Canadian Mathematical Olympiad Qualifying Repêchage 2026

A competition of the Canadian Mathematical Society.

Official Solutions

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1. [**10 points**] Do there exist four lines on the same plane, no three concurrent and no two parallel, so that four particles can travel at (possibly different) constant non-zero speeds along these lines and such that at every point in time, the positions of the four particles are concyclic?

Solution: The answer is yes. One construction is to take a point O and line ℓ_1 not containing O. Define lines ℓ_2, ℓ_3, ℓ_4 to be rotations of ℓ_1 about O such that no two are parallel; then, it is clear that no three are concurrent either. Particles initially placed at an arbitrary point on ℓ_1 and their images under rotation in ℓ_2, ℓ_3, ℓ_4 and traveling at constant speed remain equidistant from O throughout. □

2. [**10 points**] Do there exist positive real numbers a, b, c, not all equal, such that $a^b = b^c = c^a$?

Solution: The answer is no. Assume for sake of contradiction that such a, b, c exist, and that the common value of $a^b = b^c = c^a$ is x. Let $f(z) = x^{1/z}$. Then $f(a) = c, f(c) = b, f(b) = a$.

If $x < 1$, then $f(z)$ is an increasing function, which leads to a contradiction: for example, if $a < b$, then we must have $f(a) < f(b)$ and so $c < a$, and hence $b < c$, and a similar argument holds for $a > b$.

Furthermore, if $x > 1$, then $f(z)$ is a decreasing function, so $f(f(z))$ is an increasing function, and we can apply the same argument as above. □

3. [**10 points**] Let point P be outside circle Γ. The tangents from P to Γ hit Γ at A and B. A third line through P hits Γ at C and D, such that C is between P and D. Point Q is on chord CD such that $\angle DAQ = \angle PBC$. Prove that $\angle DBQ = \angle PAC$.

Solution: We see that angles CAD and CBD are supplementary, so the given angle equality implies that angles CAQ and PBD are also supplementary. Let $\angle DAQ = \angle PBC = \theta$, and $\angle CAQ = \pi - \angle PBD = \alpha$. Then, we try to use the Law of Sines to determine CQ/DQ in terms of objects that do not involve Q.

We note that $CQ = \frac{AC \sin\alpha}{\sin\angle AQC}$ and $DQ = \frac{AD\sin\theta}{\sin\angle AQC}$.

Our goal is to now write the ratio in terms of something symmetric in A and B. To that end, we write $\sin\alpha = \frac{PD}{PB}\sin\angle CDB$ and $\sin\theta = \frac{PC}{PB}\sin\angle DCB$, so we have

$$\frac{CQ}{DQ} = \frac{AC\cdot PD\cdot \sin\angle CDB}{AD\cdot PC\cdot\sin\angle DCB} = \frac{AC\cdot PD\cdot CB}{AD\cdot PC\cdot BD}.$$

This is symmetric in A and B, so if we pick a point Q' such that $\angle PAC = \angle DBQ'$, then $CQ'/DQ' = CQ/DQ$, whence $Q = Q'$, as desired. □

4. [**15 points**] Fix a prime p and natural numbers $k \geq n$. Consider choosing k vectors $v_1, \dots, v_k$ uniformly at random (possibly with repetition) from the set $\{(x_1, \dots, x_n) | x_i \in \{0, 1, \dots, p-1\}\}$. Also, let v_{k+i} be the vector with value p in the i-th coordinate and 0 elsewhere for $1 \leq i \leq n$.

These $k+n$ vectors are said to *generate* $\mathbb{Z}^n$ if for any vector of integers $v = (y_1, \dots, y_n)$, there are some integers $c_1, \dots, c_{k+n}$ such that $v = \sum_{i=1}^{k+n} c_i v_i$. Let $P(p, n, k)$ be the probability that these $k+n$ vectors generate $\mathbb{Z}^n$.

Then, there is some constant c, depending on p and n, such that $p^k(1 - P(p,n,k)) - c$ approaches 0 as $k \to \infty$. Determine c in terms of p and n.

Solution: The answer is $\frac{p^n-1}{p-1} = p^{n-1} + \cdots + 1$. As suggested by the problem statement, it is easier to count the complement.

For a vector $v \in \mathbb{Z}^n$ such that the coordinates of v are not all multiples of p, define $v^\perp$ to be

$$\left\{ w \in \mathbb{Z}^n | \sum_{i=1}^n w_i v_i \equiv 0 \pmod p \right\}.$$

Note $v^\perp$ only depends on the residue class of v mod p, and $v^\perp = (cv)^\perp$ for $c \in \{1, 2, \dots, p-1\}$, so there are $\frac{p^n-1}{p-1}$ different such sets $v^\perp$.

