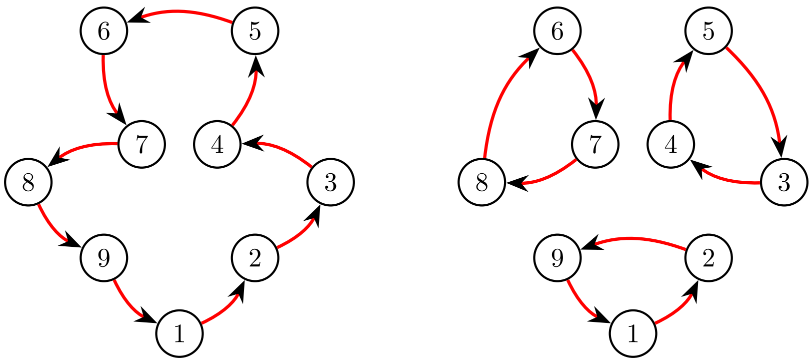


Figure 4: Optimal strategies for 9 students if $N \neq 4, 5$ (left) and $N \neq 1, 2$ (right)

consist of just one student). Veronica must then assign all of the students in the same chain the same number. Thus, Veronica will score 2. Similarly, when there are 3 students to leave the room, she can score 3 points by removing students among whom none is pointing to another. However, we should not forget that both Veronica and her students must follow the most optimal strategy. Thus, for $N = 4$, the circular strategy is no longer optimal for students because it will allow Veronica to score 4 points. However, if the students form 3 small circles of 3 students each (Figure 4, right), Veronica can score a maximum of 3 points by removing at most 2 students from each of the 3 circles. The same observation is true for $N = 5$ and $N = 6$. Then for $N = 7$, Veronica will score 2 points regardless of what students do because only 2 students will remain. Finally, for $N = 8$ Veronica will score 1 point. In summary, she will score the following number of points:

$$\begin{cases} N & \text{if } N = 1, 2, 3 \quad \text{and students form one circle of 9} \\ 3 & \text{if } N = 4, 5, 6 \quad \text{and students form 3 circles of 3} \\ 9 - N & \text{if } N = 7, 8 \quad \text{and students do anything} \end{cases}$$

Observe that if students make a circle of 4 or more members, Veronica will be able to remove 2 students and create two chains and so her score will be increased. A

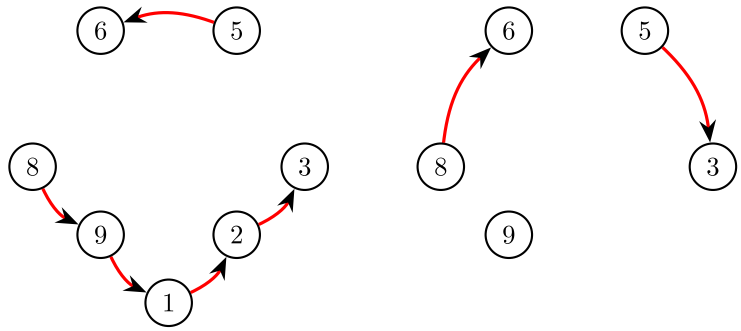


Figure 5: Optimal strategies for Veronica if $N = 2$ (left) and $N = 4$ (right)

circle of 3 does not allow her to increase the score. With this observation, we can make a hypothesis for the case where Veronica plays with M students (where M is divisible by 3). We propose that she will score the following number of points:

$$\begin{cases} N & \text{if } N = 1, 2, 3, \ldots, M/3 \quad \text{students form one circle of } M \text{ members} \\ M/3 & \text{if } N = M/3, \ldots, 2M/3 \quad \text{students form } M/3 \text{ circles of 3 members} \\ M - N & \text{if } N = 2M/3, \ldots, M-1 \quad \text{students do anything} \end{cases}$$

Applying this formula to the case $M = 2025$, we conclude that the answer in (b) is 100 and the answer in (c) is 2025/3=675.

Now, to prove our hypothesis and justify the answers, we need to show that our answers are optimal for both Veronica and her students. We will use more ideas from graph theory. A path in a graph is an ordered set of edges such that any two consecutive edges share a vertex. A graph is called *connected* if there is a path connecting any pair of its vertices. A part of a graph G which is a connected graph by itself, but is disconnected from all other vertices of G, is called a *connected component* of G. For example, Figure 5 shows a graph with two connected components (left) and a graph with three connected components (right).

In our model students initially form a graph with 2025 vertices and 2025 edges†, with the property that for every vertex there is at least one edge adjacent to it (because every student points to somebody). When Veronica removes N vertices, she also removes edges adjacent to them and her score is the number of connected components of the remaining graph. The following Lemma will be useful for our considerations.

Lemma. Any graph G with n vertices and $m < n$ edges has at least $n - m$ components.

†In some applications consideration of directed edges and directed paths can be necessary. In our case we can place an edge each time a student points to another student disregarding the direction.

Proof. A graph consisting of n vertices and no edges has n components. Every added edge connects two vertices and can decrease the number of components by at most one. Thus, after adding $m < n$ edges, the number of components will be at least $m - n$. $\square$

We first present the students' strategy. Label the students $1, 2, \ldots, 2025$.

When $N = 100$, the students point to each other in a circle, say, the person k pointing at $k + 1$ for $1 \le k \le 2024$ and the person 2025 points to person 1. Then, the resulting graph is a cycle. Suppose Veronica removes students with numbers $a_1 < a_2 < \cdots < a_{100}$ such that $1 < a_1$ and $a_{100} < 2025$. Then, the remaining graph has the following connected components (each component is a chain of vertices, which are ordered as indicated below):

Component 1: $a_1 + 1, a_1 + 2, \ldots, a_2 - 1$

Component 2: $a_2 + 1, a_1 + 2, \ldots, a_3 - 1$

Component 3: $a_3 + 1, a_3 + 2, \ldots, a_4 - 1$

$\vdots$

Component 100: $a_{100} + 1, a_{100} + 2, \ldots 2025, 1, \ldots, a_1 - 1$

Note that the number of vertices in each component is $a_{k+1} - 1 - a_k$. If $a_{k+1} = a_k + 2$, the component k contains just one vertex $a_k + 1$. If a_k and a_{k+1} are consecutive numbers, such component is empty, that it, it does not exist. Thus, there are at most 100 connected components, so the students can guarantee that Veronica scores at most $N = 100$ points.

When $N = 1000$, the students form groups of 3 and point to each other in a cycle there. Then, after Veronica removes students, each group of 3 may only contribute at most 1 connected component, so Veronica can only score at most 675 points.

In either case, Veronica scores at most $\min(N, 675)$ points.

Now, we present Veronica's strategy.

The number of edges adjacent to a vertex is called the *degree* of that vertex.

We propose that Veronica should remove a vertex of maximal degree iteratively, until she has removed N vertices.

Each time she removes a vertex, she also removes edges adjacent to it.

We consider the following cases.

<u>Case 1</u>: Veronica removes N vertices of degree at least 2. Thus, she removes $e \ge 2N$ edges. The resulting graph has $2025 - N$ vertices and $2025 - e$ edges. By the Lemma, this graph must have at least $(2025 - N) - (2025 - e) = e - N$ connected components. We note that $e - N \ge 2N - N = N$. Thus, in this case Veronica will score at least N points.

Case 2: Veronica removes N_2 vertices of degree at least 2 and N_1 vertices of degree 1. Here $N_2 + N_1 = N$. Thus, she removes $e + N_1$ edges, where $e \geq 2N_2$.

The resulting graph has $2025-N$ vertices and $2025-e-N_1$ edges. By the Lemma, this graph must have at least $(2025 - N) - (2025 - e - N_1) = e - N_2$ connected components.

Observe that after removing the first N_2 vertices with associated e edges, the remaining vertices have degrees at most 1, so the resulting graph must consist of isolated vertices and pairs of vertices connected by one edge. In any such graph, the number of vertices is greater or equal twice the number of edges. In our case the graph has $2025 - N_2$ vertices and $2025 - e$ edges, so we must have $2025-N_2 \geq 2(2025-e)$, which simplifies to $2e \geq 2025+N_2$. Then, since $e \geq 2N_2$, we see that $3e \geq 2N_2 + (2025+N_2)$, so $3e - 3N_2 \geq 2025$ and $e - N_2 \geq 675$. Thus, in this case Veronica will score at least 675 points.

In either case, Veronica scores at least $\min(N, 675)$ points, which will be her score because we assume that students follow an optimal strategy as outlined above.

