

Week 1

- [Problem](#) (posted September 3rd)

We give two entry level problems this week. Give them a try. Look for the source and the solution next week!

Problem A

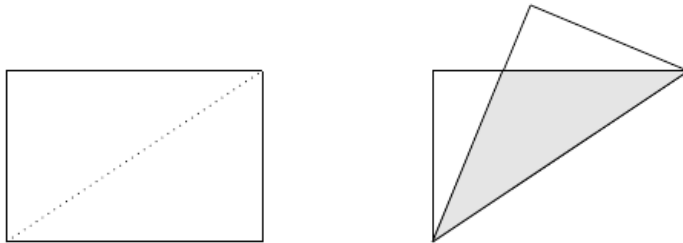
Let x be an integer of the form

$$x = \underbrace{111\dots11}_{n \text{ times}}$$

Show that, if x is prime then n is prime.

Problem B

Given a rectangle with sides a and b , with $a > b$, fold it along the diagonal. Determine the area of the overlapping triangle (the shaded area in the picture).



- [Solution](#) (posted September 10th)

Problem A

Problem 7 of the Chilean Mathematical Olympiads 1994-95, which appeared in *Crux Mathematicorum* [2001: 170]. We present the solution by Christopher J. Bradley that appeared at [2003: 294].

Note: A [video solution](#) to this problem is also available.

Note that

$$9x = \underbrace{999\dots99}_{n \text{ times}} = 10^n - 1.$$

Therefore,

$$x = \frac{10^n - 1}{9}.$$

Now, if we assume by contradiction that n is composite, $n = n_1 n_2$ for some $n_1, n_2 > 1$. Then

$$x = \frac{10^n - 1}{9} = \frac{10^{n_1} - 1}{9} (1 + 10^{n_1} + 10^{2n_1} + \dots + 10^{n_1(n_2-1)}).$$

As $\frac{10^{n_1} - 1}{9}$ is an integer greater than 1 and $1 + 10^{n_1} + 10^{2n_1} + \dots + 10^{n_1(n_2-1)} > 1$ we get that x is composite, a contradiction.

Since, we got a contradiction, our assumption that n is composite is wrong.

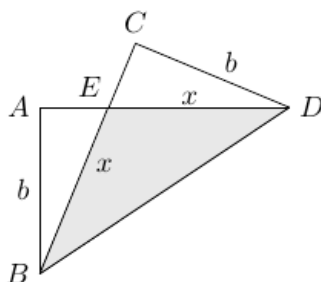
Therefore, n is not composite and $n \neq 1$ (since $x = 1$ is not prime), which proves that n is prime. ■

Editor's note: Primes of this form are sometimes called "repunit primes". So far 5 repunit primes are known, namely the values of x for $n \in \{2, 19, 23, 317, 1031\}$. The values of x for $n \in \{49081, 86453, 109297, 270343\}$ are also believed to be prime.

Problem B

Problem 1 of the XV Gara Nazionale di Matematica 1999, which appeared in *Crux Mathematicorum* [2002: 481]. We present the solution by Bob Serkey that appeared at [2005: 37].

Note: A [video solution](#) to this problem is also available.



Let E be the intersection of BC and AD , and let $x = BE$.

Note that the right triangles $\triangle AEB$ and $\triangle CED$ have $CD = AB$ and $\angle AEB = \angle CED$, and thus are congruent. Therefore, we have

$$DE = BE = x.$$

We therefore also have

$$AE = CE = a - x.$$

By Pythagoras theorem we have

$$(a - x)^2 + b^2 = x^2 \Rightarrow x = \frac{a^2 + b^2}{2a}.$$

In the triangle $\triangle BED$, $DC = b$ is the altitude to the base $BE = x$. Therefore

$$\text{Area}(\triangle BED) = \frac{bx}{2} = \frac{b}{4a}(a^2 + b^2).$$

Week 2

- [Problem](#) (posted September 10th)

How many subsets with three elements can be formed from the set $\{1, 2, 3, \dots, 20\}$ such that 4 is a factor of the product of the three numbers in the subset?

