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Editorial Board

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EDITORIAL

Welcome to issue 10 of Volume 51! It's been quite the year as in this Volume, we published:

- 50 MathemAttic problems and their featured solutions
- 50 Olympiad Corner problems and their featured solutions
- 100 numbered problems and their featured solutions
- 5 Problem Solving Vignettes
- 4 Teaching Problems columns
- 5 Competition highlights
- 9 articles (including one in French)

I am happy to say that we continue to become more and more popular. As an example, the last two issues combined spanned well over 100 pages. Behind those pages is the editors' hard work of considering 393 solutions from solvers in 25 different countries that span 5 continents. And that is the fifth of the annual total! The reach of ***Crux*** never ceases to amaze me. Thank you to the readers who continue to get involved and to the editors that continue to uplift this work. It truly takes a village.

For the first time in my tenure as Editor-in-Chief of ***Crux*** (which now spans more than a decade), I am able to truly plan ahead – this year, ***Crux*** received funding support and we will be able to continue publishing the journal indefinitely. Moreover, I have just been confirmed in my position as EIC for the next 5 years. Some of the work I am planning will be largely invisible to you, the readers: we need to fix some behind-the-scenes plumbing to ensure that the journal's operations are sustainable and scalable as we continue to grow. Other plans include enriching the content: for example, I plan to expand francophone participation and explore sustainable ways to feature contributions from Indigenous authors exploring Indigenous content. I also plan to collaborate more closely with organizers of math competitions and math camps to use ***Crux*** at their events, produce classroom-ready versions of ***Crux*** materials and to generally broaden our reach to secondary school classrooms.

I would love to hear what you would like to see happen at ***Crux*** , so please do not hesitate to drop me an email at `crux.eic@gmail.com`

Kseniya Garaschuk

MathemAttic

No. 70

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **February 15, 2026**.

MA346. Find the primes p, q, r, given that one of the numbers pqr and $p+q+r$ is 101 times the other.

MA347. There are 20 sweets on a table. A game consists of two players taking turns to choose some sweets. On each move, a player must choose at least one sweet but never more than half of what remains. The loser is the one who has no valid move. How many sweets should the first player choose on the opening move to ensure that they may always win the game? Justify your answer.

MA348. Find all positive integers $n < 200$ such that $n^2+(n+1)^2$ is a perfect square.

MA349. Let $ABCD$ be a regular tetrahedron whose edges are of length 6. Let E be the mid-point of CD so that $|CE| = |ED|$ and let F be a point on BE such that $AF \perp BE$. Find $|AF|$.

MA350. For a given arithmetic sequence, the ratio of the sum of the first m terms to the sum of the first n terms is $m^2 : n^2$. Find, in simplest form, the ratio of the m^{th} term to the n^{th} term in terms of m and n, where $m \neq n$.

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 fevrier 2026**.

MA346. Trouvez les nombres premiers p, q et r, sachant que l'un des deux nombres pqr et $p+q+r$ est égal à 101 fois l'autre.

MA347. Il y a 20 bonbons sur une table. On définit un jeu où deux joueurs prennent, à tour de rôle, un certain nombre de ces bonbons. À chaque tour, un joueur doit en prendre au moins un, mais jamais plus de la moitié de ceux qui restent. Le perdant est celui qui ne peut plus effectuer de coup valide. Combien de bonbons le premier joueur doit-il prendre au premier tour pour s'assurer de toujours pouvoir gagner la partie ? Justifiez votre réponse.

MA348. Trouvez tous les entiers positifs $n < 200$ tels que $n^2 + (n+1)^2$ soit un carré parfait.

MA349. Soit $ABCD$ un tétraèdre régulier dont les arêtes ont une longueur de 6. Soit E le milieu de CD de sorte que $|CE| = |ED|$, et soit F un point sur BE tel que $AF \perp BE$. Déterminez la longueur $|AF|$.

MA350. Pour une suite arithmétique donnée, le rapport entre la somme des m premiers termes et la somme des n premiers termes est de $m^2 : n^2$. Déterminez, sous la forme la plus simple, le rapport entre le $m^{\text{ième}}$ terme et le $n^{\text{ième}}$ terme, en fonction de m et n, où $m \neq n$.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(5), p. 208–210.

MA321. *Proposed by Daniel Rasmussen.*

Find all pairs of non-zero integers (a, b) that share no common factors other than 1 and such that $a^2 + b^2$ divides $a^2b + ab^2 + ab$.

We received 4 submissions of which 3 were correct and complete. We present an edited solution proposed by Meryem Bourget.

Let S denote the set of all integers of interest:

$$S = \{(a, b) \in \mathbb{Z}^2 : a \neq 0, b \neq 0, \gcd(a, b) = 1, \text{ and } a^2 + b^2 \mid a^2b + ab^2 + ab\}.$$

Let $d = a^2 + b^2$. We observe that if $d \mid a^2b + ab^2 + ab$, then $d \mid ab(a + b + 1)$. First, we show that d must divide $a + b + 1$. Indeed, $d \nmid ab$ as $d > ab$. Observe that,

$$a^2 + b^2 > ab, \tag{1}$$
$$\iff 2a^2 + 2b^2 - 2ab > 0, \tag{2}$$
$$\iff a^2 + b^2 + (a^2 - b^2) > 0, \tag{3}$$

and both a and b are non-zero.

Obviously, $d \mid a + b + 1$ if $a + b + 1 = 0$. We claim that $d \nmid a + b + 1$ otherwise. The only possible values of a and b for which

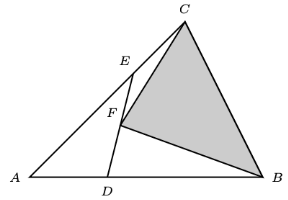
$$|a + b + 1| \geq d$$

are $a = b = 1$. In this case, it is easy to see that $d \nmid a + b + 1$ as $2 \nmid 3$. Hence, d never divides $a + b + 1$ if $a + b + 1 \neq 0$. Thus, the complete solution set is:

$$S = \{(a, -a - 1) \in \mathbb{Z}^2 : a \neq 0, a \neq -1\}.$$

MA322.

In the diagram, the area of the triangle ABC is 1, $AD = \frac{1}{3}AB$, $EC = \frac{1}{3}AC$ and $DF = FE$. Find the area of the shaded triangle.



