Week 1

- Problem *(posted September 5th)*
  We give two entry level problems this week. Give them a try. Look for the source and the solution next week!

Problem A

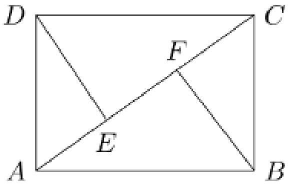
Prove that the number

$$\underbrace{111\ldots111}_{1997}\underbrace{222\ldots222}_{1998}5$$

is a perfect square.

Problem B

In the diagram, $ABCD$ is a rectangle with $AD = 1$, and both BF and DE are perpendicular to the diagonal AC. We further have $AE = EF = FC$. Find the length of the side AB.



- Solution *(posted September 12th)*

Problem A

Problem 1 of the 2nd Junior Balkan Math Olympiad, 1998, which appeared in the Skoliad Corner of Crux Mathematicorum at [2002:522]. We present the official solution that appeared at [2003:262].

$$\begin{aligned}\underbrace{111\ldots111}_{1997}\underbrace{222\ldots222}_{1998}5 &= \underbrace{111\ldots111}_{1997}\underbrace{000\ldots000}_{1999} + 2\cdot\underbrace{111\ldots111}_{1998}0 + 5\\ &= \frac{10^{1997}-1}{9}\cdot 10^{1999} + 2\cdot\frac{10^{1998}-1}{9}\cdot 10 + 5\\ &= \frac{1}{9}(10^{3996} - 10^{1999} + 2\cdot 10^{1999} - 20 + 45)\\ &= \frac{1}{9}(10^{3996} + \cdot 10^{1999} + 25)\\ &= \frac{1}{9}(10^{3996} + 2\cdot 5\cdot 10^{1998} + 5^2)\\ &= \frac{1}{9}(10^{1998}+5)^2 = \left(\frac{10^{1998}+5}{3}\right)^2 = \left(\frac{10^{1998}-1}{3}+2\right)^2\\ &= \left(\underbrace{333\ldots333}_{1998} + 2\right)^2\end{aligned}$$

Problem B

Problem 10 of the British Columbia Secondary School Mathematics Contest, 2008, Junior Final, Part A which appeared in the Skoliad Corner of Crux Mathematicorum at [2008:321-324]. We present the solution by Jixuan Wang that appeared at [2009:269].

Let x be the common length of AE, EF and FC. Then, $EC = 2x$.

By the Pythagorean Theorem, $AE^2 + DE^2 = AD^2$ and hence

$$DE = \sqrt{1 - x^2}\,.$$

Note that

$$\angle ECD = 90^\circ - \angle EDC = \angle ADE\,,$$

and therefore

$$\Delta DEA \sim \Delta CED\,.$$

We therefore get

$$\frac{EA}{DE} = \frac{ED}{CE} \quad \Rightarrow$$

$$\frac{x}{\sqrt{1-x^2}} = \frac{\sqrt{1-x^2}}{2x} \quad \Rightarrow$$

$$2x^2 = 1 - x^2 \quad \Rightarrow$$

$$x = \frac{1}{\sqrt{3}}$$

Thus, $AC = 3x = \sqrt{3}$ and by the Pythagorean Theorem in ΔABC we get

$$AB = \sqrt{2}.$$

Week 2

- Problem *(posted September 12th)*
  This week we look at an equation with integer roots.

  Determine all rational numbers r for which all the solutions of the equation

  $$rx^2 + (r+1)x + r - 1 = 0$$

  are integers.

- Solution *(posted September 19th)*
  Problem 5 of the 21^{st} *Austian Mathematical Olympiad, Final Round, 1990, which appeared in Crux Mathematicorum [1992:100]. We present the similar solutions by Joseph Ling, Pavlos Maragoudakis and Michael Selby which appeared at [1993:138].*

  If $r = 0$ then the equation becomes $x - 1 = 0$, so $x = 1$. Therefore $r = 0$ is a solution.