- [Solution](#) (posted September 17th)

Problem 1 of the Icelandic Mathematical Contest 2004-05, which appeared in Crux Mathematicorum [2008: 285]. We present the solution by John G. McLoughlin which appeared at [2009: 386].

Note: A [video solution](#) to this problem is also available.

There are $\binom{20}{3} = 1140$ subsets of three elements in total. We count how many do not meet the requirements.

We first note that a subset meets the requirements excepting in two cases:

- all elements are odd.
- two elements are odd, and the third is even but not a multiple of 4.

In the first case, there are $\binom{10}{3} = 120$ subsets consisting of only odd numbers.

In the second case, there are $\binom{10}{2}\binom{5}{1} = 225$ subsets consisting of two odd numbers and one even number not divisible by 4.

Therefore, in total there are

$$1140 - 120 - 225 = 795$$

such subsets. ■

Week 3

- Problem (posted September 17th)

Define the sequence a_1, a_2, a_3, \dots as follows

$$a_1 = a_2 = 1003$$

$$a_{n+1} = a_n - a_{n-1} \quad \text{for all } n \geq 2.$$

Calculate the sum of the first 2006 terms in the sequence.

- Solution (posted September 24th)

Problem 3 of the Mathematical Olympiad of the Asociación Venezolana de Competencias Matemáticas, 2006, which appeared in Crux Mathematicorum [2009: 380]. We present the solution by Oliver Geupel, which appeared at [2011: 278].

Note: A [video solution](#) to this problem is also available.

We have

$$a_3 = a_2 - a_1 = 0$$

$$a_4 = a_3 - a_2 = -1003$$

$$a_5 = a_4 - a_3 = -1003$$

$$a_6 = a_5 - a_4 = 0$$

$$a_7 = a_6 - a_5 = 1003 = a_1$$

$$a_8 = a_7 - a_6 = 1003 = a_2$$

It follows from here that the sequence is periodic, with period 6. Moreover, the sum of 6 consecutive terms is zero.

Therefore, as $2004 = 6 \cdot 334$ we get

$$\begin{aligned} \sum_{k=1}^{2006} a_k &= a_1 + a_2 + \sum_{j=0}^{333} (a_{6j+3} + a_{6j+4} + a_{6j+5} + a_{6j+6} + a_{6j+7} + a_{6j+8}) \\ &= a_1 + a_2 \\ &= 2006. \quad \blacksquare \end{aligned}$$

Week 4

- [Problem](#) (posted September 24th)

If

$$(x + \sqrt{x^2 + 1})(y + \sqrt{y^2 + 1}) = 1$$

show that $x + y = 0$.

- [Solution](#) (posted October 1st)

Problem 2 of the 31st Spanish Mathematical Olympiad, First Round, which appeared in *Crux Mathematicorum* [1998: 453]. We present the solution by Mohammed Aassila, which appeared at [2000: 142], modified by the editor.

Note: A [video solution](#) to this problem is also available.

Set $z = x + \sqrt{x^2 + 1}$. Then, $z > 0$ and

$$\frac{1}{z} = \frac{\sqrt{x^2 + 1} - x}{x^2 + 1 - x^2} = \sqrt{x^2 + 1} + x - 2x = z - 2x,$$

giving

$$x = \frac{z^2 - 1}{2z}.$$

Then,

$$\begin{aligned} z(y + \sqrt{y^2 + 1}) &= 1 &\Rightarrow \\ y + \sqrt{y^2 + 1} &= \frac{1}{z} &\Rightarrow \\ z = \frac{\sqrt{y^2 + 1} - y}{y^2 + 1 - y^2} &= \sqrt{y^2 + 1} + y - 2y = \frac{1}{z} - 2y \Rightarrow \\ y &= \frac{1 - z^2}{2z}. \end{aligned}$$

Hence

$$x + y = \frac{z^2 - 1}{2z} + \frac{1 - z^2}{2z} = 0.$$

Editor's note: If you are familiar with logarithms and inverse hyperbolic functions, you will probably like Michel Bataille's solution: Taking logarithms and using $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$ the equation becomes

$$\sinh^{-1}(x) + \sinh^{-1}(y) = 0.$$

Since $\sinh^{-1}(x)$ is an odd function, and one-to-one, we get

$$\sinh^{-1}(x) = \sinh^{-1}(-y) \Rightarrow x = -y.$$

Week 5

- [Problem](#) (posted October 1st)

Find all the triplets (x, y, z) consisting of positive integers, which satisfy the equation

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1$$

- [Solution](#) (posted October 8th)

Problem 3 of the 26th Belgian Mathematical Olympiad, which appeared in *Crux Mathematicorum* [2004: 268]. We present the solution by Pierre Bornshtein, which appeared at [2006: 153].