Originally Question 5 from the 2016 BC Secondary School Math Contest, Senior Final Round, Part B.

We received 10 submissions of which 9 were correct and complete. We present 2 solutions by Sicheng Du.

Solution 1. We use barycentric coordinates with reference to $\triangle ABC$. Then we have

$$\begin{aligned} D &= \frac{1}{3}B + \frac{2}{3}A = \left(\frac{2}{3}, \frac{1}{3}, 0\right), \\ E &= \frac{1}{3}A + \frac{2}{3}C = \left(\frac{1}{3}, 0, \frac{2}{3}\right), \\ F &= \frac{1}{2}D + \frac{1}{2}E = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right). \end{aligned}$$

Since $\triangle ABC$ has area 1, then the area of $\triangle BCF$ is given by the determinant

$$\begin{vmatrix} 1/2 & 0 & 0 \\ 1/6 & 1 & 0 \\ 1/3 & 0 & 1 \end{vmatrix} = \frac{1}{2}.$$

Solution 2. Let the feet from E, F, D to BC be P, Q, R respectively. Then $EPRD$ is a trapezoid and FQ is its median (or sometimes referred to as the mid-segment). Hence,

$$FQ = \frac{EP + DR}{2}.$$

Multiplying both sides by $\frac{BC}{2}$, we get

$$[FBC] = \frac{[EBC] + [DBC]}{2} = \frac{\frac{1}{3} + \frac{2}{3}}{2} = \frac{1}{2}.$$

MA323. Five people are trapped on an island. Each person can form an alliance with any of the other people on the island. The alliances are mutual, so that if A is allied with B, then B is allied with A. However, if A is allied with B and B is allied with C, it is not necessarily true that A and C are allied. If every person on the island is allied with every other person on the island, then there are 10 pairs of people on the island who each have four alliances.

1. Is it possible that no two people on the island have the same number of alliances? Explain.

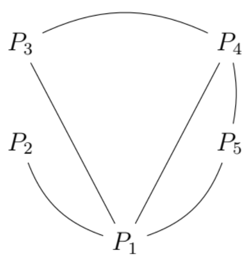
2. What is the smallest possible number of pairs of people on the island with the same number of alliances?

Originally Question 4 from the 2016 BC Secondary School Math Contest, Senior Final Round, Part B.

We received 3 submissions, two of which were correct and complete. We present the solution by Johann Peters.

(a) No, that is not possible. If it were, there'd be a person with k alliances for each $k \in \{0, 1, 2, 3, 4\}$. However, if someone has 4 alliances, no-one has none. So the hypothesis yields a contradiction.

(b) We've seen there's no solution with no pairs. Let us represent the situation as a graph, where the nodes P_i denote the five people and the edges represent alliances. Below is then an example with one pair, where P_3 and P_5 are the two people with the same number of alliances, namely two. Therefore the least number of pairs possible is one.



MA324. Let N be a 3-digit number with three distinct non-zero digits. We say that N is mediocre if it has the property that when all six 3-digit permutations of N are written down, the average is N. For example, $N = 481$ is mediocre since the average of $\{481, 148, 184, 418, 814, 841\}$ is 481. Determine the largest 3-digit mediocre number.

Originally Question 5 from the 2017 BC Secondary School Math Contest, Junior Final Part B.

We received 10 solutions of which 4 were correct and complete. We present the solution by Corneliu Manescu-Avram.

The answer is $N = 629$.

We will determine all mediocre numbers. Let $N = \overline{abc}$ be a mediocre number. Then $a, b, c \in \{1, \ldots, 9\}$, $a \neq b \neq c \neq a$, and

$$\overline{abc} + \overline{acb} + \overline{bac} + \overline{bca} + \overline{cab} + \overline{cba} = 6 \cdot \overline{abc}.$$

From this we deduce $111 \cdot 2 \cdot (a + b + c) = 6 \cdot (100a + 10b + c)$, whence, after simple calculations, we get

$$7a = 3b + 4c. \tag{1}$$

Then $7a = 3(b - c) + 7c$, so 7 divides $|b - c|$. But $0 < |b - c| < 10$, therefore $|b - c| = 7$, with solutions

$$(b, c) \in \big\{(8, 1), (1, 8), (9, 2), (2, 9)\big\}.$$

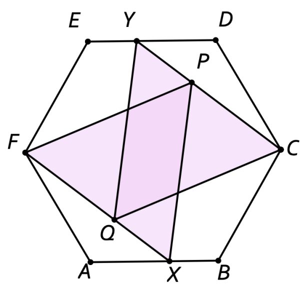
The corresponding values of a from (1) are respectively 4,5,5,6 and we find

$$N \in \{481, 518, 592, 629\}.$$

Editor's Comments. Every submission correctly determined that if $N = 100a + 10b + c$ is a mediocre number, then $7a = 3b + 4c$. However, many submissions missed the requirement that the digits were specified to be *distinct* and *non-zero.*

MA325. *Proposed by Arsalan Wares.*

Suppose hexagon $ABCDEF$ is regular with points X and Y on sides AB and DE, respectively. Two congruent overlapping equilateral triangles, FXP and CYQ, partially cover hexagon $ABCDEF$ as shown. If vertices P and Q are on sides YC and FX, respectively, determine the exact value of $AX : AB$.



We received 3 solutions, one incorrect and two incomplete. We present a solution by the editor.

We first consider the situation with one of the triangles, FXP. Since $\angle FAX + \angle FPX = 120° + 60° = 180°$, $FAXP$ is concyclic and so $\angle FAP = \angle FXP = 60° = \angle FAD$. Therefore P lies on the diagonal AD of the hexagon.

By the law of cosines, $FP^2 = PX^2 = FX^2 = 1 + x^2 - 2x\cos 120° = x^2 + x + 1$. Let $u = AP$. By the law of cosines applied to triangle PXA, we have $x^2 + x + 1 = PX^2 = u^2 + x^2 - xu$, from which

$$0 = u^2 - xu - (x + 1) = [u - (x + 1)][u + 1].$$

Since the latter factor is positive, we must have $u = x + 1$.