Case 3: Veronica removes at least one vertex of degree 0. This would mean that the graph she ends with will has no edges, so she will score $2025 - N$ points. This cannot happen for $N = 100$ and $N = 1000$ because we assume that students follow an optimal strategy as outlined above.

This completes the justification of our answers in (b) and (c).

In conclusion, I hope that the Reader unfamiliar with graphs has gained an appreciation for the usefulness of these objects in modelling some problem situations. As well, I would like to emphasize the idea that if you cannot answer a question that contains large numbers (e.g. $M = 2025$), you may start by considering much smaller cases (e.g. $M = 9$) in order to get some insight that could be generalized and verified later. Equipped with these heuristics, I encourage future participants of the COMC contest to approach problems with more confidence and a higher level of success. A shorter version of the solution discussed above as well as possible solutions to other COMC problems can be found at the contest webpage `https://cms.math.ca/competitions/comc/comc2025/`.

OLYMPIAD CORNER

No. 439

The problems in this section appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **March 15, 2026**.

OC761. At a school, there are a number of clubs. A club is a set of students. Each club contains at least one student. A student may be in more than one club, but cannot be in every club. Surprisingly, for any two clubs A and B at the school, their union $A \cup B$ is also a club. Is it guaranteed that there is a club containing an even number of students?

OC762. Consider a 2024×2024 grid of unit squares. Two distinct unit squares are *adjacent* if they share a common side. Each unit square is to be colored either black or white. Such a colouring is called *evenish* if every unit square in the grid is adjacent to an even number of black unit squares. Determine the number of evenish colorings.

OC763. In a school, there are 1000 students in each year level, from Year 1 to Year 12. The school has 12000 lockers, numbered from 1 to 12000. The school principal requests that each student is assigned their own locker, so that the following condition is satisfied:

> For every pair of students in the same year level, the difference between their locker numbers must be divisible by their year-level number.

Can the principal's request be satisfied?

OC764. The set of natural numbers from 1 to 1000 inclusive is partitioned into two groups A and B of 500 numbers each. For an integer k, let N_k be the number of pairs (a, b) of a number $a \in A$ and a number $b \in B$ such that $a - b = k$. Prove that:

(a) in any such partition there exists k such that $N_k \geq 126$;

(b) there exists a partition such that $N_k \leq 250$ for every k.

OC765. Given a family $\mathcal{F}$ of 4-element subsets (4-tuples) of a given set of 5^m elements, where m is a fixed natural number. It is known that the intersection of no two 4-tuples in $\mathcal{F}$ consists of exactly two elements. Find the maximum possible value for the number of 4-tuples in $\mathcal{F}$.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mars 2026**.

OC761. Dans une école, il existe un certain nombre de clubs. Un club est un ensemble d'élèves. Chaque club contient au moins un élève. Un élève peut appartenir à plus d'un club, mais ne peut appartenir à tous les clubs. De façon surprenante, pour tous clubs A et B de l'école, leur réunion $A \cup B$ est également un club. Est-il garanti qu'il existe un club contenant un nombre pair d'élèves ?

OC762. Considérons une grille de 2024×2024 carrés unitaires. Deux carrés unitaires distincts sont dits *adjacents* s'ils partagent un côté commun.

Chaque carré unitaire est colorié soit en noir, soit en blanc. Un tel coloriage est dit *pairisant* si chaque carré unitaire de la grille est adjacent à un nombre pair de carrés unitaires noirs.

Déterminez le nombre de coloriages pairisants.

OC763. Dans une école, il y a 1000 élèves à chaque niveau scolaire, de la 1^{re} à la 12^{e} année. L'école dispose de 12000 casiers, numérotés de 1 à 12000. La direction de l'école demande que chaque élève se voie attribuer son propre casier, de sorte que la condition suivante soit satisfaite :

> Pour toute paire d'élèves appartenant au même niveau scolaire, la différence entre les numéros de leurs casiers doit être divisible par le numéro de leur année.

La demande de la direction peut-elle être satisfaite ?

OC764. L'ensemble des nombres naturels de 1 à 1000 inclusivement est partitionné en deux sous-ensembles A et B de 500 nombres chacun. Pour un entier k, on note N_k le nombre de paires (a, b), où $a \in A$ et $b \in B$, telles que $a - b = k$.

Prouvez que :

(a) pour toute partition de ce type, il existe un entier k tel que $N_k \geq 126$;

(b) il existe une partition pour laquelle $N_k \leq 250$ pour tout entier k.

OC765. On se donne une famille $\mathcal{F}$ de sous-ensembles à 4 éléments (quadruplets) d'un ensemble de 5^m éléments, où m est un nombre naturel fixé. On sait que l'intersection de deux quadruplets quelconques de $\mathcal{F}$ ne contient jamais exactement deux éléments.

Déterminez le nombre maximal possible de quadruplets dans $\mathcal{F}$.

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Thanks to Rick Stuhan of Cleveland Ohio for forwarding us the cartoon.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2026: 51(6), p. 275–276.

OC736. Solve in $\mathbb{R}$ the equation $[\log_2 x] = \sqrt{x} - 2$, where $[x]$ denotes the integer part of x.

Originally from the 2012 Transylvanian Hungarian Mathematical Competition.

We received 18 submissions, of which 16 were correct and complete. We present the solution by C R Pranesachar, Homi Bhabha Center for Science Education, Indian Institute of Science, Bengaluru, India and a graph by Henry Diaz Bordon, Universidad Carlos III de Madrid, Madrid, Spain.

Clearly x has to be a positive real number. Since $\lfloor \log_2 x \rfloor$ is a nonnegative integer, so is $\sqrt{x} - 2$. Thus x is a square integer, say $x = n^2$, $n \geq 2$. If $x = 2^2, 3^2, 4^2, 5^2, 6^2$ then

$$\lfloor \log_2 x \rfloor = 2, 3, 4, 4, 5$$

and

$$\sqrt{x} - 2 = 0, 1, 2, 3, 4,$$

respectively. So these are not solutions of the given equation. If $x = 7^2, 8^2$ then

$$\lfloor \log_2 x \rfloor = 5 = \sqrt{x} - 2, \text{ if } x = 7^2$$

and

$$\lfloor \log_2 x \rfloor = 6 = \sqrt{x} - 2, \text{ if } x = 8^2$$

respectively. If $n \geq 9$, then it is easy to see by induction that $2^{n-2} > n^2$. For, if $n = 9$, then

$$2^{n-2} = 2^7 = 128 > n^2 = 9^2 = 81.$$

Further if $2^{n-2} > n^2$ for some $n \geq 9$, then multiplying both sides by 2, we get

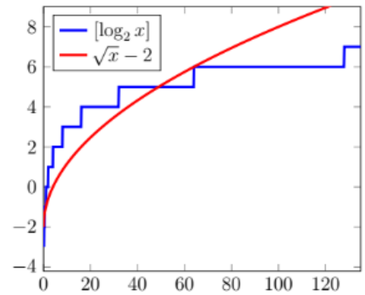
$$2^{(n+1)-2} > 2n^2 > (n+1)^2.$$

This proves our claim. Hence for $x = n^2 \geq 81$, we have

$$\lfloor \log_2 x \rfloor \leq \log_2 x = \log_2 n^2 < \log_2 2^{n-2} = n - 2 = \sqrt{x} - 2.$$

This shows that there are no other solutions and $x = 49, 64$ are the only solutions of the given equation.

The plot of the functions on the left- and right-hand sides of the equation confirms the solutions.



OC737. Find all real solutions of the equation

$$7^{\log_5\left(x^2+\frac{4}{x^2}\right)}+2\left(x+\frac{2}{x}\right)^2=25$$

Originally from the 2012 Transylvanian Hungarian Mathematical Competition.

We received 24 submissions, of which 22 were correct and complete. We present a typical solution.

Let $t=\log_5(x^2+\frac{4}{x^2})$, then $x^2+\frac{4}{x^2}=5^t$. Substituting into the given equation we obtain

$$7^t+2(5^t+4)=25 \iff (7^t-7)+2(5^t-5)=0 \qquad (1)$$

Since a^t for $a>1$ is increasing the only solution to (1) is $t=1$. Thus we have

$$x^2+\frac{4}{x^2}=5 \Longrightarrow x^4-5x^2+4=(x^2-1)(x^2-4)=0$$

Therefore there are 4 solutions for the initial equation: $x=-2$, $x=-1$, $x=1$, and $x=2$.

OC738. Prove that for each $z\in\mathbb{C}$ the following inequality holds

$$|z^2+2z+2|+|z-1|+|z^2+z|\geq 3.$$

When does the equality hold?