  If $r \neq 0$, let then $x_1 \le x_2$ be the roots of the quadratic equation. Then

  $$x_1 + x_2 = -\frac{r+1}{r} = -1 - \frac{1}{r}$$
  $$x_1 x_2 = \frac{r-1}{r} = 1 - \frac{1}{r} \,.$$

  Then

  $$x_1 x_2 - (x_1 + x_2) = 2$$
  $$(x_1 - 1)(x_2 - 1) = 3 \,.$$

  Therefore, as $x_1 \le x_2$ are integers, we either have $x_1 = 2, x_2 = 4$ or $x_1 = -2, x_2 = 0$.

  In the first case we get $r = -\frac{1}{7}$ while in the second we get $r = 1$.

  Answer: $r \in \{-\frac{1}{7}, 0, 1\}$.

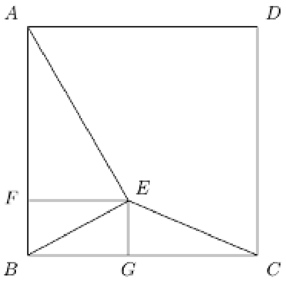
Week 3

- Problem *(posted September 19th)*
  This week we look at a point inside a square.

  A point E inside a square $ABCD$ is such that $AE = 5, BE = 2\sqrt{2}$ and $CE = 3$. Determine the area of $ABCD$.

- Solution *(posted September 26th)*
  This is problem 3 of Alberta High School Math Competition 2002, Part B. We present the solution of Keith Chung from the contest.

  Drop perpendiculars EF and EG from E onto AB and AC respectively.



Denote

$$\begin{aligned} EF &= x \\ EG &= y \\ AB &= s. \end{aligned}$$

Then

$$x^2 + y^2 = 8 \tag{1}$$

$$(s - x)^2 + y^2 = 9 \tag{2}$$

$$x^2 + (s - y)^2 = 25 \tag{3}$$

From (1) and (2) we get $s^2 - 2sx = 1$ and hence

$$x = \frac{s^2 - 1}{2s}.$$

From (1) and (3) we get $s^2 - 2sy = 17$ and hence

$$x = \frac{s^2 - 17}{2s}.$$

Plugging back in (1) we get

$$(s^2 - 1)^2 + (s^2 - 17)^2 = 32s^2,$$

or, equivalently

$$(s^2 - 5)(s^2 - 29) = 0.$$

Since E is inside $ABCD$ and $AE = 5$, we cannot have $s = \sqrt{5}$. Therefore $s^2 = 29$, so the desired area is **29**.

Week 4

- Problem *(posted September 26th)*
  Find all polynomials $P(X)$ satisfying

$$(X-16)P(2X)=16(X-1)P(X).$$

- Solution *(posted October 3rd)*
  This is problem 3 of the 10th Irish Math Olympiad 1997, which appeared in Crux Mathematicorum in [2001:6-8], with solution appearing in Crux Mathematicorum in [2003:96]. We give an alternate solution.

Setting $X=16$ in the given equation we get $P(16)=0$ and hence, we can write $P(X)=(X-16)P_1(X)$ for some polynomial $P_1(X)$. The given relation then becomes

$$(X-8)P_1(2X)=8(X-1)P_1(X).$$

Setting $X=8$ we get $P_1(8)=0$ and hence, we can write $P_1(X)=(X-8)P_2(X)$ for some polynomial $P_2(X)$. Thus, we get

$$(X-4)P_2(2X)=4(X-1)P_2(X).$$

Setting $X=4$ we get $P_2(4)=0$ and hence, we can write $P_2(X)=(X-4)P_3(X)$ for some polynomial $P_3(X)$. Thus, we get

$$(X-2)P_3(2X)=2(X-1)P_3(X).$$

Setting $X=2$ we get $P_3(2)=0$ and hence, we can write $P_3(X)=(X-2)P_4(X)$ for some polynomial $P_4(X)$. Thus, we get

$$P_4(2X)=P_4(X).$$

Then, we have $P_4(1)=P_4(2)=P_4(4)=P_4(8)=\ldots=P_4(2^n)=..$ and hence $P_4(X)=a$ for some constant a.

