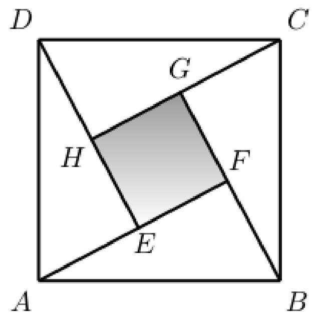
Week 1

- Problem *(posted September 6th)*
  We give two entry level problems this week. Give them a try. Look for the source and the solution next week!

  Problem A

  In the diagram, $ABCD$ is a square with side length 17 and the four triangles ABF, DAE, BCG, and CDH are congruent right triangles. Furthermore, $\overline{FB} = 8$. Find the area of the shaded quadrilateral $EFGH$.



  Problem B

  Find the number of solutions in integers (x, y) of the equation

$$x^2y^3 = 6^{12}.$$

- Solution *(posted September 13th)*

  Problem A

  This was Problem 1 of BC Colleges High School Mathematics Contest 2005, Junior Final Round B, written in May 2005, and given in the Skoliad Corner of Crux Mathematicorum in [2005:265]. We give the "official" solution which appeared at [2006:137].

  Since $FB = 8$ and $AB = 17$, the Pythagorean Theorem gives $AF = 15$. The area of each of the four congruent triangles is $\frac{8\times15}{2} = 60$. Thus the area of the shaded square $EFGH$ is $17^2 - 4 \cdot 60 = 49$.

  Problem B

  This was Problem 3 of the same contest. We give the "official" solution which appeared at [2006:138].

  First, since $6^{12} > 0$ and $x^2 \geq 0$ we must have $x^2 > 0$ and $y > 0$.

  We are given that $x^2y^3 = 6^{12} = 2^{12}3^{12}$. Since x and y both divide $2^{12}3^{12}$ we must have $x = \pm2^i3^j$ and $y = 2^k3^l$, where i, j, k and l are non-negative integers.

  Then $x^2 = 2^{2i}3^{2j}$ and $y^3 = 2^{3k}3^{3l}$. Therefore

$$x^2y^3 = 2^{2i+3k}\,3^{2j+3l}.$$

  We want $x^2y^3 = 2^{12}3^{12}$. Thus $2i + 3k = 12$ and $2j + 3l = 12$. Solving these equations we get

$$(i, k) \in \{(0, 4), (3, 2), (6, 0)\} \text{ and } (j, l) \in \{(0, 4), (3, 2), (6, 0)\}.$$

  Now, each of the three values of i can be paired with each of the three values of j. Once this is done, the values of k and l are determined. Therefore, the number of solutions in positive integers is $3 \times 3 = 9$, and the the total number of solutions is $9 \times 2 = 18$.

Week 2

- Problem *(posted September 13th)*
  This week we look at a Quintic Equation.

  Find all real solutions to the equation

  $$1 + x + x^2 + x^3 = x^4 + x^5 .$$

- Solution *(posted September 20th)*
  Problem 4 of the 21^{st} W.J. Blundon Mathematics Contest, Memorial University which was written in February 2004 and given in the Skoliad Corner of Crux Mathematicorum in [2005:354].

  The equation can be written as

  $$(1 + x)(1 + x^2) = (1 + x)x^4 .$$

  It is clear that $x = -1$ is a solution.

  If $x \neq -1$ we can divide by $1 + x$ to get

  $$1 + x^2 = x^4 ,$$

  or

  $$x^4 - x^2 - 1 = 0 .$$

  Substituting $y = x^2$ we get the quadratic equation

  $$y^2 - y - 1 = 0$$

  with solutions

  $$y_1 = \frac{1 + \sqrt{5}}{2} \text{ and } y_2 = \frac{1 - \sqrt{5}}{2} .$$

  Since $y_2 < 0$ the equation $x^2 = y_2$ has no solution.