Originally from the 2012 Transylvanian Hungarian Mathematical Competition.

We received 9 submissions, of which 7 were correct and complete. We present the solution by Theo Koupelis, Clark College, Washington, USA.

The triangle inequality for complex numbers says that if u and v are complex numbers then $|u|+|v|\geq|u+v|$. Using this inequality for $u=z^2+2z+2$ and $v=-(z^2+z)$ we get

$$|z^2+2z+2|+|z^2+z|\geq|(z^2+2z+2)-(z^2+z)|=|z+2|.$$

Thus, it suffices to show that

$$|z+2|+|z-1| \geq 3.$$

Setting $z = a + ib$, where a, b are real numbers, we get

$$\begin{aligned} |z+2|+|z-1| &= \sqrt{(a+2)^2+b^2}+\sqrt{(a-1)^2+b^2} \\ &\geq |a+2|+|a-1| \geq |(a+2)-(a-1)| = 3, \end{aligned}$$

with equality possible when $b = 0$. Thus, setting $S = |z^2+2z+2|+|z-1|+|z^2+z|$ with $z = a$, a a real number we get

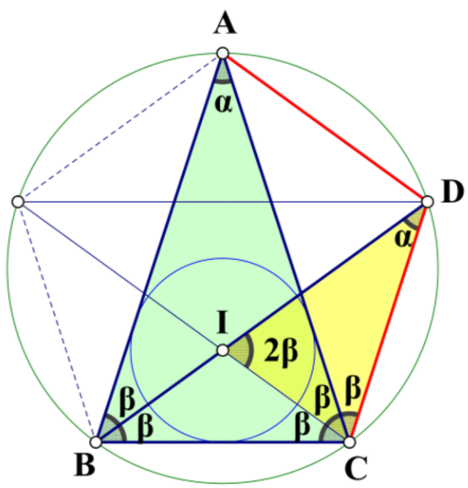
$$\begin{aligned} S &\geq |(a+1)^2+1|+|a-1|+|a(a+1)| \\ &= \begin{cases} (a+1)^2+1+(a-1)+a(a+1) = 2a^2+4a+1 \geq 7, & \text{when } a \geq 1; \\ (a+1)^2+1+(1-a)+a(a+1) = 2a^2+2a+3 > 3, & \text{when } 0 < a < 1; \\ (a+1)^2+1+(1-a)-a(a+1) = 3, & \text{when } -1 \leq a \leq 0; \\ (a+1)^2+1+(1-a)+a(a+1) = 2a^2+2a+3 > 3, & \text{when } a < -1. \end{cases} \end{aligned}$$

Therefore, the inequality holds as an equality when $z = a$, where a is a real number with $-1 \leq a \leq 0$.

OC739. In triangle ABC with $AB = AC$ let I denote the incenter of the triangle. Line BI meets the circumcircle a second time in point D. Find the measures of the angles of the triangle if $BC = ID$.

Originally from the 2012 Transylvanian Hungarian Mathematical Competition.

We received 22 submissions, of which 21 were correct and complete. We present the solution by José Luis Díaz-Barrero.



Since the points A, B, C, D lie on the same circle, we have $\angle BAC = \angle BDC = \alpha$. On the other hand, $\triangle BIC$ is isosceles with $\angle IBC = \angle ICB = \beta$, and therefore $\angle DIC = 2\beta$.

Moreover, $\angle ICA = \angle ICB = \beta$ and $\angle ACD = \angle ABD = \beta$. Thus, $\angle ICD = 2\beta$, and triangle ICD is isosceles. Consequently, $BC = ID = CD = DA$ and $\alpha = \beta$ on account that $\triangle BCD$ is isosceles.

Finally, we compute the angles of triangle ABC. We have $\angle BAC = \alpha$ and $\angle ABC = \angle ACB = 2\alpha$, and since $\alpha+2\alpha+2\alpha = 180^\circ$, it follows that $\angle BAC = 36^\circ$ and $\angle ABC = \angle ACB = 72^\circ$.

OC740. Find the 73rd digit from the end of $111\ldots1^2$, where the number of ones is 2012.

Originally from the 2012 Transylvanian Hungarian Mathematical Competition.

We received 9 submissions, of which 8 were correct and complete. We present two solutions.

Solution 1, by Theo Koupelis.

Let $n = 111\ldots1$, where the number of ones is 2012. We refer to the digits of the number n^2 from right to left. Because 73 is much smaller than 2012, finding the pattern in the digits of n^2 is straightforward. Clearly the first 9 digits of n^2 are $\ldots987654321$, from right to left. The tenth digit is 0, but we now have a carry of one when finding the digits at positions 11 to 18. Thus, the digits at positions 10 to 18 are $\ldots987654320\ldots$. The digit at position 19 is 0, but now we have a carry of two when finding the digits at positions 20 to 27. Therefore, the digits at positions 19 to 27 are $\ldots987654320\ldots$, from right to left. This is the same pattern as before, and this pattern continues, every time with a carry that increases by one. That is, the digits at positions 28 to 36 are $\ldots987654320\ldots$, from right to left, as are the digits at positions 37 to 45, 46 to 54, 55 to 63, and 64 to 72. Thus, the digit of the number n^2 at position 73, from right to left, is zero.

Solution 2 by The Ring Lords - Problem Solving Group.

Let R be the number consisting of 2012 ones. This can be expressed as

$$R = (10^{2012} - 1)/9.$$

The square of this number is:

$$R^2 = \left(\frac{10^{2012}-1}{9}\right)^2 = \frac{(10^{2012}-1)^2}{81}$$

To determine the 73rd digit from the right, we analyze R^2 modulo 10^{73}. Let x be the inverse of 81 modulo 10^{73}, that is

$$81x \equiv 1 \pmod{10^{73}}.$$

Then $R^2 \equiv (10^{2012}-1)^2 x \equiv x \pmod{10^{73}}$. So the 73rd digit from the end of R^2 is equal to the 73rd digit from the end of x.

Since 81 and 10^{73} are relatively prime, there are integers x, y such that

$$81x + 10^{73}y = 1.$$

We use Euclid's algorithm to find a and b:

$$10^{73} = \left\lfloor \frac{10^{73}}{81} \right\rfloor 81 + 10,$$

$$81 = 8 \cdot 10 + 1.$$

So

$$1 = 81 - 8 \cdot 10 = 81 - 8\left(10^{73} - \left\lfloor \frac{10^{73}}{81} \right\rfloor 81\right) = 81\left(1 + 8\left\lfloor \frac{10^{73}}{81} \right\rfloor\right) - 8 \cdot 10^{73}$$

and $x = 1 + 8\left\lfloor \frac{10^{73}}{81} \right\rfloor$ and $y = -8$. Clearly, $0 < x < 10^{72}$, so the 73rd digit from the end of x is a zero. Therefore the 73rd digit from the end of R^2 is a zero.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 15, 2026**.

5101. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with the orthocenter H and the incenter I. Let H_a, H_b and H_c be the orthocenters of $\triangle IBC$, $\triangle IAC$ and $\triangle IAB$, respectively, and let

$$HA + HB + HC = IH_a + IH_b + IH_c.$$

Show that $\triangle ABC$ is an equilateral triangle.

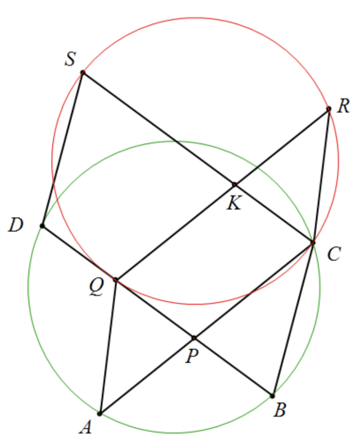
5102. *Proposed by Michel Bataille.*

Let n be a positive integer. Evaluate

$$\sum_{k=1}^{n} \binom{n+k}{2k} (-1)^{k-1} k.$$

5103. *Proposed by Xicheng Peng.*

As shown in the figure, diagonals AC and BD of quadrilateral $ABCD$ intersect at point P. Let Q be a point on line BD (not coinciding with B) such that $PQ = PB$. Construct parallelograms $CAQR$ and $DBCS$. Prove that A, B, C, D are concyclic if and only if Q, C, R, S are concyclic.



5104. *Proposed by Vasile Cirtoaje.*

Prove that 8 is the smallest positive value of k such that

$$\frac{a^2+b^2+c^2+d^2}{4} \le \left(\frac{a+b+c+d}{4}\right)^k$$

for all nonnegative real numbers a, b, c, d with $a \ge b \ge c \ge d$ and $(a+b)(c+d) = 4$.