Note: A [video solution](#) to this problem is also available.

Let (x, y, z) be any solution to the equation. Then

$$yz + 2xz = 3xy + xyz. \quad (1)$$

If $x \geq 3$ and $y \geq 3$ then $3(x-1) \geq 2x$ and hence

$$3xy + xyz > xyz = yz + (x-1)yz \geq yz + 3(x-1)z \geq yz + 2xz,$$

which contradicts (1). Therefore, $x < 3$ or $y < 3$.

- First suppose that $y \geq 3$. Then $x = 1$ or $x = 2$.
If $x = 2$ then

$$yz + 4z = 6y + 2yz \Rightarrow yz + 6y - 4z = 0 \Rightarrow (y-4)(z+6) = -24.$$

Since $y-4 \geq -1$ and $z+6 \geq 7$ we get $y-4 = -1$ and $z+6 = 24$ giving the solution

$$(x, y, z) = (2, 3, 18).$$

It is easy to check that this is a solution.

If $x = 1$ then

$$yz + 2z = 3y + yz \Rightarrow 3y = 2z$$

giving the solutions

$$(x, y, z) = (1, 2a, 3a),$$

for an integer $a \geq 2$ (since $y \geq 3$).

It is easy to check that all these are solutions.

- If $y = 2$ then the equation becomes

$$2z + 2xz = 6x + 2xz \Rightarrow z = 3x.$$

This gives the solutions

$$(x, y, z) = (b, 2, 3b),$$

for an integer $b \geq 1$. Note that for $b = 1$ we get the solution $(1, 2, 3)$, which corresponds to $a = 1$ in the previous paragraph.

- If $y = 1$ then the equation becomes

$$z + 2xz = 3x + xz \Rightarrow xz - 3x + z = 0 \Rightarrow (x+1)(z-3) = -3.$$

Since $x+1 \geq 2$ we have $x+1 = 3, z-3 = -1$, giving the solution

$$(x, y, z) = (2, 1, 2).$$

Conversely, the triple $(2, 1, 2)$ is a solution.

Therefore, the solutions are all the triples of the form $(1, 2a, 3a), (a, 2, 3a)$ for positive integer a , together with $(2, 1, 2)$ and $(2, 3, 18)$.

Week 6

- [Problem](#) (posted October 8th)

Find all the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with the property that for all $x, y \in \mathbb{R}$ we have

$$f(xf(x) + f(y)) = (f(x))^2 + y. \quad (1)$$

- [Solution](#) (posted October 15th)

Note: A [video solution](#) to this problem is also available.

Problem 1 of the 17th Balkan Mathematical Olympiad, which appeared in *Crux Mathematicorum* [2004: 84]. We present the solution by José Luis Díaz-Barrero, which appeared at [2005: 523-524].

It is easy to see that the function $f(x) = x$ for all $x \in \mathbb{R}$ and the function $f(x) = -x$ for all $x \in \mathbb{R}$ satisfy (1). We show that these are the only solutions.

Let $a = f(0)$. Setting $x = 0$ in (1) we get

$$f(f(y)) = y + a^2. \quad (2)$$

Pick $b = f(-a^2)$. Then $f(b) = f(f(-a^2)) = -a^2 + a^2 = 0$.