  Solving $x^2 = y_1$ we get two more real solutions

  $$x = \pm\sqrt{\frac{1 + \sqrt{5}}{2}} .$$

  Therefore, the equation has 3 real solutions:

  $$x = -1, x = \sqrt{\frac{1 + \sqrt{5}}{2}} \text{ and } x = -\sqrt{\frac{1 + \sqrt{5}}{2}} .$$

Week 3

- Problem *(posted September 20th)*
  Let p be the number of functions defined on the set $\{1, 2, 3, \ldots, m\}$, $m \in N^*$, with values in the set $\{1, 2, \ldots, 35, 36\}$ and q be the number of functions defined on the set $\{1, 2, 3, \ldots, n\}$, $n \in N^*$, with values in the set $\{1, 2, 3, 4, 5\}$. Find the least possible value for the expression $|p - q|$.

- Solution *(posted September 27th)*
  Problem 5 of the Republic of Moldova XL Mathematical Olympiad, form 11, which appeared in Crux Mathematicorum [1999:326-327]. We present the solution by Pierre Bornsztein which appeared at [2001:369-370], slightly modified.

  Let $m, n \in N^*$. We have $p = 36^m$ and $q = 5^n$. The problem is then to find the least possible value of $|36^m - 5^n|$ over all $m, n \in N^*$.

  The last two digits of 36^m can be $36, 96, 56, 16$ or 76. The last two digits of 5^n are either 5 or 25.

  This shows that

  $$36^m - 5^n \in \{\pm 9, \pm 11, \pm 29, \pm 31, \pm 49\} \pmod{100}$$

  Let us also observe that since $9|36^m$ and $9 \nmid 5^n$ we have $9 \nmid 36^m - 5m$. Therefore $36^m - 5^n \neq \pm 9$.

  This shows that the smallest possible value $|36^m - 5^n|$ could take is 11. This value is achieved when $m = n = 2$.

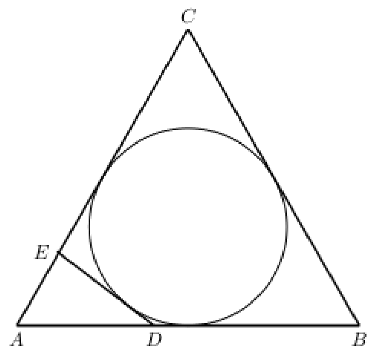
  Therefore, the least possible value of $|p - q|$ is 11.

Week 4

- Problem *(posted September 27th)*
  This week we will look at an inscribed circle problem.

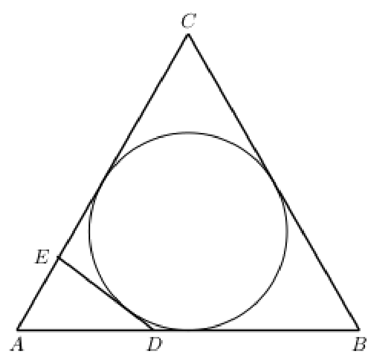
  Let ABC be an equilateral triangle and Γ its incircle. If D and E are points on the sides AB and AC, respectively, such that DE is tangent to Γ, show that

$$\frac{AD}{DB}+\frac{AE}{EC}=1$$



- Solution *(posted October 4th)*
  Problem 4 of of the 8th IberoAmerican Mathematical Olympiad 1993, which appeared in Crux Mathematicorum [1996:159-160]. We present the solution by Sĕfket Arslanagić which appeared at [1997:465-466].

  Let $AB = AC = BC = a, BD = p$ and $CE = q$. Then $AD = a - p$ and $AE = a - q$.



Since the circle Γ is inscribed in the quadrilateral $BCED$ we have

$$ED + BC = BD + CE\,,$$

or

$$ED = p + q - a\,. \qquad (1)$$

By the Law of cosines in the triangle ADE we get

$$ED^2 = AD^2 + AE^2 - 2AD \cdot AE \cdot \cos(60°)\,,$$

and hence, by (1)

$$(p+q-a)^2 = (a-p)^2 + (a-q)^2 - (a-p)(a-q)\,.$$

This gives

$$3pq = ap + aq\,.$$

Then

$$\begin{aligned}\frac{AD}{DB}+\frac{AE}{EC} &= \frac{a-p}{p}+\frac{a-q}{q}\\ &= \frac{aq-pq+ap-qp}{pq}\\ &= \frac{ap+aq-2pq}{pq}\\ &= 1\,.\end{aligned}$$

Week 5

- Problem *(posted October 4th)*
  This week we look at a functional equation.