Let $\theta = \angle AFX$. Since $FAXP$ is concyclic, $\angle APX = \angle AFX = \theta$. Applying the law of cosines to triangle AXP we obtain

$$x^2 = u^2 + x^2 + x + 1 - 2u\sqrt{1 + x + x^2}\cos\theta,$$

whence

$$2(x + 1)\sqrt{1 + x + x^2}\cos\theta = x^2 + 2x + 1 + x + 1 = (x + 1)(x + 2).$$

Therefore

$$\cos\theta = \frac{x+2}{2\sqrt{x^2+x+1}}.$$

Applying the law of sines to triangle FAX, we find that

$$\sin\theta = \frac{x\sin 120^\circ}{\sqrt{x^2+x+1}} = \frac{\sqrt{3}x}{2\sqrt{x^2+x+1}}.$$

We determine PQ, where $Q = PA \cap FX = DA \cap FX$ by applying the law of sines to triangle PQX. Observe that $\angle PQX = \angle FQA = 120^\circ - \theta$ and that

$$\begin{aligned}\sin(120^\circ - \theta) &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{x+2}{2\sqrt{x^2+x+1}}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}x}{2\sqrt{x^2+x+1}}\right)\\ &= \frac{\sqrt{3}(x+1)}{2\sqrt{x^2+x+1}}.\end{aligned}$$

Therefore

$$PQ = \frac{\sqrt{x^2+x+1}\sin 60^\circ}{\sin(120^\circ-\theta)} = \frac{x^2+x+1}{1+x},$$

and

$$AQ = (x+1) - \frac{x^2+x+1}{x+1} = \frac{x}{x+1}.$$

For the configuration described in the problem, we produce CP to meet the circumference of the hexagon at Y. Y must lie on DE, otherwise the equilateral triangle constructed on CY is too large to fit strictly within the interior of the hexagon. Let $DE = y$ and follow the same arguments for triangle CYQ as for triangle FXP. We know that the point Q defined in the problem is consistent with our definition. Evaluating PQ in terms of x and y, we find that

$$\frac{x^2+x+1}{x+1} = \frac{y^2+y+1}{y+1},$$

whence

$$0 = (y+1)x^2 - (x+1)y^2 = (x-y)(xy+x+y).$$

Therefore $x = y$ and $AP = DQ = x+1$. Since

$$\frac{x^2+x+1}{x+1} = PQ = AP + DQ - AD = 2(1+x) - 2 = 2x,$$

$x^2 + x - 1 = 0$ and $x = (\sqrt{5}-1)/2 = 1/\phi$, where ϕ is the golden ratio.

Comment by the editor. The condition that the two equilateral triangles are congruent seems redundant. With this condition, it can be argued that the configuration is stable under a 180° rotation about the centre of the hexagon, so that FX and CY are parallel. But this does not appear to shorten the necessary computation significantly.

A Method Involving Muirhead's Inequality Applied on Five Inequality Problems from Around The World

Hüseyin Yiğit Emekçi

1 Introduction

Inequalities were and are very popular in mathematical olympiads, in international and national. Many different inequality problems have been put forward and also are being proposed day by day as the olympiad culture spreads out all around the world. However, there can be a group of olympiad inequality problems that can be solved with a common trick. In this article, we will introduce a method, after providing majorization and Muirhead's inequality.

The method is

> Once we have a cyclic sum, with denominator having higher exponents than the numerator, then we may consider applying Muirhead's inequality into the denominator, in order to maximize the cyclic sum.

and the method will generally require algebraic manipulations after being used. With the help of the method, we will solve five inequality problems from IMO Shortlist, USAMO, Junior Balkan MO, Balkan MO and Tournament of Towns contest, respectively.

Before passing through examples, let us look into the theoretical part, in other words symmetric sums, majorization and Muirhead's inequality.

Symmetric sums and majorization. Denote $\sum_{sym}$ as symmetric sum. For instance, in a problem with three variables x, y, z we write

$$\sum_{sym} x^2y = x^2y + x^2z + y^2z + y^2x + z^2x + z^2y$$

$$\sum_{sym} x^3 = x^3 + x^3 + y^3 + y^3 + z^3 + z^3$$

which goes through all 3! permutations. Then for $a_i, b_i \geq 0$ for $i = 1, 2, \cdots, n$ we say (a) majorizes (b) and express it as

$$(b) \prec (a)$$

if and only if both (a) and (b) can be arranged so that the following three criteria are met in the same time ([2], pp. 32-33.):

i) $b_1 + b_2 + \cdots + b_n = a_1 + a_2 + \cdots + a_n$

ii) $b_1 \geq b_2 \geq \cdots \geq b_n, \quad a_1 \geq a_2 \geq \cdots \geq a_n$

iii) $b_1 + b_2 + \cdots + b_k \leq a_1 + a_2 + \cdots + a_k \quad (k = 1, 2 \cdots, n)$

Majorization is defined for finite ordered sums. To provide examples, for instance

- $(2, 1, 0) \succ (1, 1, 1)$;
- $(3, 0) \not\succ (1, 1, 1)$ since the number of elements differs;
- $(4, 2, 0, -1) \not\succ (3, 2, 0, 0)$ since there is a negative element;
- $(2, 1, 1) \not\succ (1, 1, 1)$ since $2 + 1 + 1 \neq 1 + 1 + 1$;
- $(3, 1, 1, 1) \not\succ (3, 2, 1, 0)$ since $3 + 1 \ngeq 3 + 2$.

Muirhead's inequality. If $(b) \prec (a)$, then

$$\sum_{sym} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \prec \sum_{sym} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

with the equality when $(a) \equiv (b)$ or $x_1 = x_2 = \cdots = x_n$ ([1], pp. 44-48).

To make it more clear, for instance $(3, 3, 0) \succ (2, 2, 2)$ expresses

$$\sum_{sym} x^3 y^3 = 2(x^3y^3 + y^3z^3 + z^3x^3) \geq 6x^2y^2z^2 = \sum_{sym} x^2y^2z^2$$

implying $x^3y^3 + y^3z^3 + z^3x^3 \geq 3x^2y^2z^2$ for all positive real numbers x, y, z.

Remark. We can prove the AM-GM inequality by Muirhead's Inequality. Note that for $n \geq 1$

$$(1, 0, 0, \cdots, 0) \succ \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right),$$

so then

$$\sum_{sym} a_1 = (n-1)!(a_1 + a_2 + \cdots + a_n) \geq n!(a_1a_2 \cdots a_n)^{1/n} = \sum_{sym} (a_1a_2 \cdots a_n)^{1/n},$$

implying $a_1 + a_2 + \cdots + a_n \geq n(a_1a_2 \cdots a_n)^{1/n}$ which is the AM-GM inequality.