5105. *Proposed by Tatsunori Irie.*

Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for every $m, n \in \mathbb{N}_0$,

$$f(m^2+mn+n^2) = f(m)^2 + f(m)f(n) + f(n)^2.$$

5106. *Proposed by Minh Ha Nguyen, modified by the Editorial Board.*

Let ABC be a scalene triangle and let (I) be its incircle. A point M varies on (I). Denote by H, K, L the perpendicular projections of M onto BC, CA, AB, respectively. Find the position of M such that the expression $MH + MK + ML$ attains its maximum (or minimum) value.

5107. *Proposed by Nikolai Osipov and Alex Chen.*

The base-b repunits are defined as

$$R_d(b) = 1 + b + \ldots + b^{d-1} = \frac{b^d-1}{b-1}.$$

For positive integers m, k, d, a such that $d \geqslant 2$, $a \geqslant 2$, prove that $R_d(a^m)$ is divisible by $R_d(a^k)$ if and only if m is divisible by k and $\gcd(m/k, d) = 1$.

5108. *Proposed by Huseyin Yigit Emekci.*

Let a, b, c, d be positive real numbers such that $a+b+c+d = 16$. Find the smallest value of

$$T = \frac{a}{b^3+32} + \frac{b}{c^3+32} + \frac{c}{d^3+32} + \frac{d}{a^3+32}$$

5109. *Proposed by Ion Patrascu.*

Let ABC be an isosceles right-angled triangle with $BA = BC$. On the segment BC we consider the point M to be mobile and denote by N its symmetry with respect to C. The points P and Q are the projections of B onto AM and AN respectively. Prove that the line PQ passes through a fixed point.

5110. *Proposed by Paul Bracken.*

Define the sequence of integrals I_n for $n \in \mathbb{N}$ as

$$I_n = \int_0^{\pi/2} (\sin^n x + \cos^n x)^{1/n} dx$$

and let $\Lambda = \lim_{n\to\infty} I_n$. Determine the limit

$$\lim_{n\to\infty} n^2(I_n - \Lambda).$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mars 2026**.

5101. *Soumis par Mihaela Berindeanu.*

Soit ABC un triangle acutangle, d'orthocentre H et de centre du cercle inscrit I. Si $H_a,\ H_b,\ H_c$ sont les orthocentres des triangles IBC, IAC et IAB, respectivement, et si

$$HA + HB + HC = IH_a + IH_b + IH_c,$$

montrez alors que le triangle ABC est équilatéral.

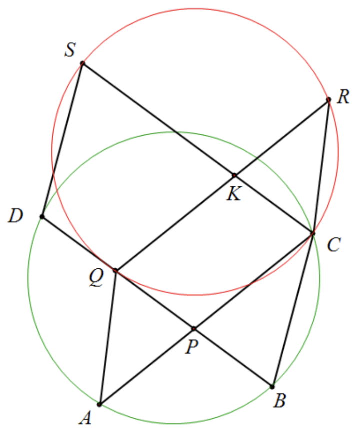
5102. *Soumis par Michel Bataille.*

Soit n un entier strictement positif. Évaluez

$$\sum_{k=1}^{n} \binom{n+k}{2k} (-1)^{k-1} k.$$

5103. *Soumis par Xicheng Peng.*

Comme sur la figure, les diagonales AC et BD du quadrilatère $ABCD$ se coupent en P. Soit Q un point de la droite BD (distinct de B) tel que $PQ = PB$. On construit les parallélogrammes $CAQR$ et $DBCS$. Montrez que A, B, C et D sont concycliques si et seulement si Q, C, R, S sont concycliques.



5104. *Soumis par Vasile Cirtoaje.*

Montrez que 8 est la plus petite valeur positive de k telle que

$$\frac{a^2+b^2+c^2+d^2}{4} \le \left(\frac{a+b+c+d}{4}\right)^k$$

pour tous nombres réels $a, b, c, d \ge 0$ vérifiant $a \ge b \ge c \ge d$ et $(a+b)(c+d) = 4$.

5105. *Soumis par Tatsunori Irie.*

Déterminez toutes les fonctions $f : \mathbb{N}_0 \to \mathbb{N}_0$ telles que, pour tous $m, n \in \mathbb{N}_0$,

$$f(m^2+mn+n^2) = f(m)^2 + f(m)f(n) + f(n)^2.$$

5106. *Soumis par Minh Ha Nguyen, modifié par le comité de rédaction.*

Soit ABC un triangle scalène et soit (I) son cercle inscrit. Un point M varie sur (I). On note H, K et L les projections orthogonales de M sur BC, CA et AB, respectivement. Déterminez la position de M pour laquelle l'expression $MH + MK + ML$ atteint sa valeur maximale (ou minimale).

5107. *Soumis par Nikolai Osipov et Alex Chen.*

Les *répunits* en base b sont définis par

$$R_d(b) = 1 + b + \ldots + b^{d-1} = \frac{b^d - 1}{b - 1}.$$

Pour des entiers strictement positifs m, k, d, a tels que $d \geq 2$ et $a \geq 2$, montrez que $R_d(a^m)$ est divisible par $R_d(a^k)$ si et seulement si m est divisible par k et $\mathrm{PGCD}(m/k, d) = 1$.

5108. *Soumis par Huseyin Yigit Emekci.*

Soient a, b, c et d des nombres réels strictement positifs tels que $a + b + c + d = 16$. Déterminez la plus petite valeur de

$$T = \frac{a}{b^3 + 32} + \frac{b}{c^3 + 32} + \frac{c}{d^3 + 32} + \frac{d}{a^3 + 32}.$$

5109. *Soumis par Ion Patrascu.*

Soit ABC un triangle rectangle isocèle en B, avec $BA = BC$. Sur le segment BC, on considère un point mobile M et l'on note N son symétrique par rapport à C. Les points P et Q sont les projections orthogonales de B sur les droites AM et AN, respectivement. Montrez que la droite PQ passe par un point fixe.

5110. *Soumis par Paul Bracken.*

On définit, pour $n \in \mathbb{N}$, la suite d'intégrales I_n par

$$I_n = \int_0^{\pi/2} (\sin^n x + \cos^n x)^{1/n} dx$$

et l'on pose $\Lambda = \lim_{n\to\infty} I_n$. Déterminez la limite

$$\lim_{n\to\infty} n^2(I_n - \Lambda).$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2025: 51(6), p. 284–287.

5032. *Proposed by Michael Friday, modified by the Editorial Board.*

Let ABC be a triangle with side-lengths a, b, c.

(a) Prove that the associated Euler line is parallel to side BC if and only if

$$(b^2 - c^2)^2 = 2a^4 - a^2(b^2 + c^2).$$

(b) Show that there are non-isosceles triangles satisfying this condition.

(c)* Are there triangles with integer sides satisfying this condition?

We thank M. Bello, M. Benito, Ó. Ciaurri and E. Fernández for pointing out that the featured solution for the starred part (c) of this problem [51(9), p. 435–436] is incomplete. Below is their reasoning.

From the last two lines on the page 435, we read:

> *From the familiar theory of Pythagorean triples we know that the general solution of the equation $x^2 + y^2 = z^2$ over the integers (with y an even number) is given by $(x, y, z) = (m^2 - n^2, 2mn, m^2 + n^2)$.*

The solver applies this to the relation

$$[3(b^2 - c^2)]^2 + (2bc)^2 = k^2$$

previously obtained, where $b > c$ and $(b, c) = 1$, and concludes (p. 436, line 4):

> *Therefore, we must have*
>
> $$\text{(i) } m^2 - n^2 = 3(b^2 - c^2) \quad \text{and} \quad \text{(ii) } mn = bc$$
>
> *where m, n are, without loss of generality, positive integers with $m > n$.*

This is unfortunately inaccurate. The formula

$$(x, y, z) = (m^2 - n^2, 2mn, m^2 + n^2),$$

when $m > n$ are coprime and of opposite parity yields only the primitive Pythagorean triples – that is, the solutions of $x^2 + y^2 = z^2$ in positive integers with x, y, z mutually coprime. But the general solution of the equation $x^2 + y^2 = z^2$ over the positive integers is given by

$$(x, y, z) = \big(\lambda(m^2 - n^2), 2\lambda mn, \lambda(m^2 + n^2)\big),$$

with arbitrary λ and $m > n$ coprime and of opposite parity. For instance, when λ is an even number, both x and y are even numbers.