  Find all functions $f : N^* \to N^*$ such that

  $$f\,(f(m) + f(n)) = m + n\,;\ \forall m, n \in N^*\,,$$

  where N^* denotes the set $\{1, 2, 3, \ldots\}$ of positive integers.

- Solution *(posted October 11th)*
  Problem 4 of the 44th Lithuanian Mathematical Olympiad, which appeared in Crux Mathematicorum [1998:196-197]. We present the solution by Pierre Bornsztein which appeared at [1999:334-335].

  For all $m, n \in N^*$, we have

  $$f\,(f(m) + f(n)) = m + n \tag{1}$$

  $$f\,(f(m) + f(n)) + f\,(f(m) + f(n)) = 2(m + n)\,.$$

  Therefore

  $$\begin{aligned} f\,[f\,(f(m) + f(n)) + f\,(f(m) + f(n))] &= f(m) + f(n) + f(m) + f(n) \\ &= 2\,(f(m) + f(n)) \end{aligned}$$

  and

  $$f\,[f\,(f(m) + f(n)) + f\,(f(m) + f(n))] = f(2m + 2n)\,.$$

  It follows that

  $$f(2m + 2n) = 2\,(f(m) + f(n))\ . \tag{2}$$

  Setting $m = n$, we get $4f(n) = f(4n)$.

  Setting $m = 2p + 1, n = 2p - 1$ we also get

  $$2f(2p + 1) + 2f(2p - 1) = f(8p) = 4f(2p)$$

  for all $p \geq 1$, and hence

  $$f(2p + 1) = 2f(2p) - f(2p - 1)\,. \tag{3}$$

  Setting $m = 2p + 2, n = 2p - 2, (p \geq 2)$ we get

  $$f(2p + 2) = 2f(2p) - f(2p - 2)\,. \tag{4}$$

  Let us set $f(1) = a, f(2) = b$.

  Using (3) and (4) we get

  $$\begin{aligned} &f(3) = 2b - a \\ &f(4) = f(4 \cdot 1) = 4f(1) = 4a \\ &f(5) = 9a - 2b \\ &f(6) = 8a - b \end{aligned}$$

  Now, using $m = 2, n = 1$ in (2) we get

  $$f(6) = 2f(2) + 2f(1) = 2a + 2b\,.$$

  Thus $8a - b = 2a + 2b,$ hence $b = 2a$. It follows that for $n \in \{1, 2, 3, 4, 5, 6\}$ we have

  $$f(n) = an\,.$$

  Now, using (3) and (4) by induction we get immediately

  $$f(n) = an,\ \text{ for all } n \in N^*.$$

  Now, by (1), for all $m, n \in N^*$ we have

  $$m + n = f\,(f(m) + f(n)) = a(f(m) + f(n)) = a^2(m + n)\,.$$

  Therefore $a = 1$ and

  $$f(n) = n \ \text{ for all } n \in N^*\,.$$

Conversely, $f(n) = n$ works.

Week 6

- Problem *(posted October 11th)*
  Find all the natural numbers n such that the number

$$n(n+1)(n+2)(n+3)$$

  has exactly 3 prime divisors.

- Solution *(posted October 18th)*
  Problem 5 of the 30th Spanish Mathematical Olympiad, Final Round, 1993, which appeared in Crux Mathematicorum [1998:69-70]. We present the solution by Edward T.H. Wang which appeared at [1999:203-204], slightly modified.

  We prove that the only such integers are $n = 2, 3$ and 6.