The key input is the following linear algebra lemma: If the $k+n$ vectors fail to generate $\mathbb{Z}^n$, then there exists some v for which they are all in $v^\perp$. We defer the proof of this lemma to the end of this solution. Assuming this lemma, we can finish this problem by inclusion-exclusion. We see that

$$1 - P(p,n,k) = \sum_w \mathbb{P}(v_1, \dots, v_k \in w^\perp) - \sum_{w_1 \neq cw_2} \mathbb{P}(v_1, \dots, v_k \in w_1^\perp \cap w_2^\perp) + \dots$$

$$+(-1)^{m-1} \sum_{\substack{w_1,\ldots,w_m \\ \text{distinct up to scaling}}} \mathbb{P}(v_i \in w_j^{\perp} \forall i, j) - \ldots,$$

where this sum is finite since there are only $\frac{p^n-1}{p-1}$ different possibilities for $v^{\perp}$ up to scaling.

We also note that the first term in the above probability is $\mathbb{P}(v_1, \ldots, v_k \in w^{\perp}) = \frac{1}{p^k}$, and higher terms satisfy $\mathbb{P}(v_1, \ldots, v_k \in w_1^{\perp}, w_2^{\perp}, \ldots, w_m^{\perp}) \leq \mathbb{P}(v_1, \ldots, v_k \in w_1^{\perp}, w_2^{\perp}) = \frac{1}{p^{2k}}$, so we have $p^k(1 - P(p, n, k)) = \frac{p^n-1}{p-1} + O(p^{-k})$, where the implied constant depends on n. This allows us to conclude that $c = \frac{p^n-1}{p-1}$.

Proof of lemma: We now prove the lemma above. The main idea is to build a *spanning subset* of the vectors $v_1, \ldots, v_k$. For $1 \leq \ell \leq k$, let T_ℓ be the set spanned by $v_1, \ldots, v_\ell, v_{k+1}, \ldots, v_n$. We call v_ℓ a spanning vector if $T_\ell \neq T_{\ell-1}$ (so that v_1 is always a spanning vector).

Furthermore, define the *size* of T_ℓ as the number of residue classes in $\mathbb{Z}^n$ that it hits. We now have the following fact: if v_ℓ is the m-th spanning vector, then the size of T_ℓ is p^m. Assuming this fact, we can see that we have at most $n-1$ spanning vectors. Call them $x_1, \ldots, x_h$. Then, we can find a vector v such that $x_1, \ldots, x_h \in v^{\perp}$ since this is solving a system of h equations in n variables. □

5. **[15 points]** Determine whether there exists a surjective function $f : (0, \infty) \to \mathbb{N}$, such that for any positive numbers $a < b$, the image of the open interval (a, b) under f is a set of the form $\{1, 2, \ldots, N\}$ for some finite positive integer N.
Recall that a function $f : (0, \infty) \to \mathbb{N}$ is *surjective*, if for every natural number M, there exists some positive real number r such that $f(r) = M$.

Solution: The answer is yes. We provide an explicit construction below. For all numbers of the form $m/2^k$, where m and k are positive integers, let $d(m)$ be the number of 1s in the binary expansion of m. Note that this is independent of the representation of m. Then, define

$$f\left(\frac{m}{2^k}\right) = \max\left(1, \left\lfloor \log_2 \frac{m}{2^k} \right\rfloor + 1 - d(m)\right).$$

For all real numbers r such that there exists no positive integer k for which $2^k r$ is an integer, set $f(r) = 1$.

Now, we can check that this function is surjective; indeed, $f(2^n) = n$ for every positive integer n. Furthermore, say that an open interval (a, b) contains some r such that $f(r) = c$. Then, if $c > 1$, by adding a sufficiently small power of two to r, we get some $r + \frac{1}{2^m} \in (a, b)$ such that $f(r + \frac{1}{2^m}) = c - 1$. We have thus shown that if a positive integer c is in the image of (a, b) under f, then so is $c - 1$. Further, we see that $f(r) \leq \max(1, \log_2 r)$, by definition, so f is bounded on any interval (a, b), and hence the image is of the form $\{1, 2, \ldots, N\}$ for some N, as desired. □

6. **[20 points]** A set of positive integers T is given. For a positive integer n, Alice and Bob play a game, where there are initially n stones in a pile, they take turns taking m stones from the pile, where $m \in T, m \leq n$, and whoever takes the last stone wins. If at some point the number of stones in the pile is less than $\min T$, then the game is a draw. Call a number n *good* if Alice can guarantee a win, *bad* if Bob can guarantee a win, and *neutral* if neither Alice nor Bob can guarantee a win.

