1. Week 1

We give two entry level problems this week.

Problem A

Find all 3-digit integers that are 34 times the sum of their digits.

Solution:

Problem 1 of the XVIII Italian Mathematical Olympiad , which appeared in Crux at [2005:217] . We present the solution by Robert Bilinski that appeared at [2006:386] .

Let abc be the three digit number. Then

$$100a + 10b + c = 34(a + b + c) \implies$$
$$22a - 11c = 8b\,.$$

It follows that b is divisible by 11 and hence $b = 0$.

Next, since $b = 0$ we get

$$c = 2a\,.$$

To get a 3 digit number we must have $a \geq 1$. Moreover, since $c = 2a$ is a digit, $a \leq 4$.

There are 4 such numbers:

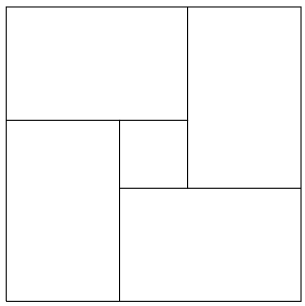
$$102, 204, 306, 408\,.$$

Problem B

Four identical rectangles are arranged in a square pattern so that they enclose a smaller square. Let S be the area of the outer square and Q be the area of the inner square. If

$$\frac{S}{Q} = 9 + 4\sqrt{5}$$

determine the ratio of the sides of the rectangle.



Solution:

Problem M282 of Mayhem that appeared in Crux at [2007:73-74]. We present the solution by multiple solvers that appeared at [2008:71-72].

Let $x > y$ represent the two sides of the rectangle. Then, the outer square has side $x + y$ and the inner square has side $x - y$. It follows that

$$\frac{(x+y)^2}{(x-y)^2} = 9 + 4\sqrt{5} \qquad \Longrightarrow$$

$$\frac{x+y}{x-y} = \sqrt{9+4\sqrt{5}} = \sqrt{4+4\sqrt{5}+(\sqrt{5})^2} = 2+\sqrt{5} \qquad \Longrightarrow$$

$$x + y = (2+\sqrt{5})x - (2+\sqrt{5})y \qquad \Longrightarrow$$

$$(3+\sqrt{5})y = (1+\sqrt{5})x \qquad \Longrightarrow$$

$$\frac{x}{y} = \frac{3+\sqrt{5}}{1+\sqrt{5}} = \frac{(3+\sqrt{5})(\sqrt{5}-1)}{4} = \frac{\sqrt{5}+1}{2}$$

which is the golden mean.

2. Week 2

Problem

Given an alphabet of three letters a, b, c, find the number of words of n letters which contain an even number of a's.

Solution: Problem 4 of the XII Italian Mathematical Olympiad 1996, which appeared in Crux Mathematicorum at [1999:390]. We present the solution by Pierre Bornzstein, which appeared at [2001:430-431], slightly modified.

For each integer k, with $0 \leq 2k \leq n$, let us count the number of words of length n which contain exactly $2k$ a's. The positions of the a's can be chosen in $\binom{n}{2k}$ ways. After filling in the a's, for each of the remaining $n - 2k$ spots we have two choices of letter, b or c.

Therefore, there are exactly

$$\binom{n}{2k} 2^{n-2k}$$

words of length n which contain exactly $2k$ a's. It follows that the number of words of length n with an even number of a's is

$$\sum_{k=0}^{n/2} \binom{n}{2k} 2^{n-2k} .$$

Next, the binomial formula gives

$$3^n = (2+1)^n = \sum_{j=0}^{n} \binom{n}{j} 2^{n-j}$$

$$1^n = (2-1)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j 2^{n-j} .$$

Adding these we get

$$3^n + 1 = 2 \sum_{k=0}^{n/2} \binom{n}{2k} 2^{n-2k} .$$

Therefore, the number of words of length n with an even number of a's is

$$\frac{3^n + 1}{2} .$$

Second Solution by the editor: Let a_n, b_n be the number of n letter words which contain an even and odd number of $a's$, respectively. Then,

$$a_n + b_n = 3^n .$$

Now, consider a word of length $n+1$ which contains an even number of a's. There are two possibilities:

Case 1: The last letter is a.

Then, the word comes from one of the b_n words of length n which contain an odd number of a's, by the addition of an a at the end.

It follows that there are b_n words of length $n+1$ which contain an even number of a's and end in a.

Case 2: The last letter is not a.

Then, the word comes from one of the a_n words of length n which contain an even number of a's, by the addition of a b or a c at the end.

It follows that there are $2a_n$ words of length $n+1$ which contain an even number of a's and do not end in a.