  Let $P(n) = n(n+1)(n+2)(n+3)$. Then $P(1) = 2^3 \cdot 3$ and thus $n = 1$ is not a solution. Hence we assume $n \geq 2$.

  Note first that for all $k \in N$ we have $(k, k+1) = (2k-1, 2k+1) = 1$.

  We have two cases:

  Case 1: n is odd. Then the numbers $n, n+1, n+2$ are by the above pairwise relatively prime, and hence each needs to be a power of a prime.

  Since $n+1$ is even, we must have

$$n = p^a \text{ ; } n+1 = 2^b \text{ and } n+2 = q^c$$

  where $a, b, c, p, q \in N$ with p, q distinct odd primes.

  Note now that $n+3 = 2^b + 2 = 2(2^{b-1}+1)$ where $b \geq 2$. Since the only possible prime divisors of $n+3$ are $2, p$ and q and $(n+2, n+3) = 1$ we must have

$$2^{b-1} + 1 = p^d ,$$

  with $d \geq 1$.

  Therefore,

$$2p^d = n+3 = p^a + 3 .$$

  This shows that $p|3$ and hence $p = 3$. Dividing by 3, we get

$$2 \cdot 3^{d-1} = 3^{a-1} + 1 .$$

  Now, we cannot have $d-1 > 0$ and $a-1 > 0$, as in this case we would get $3|1$. Therefore one of $d-1$ or $a-1$ must be 0, and it is immediate that the other one is also zero.

  This shows that $a = 1$, and hence $n = 3$.

  We showed that the only possible solution with n odd is $n = 3$, and it is easy to check that this is indeed a solution.

  Case 2: n is even. Then, we have $(n+1, n+2) = (n+1, n+3) = (n+2, n+3) = 1$. By the same argument as in Case 1 we must have

$$n+1 = p^a \text{ ; } n+2 = 2^b \text{ and } n+3 = q^c$$

  where $a, b, c, p, q \in N$ with p, q distinct odd primes.

  Now,

$$n = 2^b - 2 = 2(2^{b-1} - 1) .$$

  We know that $n \geq 2$ and hence $b \geq 2$.

  If $b = 2$ we get $n = 2$ which is a solution.

  Otherwise $b \geq 3$ and hence $2^{b-1} - 1$ is an odd number greater or equal than 3. The only primes which can divide n are $2, p, q$ and as $(n, p^a) = 1$ and $2^{b-1} - 1$ is odd, we must have

$$2^{b-1} - 1 = q^d ,$$

  with $d \geq 1$.

  Therefore,

$$2q^d = n = q^c - 3 .$$

This shows that $q|3$ and hence $q = 3$. Dividing by 3, we get

$$2 \cdot 3^{d-1} = 3^{c-1} - 1 .$$

We must have $d - 1 \leq c - 1$ and hence $3^{d-1} | 1$. This shows that $d = 1$ and hence

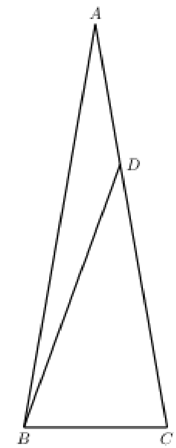
$$n = 2 \cdot 3 = 6 .$$

It is easy to check that $n = 6$ is indeed a solution.

In conclusion, $n(n+1)(n+2)(n+3)$ has exactly three prime divisors if and only if $n \in \{2, 3, 6\}$.

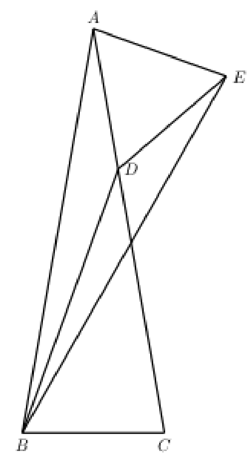
Week 7

- Problem *(posted October 18th)*
  ABC is a triangle with $AB = AC$ and $\angle BAC = 20^\circ$. On the side AC we pick inside a point D such that $AD = BC$. Find the measure of the angle $\angle CDB$.