2 Problems

Problem 1 (IMO Shortlist 1996/A.1). Let a, b and c be positive real numbers with $abc = 1$. Then

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1$$

Proof. The method is: minimize the denominator with Muirhead's Inequality and then manipulate it on the cyclic sum. Then we will get the desired maximum value.

By majorization, we have $(5,0) \succ (3,2)$. Hence by Muirhead's inequality

$$\sum_{sym} a^5 = a^5 + b^5 \geq a^3b^2 + a^2b^3 = \sum_{sym} a^3b^2$$

so $a^5 + b^5 \geq a^2b^2(a+b)$. Hence,

$$\sum_{cyclic} \frac{ab}{a^5+b^5+ab} \leq \sum_{cyclic} \frac{ab}{a^2b^2(a+b)+ab} = \sum_{cyclic} \frac{1}{ab(a+b)+1}$$

Since $abc = 1$, we have $ab(a+b)+1 = ab(a+b+c)$ implying

$$\sum_{cyclic} \frac{ab}{a^5+b^5+ab} \leq \sum_{cyclic} \frac{1}{ab(a+b)+1} = \frac{1}{a+b+c} \sum_{cyclic} \frac{1}{ab} = 1$$

as desired. Equality occurs for $a = b = c = 1$.

Problem 2 (USAMO 1997/5). Let a, b and c be positive real numbers. Then

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}$$

always holds.

Proof. Since $(3,0) \succ (2,1)$, by Muirhead's inequality we have

$$\sum_{sym} a^3 = a^3 + b^3 \geq a^2b + ab^2 = \sum_{sym} a^2b$$

so $a^3 + b^3 \geq ab(a+b)$. Then

$$\sum_{cyclic} \frac{1}{a^3+b^3+abc} \leq \sum_{cyclic} \frac{1}{ab(a+b)+abc} = \frac{1}{a+b+c} \sum_{cyclic} \frac{1}{ab} = \frac{1}{abc}$$

as desired. Equality holds just for $a = b = c$.

Exercise 1. Try to prove the general case of USAMO 1997/5 below with the method that has been introduced.

Generalization 1. Let $n \geq 1$ be an integer. Prove the inequality

$$\sum_{i=1}^{n} \left(\frac{1}{\sum_{j=1}^{n} (a_j^n) - a_i^n + \prod_{k=1}^{n} a_k} \right) \leq \frac{1}{\prod_{k=1}^{n} a_k}$$

for all positive real numbers $a_1, a_2, \cdots, a_n$.

Problem 3 (Junior Balkan MO Shortlist 2022/A.4) Prove that if positive real numbers a, b, c satisfy $a + b + c \geq 1/a + 1/b + 1/c$, then

$$\frac{a+b-c}{a^3+b^3+abc} + \frac{b+c-a}{b^3+c^3+abc} + \frac{c+a-b}{c^3+a^3+abc} \leq 1$$

must hold.

Proof. Using $(3, 0) \succ (2, 1)$ or equivalently $a^3+b^3 \geq ab(a+b)$ in the denominator, and manipulating the cyclic sum, we get

$$\sum_{cyclic} \frac{a+b-c}{a^3+b^3+abc} \leq \sum_{cyclic} \frac{a+b-c}{ab(a+b)+abc} = \frac{1}{a+b+c} \sum_{cyclic} \frac{a+b-c}{ab}$$
$$= \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{abc(a+b+c)}$$

On the other hand, note that $(2, 0, 0) \succ (1, 1, 0)$ gives

$$\sum_{sym} a^2 = 2(a^2+b^2+c^2) \geq 2(ab+bc+ca) = \sum_{sym} ab$$

so $a^2 + b^2 + c^2 \geq ab + bc + ca$. Using this fact as well as the problem condition $a + b + c \geq 1/a + 1/b + 1/c$, we conclude

$$\sum_{cyclic} \frac{a+b-c}{a^3+b^3+abc} \leq \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{abc(a+b+c)} \leq \frac{ab+bc+ca}{abc(a+b+c)}$$
$$= \frac{1/a+1/b+1/c}{a+b+c} \leq 1$$

as desired. Equality occurs for $a = b = c = 1$.

Problem 4 (Balkan MO Shortlist 2017/A.1) Let a, b and c be positive real numbers with $abc = 1$. Then

$$\frac{1}{a^5+b^5+c^2} + \frac{1}{b^5+c^5+a^2} + \frac{1}{c^5+a^5+b^2} \leq 1$$

always holds.

Proof. By majorization, $(5, 0) \succ (4, 1)$. Then Muirhead's inequality gives

$$\sum_{sym} a^5 = a^5 + b^5 \geq a^4b + ab^4 = \sum_{sym} a^4b$$

so $a^5 + b^5 \geq ab(a^3 + b^3)$. Hence,

$$\sum_{cyclic} \frac{1}{a^5+b^5+c^2} \leq \sum_{cyclic} \frac{1}{ab(a^3+b^3)+c^2}$$

Manipulating the sum with $abc = 1$ gives

$$\sum_{cyclic} \frac{1}{ab(a^3+b^3)+c^2} = \sum_{cyclic} \frac{1}{ab(a^3+b^3)+abc^3} = \frac{a+b+c}{a^3+b^3+c^3}$$

By Power-Mean Inequality and AM-GM,

$$\frac{a+b+c}{a^3+b^3+c^3} \leq \frac{9}{(a+b+c)^2} \leq 1$$

as desired since $a+b+c \geq 3\sqrt[3]{abc} = 3$. Equality case holds if and only if $a = b = c = 1$.

Problem 5 (Tournament of Towns 1997/7). Prove that if the positive real numbers x_1, x_2, x_3 satisfy $x_1x_2x_3 = 1$, then

$$\frac{1}{x_1+x_2+1} + \frac{1}{x_2+x_3+1} + \frac{1}{x_3+x_1+1} \leq 1$$

Proof. The problem is trivial, you can see it by homogenizing the problem and giving $x_1 = a^3, x_2 = b^3$ and $x_3 = c^3$ in Problem 2.