For example, if T is the set of prime numbers, then $n = 4$ is neutral and $n = 5$ is good; in the first game, Alice can guarantee a draw by taking 3 stones on her first move, and cannot do better, and in the second game, she can take all the stones on her first move to win.

(a) Prove that if T is finite, then the set of good numbers is eventually periodic: that is, there exists some positive integers N and p such that for all positive integers $n > N$, we have that n is good if and only if $n + p$ is good.

(b) Construct an infinite set T such that the set of good numbers is not eventually periodic: that is, for every pair of positive integers N and p, there exists some $n > N$ such that exactly one of n and $n + p$ is good.

[note]: the word 'eventually' was originally missing from this question part b but we added it and notified students during the exam.

Solution:

(a) Let $M = \max T$. We see that the status of n only depends on the statuses of $n - M, n - M + 1, \ldots, n - 1$. There are a finite number of such states, so by Pigeonhole, there exists some $n_1 < n_2$ such that the statuses of $n_1 - c$ and $n_2 - c$ are identical for $c = M, M - 1, \ldots, 1$. Then, by induction, they are also equal for $c \leq 0$, so we may choose $N = n_1$ and $p = n_2 - n_1$.

(b) We select T such that $1, 2 \notin T$, $3, 4, 5 \in T$, and for every $n \in \mathbb{N}$, at least one of $n, n + 1, n + 2 \in T$. The main claim for this kind of set T is that a positive integer $n \geq 5$ is good if and only if $n \in T$, and all integers that are not good are neutral.

We first see that if $n \in T$, then Alice may take all the stones and win. Similarly, if $n \notin T$, then either $n - 1 \in T$ or $n - 2 \in T$. Alice may take this number of stones, ending the game in a draw, so all $n \notin T$ are either neutral or good. Furthermore, after Alice's turn, if she takes fewer than $n - 3$ stones, then Bob will be left in the same situation and can force a draw at worst, so $n \notin T$ cannot be good; they must be neutral.

Now it suffices to construct such a set T that is also not eventually periodic. We may choose $T = \mathbb{N} \backslash \{n! | n \in \mathbb{N}\}$, for example.

□

7. [20 points] Determine the least positive integer that is not a perfect square that can be written in the form $\frac{a^2+b^2}{ab+1}$ for some rational numbers a and b such that $ab \neq -1$.

Solution: The answer is 10. We see that the pair $(a, b) = (\frac{13}{2}, \frac{1}{2})$ gives us $\frac{a^2+b^2}{ab+1} = 10$. It remains to show that for positive integers $k \in \{2, 3, 5, 6, 7, 8\}$, it is impossible to obtain $\frac{a^2+b^2}{ab+1} = k$ with rational numbers a and b.

First, we see if that either $a = 0$ or $b = 0$, then we get a perfect square, so we can safely assume that a, b are both non-zero.

For a prime p and nonzero rational number r, define $v_p(r)$ to be the number of prime factors of p in the factorization of r. It is clear from definition that $v_p(rs) = v_p(r) + v_p(s)$ for all r, s, and we can also see that $v_p(r + s) \geq \min(v_p(r), v_p(s))$, where equality holds unless possibly $v_p(r) = v_p(s)$.

Now, we see that if $p \equiv 3 \pmod 4$, $p|k$, and $p \not| k^2$, then $v_p(a^2 + b^2) = 2\min(v_p(a), v_p(b))$. Thus, $v_p(ab + 1)$ must be odd, and so $v_p(a) + v_p(b) = 0$. But then $v_p(a^2 + b^2) \leq 0$ and $v_p(k(ab + 1)) > 0$, which is a contradiction. This eliminates the cases $k = 3, 6, 7$.

Now, for $k = 5, 8$, we consider the equation $(a - b)^2 = (k - 2)ab + k$. If $v_2(ab) \geq 0$, then we have a contradiction as the right-hand side is $2 \pmod 3$. Otherwise, $v_2(ab)$ must be odd, since $v_2((k - 2)ab)$ must be a non-positive even number. This implies that $v_2(a) \neq v_2(b)$. Assume that $v_2(a) < v_2(b)$. Then, $v_2((a - b)^2) = 2v_2(a)$ and $v_2((k - 2)ab + k) = 1 + v_2(a) + v_2(b)$, which is a contradiction.

Finally, for $k = 2$, our equation is $(a - b)^2 = 2$, which has no rational solutions.

Remark: By the Hasse-Minkowski theorem, we can see that if for some k such that no rational a, b satisfy $\frac{a^2+b^2}{ab+1} = k$, then there will always be an argument considering v_p modulo some prime that would exhibit the contradiction, as in above. □