One cannot assume that, although $b > c$ and $(b, c) = 1$, the Pythagorean triple $\big(3(b^2-c^2), 2bc, k\big)$ will be primitive. Thus the possibility discussed in the published solution eliminates only the case in which $3(b^2 - c^2)$ and $2bc$ are coprime, let us denote it as case (1). In the work we had done with this problem, two further cases still remain to be discussed, namely,

(2) 2 divides $b^2 - c^2$, with b and c both odd, and

(3) 6 divides bc.

Editor's Comments. Of the three submissions that dealt with part (c) of problem 5032, two were incomplete. The third appears to be a correct proof of the nonexistence of an integer-sided triangle whose Euler line is parallel to a side; it will be published at a later date.

5051. *Proposed by Giuseppe Fera.*

A climber starts at altitude 0 at time 0. Until he reaches the mountain top, every second he tosses a biased coin that gives heads with probability p such that $0 < p < 1/2$ and tails with probability $1 - p$. If the coin shows heads, the climber moves up one meter; otherwise, he either moves down one meter or he remains at altitude 0. The mountain top is at an altitude N meters, where N is a positive integer. Find the average climbing time to the top.

We received 8 solutions, all but one correct. We present the solution by Michał Adamaszek.

Denote $q := 1 - p$. For $k \ge 1$ let t_k denote the expected time for advancing from level $k-1$ to level k. Clearly $t_1 = 1/p$ is the expected waiting time for first success in a sequence of Bernoulli trials, and for $k > 1$ we have

$$t_k = 1 + q(t_{k-1} + t_k)$$

as after falling from level k we need to climb again from $k - 1$ and continue the attempts. That leads to

$$t_k = \frac{1}{p} + \frac{q}{p} t_{k-1}$$

and in consequence

$$\begin{aligned} t_k &= \frac{1}{p} + \frac{q}{p}\left(\frac{1}{p} + \frac{q}{p}\left(\cdots \frac{q}{p}\left(\frac{1}{p}\right)\right)\right) = \sum_{i=1}^{k} \frac{q^{i-1}}{p^i} = \frac{1}{p}\sum_{i=0}^{k-1}\left(\frac{q}{p}\right)^i = \\ &= \frac{1}{q-p}\left(\left(\frac{q}{p}\right)^k - 1\right) \end{aligned}$$

This formula covers also $k = 1$. The expected time to advance from level 0 to N is therefore

$$t_1 + \ldots + t_N = \frac{1}{q-p}\left(\sum_{k=1}^{N}\left(\frac{q}{p}\right)^k - N\right) = \frac{1}{q-p}\left(\frac{q}{q-p}\left(\left(\frac{q}{p}\right)^N - 1\right) - N\right).$$

Editor's Comments. All correct solutions recognized (if implicitly) that the problem can be modelled by an absorbing Markov chain with states $0, \ldots, N$ and transition matrix

$$\begin{bmatrix} q & p & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & p & 0 \\ 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

5052. *Proposed by Tatsunori Irie.*

Let p be a prime number with $p \geq 3$ and let m be a natural number that is relatively prime to p and that is not congruent to 1 modulo p. Also, let n be an integer with $n \geq 2$. Define

$$N = \frac{(1+p)^{p^{m-1}} - 1}{p^{m-1}} + p(m-1).$$

Determine whether it is possible for N, when expressed in base n, to be a p-digit number consisting solely of the digit 1.

We did not receive any submission. We present the solution by the proposer, modified by the editor.

For each nonzero integer x, we use $v_p(x)$ to denote the largest integer d such that $p^d \mid x$. Observe that if $1 \leq x < p^{m-1}$, then

$$v_p\big(p^{m-1} - x\big) = v_p(x).$$

By the binomial theorem,

$$(1+p)^{p^{m-1}} - 1 = \sum_{j=1}^{p^{m-1}} \binom{p^{m-1}}{j} p^j. \tag{1}$$

For each $1 \le j \le p^{m-1}$, we have

$$\begin{aligned} v_p\left(\binom{p^{m-1}}{j}p^j\right) &= j + v_p\left(\binom{p^{m-1}}{j}\right) \\ &= j + v_p\left(\frac{\prod_{k=0}^{j-1}(p^{m-1}-k)}{j!}\right) \\ &= j + \sum_{k=0}^{j-1} v_p(p^{m-1}-k) \ - \ \sum_{k=1}^{j} v_p(k) \\ &= j + v_p(p^{m-1}) + \sum_{k=1}^{j-1} v_p(k) \ - \ \sum_{k=1}^{j} v_p(k) \\ &= j + (m-1) - v_p(j) \end{aligned}$$

We claim that for $1 \le j \le p^{m-1}$,

$$m + j - 1 - v_p(j) \ge m,$$

with equality if and only if $j = 1$. Indeed, write $j = p^d s$ with $d = v_p(j)$ and $p \nmid s$. Then

$$m + j - 1 - v_p(j) = m - 1 + p^d s - d \ge m - 1 + p^d - d \ge m,$$

and equality holds only when $d = 0$ and $s = 1$, i.e. $j = 1$.

Thus, by equation (1), we have

$$(1+p)^{p^{m-1}} - 1 \equiv \binom{p^{m-1}}{1} p \equiv p^m \pmod{p^{m+1}}.$$

It follows that

$$N = \frac{(1+p)^{p^{m-1}} - 1}{p^{m-1}} + p(m-1) \equiv p + p(m-1) \equiv pm \pmod{p^2}. \tag{2}$$

We now show the answer to the proposed question is negative. Suppose otherwise that there exists an integer $n \ge 2$ such that N, when expressed in base n, is a p-digit number consisting solely of the digit 1. Write $n = t + 1$. Then

$$N = \frac{n^p - 1}{n - 1} = \frac{(t+1)^p - 1}{t} = \sum_{j=1}^{p} \binom{p}{j} t^{j-1} = t^{p-1} + \sum_{j=2}^{p-1} \binom{p}{j} t^{j-1} + p. \tag{3}$$

Combining (2) and (3), we obtain

$$t^{p-1} + \sum_{j=2}^{p-1} \binom{p}{j} t^{j-1} \equiv p(m-1) \pmod{p^2}. \tag{4}$$

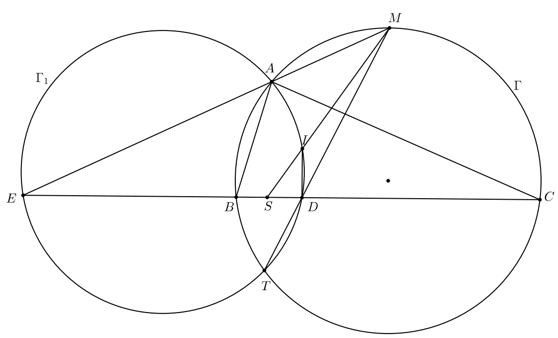
Reducing modulo p gives $t^{p-1} \equiv 0 \pmod{p}$, hence $p \mid t$. Then for $2 \leq j \leq p-1$, since $\binom{p}{j}$ is divisible by p and $p \mid t$, we have

$$v_p\left(t^{p-1}\right) \geq p-1 \geq 2, \qquad v_p\left(\binom{p}{j} t^{j-1}\right) \geq 1+(j-1) \geq 2,$$

so the entire left-hand side of equation (4) is divisible by p^2. Therefore $p^2 \mid p(m-1)$, i.e. $p \mid (m-1)$, so $m \equiv 1 \pmod{p}$, contradicting the assumption $m \not\equiv 1 \pmod{p}$. This contradiction completes the proof.

5053. *Proposed by Michel Bataille.*

Let triangle ABC with $AB \neq AC$ be inscribed in circle Γ and let D be the projection of its incenter I onto BC. Let M be the midpoint of the arc BC of Γ containing A and let the line MI intersect BC at S. Prove that the line AS, the line MD, Γ and the circumcircle of ΔAID have a common point.



There were 10 solutions submitted, of which 8 were established without electronic aid and 2 relied on computational software to crunch through an argument based on cartesian geometry. The solution follows that of Michal Adadamzek.

Without loss of generality, let $AC > AB$. Let N be the midpoint of the arc BC not containing A, so that MN is a diameter of Γ that passes through the midpoint K of BC. Note that AI produced passes through N.

Let Γ_1 be the circumcircle of triangle AID. Suppose that T is the intersection of Γ and Γ_1. Then, since $AIDT$ is concyclic and $DI \| MN$,

$$\angle ATD = 180^\circ - \angle AID = \angle DIN = \angle ANM = \angle ATM.$$

Therefore M, D, T are collinear, and $T = MD \cap \Gamma \cap \Gamma_1$. It remains to show that S lies on the radical axis AT of Γ and Γ_1, so that A, S, T are collinear.