- Solution *(posted October 25th)*
  This is one of my favourite geometry problems, with a very nice solution. Unfortunately we lost the source of this problem and of the solution below.

  On side AD we construct outside an equilateral triangle ADE. Connect E to B.



Now, by SAS, triangles ABC and BAE are congruent. Indeed

$$AB = AC\ ;\ \angle ABC = 80^\circ = \angle BAE\ ;\ BC = AE\ .$$

Then, it follows that triangle BAE is isosceles and hence $AB = BE$.

Then, by SSS, triangles ADB and EDB are congruent. Indeed

$$AD = ED\ ;\ DB = DB\ ;\ AB = EB\ .$$

It follows that $\angle ADB = \angle EDB$.

Since

$$\angle ADB + \angle EDB + 60^\circ = 360^\circ$$

we get

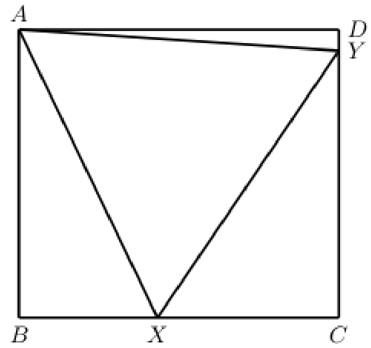
$$\angle ADB = 150^\circ\ .$$

Then

$$\angle CDB = 30^\circ\ .$$

Week 8

- Problem *(posted October 25th)*
  We cut an equilateral triangle AXY from the rectangle $ABCD$ in such a way that the vertex X is on side BC and that vertex Y is on side CD. Prove that among the remaining three right triangles there are two, the sum of whose areas equals the area of the third.



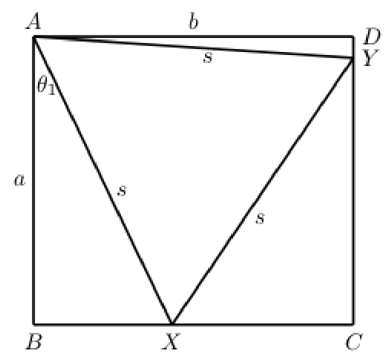
- Solution *(posted November 1st)*
  This was Problem 2 of the Hungarian National Olympiad, 1987, which appeared in Crux Mathematicorum [1989:100-101]. We present the solution by Michael Selby and D.J. Smeenk which appeared at [1991:68-69].

  First, this is not possible in every rectangle. If the rectangle has sides $a \leq b$, then a necessary condition is that $a \geq \frac{\sqrt{3}}{2}b$.

  Now, if we can cut such a triange, let s denote the side of the triangle. We claim that

$$[XYC] = [ADY] + [ABX],$$

  where $[T]$ denotes the area of the triangle T. Let us denote by θ_1 and θ_2, respectively the angles $\angle BAX$ and $\angle DAY$, respectively.



  Then

$$\begin{aligned}[ABX] &= \frac{1}{2}AB \cdot BX \\ &= \frac{1}{2}s\cos(\theta_1)s\sin(\theta_1) \\ &= \frac{1}{2}s^2\cos(\theta_1)\sin(\theta_1) \\ &= \frac{1}{4}s^2\sin(2\theta_1),\end{aligned}$$

  and same way

$$[ADY] = \frac{1}{4}s^2\sin(2\theta_2).$$

  Therefore

$$
\begin{aligned}
[ADY] &= \frac{1}{4}s^2\sin(2\theta_2) \\
&= \frac{1}{4}s^2\sin(2(30^\circ - \theta_1)) \\
&= \frac{1}{4}s^2\sin(60^\circ - 2\theta_1) \\
&= \frac{1}{4}s^2\left(\frac{\sqrt{3}}{2}\cos(2\theta_1) - \frac{1}{2}\sin(2\theta_1)\right) \\
&= \left(\frac{\sqrt{3}}{8}s^2\cos(2\theta_1)\right) - \left(\frac{1}{8}s^2\sin(2\theta_1)\right).
\end{aligned}
$$