This shows that

$$a_{n+1} = b_n + 2a_n = (3^n - a_n) + 2a_n = 3^n + a_n \,.$$

A simple recursion then gives

$$\begin{aligned} a_n &= 3^{n-1} + a_{n-1} = 3^{n-1} + 3^{n-2} + a_{n-2} \\ &= \ldots \\ &= 3^{n-1} + 3^{n-2} + \ldots + 3^1 + a_1 \\ &= 3^{n-1} + 3^{n-2} + \ldots + 3^1 + 2 \,. \end{aligned}$$

Recalling the geometric formula

$$3^{n-1} + \ldots + 3 + 1 = \frac{3^n - 1}{3 - 1}$$

we get

$$a_n = \frac{3^n + 1}{2} \,.$$

3. Week 3

Problem

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$2f(x) = f(x+y) + f(x+2y)$$

for all real numbers x and all non-negative real numbers y.

Solution:

Problem 1 of the Romania Team Selection Test 2011, Day 1, which appeared in Crux Mathematicorum [2012:405-406]. We present the common solution by Titu Zvonaru and Neculai Stanciu which appeared at [2014:60-61].

Replacing $y \geq 0$ by $2y$ we get

$$2f(x) = f(x+2y) + f(x+4y)\,.$$

Therefore, for all real x and all non-negative y we have

$$f(x+y) + f(x+2y) = 2f(x) = f(x+2y) + f(x+4y) \Longrightarrow f(x+y) = f(x+4y)\,.$$

Now, let $a < b$ be any two real numbers. Solving

$$\begin{cases} x & +y & = a \\ x & +3y & = b \end{cases}$$

we get

$$\begin{cases} y & = \frac{b-a}{2} > 0 \\ x & = \frac{3a-b}{2}\,. \end{cases}$$

Setting these values in $f(x+y) = f(x+4y)$ we get that

$$f(a) = f(b) \qquad \forall a < b\,.$$

It follows that f must be a constant function.

It is easy to check that any constant function $f(x) = c$ satisfies the given equation.

4. Week 4

Problem Find all polynomials of the form x^3+mx+6 who have 3 integer roots.

Solution: Problem 2 of the Mathematics Association of Quebec Contest, Secondary level, 2010, which appeared in Crux Mathematicorum at [2010:417–419]. We present the solution suggested in the editor's comment, which appeared at [2011:262].

Let $a, b, c \in \mathbb{Z}$ be the three solutions. Then,

$$\begin{aligned} x^3+mx+6 &= (x-a)(x-b)(x-c) \\ &= x^3-(a+b+c)x^2+(ab+ac+bc)x-abc\,. \end{aligned}$$

It follows that

$$\begin{aligned} a+b+c &= 0 \\ abc &= -6\,. \end{aligned}$$

Since abc is negative, exactly one or three of a, b, c are negative. Since the sum is zero, it follows that exactly one of a, b, c are negative.

By eventually relabeling them, we can assume without loss of generality that

$$a < 0 < b \leq c\,.$$

Since $abc = -6$ we have $a \in \{-6, -3, -2, -1\}$ and $b, c \in \{1, 2, 3, 6\}$.

Case 1: $a = -1$. Then $bc = 6$ and $a+b+c \neq 0$.

Case 2: $a = -2$. Then $bc = 3$ and hence $b = 1, c = 3$ and again $a+b+c \neq 0$.

Case 3: $a = -3$. Then $bc = 2$ and hence $b = 1, c = 2$. It follows that

$$P(x) = (x-1)(x-2)(x+3) = x^3-7x+6$$

and hence $m = 7$.

Case 4: $a = -6$. Then $bc = 1$ and hence $b = c = 1$ and again $a+b+c \neq 0$.

Thus, $m = -7$.

5. Week 5

Problem

Determine all pairs (x, k) of positive integers which satisfy

$$3^k - 1 = x^3 \, .$$

Solution: Problem 6(a) of the XV Gara Nazionale di Matematica 1999, which appeared in Crux Mathematicorum [2002:482]. We present the solution by Michael Bataille, which appeared at [2005:39], modified by the editor.

We have

$$3^k = x^3 + 1 = (x+1)(x^2 - x + 1) \, .$$

Therefore, there exists some positive integer n so that

$$3^n = x^2 - x + 1 \, .$$

Let r be the reminder when x is divided by 3. We split the problem into two cases:

Case 1: $r = 0$ or $r = 1$. Then 3 divided $x(x-1) = x^2 - x$ and hence $x^2 - x + 1$ is not divisible by 3.