Problem 6. Let a, b, c be positive real numbers such that $abc = 2$. Prove that

$$\frac{a^2b^2}{a^7+b^7+2a^2b^2} + \frac{b^2c^2}{b^7+c^7+2b^2c^2} + \frac{c^2a^2}{c^7+a^7+2c^2a^2} \leq \frac{1}{2}$$

must hold.

Problem 7. Prove the inequality

$$\frac{1}{a^4+b^4+c^4+k} + \frac{1}{b^4+c^4+d^4+k} + \frac{1}{c^4+d^4+a^4+k} + \frac{1}{d^4+a^4+b^4+k} \leq \frac{1}{k}$$

for all positive real numbers a, b, c, d where $abcd = k$.

Problem 8 Let a, b, c be positive real numbers. Let $abc = X$. Prove that

$$\frac{(ab)^p}{a^{2p+3}+b^{2p+3}+X(ab)^p} + \frac{(bc)^p}{b^{2p+3}+c^{2p+3}+X(bc)^p} + \frac{(ca)^p}{c^{2p+3}+a^{2p+3}+X(ca)^p} \leq \frac{1}{X}$$

always holds for any real number $p \geq -1$.

3 Hints and comments for exercises

Exercise 1. Note that $(n, 0, \cdots, 0) \succ (2, 1, 1, \cdots, 1)$, where there are $n-1$ elements in both sequences/lists. Then Muirhead's Inequality gives

$$\sum_{sym} a_1^n = (n-2)!(a_1^n + a_2^n + \cdots + a_{n-1}^n) \geq (n-2)! \left(\sum_{i=1}^{n-1} a_i \prod_{k=1}^{n-1} a_k \right) = \sum_{sym} a_1^2 a_2 \cdots a_{n-1}$$

or equivalently

$$\sum_{i=1}^{n-1} a_i^n \geq \left(\prod_{k=1}^{n-1} a_k \right) \left(\sum_{i=1}^{n-1} a_i \right)$$

Hence, for all $i = 1, 2, \cdots, n$ we have

$$\sum_{j=1}^{n} \left(a_j^n \right) - a_i^n \geq \left[\sum_{j=1}^{n} (a_j) - a_i \right] \left(\frac{\prod_{k=1}^{n} a_k}{a_i} \right)$$

Applying the above inequality in the problem, we get

$$\begin{aligned}
\sum_{i=1}^{n} \left(\frac{1}{\sum_{j=1}^{n} \left(a_j^n \right) - a_i^n + \prod_{k=1}^{n} a_k} \right) &\leq \sum_{i=1}^{n} \left[\frac{1}{\left[\sum_{j=1}^{n} (a_j) - a_i \right] \left(\frac{\prod_{k=1}^{n} a_k}{a_i} \right) + \prod_{k=1}^{n} a_k} \right] \\
&= \sum_{i=1}^{n} \left[\frac{1}{\left[\sum_{j=1}^{n} (a_j) - a_i \right] \left(\frac{\prod_{k=1}^{n} a_k}{a_i} \right) + a_i \left(\frac{\prod_{k=1}^{n} a_k}{a_i} \right)} \right] \\
&= \sum_{i=1}^{n} \left[\frac{a_i}{\left[\sum_{j=1}^{n} a_j \right] \left(\prod_{k=1}^{n} a_k \right)} \right] = \frac{1}{\prod_{k=1}^{n} a_k}
\end{aligned}$$

as desired proving the problem. Equality holds for $a_1 = a_2 = \cdots = a_n$.

Problem 6. Use $(7, 0) \succ (4, 3)$, then manipulate the cyclic sum with the fact that $abc = 2$.

References

[1] Hardy, G.H., Littlewood, J.E. and Polya, G., *Inequalities.*, Cambridge University Press, Cambridge, 1934.

[2] Kadelburg, Z., Dukic, D., Lukic, M., & Matic, I. *Inequalities of Karamata, Schur and Muirhead, and some applications.*, The Teaching of Mathematics, 8(1),pp. 31-45, 2005.

OLYMPIAD CORNER

No. 438

The problems in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **February 15, 2026**.

OC756. Let $A \in \mathcal{M}_2(\mathbb{R})$ be a matrix with real entries such that

$$\det(A^{2014} - I_2) = \det(A^{2014} + I_2)$$

and

$$\det(A^{2016} - I_2) = \det(A^{2016} + I_2).$$

Prove that $\det(A^n - I_2) = \det(A^n + I_2)$, for any $n \in \mathbb{N}$. Above $\mathbb{N}$ is the set of positive integers and I_2 is the 2×2 identity matrix.

OC757. Find all continuous bijective functions $f : [0, 1] \to [0, 1]$ such that

$$\int_0^1 g(f(x))dx = \int_0^1 g(x)dx,$$

for any continuous function $g : [0, 1] \to \mathbb{R}$.

OC758. Consider $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $AB = BA$ and $\det B \neq 0$.

a) If $|\det(A + zB)| = 1$, for all $z \in \mathbb{C}$ with $|z| = 1$, prove that $A^n = 0_n$;

b) Is the conclusion true if the commutative condition is dropped?

OC759. Let ABC be a scalene triangle, let I be its incentre, and let A_1, B_1, and C_1 be the points of contact of the excircles with the sides BC, CA, and AB, respectively. Prove that the circumcircles of the triangles AIA_1, BIB_1, and CIC_1 have a common point different from I.

OC760. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$\gcd(f(x), y)f(xy) = f(x)f(y)$$

for all x and y in $\mathbb{N}$.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 fevrier 2026**.

OC756. Soit $A \in \mathcal{M}_2(\mathbb{R})$ une matrice á entrées réelles telle que

$$\det(A^{2014} - I_2) = \det(A^{2014} + I_2)$$

and

$$\det(A^{2016} - I_2) = \det(A^{2016} + I_2).$$

Montrez que $\det(A^n - I_2) = \det(A^n + I_2)$ pour tout $n \in \mathbb{N}$. Notons que $\mathbb{N}$ désigne l'ensemble des nombres entiers positifs et I_2 est la matrice identité 2×2.

OC757. Trouvez toutes les fonctions bijectives continues $f : [0, 1] \to [0, 1]$ telles que

$$\int_0^1 g(f(x))dx = \int_0^1 g(x)dx,$$

pour toute fonction continue $g : [0, 1] \to \mathbb{R}$.