Setting $x = b$ in (1) we get

$$f(f(y)) = y. \quad (3)$$

By combining (2) and (3) we get $a = 0$, and hence $f(0) = 0$. Setting $y = 0$ in (1) we get

$$f(xf(x)) = (f(x))^2. \quad (4)$$

Now, let $t \in \mathbb{R}$ be arbitrary. Setting $x = f(t)$ in (4), and using the fact that by (3) we have $f(x) = t$ we get

$$f(tf(t)) = (f(f(t)))^2 = t^2. \quad (5)$$

Therefore, for all $x \in \mathbb{R}$ we have by (4) and (5)

$$x^2 = f(xf(x)) = (f(x))^2. \quad (6)$$

This gives that $f(x) = \pm x$ for each $x \in \mathbb{R}$. Therefore, for each $x \in \mathbb{R}$ we have either $f(x) = x$ or $f(x) = -x$. To complete the solution we need to show that the sign is the same for all x .

Assume by contradiction that there exist non-zero numbers α, β such that $f(\alpha) = \alpha$ and $f(\beta) = -\beta$. Then, setting $x = \alpha, y = \beta$ in (1) we get

$$f(\alpha^2 - \beta) = \alpha^2 + \beta,$$

and hence, by (6)

$$(\alpha^2 - \beta)^2 = (\alpha^2 + \beta)^2.$$

Expanding, we get $\alpha^2\beta = 0$ which contradicts the fact that α, β are non-zero.

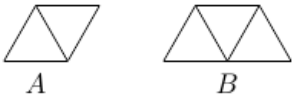
Since we got a contradiction, our assumption was wrong. Therefore, the only solutions are the function $f(x) = x$ for all $x \in \mathbb{R}$ and the function $f(x) = -x$ for all $x \in \mathbb{R}$.

Editor's note: This problem is similar to the 2014 COMC Problem of the Week 4, see <https://comc.math.ca/2014/potw.html>

Week 7

- [Problem](#) (posted October 15th)

An equilateral triangle of side length 6 is divided into 36 small equilateral triangles of side length 1. We construct pieces of type A and type B by joining triangles of side length 1, as in the diagram below. By using m pieces of type A and n pieces of type B, with no overlapping and no gaps, like the pieces of a jigsaw puzzle, we can construct the original large triangle. Determine all possible values of m .



- [Solution](#) (posted October 22nd)

Problem 2 Olimpiada Nacional Escolar de Matematica 2009, Level 2, which appeared in *Crux Mathematicorum* at [2010: 373]. We present the solution by Chip Curtis, which appeared at [2011: 425-426], modified by the editor.

Note: A [video solution](#) to this problem is also available.

Since each piece of type A covers two triangles, and each piece of type B covers three triangles, we have