It follows that 3^n is not divisible by 3 and hence $n = 0$. This implies that $x = 0$ or $x = 1$. Solving for k, we get that k is not integer.

There is no solution in this case.

Case 2: $r = 2$. Then, there exists an integer m so that $x = 3m + 2$. It follows that

$$3^n = (3m+2)^2 - (3m+2) + 1 = 9m^2 + 12m + 4 - 3m - 2 + 1 = 9(m^2 + m) + 3 \, .$$

Since the right hand side is divisible by 3 but not divisible by 9, it follows that $n = 1$. Therefore

$$3 = 3^1 = x^2 - x + 1 \Longrightarrow x = -1 \text{ or } x = 2 \, .$$

Since x is am positive integer, we get $x = 2$. Plugging in into the original equation we get

$$3^k - 1 = 2^3 = 8 \Longrightarrow k = 2 \, .$$

It follows that the only solution is $k = 2, x = 2$.

6. Week 6

Problem

Let m, n be two positive integers so that m^2+n^2-m is divisible by $2mn$. Prove that m is the square of an integer.

Solution:

Problem 1 of the second day of Swiss Mathematical Olympiad (1999), which appeared in Crux Mathematicorum [2002:130]. We present the solution by Pierre Bornsztein which appeared at [2004:282], expanded by the editor.

There exists an integer k such that

$$m^2+n^2-m=2kmn$$

or equivalently

$$n^2-2kmn+m^2-m=0\,.$$

Solving for n we get

$$n_{1,2}=\frac{2km\pm\sqrt{4k^2m^2-4m^2+4m}}{2}\in\mathbb{Z}\,.$$

It follows that

$$4k^2m^2-4m^2+4m=4m(k^2m-m+1)\,,$$

is a perfect square.

To show that m is the square of an integer, we need to show that every prime p appears at an even power in m.

Let p be any prime divisor of m and let $j\geq 1$ be the power of p in m.

We split the problem into two cases.

Case 1: $p=2$.

In this case m is even and hence k^2m-m+1 is odd. Therefore, the power of 2 in $4m(k^2m-m+1)$ is $2+j$. Since $4m(k^2m-m+1)$ is a perfect square, we get that $j+2$ is even and hence j is even.

Case 2: $p\neq 2$.

Then, p does not divide 4. Moreover, p divides m and hence p divides k^2m-m. It follows that p does not divide k^2m-m+1.

Therefore, the power of p in $4m(k^2m-m+1)$ is j. Since $4m(k^2m-m+1)$ is a perfect square, j is even.

This proves the claim.

Editor's note: In the second part of the above prove, we argue that for integers a, b if $4ab$ is a perfect square and a, b are relatively prime, then a (and b) must be perfect square.

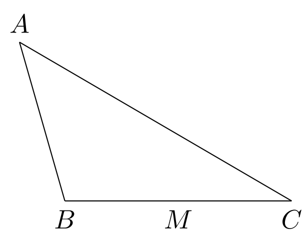
7. Week 7

Problem

In triangle ABC we have $\angle BAC = 45°$ and $\angle ACB = 30°$. Let M be the midpoint of the side BC.

Prove that $\angle AMB = 45°$ and that

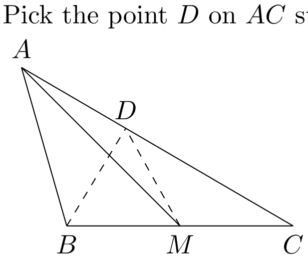
$$BC \cdot AC = 2AM \cdot AB\,.$$



Solution:

Problem 5 of the Olimpiada Matemática Española (2005), which appeared in Crux Mathematicorum [2008:342]. We present the solution by George Apostolopoulos which appeared at [2009:442], slightly modified.

Pick the point D on AC such that $BD \perp AC$.



Since $\angle ADB = 90°$ and $\angle BAD = 45°$, the triangle ABD is isosceles. Therefore

$$AD = DB\,.$$

Next, since triangle BDC is a right triangle, the centre of the circumcircle is the midpoint M of the hypotenuse. Therefore

$$MB = MC = MD\,.$$

Moreover, since BDC is a right triangle and $\angle BCD = 30°$ we have

$$BD = \frac{1}{2}BC = BM\,.$$

Therefore

$$AD = BD = MB = MC = MD\,.$$

As BMD is an equilateral triangle, we get

$$\begin{aligned}
\angle BDM &= 60^\circ \\
\angle ADM &= 150^\circ \\
\angle DMA &= \frac{1}{2}(180^\circ - 150^\circ) = 15^\circ \\
\angle AMB &= 60^\circ - 15^\circ = 45^\circ\,.
\end{aligned}$$

This shows the first part of the problem.