OC758. Considérons $A, B \in \mathcal{M}_n(\mathbb{C})$ tels que $AB = BA$ et $\det B \neq 0$.

a) Si $|\det(A + zB)| = 1$ pour tout $z \in \mathbb{C}$ avec $|z| = 1$, montrez que $A^n = 0_n$;

b) La conclusion est-elle vraie si l'on supprime la condition de commutativité ?

OC759. Soit ABC un triangle scalêne, et soit I le centre du son cercle inscrit. Soient A_1, B_1 et C_1 les points de contact des cercles exinscrits avec les côtés BC, CA et AB, respectivement. Montrez que les cercles circonscrits aux triangles AIA_1, BIB_1 et CIC_1 ont un point commun distinct de I.

OC760. Soit $\mathbb{N}$ l'ensemble des nombres entiers positifs. Trouvez toutes les fonctions $f : \mathbb{N} \to \mathbb{N}$ telles que

$$\text{PGCD}(f(x), y)f(xy) = f(x)f(y)$$

pour tous les x et y dans $\mathbb{N}$.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(5), p. 221–222.

OC731. 12 children came to a physical education lesson, all of different strengths. The teacher divided them into two teams of 6 people 10 times, each time in a new way, and held a tug-of-war competition. Could it be that all 10 times the competition ended in a draw (that is, the sum of the strengths of the children in the teams was equal)?

Originally from Moscow Mathematical Olympiad 2024 - Grade 8, Problem 2.

We received 6 solutions. We present the solution by Roy Barbara.

The answer is yes. Denote the children by $1, 2, 3, \ldots, 12$, where child j has precisely strength j. Here are 10 different partitions. where in each case, the sum of the strengths in each team is 39:

$$\begin{aligned}
(1,3,4,9,10,12) &\text{ vs } (2,5,6,7,8,11)\\
(1,3,5,8,10,12) &\text{ vs } (2,4,6,7,9,11)\\
(1,3,6,7,10,12) &\text{ vs } (2,4,5,8,9,11)\\
(1,3,6,8,9,12) &\text{ vs } (2,4,5,7,10,11)\\
(1,3,7,8,9,11) &\text{ vs } (2,4,5,6,10,12)\\
(1,4,5,7,10,12) &\text{ vs } (2,3,6,8,9,11)\\
(1,4,5,8,9,12) &\text{ vs } (2,3,6,7,10,11)\\
(1,4,6,7,9,12) &\text{ vs } (2,3,5,8,10,11)\\
(1,4,6,8,9,11) &\text{ vs } (2,3,5,7,10,12)\\
(1,5,6,7,8,12) &\text{ vs } (2,3,4,9,10,11)
\end{aligned}$$

Remark. The strengths $1, 2, \ldots, 12$ may be replaced by $k+1, k+2, k+3, \ldots, k+12$, where k is any real number greater than -1.

OC732. Prove that among the vertices of a convex nonagon, three can be found that form an obtuse triangle, none of whose sides coincide with the sides of the nonagon.

Originally from Moscow Mathematical Olympiad 2024 - Grade 10, Problem 2.

We received 6 solutions. We present the solution by Srikanth Pai Ashwath.

Label the vertices of the convex nonagon $A, B, C, D, E, F, G, H, I$ in clockwise order and suppose, for the sake of contradiction, that every triangle whose three sides are diagonals of the nonagon is non-obtuse.

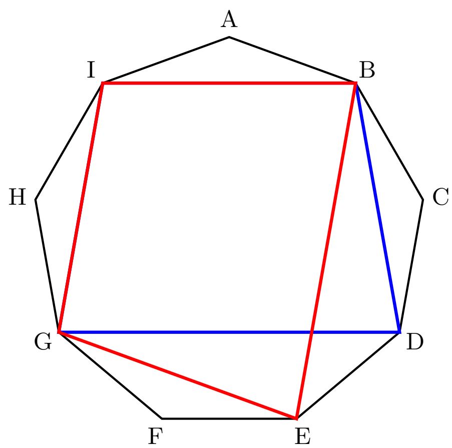


Figure 1: The rectangles $BIGD$ (blue) and $BEGI$ (red).

Consider the quadrilateral $BIGD$. All four of its sides BI, IG, GD and DB are diagonals, so any triangle formed from three of these segments falls under the hypothesis. If one corner of $BIGD$ were obtuse, the diagonal opposite that corner would give a triangle with three diagonal sides and an obtuse angle, contradicting our assumption. Hence every corner of $BIGD$ is at most $90°$. Conversely, if one corner were acute, the remaining three would sum to more than $270°$; with each already at most $90°$, one of them would then have to exceed $90°$, again impossible. Therefore no corner is acute either, and all four corners are right angles: $BIGD$ is a rectangle.

Similarly $BEGI$ is also a rectangle. These two rectangles share the vertices B, E, and I; but a parallelogram (and thus a rectangle) is uniquely determined by any three of its vertices, forcing their fourth vertices to coincide, i.e. $D = G$. Such an identification is impossible in a nonagon, whose nine vertices are distinct.

The contradiction proves that our initial assumption is false. Consequently, the nonagon must contain at least one obtuse triangle all of whose sides are diagonals.

OC733. In an acute triangle ABC, the altitude AH is drawn. Points M and N are the midpoints of segments BH and CH. Prove that the intersection point of the perpendiculars dropped from points M and N to lines AB and AC, respectively, is equidistant from points B and C.

Originally from Moscow Mathematical Olympiad 2024 - Grade 11, Problem 2.

We received 8 solutions. We present the solution by Theo Koupelis.

Let $(a, b, c) = (BC, AC, AB)$ and $(\angle A, \angle B, \angle C) = (\alpha, \beta, \gamma)$. Let E, F be points on the sides AB, AC, respectively, so that $ME \perp AB$ and $NF \perp AC$. Let D be the intersection point of lines ME and NF, and let K be the foot of the perpendicualr from D on the side BC. Such points exist because the triangle is acute. The right triangles DKM and BEM are similar, and thus $MK = DK \cdot \tan\beta$. The right triangles DKN and CEN are also similar, and thus $NK = DK \cdot \tan\gamma$. We have

$$MN = a - (BM + CN) = a - \frac{1}{2}(BH + CH) = a/2$$

and

$$MN = MK + NK = DK \cdot (\tan\beta + \tan\gamma) = DK \cdot \frac{\sin\alpha}{\cos\beta \cdot \cos\gamma}.$$

Thus,

$$DK = \frac{a}{2} \cdot \frac{\cos\beta \cdot \cos\gamma}{\sin\alpha}.$$

Therefore, using $BH = c \cdot \cos\beta$, $CH = b \cdot \cos\gamma$, and the law of sines in the triangle ABC, we get

$$BM = \frac{c \cdot \cos\beta}{2} = \frac{a \cdot \sin\gamma}{\sin\alpha} \cdot \frac{\cos\beta}{2} = DK \cdot \tan\gamma = NK,$$

and similarly

$$CN = \frac{b \cdot \cos\gamma}{2} = \frac{a \cdot \sin\beta}{\sin\alpha} \cdot \frac{\cos\gamma}{2} = DK \cdot \tan\beta = MK.$$

Thus, $BK = BM + MK = NK + CN = CK$, and therefore $DB = DC$.