Answer: $P(X)=a(X-16)(X-8)(X-4)(X-2)$.

Week 5

- Problem *(posted October 3rd)*

  The sequence a_n is defined by $a_1 = 1$ and for all $n \geq 1$

  $$a_{n+1} = \frac{a_n}{1 + na_n} .$$

  Find a_{1996}.

- Solution *(posted October 10th)*

  This is problem 5 of the 1999 Íslenzka Staerðfræðikeppni Framhaldsskólanema which appeared in Crux Mathematicorum in [2001:232-233]. We give the solution of Michel Bataille that appeared in Crux Mathematicorum in [2003:304].

  It is easy to see that $a_n > 0$ for all n. Then, for all n we have

  $$\frac{1}{a_{n+1}} = \frac{1}{a_n} + n .$$

  Therefore,

  $$\begin{aligned} \frac{1}{a_{1996}} - 1 &= \frac{1}{a_{1996}} - \frac{1}{a_1} = \sum_{k=1}^{1995} \left(\frac{1}{a_{k+1}} - \frac{1}{a_k}\right) \\ &= \sum_{k=1}^{1995} k = \frac{1995 \cdot 1996}{2} = 1991010 \end{aligned}$$

  Therefore

  $$a_{1996} = \frac{1}{1991011} .$$

Week 6

- Problem *(posted October 10th)*
  This week we look at a problem about two polynomials.

  Let $P(X)$ and $Q(X)$ be two polynomials with integer coefficients. If $\frac{P(n)}{Q(n)}$ is an integer for every integer n, prove that there exists a polynomial $S(X)$ with rational coefficients such that $P(X) = Q(X)S(X)$.

- Solution *(posted October 17th)*
  This is problem 5 of Alberta High School Math Competition 1995, Part B. We present the official solution.

  We divide $P(X)$ by $Q(X)$ and obtain a quotient $S(X)$ and a remainder $R(X)$. Then, we have

  $$P(X) = Q(X)S(X) + R(X),$$

  with $\deg(R(X)) < \deg(Q(X))$. We then have:

  $$\frac{P(X)}{Q(X)} = S(X) + \frac{R(X)}{Q(X)}. \tag{1}$$

  Since both $P(X)$ and $Q(X)$ have integer coefficients, $S(X)$ has rational coefficients. To complete the proof we show that $R(X) = 0$.

  We can write

  $$S(X) = \frac{1}{m}T(X),$$

  for some integer $m > 0$ and some polynomial with integer coefficients. By combining the hypothesis with (1) we then get that for all integers n we have

  $$m\frac{R(n)}{Q(n)} = m\frac{P(n)}{Q(n)} - T(n) \in Z.$$

  Since $\deg(Q(x)) > \deg(R(x))$, for x sufficiently large we have

  $$\left|m\frac{R(x)}{Q(x)}\right| < 1.$$

  Therefore, for all integers n which are sufficiently large we get that $\left|m\frac{R(n)}{Q(n)}\right| = 0$ and hence, that $R(n) = 0$. This shows that R has infinitely many roots, and thus $R = 0$, as claimed.

  Editor note: Note that we cannot conclude that $S(x)$ has integer coefficients. Indeed, if $P(X) = X^2 + X$ and $Q(X) = 2$ then $\frac{P(n)}{Q(n)} = \frac{n(n+1)}{2}$ is an integer for all integers n but $S(X) = \frac{1}{2}X^2 + \frac{1}{2}$.

  Pólya proved that a polynomial $P(X)$ takes integer values at all integers if and only if there exists some n and integers $a_0, . . , a_n$ such that

  $$P(X) = a_n\frac{X(X-1)...(X-n+1)}{n!} + a_{n-1}\frac{X(X-1)+...+(X-n+2)}{(n-1)!} + ...$$
  $$+ a_2\frac{X(X-1)}{2!} + a_1X + a_0.$$

  See if you can prove this!