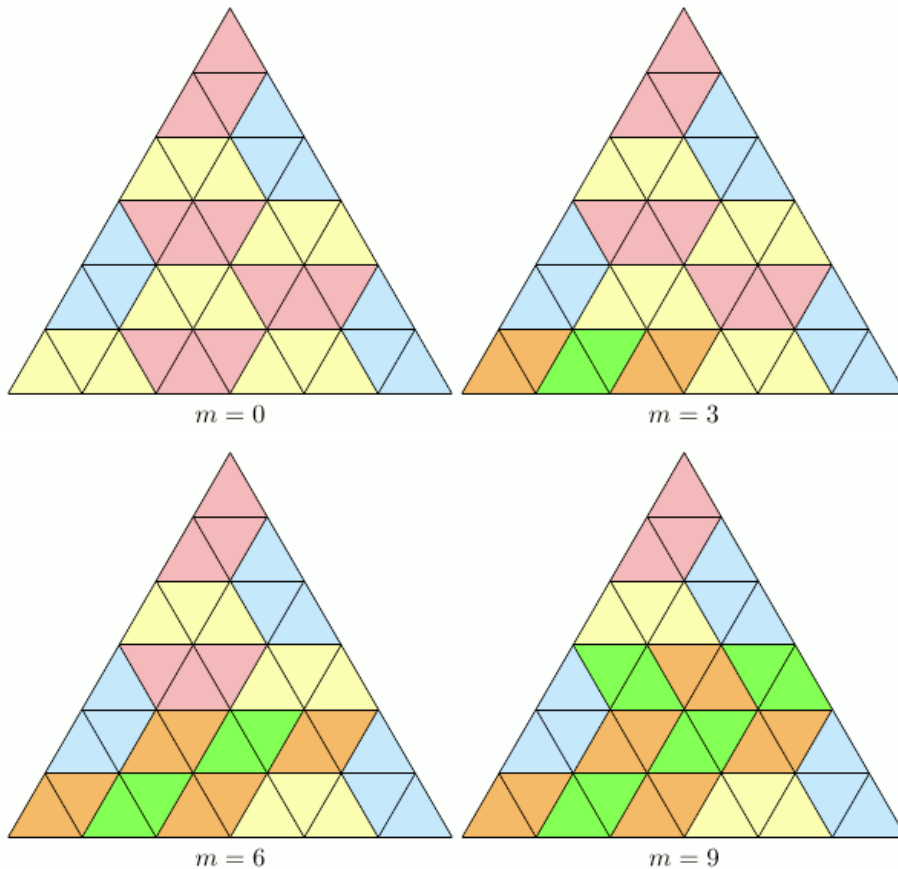
$$2m + 3n = 36.$$

This yields

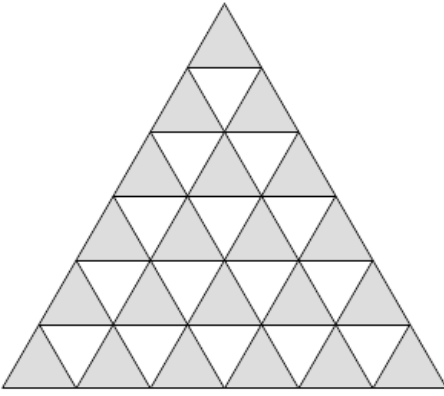
$$m \in \{0, 3, 6, 9, 12, 15, 18\}.$$

We prove that $m = 0, 3, 6, 9$ are possible, while $m = 12, 15, 18$ are impossible.

The following pictures show that $m = 0, 3, 6, 9$ are possible.



To see that $m = 12, 15, 18$ are impossible, color the small triangles with two colors alternately.



Note that any piece of type A covers exactly one white and one gray triangle. Any piece of type B covers either two white and one gray triangles, or one white and two gray triangle.

Since there are 6 more gray triangle than white triangles, we need to use at least 6 pieces of type B, that is $n \geq 6$. Therefore

$$m = \frac{36 - 3n}{2} \leq \frac{36 - 18}{2} = 9 .$$

This shows that

$$m \in \{0, 3, 6, 9\} .$$

Week 8

- [Problem](#) (posted October 22nd)

Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be $2n$ real numbers such that a_1, a_2, \dots, a_n are distinct. If there exists a real number α such that for each $1 \leq j \leq n$ we have

$$(a_j + b_1) \cdot (a_j + b_2) \cdot \dots \cdot (a_j + b_n) = \alpha,$$

show that there exists a number β such that for each $1 \leq k \leq n$ we have

$$(a_1 + b_k) \cdot (a_2 + b_k) \cdot \dots \cdot (a_n + b_k) = \beta.$$

- [Solution](#) (posted October 29th)

*Problem 2 of the Second paper, Sixth Irish Mathematical Olympiad, which appeared in *Crux Mathematicorum* [1995: 152]. We present the solution by Michael Selby, which appeared at [1997: 141-142].*

Note: A [video solution](#) to this problem is also available.

Define

$$P(X) = (X + b_1) \cdot (X + b_2) \cdot \dots \cdot (X + b_n) - \alpha.$$

Then, $P(a_1) = P(a_2) = \dots = P(a_n) = 0$. Therefore, since a_1, a_2, \dots, a_n are distinct, by the Factor Theorem we have

$$P(X) = (X - a_1) \cdot (X - a_2) \cdot \dots \cdot (X - a_n).$$

This gives

$$(X + b_1) \cdot (X + b_2) \cdot \dots \cdot (X + b_n) - \alpha = (X - a_1) \cdot (X - a_2) \cdot \dots \cdot (X - a_n). \quad (1)$$

Now, for each $1 \leq k \leq n$, setting $X = -b_k$ in (1) gives

$$0 - \alpha = (-1)^n (a_1 + b_k) \cdot (a_2 + b_k) \cdot \dots \cdot (a_n + b_k).$$

Therefore, for all $1 \leq k \leq n$ we have

$$(a_1 + b_k) \cdot (a_2 + b_k) \cdot \dots \cdot (a_n + b_k) = (-1)^{n+1} \alpha.$$

Setting $\beta = (-1)^{n+1} \alpha$ gives the claim.