OC734. A natural number has exactly 50 divisors. Is it possible that no difference between two of its distinct divisors is divisible by 100?

Originally from All Russian Mathematical Olympiad 2024 - Final Round, Grade 9, Problem 2.

We received 6 solutions. We present the solution by C. R. Pranesachar.

The answer is no, that is, the difference of some two divisors of n with 50 divisors is divisible by 100. In what follows, we use Euler's theorem on Totient function and the Pigeonhole principle. We also use the fact that if b is coprime to a, then the remainder obtained when b is divided by a is also coprime to a. Suppose a number n has 50 positive divisors. Then there are four possibilities: (1) $n = q^{49}$; (2) $n = pq^{24}$; (3) $n = p^4q^9$; (4) $n = pq^4r^4$, where p, q, r are primes. Since

$$100 = 2^2 \cdot 5^2,$$

we consider three cases:

(a) 5 divides n;

(b) 5 does not divide n, but 2 divides n;

(c) neither 5 divides n nor 2 divides n.

(a) **(i)** If $q = 5$ in cases (1), (2), (3), (4) above, we observe that

$$100 \mid 5^3 - 5^2.$$

The same is the case if $p = 5$ in case (3), or $r = 5$ in case (4).

(ii) If $p = 5$ in (2), then

$$n = 5q^{24}, \ q \neq 5.$$

If $q = 2$, then

$$n = 5 \cdot 2^{24},$$

and we have

$$100 \mid 5 \cdot 2^6 - 5 \cdot 2^2.$$

If $q \neq 2$, then

$$\gcd(q, \ 20) = 1$$

and so

$$q^8 \equiv 1 \pmod{20},$$

because $\phi(20) = 8$. So

$$100 \mid 5 \cdot q^8 - 5.$$

(iii) If $p = 5$ in (4), then $n = 5q^4r^4, \ q \neq 5, \ r \neq 5$. Now q^4r^4 has $5 \cdot 5 = 25$ divisors. So some set of two of them, say m_1, m_2, satisfy the condition $20 \mid m_1 - m_2$. Hence

$$100 \mid 5m_1 - 5m_2.$$

(b) (1) If $n = 2^{49}$, then

$$2^{20} \equiv 1 \pmod{25}$$

as $\phi(25) = 20$. So

$$25 \mid 2^{20} - 1$$

and

$$100 \mid 2^{22} - 2^2.$$

(2) In case (2), either $p = 2$ or $q = 2$.

If $p = 2$, then

$$n = 2 \cdot q^{24}, \ q \neq 5.$$

As above $25 \mid q^{20} - 1$. But q is odd. So

$$50 \mid q^{20} - 1.$$

Hence

$$100 \mid 2 \cdot q^{20} - 2;$$

If $q = 2$, then $n = p \cdot 2^{24}, \ p \neq 5$, and we have $100 \mid 2^{22} - 2^2$ as above.

(3) In case (3), either $p = 2$ or $q = 2$.

If $p = 2$, then

$$n = 2^4 \cdot q^9,\ \ q \neq 5.$$

Then

$$n = 2^2(2^2 \cdot q^9).$$

Now $2^2 \cdot q^9$ has 30 divisors. When these are divided by 25, some two divisors m_1 and m_2 leave the same remainder. Hence

$$25 \mid m_1 - m_2$$

and so

$$100 \mid 2^2 m_1 - 2^2 m_2.$$

If $q = 2$, then

$$n = p^4 \cdot 2^9 = 2^2(2^7 \cdot p^4).$$

Here $2^7 \cdot p^4$ has 40 divisors. When these are divided by 25, some two divisors m_1 and m_2 leave the same remainder. Hence

$$25 \mid m_1 - m_2$$

and so

$$100 \mid 2^2 m_1 - 2^2 m_2.$$

(4) In case (4) either $p = 2$ or, say $q = 2$

If $p = 2$, then

$$n = 2q^4 r^4,\ \ q \neq 5,\ \ r \neq 5.$$

Here $q^4 r^4$ has 25 divisors. Since

$$\gcd(q, 10) = 1,\ \ \gcd(r, 10) = 1,$$

each of these 25 divisors when divided by 50, leave remainders coprime to 2 as well as 5. There are only $\phi(50) = 20$ such remainders. Hence some two divisors m_1 and m_2 leave the same remainder. Therefore

$$50 \mid m_1 - m_2$$

and so

$$100 \mid 2m_1 - 2m_2.$$

If $q = 2$, then

$$n = p \cdot 2^4 \cdot r^4 = 2^2(2^2 \cdot p \cdot r^4).$$

Here $2^2 \cdot p \cdot r^4$ has 30 divisors. When these are divided by 25, some two divisors m_1 and m_2 leave the same remainder. Hence

$$25 \mid m_1 - m_2$$

and so

$$100 \mid 2^2 m_1 - 2^2 m_2.$$

(c) We have

$$\gcd(n,\ 10) = 1.$$

So in this case the 50 divisors of n when divided by 100 leave remainders coprime to 2 as well as 5. The number of these remainders is 40, since $\phi(100) = 40$. Hence two divisors m_1, m_2 among the 50 divisors of n leave the same remainder when divided by 100. Thus

$$100 \mid m_1 - m_2.$$

This completes the proof.

OC735. An odd number $n \geq 3$ is given. In a $2n \times 2n$ grid, $2(n-1)^2$ cells are painted. What is the greatest number of L-trominos that can be guaranteed to be cut out of an unpainted grid?

Originally from All Russian Mathematical Olympiad 2024 - Final Round, Grade 10, Problem 2.