Week 7

- Problem *(posted October 17th)*

  Do there exist a number $q \in \mathrm{N}$ and a prime number $p \in \mathrm{N}$ such that

  $$3^p + 7^p = 2 \cdot 5^q?$$

- Solution *(posted October 24th)*

  This is problem 4 of the Ukrainian Math Olympiad, 11th Grade, which appeared in Crux Mathematicorum in [2006:217-218]. We give the solution of Michel Bataille that appeared in Crux Mathematicorum in [2007:227-228].

  There is no such pair (p, q). To prove this, we argue by contradiction.

  Assume by contradiction that a pair (p, q) exists. It is straightforward to check that $p = 2$ doesn't work. Then, p is odd, and hence

  $$2 \cdot 5^q = 3^p + 7^p = 10 \cdot (3^{p-1} - 3^{p-2} \cdot 7 + \ldots - 3 \cdot 5^{p-2} + 7^{p-1}).$$

  It follows that

  $$3^{p-1} - 3^{p-2} \cdot 7 + \ldots - 3 \cdot 7^{p-2} + 7^{p-1} = 5^{q-1}. \tag{1}$$

  A fast computation shows that $q = 1$ cannot satisfy the initial equation, and hence $5^{q-1} \equiv 0 \pmod 5$. Now, using $3 \equiv -2 \pmod 5$ and $7 \equiv 2 \pmod 5$, and p odd, in (1) we get

  $$2^{p-1} + 2^{p-1} + \ldots + 2^{p-1} = p2^{p-1} \equiv 0 \pmod 5.$$

  This implies that $p = 5$ and hence

  $$2 \cdot 5^2 \cdot 341 = 3^5 + 7^5 = 2 \cdot 5^q,$$

  which is a contradiction.

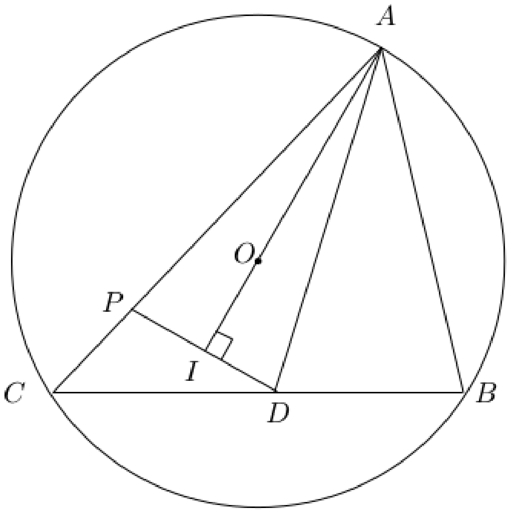
Week 8

- Problem *(posted October 24th)*

  An acute triangle ABC is inscribed in a circle with the center at O. The bisector of $\angle A$ meets the side BC at D. The perpendicular from D to AO meets the side AC at a point P which is interior to AC. Show that $AB = AP$.

- Solution *(posted October 31st)*

  This is problem 5 of the 1995 Italian Math Olympiad which appeared in Crux Mathematicorum in [1998:323-324]. We give the solution of Pierre Bornsztein that appeared in Crux Mathematicorum in [2000:79-80].



Since ABC is acute, O is interior to ABC. Since P is interior to AC, O is interior to ADC.

Let us denote as usual $\angle A = \alpha, \angle B = \beta$ and $\angle C = \gamma$. We have

$$\begin{aligned} \angle AOC &= 2\beta \\ AO &= OC \end{aligned}$$

and therefore,

$$\angle CAO = \frac{1}{2}(180^\circ - 2\beta) = 90^\circ - \beta.$$

This yields

$$\angle OAD = \frac{\alpha}{2} - (90^\circ - \beta) = \frac{\alpha}{2} + \beta - 90^\circ.$$

Let us denote by I the intersection between AO and PD. Then, the triangle ΔAID is right at I and hence

$$\begin{aligned} \angle PDA = \angle IDA &= 90^\circ - \angle OAD \\ &= 180^\circ - \frac{\alpha}{2} - \beta \\ &= 180^\circ - \angle DAB - \angle DBA = \angle ADB. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \angle PDA &= \angle ADB \\ AD &= AD \\ \angle PAD &= \angle BAD. \end{aligned}$$