Week 9

- [Problem](#) (posted October 29th)

In triangle $\triangle ABC$ we denote as usual $AB = c, AC = b, BC = a$. Let D be the point on the side BC such that AD is the bisector of $\angle BAC$. Let E be point symmetric to D with respect to the midpoint M of BC , and let F be the point inside the segment BC , such that $\angle BAF = \angle EAC$. Show that

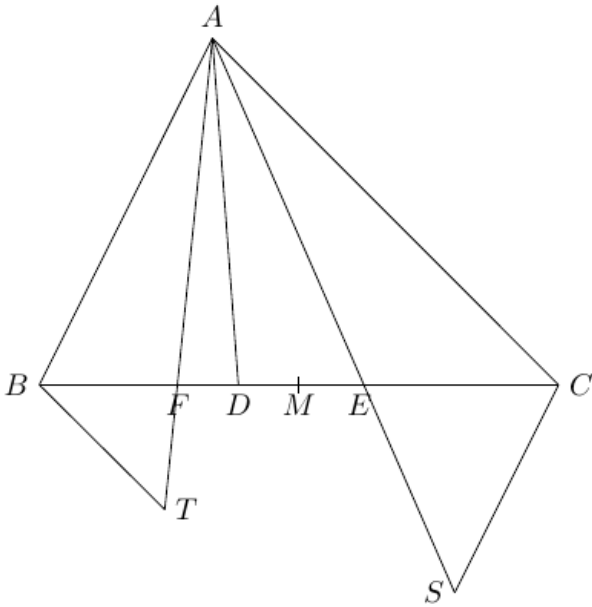
$$\frac{BF}{FC} = \frac{c^3}{b^3}.$$

- [Solution](#) (posted November 5th)

Problem 4 of the 30th Spanish Mathematical Olympiad, First Round, 1993, which appeared in *Crux Mathematicorum* [1998: 69]. We present the solution by Toshio Seimiya, which appeared at [1999: 201-203].

Note: An extended [video solution](#) to this problem is also available and includes an alternative solution involving the Law of Sines.

Let T be the point on AF so that $BT \parallel AC$ and let S be the point on AE so that $CS \parallel AB$.



Then

$$\begin{aligned}\angle ABT + \angle BAC &= 180^\circ \\ \angle ACS + \angle BAC &= 180^\circ.\end{aligned}$$

Then, $\angle ABT = \angle ACS$ and $\angle BAT = \angle CAS$ showing that $\triangle BAT \sim \triangle CAS$. This gives

$$\frac{AT}{AS} = \frac{BT}{CS} = \frac{c}{b}. \quad (1)$$

As $BT \parallel AC$ we get $\triangle BFT \sim \triangle CFA$ and hence

$$\frac{BF}{FC} = \frac{FT}{AF} = \frac{BT}{b}. \quad (2)$$

Similarly, $CS \parallel AB$ implies $\triangle ABE \sim \triangle CSE$ and hence

$$\frac{BE}{CE} = \frac{AE}{SE} = \frac{c}{CS}. \quad (3)$$

Since E is the symmetric of D with respect to M we have $EC = BD$ and $BE = CD$, and hence

$$\frac{BE}{EC} = \frac{DC}{DB}.$$

Finally, since AD is the bisector of $\angle BAC$, the Bisector Theorem gives

$$\frac{DC}{DB} = \frac{b}{c}. \quad (4)$$

Combining (3) and (4) we get

$$\frac{c}{CS} = \frac{b}{c} \Rightarrow CS = \frac{c^2}{b}.$$

Therefore, by (1) we get

$$BT = \frac{c^3}{b^2}.$$

Finally, (2) gives

$$\frac{BF}{FC} = \frac{BT}{b} = \frac{c^3}{b^3}.$$

■