Let Γ_2 be the circle with centre N and radius $NB = NC$. Since $\angle NBC = \angle NAC$ and angle BIN is external to triangle IBA,

$$\angle IBN = \angle NBC + \angle IBC = \angle NAC + \angle IBA = \angle BAI + \angle IBA = \angle BIN,$$

so that $NB = NI$ and I lies on Γ_2. Note that BC is the radical axis of Γ and Γ_2.

Since the right triangles MDK and MNT are similar, as are MNC and MCK,

$$MT \cdot MD = MN \cdot MK = MC^2.$$

Since $\angle MCN = 90^\circ$, MC is tangent to Γ_2, it follows that M has the same power with respect to Γ_1 and Γ_2, and so lies on the radical axis MI of these circles.

The radical centre of the circles Γ, Γ_1 and Γ_2 is common to the lines AT, BC and MI, and must equal S. Hence T belongs to AS produced, and the problem is solved.

Comments by the editor. Alternative arguments can start with different definitions of T and E. Let MD intersect Γ at T and MA intersect CB at E.

Observe that angles MKE and NAM are both right. Hence

$$\begin{aligned}\angle AED = \angle MEK = 90^\circ - \angle EMK = 90^\circ - \angle AMN \\ = \angle ANM = \angle ATM = \angle ATD,\end{aligned}$$

and so A, D, T, E are concyclic.

Also, $\angle EAI = \angle MAI = 90^\circ = \angle EDI$, so that A, I, D, E are concyclic. Therefore T lies on Γ_1.

To show that A, S, T are collinear, the proposer used barycentric coordinators on the triangle with sides a, b, c and semi-perimeter s. Thus, $I = (a : b : c)$, $E = (0 : b : -c)$, $D = (0 : s - c : s - b)$ and $S = (0; b(s - c) : c(s - b))$. Since AT is the radical axis of Γ and Γ_1, it suffices to show that S has the same power with respect to both circles, namely $\overrightarrow{SB} \cdot \overrightarrow{SC} = \overrightarrow{SD} \cdot \overrightarrow{SE}$.

Let $m = b(s - c) + c(s - b)$. Then, since $\overrightarrow{BC} \cdot \overrightarrow{CB} = -a^2$,

$$m\overrightarrow{SB} \cdot m\overrightarrow{SC} = c(s - b)\overrightarrow{CB} \cdot b(s - c)\overrightarrow{BC} = -a^2bc(s - b)(s - c).$$

Since $(s - b) + (s - c) = a$, $m = (b - c)(s - c) + ac = (c - b)(s - b) + ab$, $am\overrightarrow{SD} = (b - c)(s - b)(s - c)\overrightarrow{BC}$, $m(b - c)\overrightarrow{SE} = abc\overrightarrow{CB}$ and

$$m^2\overrightarrow{SD} \cdot \overrightarrow{SE} = -a^2bc(s - b)(s - c).$$

Sicheng Du showed that A, S, T are collinear by using the linearity of the function

$$f(P) = \mathrm{Pow}_{\Gamma_1}(P) - \mathrm{Pow}_{\Gamma}(P)$$

introduced in his paper *Exploring the linearity of power differences in Olympiad geometry* appearing in *Crux* 51:5 (May, 2025), 227-234, Using the fact that the power of a point is equal to $|R^2 - d^2|$, where R is the radius of the circumcircle and d is the distance to the circumcentre and that $d = \sqrt{R^2 - 2Rr}$ when the point is the incentre (and R and r the circumradius and inradius of Γ), he obtained

$$\begin{aligned} f(S) &= \frac{SM}{IM} f(I) - \frac{SI}{IM} f(M) = \frac{SM}{IM}\left(f(I) - \frac{ID}{MN} f(M)\right) \\ &= \frac{SM}{IM}\left(-\mathrm{Pow}_{\Gamma}(I) - \frac{ID}{MK}\mathrm{Pow}_{\Gamma_1}(M)\right) \\ &= \frac{SM}{IM}\left(2Rr - \frac{r}{MK}\cdot MA\cdot ME\right) = 2Rr\cdot\frac{SM}{IM}\cdot\frac{ME}{MK}\left(\frac{MK}{ME} - \frac{MA}{2R}\right) \\ &= 0. \end{aligned}$$

5054. *Proposed by Eugen J. Ionascu, modified by the Editorial Board.*

Find the smallest possible number of 1's in the binary representation of a positive integer which is a multiple of 2025.

We received 14 solutions. We present the solution by Sicheng Du.

Let $t = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_n}$ be a multiple of 2025, where $a_1 < a_2 < \cdots < a_n$ are natural numbers.

Note that t is also a multiple of 15. The set of possible remainders of 2^{a_i} modulo 15 is $\{1, 2, 4, 8\}$. It is easy to check that no combination of 3, 2, or 1 (not necessarily distinct) numbers from the set sum to a multiple of 15. So $n \geq 4$.

For $n = 4$, we have $2^0 + 2^{14} + 2^{15} + 2^{17} = 180225 = 2025 \times 89$.

In conclusion, the smallest possible number of 1's in the binary representation of a positive multiple of 2025 is 4.

Editor's comments. Peter Dombi proposed the following generalization:

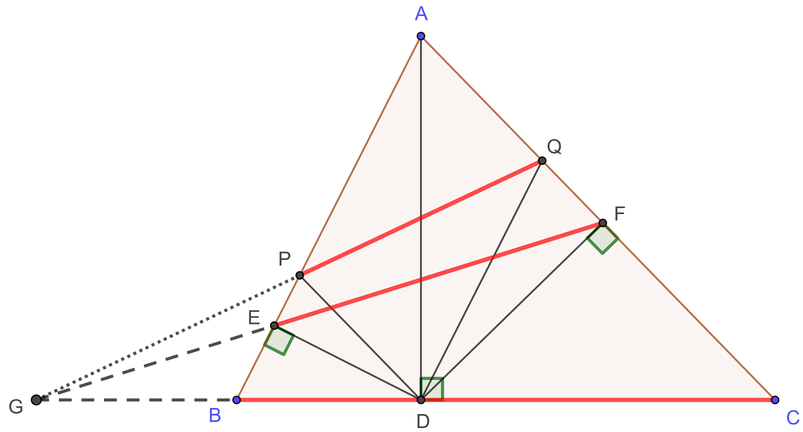
> If $w(n)$ denote the number of 1's in the binary representation of the number n, then
>
> $$w\big((2^k - 1)n\big) \geq k, \qquad (k, n \in N).$$

5055. *Proposed by Bing Jian.*

In triangle ABC, let AD be the altitude from vertex A to side BC. Let $DE \perp AB$ and $DF \perp AC$, with points E and F lying on AB and AC, respectively. Let points P and Q lie on the line AB and the line AC, respectively, such that $DP \parallel AC$ and $DQ \parallel AB$. Prove that the lines PQ, EF, and BC are concurrent or parallel.

We received 16 submissions, of which 15 were correct and complete.

We present the solution by Michal Adamaszek.



Since DF is a height in the right-angled $\triangle ADC$, from $\triangle DFC \sim \triangle ADC$ and $\triangle DFA \sim \triangle CDA$ we get that $\frac{CF}{FA} = \frac{CD^2}{AD^2}$. Similarly, $\frac{AE}{EB} = \frac{AD^2}{BD^2}$.

Let $G = EF \cap BC$; if instead $EF \,||\, BC$ then the calculations below work by replacing $\frac{BG}{GC}$ by 1 in all instances, and replacing the appeal to Menelaus' theorem by the fact that a line meeting two sides of a triangle splits them proportionally if and only if it is parallel to the third side.