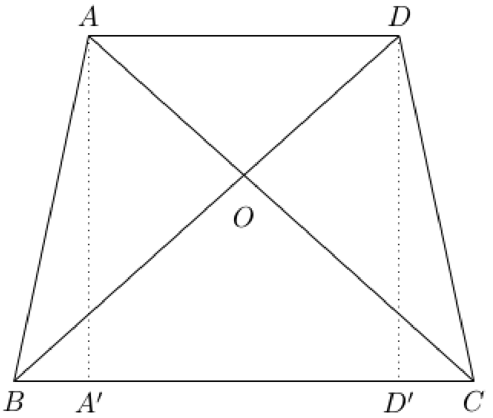
This shows that the triangles ΔPDA and ΔBDA are congruent, and hence $PA = BA$.

Week 9

- Problem *(posted October 31st)*
  The area of a trapezoid is 2 cm^2 and the sum of its diagonal is 4 cm. Find the altitude of the trapezoid.

- Solution *(posted November 7th)*
  This is problem 4 of the Georgian Math Olympiad 1997, Grade X, which appeared in Crux Mathematicorum in [2000:193]. We give the solution of Amengual Covas that appeared in Crux Mathematicorum in [2002:204-205], modified by the editor.

  Let AD be the small base of $ABCD$ and CD the large base.



We first prove that the diagonals are equal and perpendicular at O.

We have

$$\begin{aligned} \mathrm{Area}(ABD) &= \frac{1}{2} BD \cdot AO \cdot \sin(\angle AOB) \\ \mathrm{Area}(CBD) &= \frac{1}{2} BD \cdot CO \cdot \sin(\angle COD) \\ 2 = \mathrm{Area}(ABCD) &= \frac{1}{2} BD \cdot AC \cdot \sin(\angle AOB) \end{aligned}$$

Let us denote $BD = x$. Then $AC = 4 - x$, and hence

$$4 = x(4-x)\sin(\angle AOB) \leq x(4-x) = 4x - x^2 = 4 - (x-2)^2 \leq 4\,.$$

Therefore, all inequalities in the above line are equalities. This implies that

$$\begin{gathered} 4 - (x-2)^2 = 4 \qquad \text{and} \\ x(4-x)\sin(\angle AOB) = x(4-x). \end{gathered}$$

The first equality gives $x = 2$, and therefore $BD = x = 2 = 4 - x = AC$. The second equality gives $\angle AOB = 90^\circ$ and hence $AC \perp BD$.

Next, we show that $AB = CD$, that is the trapezoid is isosceles. To see this, drop perpendiculars AA', DD' from A and D, respectively, on the side BC .

We have

$$\begin{gathered} AC = BD \\ AA' = DD' \\ \angle AA'C = \angle DD'B = 90^\circ \end{gathered}$$

This implies that the right triangles $AA'C$ and $DD'B$ are congruent, and hence $A'C = BD'$. From here, we get

$$A'B = CD'\,.$$

Now, the right triangles $AA'B$ and $DD'C$ are congruent, since

$$\begin{gathered} A'B = D'C \\ AA' = DD' \\ \angle AA'B = \angle DD'C = 90^\circ \end{gathered}$$

and hence $AB = CD$.

Since $ABCD$ is an isosceles trapezoid, it follows immediately that BOC is an isosceles right triangle, and hence $\angle OCB = 45^\circ$. From here, we get that $AA' = A'C$.

Finally

$$\begin{aligned} 2 &= \mathrm{area}(ABCD) = \mathrm{area}(ABA') + \mathrm{area}(AA'D'D) + \mathrm{area}(CDD') \\ &= \mathrm{area}(AA'D'D) + 2\mathrm{area}(CDD') \\ &= AA' \cdot A'D' + DD' \cdot D'C = AA' \cdot A'C \end{aligned}$$

Since $AA' = A'C$ we get that $A'A^2 = 2$ and hence

$$AA' = \sqrt{2}.$$