This shows that

$$[ADY] + [ABX] = \frac{\sqrt{3}}{8}s^2\cos(2\theta_1) + \frac{1}{8}s^2\sin(2\theta_1).$$

Now, for XCY the angle $\angle CXY = 30^\circ + \theta_1$. Therefore

$$
\begin{aligned}
[CXY] &= \frac{1}{2}CX \cdot CY \\
&= \frac{1}{2}s^2\sin(30^\circ + \theta_1)\cos(30^\circ + \theta_1) \\
&= \frac{1}{4}s^2\sin(60^\circ + 2\theta_1) \\
&= \frac{\sqrt{3}}{8}s^2\cos(2\theta_1) + \frac{1}{8}s^2\sin(2\theta_1) \\
&= [ADY] + [ABX].
\end{aligned}
$$

Week 9

- Problem *(posted November 1st)*
  Find all polynomials $f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ with the following properties:

  1. all the coefficients $a_1, \ldots, a_n$ belong to the set $\{-1, 1\}$;
  2. all the roots of the equation $f(x) = 0$ are real.

- Solution *(posted November 8th)*
  This was Problem 5 of the Fourth Irish Mathematical Olympiad, 1991, which appeared in Crux Mathematicorum [1993:192-193]. We present the solution by Michael Selby which appeared at [1995:12-13].

  Let $r_1, r_2, \ldots, r_n$ represent the roots of this polynomial. Then

$$
\begin{aligned}
r_1 + r_2 + \ldots + r_n &= -a_1 \\
r_1 r_2 + r_1 r_3 + \ldots + r_{n-1} r_n &= a_2 \\
r_1 r_2 \ldots r_n &= (-1)^n a_n .
\end{aligned}
$$

We therefore get

$$
\begin{aligned}
r_1^2 + r_2^2 + \ldots + r_n^2 &= (r_1 + \ldots + r_n)^2 - 2(r_1 r_2 + r_1 r_3 + \ldots + r_{n-1} r_n) \\
&= 1 - 2a_2 .
\end{aligned}
$$

This implies that, as long as $n \geq 2$, we have $1 - 2a_2 \geq 0$ and hence $a_2 = -1$.

Next, by the Arithmetic Mean-Geometric Mean inequality we get

$$
\begin{aligned}
\frac{r_1^2 + \ldots + r_n^2}{n} &\geq \sqrt[n]{r_1^2 r_2^2 \ldots r_n^2} \\
&= \sqrt[n]{((-1)^n a_n)^2} = 1
\end{aligned}
$$

with equality if and only if $r_1^2 = r_2^2 = \ldots = r_n^2 = 1$.

Therefore

$$1 - 2a_2 \geq n ,$$

with equality if and only if $r_1^2 = r_2^2 = \ldots = r_n^2 = 1$.

Therefore $n \leq 3$.

Case 1: $n = 1$. It is straightforward to see that in this case both $X + 1$ and $X - 1$ work.

Case 2: $n = 2$. Then we must have $a_2 = -1$, therefore, the possible polynomials are $X^2 - X - 1$ and $X^2 + X - 1$, and both work.

Case 3: $n = 3$. Then we have $a_2 = -1$ and equality in the AM-GM inequality, hence $r_j \in \{\pm 1\}$. Moreover, since $r_1 r_2 + r_1 r_3 + r_2 r_3 = -1$, it is not possible for all three roots to have the same sign. This leaves us with two possible polynomials:

$$
\begin{aligned}
(X-1)^2(X+1) &= X^3 - X^2 - X + 1 \\
(X-1)(X+1)^2 &= X^3 + X^2 - X - 1
\end{aligned}
$$

Answer: There are 6 such polynomials:

$$X - 1 ; X + 1 ; X^2 - X - 1 ; X^2 + X - 1 ; X^3 - X^2 - X + 1 \text{ and } X^3 + X^2 - X - 1 .$$