Next, let us note that

$$\angle AMC = 180^\circ - 15^\circ - 30^\circ = 135^\circ\,.$$

Then, the Law of Sines gives

$$\begin{aligned}
\frac{AB}{\sin(30^\circ)} &= \frac{BC}{\sin(45^\circ)} \qquad \text{and} \\
\frac{AC}{\sin(135^\circ)} &= \frac{AM}{\sin(30^\circ)}\,.
\end{aligned}$$

These simplify to

$$\begin{aligned}
\frac{AB}{BC} &= \frac{1}{\sqrt{2}} \\
\frac{AM}{AC} &= \frac{1}{\sqrt{2}}\,.
\end{aligned}$$

Therefore,

$$BC \cdot AC = (AB\sqrt{2})(AM\sqrt{2}) = 2AB \cdot AM\,.$$

8. Week 8

Problem Find all integers n with the property that we can colour all the edges

and diagonals of a convex n-gon with n given colours, subject to satisfying both following conditions:

(i) Every one of the edges or diagonals is coloured by only one colour;
(ii) For any three distinct colours, there exists a triangle whose vertices are vertices of the n-gon and the three edges are coloured by the three colours, respectively.

Solution:

Problem 5 of the 2009 Chinese Mathematical Olympiad, which appeared in Crux Mathematicorum [2011:353-354]. We present the solution by Oliver Geupel which appeared at [2012:318-319], modified by the editor.

We prove that an integer $n \geq 3$ has the desired property if and only if n is odd.

First, let n be an integer which satisfies this property. Since there are $\binom{n}{3}$ choices of three colours and $\binom{n}{3}$ triangles made by the vertices of the polygon, any two such triangles must have different colours of the edges. Moreover, no triangle can contain two edges of the same colour.

Now, fix one colour. There are exactly $\binom{n-1}{2}$ triples of colours containing this fixed colour. This means that this fixed colour appears in exactly $\binom{n-1}{2}$ triangles.

Now, let k be the number of edges of this fixed colour. Since no triangle can contain two edges of the same colour, for each of the k edges we get exactly $n-2$ triangles containing that edge. Since these triangles are distinct, in total we have $k \cdot (n-2)$ triangles containing an edge of this fixed colour. It follows that

$$\binom{n-1}{2} = k \cdot (n-2) \qquad \Longrightarrow$$
$$n-1 = 2k$$

and hence n is odd as claimed.

Next, assume that n is odd.

We will use the numbers $\pmod{n}$ as the colours: we colour the edge/diagonal P_iP_j with colour $i+j \pmod{n}$.

Now, let $i, j, k \pmod{n}$ be distinct numbers and let $n = 2p-1$. Note here that

$$2p = n+1 \equiv 1 \pmod{n}.$$

Pick $1 \leq l, m, n$ so that

$$\begin{cases} l & \equiv p(i+j-k) \pmod{n} \\ m & \equiv p(i+k-j) \pmod{n} \\ n & \equiv p(j+k-i) \pmod{n} \end{cases} .$$

Then, l, m, n are distinct and the edges of the triangle $P_l P_m P_n$ are coloured with colours:

$$\begin{aligned} l + m &\equiv 2pi \equiv i \pmod{n} \\ l + n &\equiv 2pj \equiv j \pmod{n} \\ m + n &\equiv 2pk \equiv k \pmod{n}. \end{aligned}$$

This completes the proof.

Editor note: There is a nice geometrical way of interpreting this colouring for odd n: when n is odd, the regular n-gon $P_1 P_n \dots P_n$ has the property that each diagonal is parallel to exactly one edge of the n-gon. We can then colour the n edges $P_1P_2, P_2P_3, \dots, P_nP_1$ with n distinct colours, and colour each diagonal with the same colour as the unique edge to which it is parallel.

In this colouring scheme, P_iP_j and P_kP_l have the same colour if and only if $P_iP_j \parallel P_kP_l$.

This immediately implies that each triangle $P_iP_kP_j$ must have edges of three different colours. Moreover, two distinct triangles $P_iP_jP_k$ and $P_lP_mP_n$ cannot have the edges coloured same way since in this case they would be parallel (*Try to figure out why this is impossible for odd* n). Therefore, each of the $\binom{n}{3}$ triangles must use a different one of the $\binom{n}{3}$ potential colour triple. It follows that all colour triples are actually used.