By Menelaus' theorem we have

$$\begin{aligned} 1 &= \frac{BG}{GC} \cdot \frac{CF}{FA} \cdot \frac{AE}{EB} \\ &= \frac{BG}{GC} \cdot \frac{CD^2}{AD^2} \cdot \frac{AD^2}{BD^2} = \frac{BG}{GC} \cdot \frac{CD^2}{BD^2}. \end{aligned} \tag{1}$$

From $DQ \,||\, AB$ and $DP \,||\, AC$ we have the proportions

$$\frac{CQ}{QA} = \frac{CD}{DB} \text{ and } \frac{AP}{PB} = \frac{CD}{DB}.$$

Hence

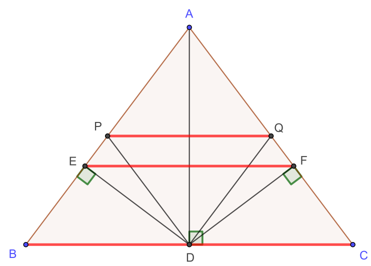
$$\frac{BG}{GC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = \frac{BG}{GC} \cdot \frac{CD^2}{DB^2} = 1,$$

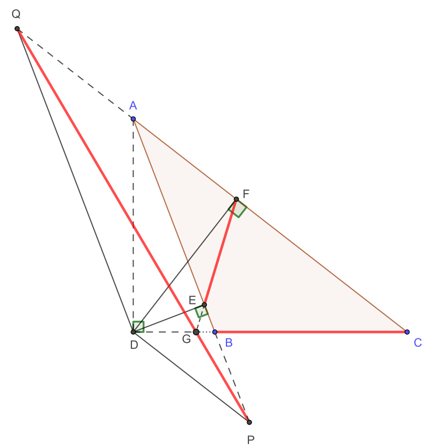
where for the last equality we used (1). Therefore by Menelaus' theorem we conclude that G, P and Q are collinear, and thus the lines BC, EF and PQ all go through the point G (or, in the case when $EF \,||\, BC$, we get that $PQ \,||\, BC$ as well).

Editor's Comments. As pointed out by Chikara Tsugawa, the result holds more generally whenever points E, P on side AB and F, Q on side AC satisfy

$$\frac{AE}{BE} \Big/ \frac{AP}{BP} = \frac{AF}{CF} \Big/ \frac{AQ}{CQ}.$$

We include diagrams for two more possible versions of $\triangle ABC$: when $\triangle ABC$ is isosceles (and $PQ||EF||BC$) and when $\triangle ABC$ is obtuse (and B and C are both on the same side of D).





5056. *Proposed by Mihaela Berindeanu.*

If $a_n = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{n^3}$ $\forall n \in \mathbb{N}^*$, show that $\frac{1}{a_1^2} + \frac{1}{8a_2^2} + \frac{1}{27a_3^2} + \cdots + \frac{1}{n^3 a_n^2} < \frac{6}{5}$.

We received 12 solutions, of which 9 were correct. The remaining involved numerical estimates that were not clearly set out.

Solution 1, by Michal Adamaszek, Oliver Geupel, C.R. Pranesachar, and Chikara Tsugawa (all independently).

Since $a_k \geq a_2 = 9/8$ for $k \geq 2$,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k^2 a_k^2} < 1 + \frac{1}{a_2^2} \sum_{k=2}^{n} \frac{1}{k^3} &< 1 + \left(\frac{8}{9}\right)^2 \left(\frac{1}{8} + \sum_{k=3}^{n} \frac{1}{k^3}\right) \\
&< 1 + \left(\frac{64}{81}\right) \left(\frac{1}{8} + \int_2^\infty \frac{1}{x^3}\, dx\right) \\
&< 1 + \left(\frac{64}{80}\right) \left(\frac{1}{8} + \frac{1}{8}\right) \\
&= 1 + \frac{16}{80} = \frac{6}{5}.
\end{aligned}$$

Solution 2, by Arkady Alt.

Since

$$\frac{1}{k^3} < \frac{1}{(k-1)k(k+1)} = \frac{1}{2}\left(\frac{1}{(k-1)k} - \frac{1}{k(k+1)}\right),$$

for $k \geq 2$,

$$a_n < 1 + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n(n+1)}\right) < \frac{5}{4}.$$

Since $a_k > a_{k-1}$, for $k \geq 2$, we have

$$\frac{1}{k^3 a_k^2} = \frac{a_k - a_{k-1}}{a_k^2} < \frac{a_k - a_{k-1}}{a_k a_{k-1}} = \frac{1}{a_{k-1}} - \frac{1}{a_k}.$$

It follows that

$$S_n = 1 + \sum_{k=2}^{n} \frac{1}{k^3 a_k^2} < 1 + (1 - \frac{1}{a_n}) < 2 - \frac{4}{5} = \frac{6}{5}.$$

Comments by the editor. Alt's solution can be carried a little further to obtain what was originally proposed, namely that

$$a_n < \frac{5n^2+5n-2}{4n(n+1)} \qquad \text{and} \qquad S_n < \frac{6n^2+6n-4}{5n^2+5n-2}.$$

Brian Beasley found by computation that $\lim_{n\to\infty} S_n$ is about 1.155.

5057. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Evaluate

$$\lim_{n\to\infty} n(-1)^n \sum_{k=1}^{n} \frac{(-1)^k}{k(n-k)!}.$$

We received 12 solutions, of which 10 were correct and complete. We present the solution by Michel Bataille, modified by the editor.

Let $S_n = n(-1)^n \sum_{k=1}^{n} \frac{(-1)^k}{k(n-k)!}$. For $n \geq 2$, we have

$$S_n = \sum_{j=0}^{n-1} \frac{n(-1)^j}{j!(n-j)} = \sum_{j=0}^{n-1} \frac{(-1)^j}{j!}\left(1 + \frac{j}{n-j}\right) = \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} + T_n$$

where

$$T_n = \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \frac{j}{n-j} = \sum_{j=1}^{n-1} \frac{(-1)^j}{j!} \frac{j}{n-j} = \sum_{k=0}^{n-2} \frac{(-1)^{k+1}}{k!(n-1-k)} = -\frac{S_{n-1}}{n-1}.$$

At this point, we notice that

$$|S_n| \le \sum_{j=0}^{n-1} \frac{1}{j!} + \sum_{j=1}^{n-1} \frac{1}{(j-1)!(n-j)} \le \sum_{j=0}^{\infty} \frac{1}{j!} + \sum_{j=1}^{\infty} \frac{1}{(j-1)!} \le 2e,$$

so that $(S_n)_{n=1}^{\infty}$ is a bounded sequence. It follows that $T_n = -\frac{S_{n-1}}{n-1} \to 0$ as $n \to \infty$, and consequently

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} = \frac{1}{e}.$$

5058. *Proposed by Vasile Cîrtoaje.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that at most one of them is less than 1 and $a_1^3 + a_2^3 + \cdots + a_n^3 = n$. Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge a_1 + a_2 + \cdots + a_n.$$

We received 10 solutions, of which 9 were correct and complete. We present the solution by Peter Dombi.

We may assume that

i) there is no $a_i = 1$ term, otherwise it can be omitted without affecting the statement;

ii) the case $a_i < 1$ does occur among the terms, otherwise all $a_i > 1$ and the statement is trivial.

Let $a_i = 1 + x_i$, $x_i > 0$ for $i = 1, \ldots, n-1$, and let $a_n = 1 + x_n$ with $-1 < x_n < 0$. We first show that

$$\sum_{i=1}^{n} x_i < 0. \tag{1}$$

Using Jensen's inequality on $f(t) = (1+t)^3$, which is strictly convex on $(-1, \infty)$, we get

$$\left(1 + \frac{\sum_{i=1}^n x_i}{n}\right)^3 < \frac{1}{n}\sum_{i=1}^{n}(1+x_i)^3 = 1$$

from the problem's conditions, hence (1) indeed holds.

From (1), we know $\sum_{i=1}^{n-1} x_i < -x_n$, so

$$\sum_{i=1}^{n-1} x_i^3 < \left(\sum_{i=1}^{n-1} x_i\right)^3 < -x_n^3,$$

as x_i's are positive for $i = 1, \ldots, n-1$. From this, we can conclude that

$$\sum_{i=1}^{n} x_i^3 < 0. \tag{2}$$

To complete the solution, consider the following identity

$$(1+x_i)^2-(1+x_i)=x_i^2+x_i=\frac{1}{3}\big((1+x_i)^3-1-x_i^3\big)\,.$$

Summing these up, we get

$$\sum_{i=1}^{n}\big((1+x_i)^2-(1+x_i)\big)=\frac{1}{3}\left(\sum_{i=1}^{n}(1+x_i)^3-n-\sum_{i=1}^{n}x_i^3\right).$$

Since $\sum_{i=1}^{n}(1+x_i)^3=n$, applying (2), we get

$$\sum_{i=1}^{n}\big((1+x_i)^2-(1+x_i)\big)=-\frac{1}{3}\sum x_i^3>0,$$

which is exactly the required inequality. Moreover, the inequality is strict unless all a_i are equal to 1.

5059. *Proposed by Tatsunori Irie, modified by the Editorial Board.*

[*Corrected version.*] Let T be the Fermat-Torricelli point of triangle ABC with angles less than 120 degrees, that is T is the point such that the sum of the three distances from each of the three vertices of the triangle to the point is the smallest possible. Prove that 3 times Area(ABC) is less than or equal to the area of an equilateral triangle with sides equal to $TA+TB+TC$.

The published statement of the problem (corrected above) was missing the factor 3. Most of the 11 submissions corrected the omission. We feature one of these solutions, a generalization and an upper bound.

Solution 1 is a composite of solutions submitted independently by Michal Adamaszek and by Theo Koupelis.

We use the following three familiar facts:

1. The Fermat-Torricelli point T of a triangle (with angles less that $120°$) is the unique point inside the triangle so that $\angle ATB=\angle ATC=\angle BTC=120°$. [See, for example, Coxeter's *Introduction to Geometry*, sec. 1.8, pp. 21-22.]

2. Setting $x=TA, y=TB, z=TC$ we have the inequality (for all positive numbers x,y,z),
$$3(xy+yz+zy)\le(x+y+z)^2,$$
with equality if and only if $x=y=z$.

3. The area of an equilateral triangle of side length $x+y+z$ is
$$\mathbf{E}=\frac{\sqrt{3}}{4}(x+y+z)^2.$$

Using square brackets to denote areas, we have

$$\begin{aligned}[ABC] &= [ATB] + [ATC] + [BTC] = \frac{1}{2} \cdot \sin 120^\circ \cdot (xy + yz + zx) \\ &= \frac{\sqrt{3}}{4} \cdot (xy + yz + zx) \le \frac{1}{3} \cdot \frac{\sqrt{3}}{4} \cdot (x + y + z)^2 = \frac{1}{3}\mathbf{E},\end{aligned}$$

with equality if and only if the given triangle ABC is equilateral.

Solution 2, by Walther Janous (with the notation changed to agree with Solution 1).

More generally, we shall see that for all points P in the plane of ΔABC, we have $[ABC] \le \frac{1}{3}\mathbf{E}$, where $\mathbf{E}$ is the area of the equilateral triangle with sides $PA + PB + PC$. Indeed, with a, b, c the sides of ΔABC and P any point in its plane, item 12.55 of Bottema et al., *Geometric Inequalities*, says that

$$(PA + PB + PC)^2 \ge \frac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}[ABC],$$

with equality occurring when P is the Fermat-Torricelli point. Together with $a^2 + b^2 + c^2 \ge 4\sqrt{3}[ABC]$ (proof of which will follow) this becomes

$$\mathbf{E} = \frac{\sqrt{3}}{4}(PA + PB + PC)^2 \ge 3[ABC],$$

as claimed. Finally, for the proof that $a^2 + b^2 + c^2 \ge 4\sqrt{3}[ABC]$, start with $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$. The arithmetic-geometric mean inequality then gives us

$$[ABC] \le \sqrt{s}\sqrt{\left(\frac{(s-a) + (s-b) + (s-c)}{3}\right)^3} = \sqrt{\frac{s^4}{27}} = \frac{\sqrt{3}}{4}\left(\frac{a+b+c}{3}\right)^2.$$

The arithmetic mean-root mean square inequality leads to

$$[ABC] \le \frac{\sqrt{3}}{4} \cdot \frac{a^2 + b^2 + c^2}{3},$$

which completes the proof.

Comment by the proposer. Here is the proof of an upper bound, namely $a^2 + b^2 + c^2 \ge (x + y + z)^2$, where x, y, z are the distances from the Fermat-Torricelli point to the vertices A, B, C. Note that with the notation from the previous solutions, this implies that

$$a^2 + b^2 + c^2 \ge \frac{4}{\sqrt{3}}\mathbf{E} \ge 4\sqrt{3}[ABC].$$

With T the Fermat-Torricelli point, the Law of Cosines applied to ΔATB, ΔBTC, ΔCTA (and recalling that $\cos \angle ATB = \cos 120^\circ = -\frac{1}{2}$, etc.) gives us

$$c^2 = x^2 + y^2 + xy, \quad a^2 = y^2 + z^2 + yz \quad b^2 = z^2 + x^2 + zx,$$

whence,

$$a^2+b^2+c^2 = 2(x^2+y^2+z^2)+xy+yz+zx = (x+y+z)^2+(x^2+y^2+z^2)-(xy+yz+zx).$$

Thus,

$$a^2+b^2+c^2-(x+y+z)^2 \geq 0 \Leftrightarrow (x^2+y^2+z^2)-(xy+yz+zx) \geq 0 \Leftrightarrow \frac{1}{2}\sum_{cyclic}(x-y)^2 \geq 0,$$

with equality if and only if $x = y = z$. That is, $a^2 + b^2 + c^2 \geq (x + y + z)^2$, as claimed.

Editor's comments. The proposer also provided a sharper bound, namely

$$ab + bc + ca \geq (x + y + z)^2,$$

but without a satisfactory proof.

5060. *Proposed by Nguyen Van Huyen.*

Find the smallest constant k such that the inequality

$$2(a + b + c + abc) + k[(ab - 1)^2 + (bc - 1)^2 + (ca - 1)^2] \geqslant (a + 1)(b + 1)(c + 1),$$

holds for all non-negative real numbers $a,\ b,\ c$.

We received 7 submissions, 5 of which were correct and complete. We present the solution by Du Sicheng.

When $a = b = c = 0$, we need $3k \geq 1$, or $k \geq \frac{1}{3}$. We will show two proofs of the inequality when $k = \frac{1}{3}$.

Proof 1. Let $p = a + b + c$, $q = ab + bc + ca$, $r = abc$, then the inequality rewrites as

$$\begin{aligned} 2(p + r) + \frac{1}{3}\left(q^2 - 3pr - 2q + 3\right) &\geq 1 + p + q + r \\ F := q^2 - 5q + 3p + (3 - 2p)r &\geq 0. \end{aligned}$$

If $p \leq \frac{3}{2}$, then $3 - 2p \geq 0$. Since $p \geq \sqrt{3q}$, then

$$\begin{aligned} F &\geq q^2 - 5q + 3\sqrt{3q} + 0 = q^2 + 2\cdot\frac{3}{2}\sqrt{3q} - 5q \\ &\geq 3\sqrt[3]{q^2\left(\tfrac{3}{2}\sqrt{3q}\right)^2} - 5q = \frac{9\sqrt[3]{2} - 10}{2}q \geq 0. \end{aligned}$$

If $p > \frac{3}{2}$, then $2p - 3 > 0$. Using $r \leq \frac{q^2}{3p}$ (due to AM-GM inequality), we have

$$F \geq q^2 - 5q + 3p - (2p - 3)\frac{q^2}{3p} \equiv \frac{q^2\left(p^2 - 3q\right) + 3(p - q)^2(3p + q)}{3p^2} \geq 0.$$

The proof is complete.

Proof 2. According to *Proof 1*, we need to prove that

$$q^2 - 5q + 3p + (3 - 2p)r \geq 0 \ (p = a + b + c, q = ab + bc + ca, r = abc),$$

which is linear in r. By the *uvw* theorem (see, for examples, `https://brilliant.org/wiki/the-uvw-method/`, it suffices to prove the inequality when either $a = b$ or $c = 0$.

If $a = b$, we get

$$2a^2c^2 + \left(3a^2 - 10a + 3\right) c + a^4 - 5a^2 + 6a \geq 0.$$

By AM-GM inequality,

$$\begin{aligned} a^4 - 5a^2 + 6a = a^4 + 2 \cdot 3a - 5a^2 &\geq 3\sqrt[3]{a^4 \left(3a^2\right)^2} - 5a^2 \\ &= \left(3\sqrt[3]{9} - 5\right)a \geq 0. \end{aligned} \tag{1}$$

When $a \geq 3$ or $a \leq \frac{1}{3}$ we also have $3a^2 - 10a + 3 \geq 0$, so the inequality holds. If $a \in \left(\frac{1}{3}, 3\right)$, then the discriminant

$$\Delta_c = -(a-1)^2 \left[8a^4 + 15a(a-1)^2 + a^3 + 5a^2 + 9(3a-1)\right] \leq 0.$$

If $c = 0$, we get $a^2b^2 - 5ab + 3a + 3b \geq 0$. By AM-GM inequality,

$$\text{LHS} \geq a^2b^2 + 6\sqrt{ab} - 5ab = \sqrt{ab}^4 - 5\sqrt{ab}^2 + 6\sqrt{ab} \geq 0.$$

The last inequality is the result of substituting $a \to \sqrt{ab}$ in (1).

The proof is complete.