We received no solutions to this problem.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **February 15, 2026**.

5091. *Proposed by Tatsunori Irie.*

On a circle there are 2025 places. The numbers $1, 2, \ldots, 2025$ are written on them in some order (one number per place). A move consists of choosing three consecutive places with entries (x, y, z) in this cyclic order and replacing them by $(x+1, 2-y, z+1)$. In other words, we give 1 to each neighbour and then replace the middle entry by the negative of what remains. Show that, no matter how the numbers are arranged at the start, it is possible by finitely many moves to make all 2025 entries equal. Decide whether the common value is uniquely determined.

5092. *Proposed by Khuong Trang Tran Ngoc.*

Prove that the following inequality

$$\frac{1}{x^2+y^2+3z^2}+\frac{1}{y^2+z^2+3x^2}+\frac{1}{z^2+x^2+3y^2} \leq \frac{3}{5}$$

holds for all positive real numbers $x \geq y \geq z$ such that $x^2+y^2+z^2+xyz=4$. When does equality occur?

5093. *Proposed by Michel Bataille.*

Let ABC be a triangle with $\angle A \neq 2\angle C$ and let D on side BC be such that AD bisects $\angle A$. Let Γ_1 be the circumcircle of ΔADB, Γ_2 be the circle through C and D orthogonal to Γ_1 and Γ_3 be the circle through A and C orthogonal to Γ_2. Let Γ_3 intersect the line BC at $M \neq C$ and its diameter through D intersect Γ_2 at $N \neq D$. Prove that the circumcenter of ΔDNM is on the line AC.

5094. *Proposed by Benjamin Braiman.*

Find a dense sequence of real numbers $\{x_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} \frac{x_1+\cdots+x_n}{n} = 0,$$

or show no such sequence exists. (Recall that a sequence is said to be dense if every open interval (a, b) contains an element of the sequence.)

5095. *Proposed by Nikolai Osipov.*

(a) Let x, y, z be positive integers such that $(x^2-1)(y^2-1)=z^2-1$. Show that x, y, z are pairwise coprime.

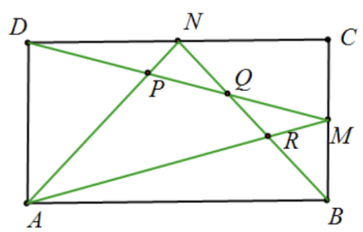
(b) Let $a \geqslant 2$ and $b \geqslant 1$ be integers such that b is a divisor of a^2-2. Prove that the equation
$$x^2-(a^2-1)y^2=\frac{2-a^2}{b}$$
is solvable in integers x, y if and only if $b=1$.

5096. *Proposed by Vasile Cîrtoaje.*

Let a,b,c,d be nonnegative real numbers such that at most one of them is larger than 1 and $ab+ac+ad+bc+bd+cd=6$. Prove that
$$\frac{1}{(a+b+c)^2}+\frac{1}{(b+c+d)^2}+\frac{1}{(c+d+a)^2}+\frac{1}{(d+a+b)^2}\geq\frac{4}{9}.$$

5097. *Proposed by Xicheng Peng.*

Let $ABCD$ be a parallelogram. Let M and N be the midpoints of BC and CD, respectively. Let AM intersect BN at R, DM intersect BN at Q, and AN intersect DM at P. Prove that points A,P,Q,R are concyclic if and only if $BA\perp BC$.



5098. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Given a cyclic quadrilateral $BCAD$ (with A and B separating C from D) such that $CB=CD$, define A_1 and B_1 to be the feet of the altitudes from A and B in triangle ABC. Prove that the orthocenter H of ΔABC is the midpoint of AA_1 if and only if DB_1 is perpendicular to DB.

5099. *Proposed by Michel Bataille.*

Let s be a positive integer and let $a_0=1,\ a_k=\prod_{j=1}^{k}(s+j)$ for $k\geq 1$. Evaluate
$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{(2ns)^k}.$$

5100. *Proposed by Huseyin Yigit Emekci.*

Let $n \geq 2$ be an integer and let $a_i, \ldots, a_n$ be positive real numbers such that $a_1 + \cdots + a_n = 1$. Prove that

$$\sum_{k=2}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^4 < \frac{1}{5}.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 fevrier 2026**.

5091. *Soumis par Tatsunori Irie.*

Sur un cercle, on dispose de 2025 emplacements. Les nombres $1, 2, \ldots, 2025$ y sont inscrits dans un certain ordre (un nombre par emplacement). Un coup consiste à choisir trois emplacements consécutifs portant les valeurs (x, y, z) dans cet ordre cyclique, et à les remplacer par $(x+1, 2-y, z+1)$. Autrement dit, on ajoute 1 à chacun des voisins et on remplace la valeur du milieu par l'opposé de ce qui reste.

Montrez que, quelle que soit la disposition initiale des nombres, il est possible, en un nombre fini de coups, de rendre toutes les 2025 valeurs égales. Déterminez si la valeur commune obtenue est unique.

5092. *Soumis par Khuong Trang Tran Ngoc.*

Montrez que l'inégalité suivante

$$\frac{1}{x^2+y^2+3z^2} + \frac{1}{y^2+z^2+3x^2} + \frac{1}{z^2+x^2+3y^2} \leq \frac{3}{5}$$

est vérifiée pour tous les réels positifs $x \geq y \geq z$ tels que $x^2 + y^2 + z^2 + xyz = 4$. Quand a-t-on égalité ?

5093. *Soumis par Michel Bataille.*

Soit ABC un triangle tel que $\angle A \neq 2\angle C$, et soit D un point du côté BC tel que AD soit la bissectrice de l'angle $\angle A$. Soit Γ_1 le cercle circonscrit au triangle ADB, Γ_2 le cercle passant par C et D et orthogonal à Γ_1, et Γ_3 le cercle passant par A et C et orthogonal à Γ_2. Soit $M \neq C$ le point d'intersection de Γ_3 avec la droite

BC, et soit $N \neq D$ le point d'intersection de Γ_2 avec le diamêtre de Γ_3 passant par D. Montrez que le centre du cercle circonscrit au triangle DNM appartient à la droite AC.

5094. *Soumis par Benjamin Braiman.*

Trouvez une suite dense de nombres réels $\{x_n\}_{n\geq 1}$ telle que

$$\lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} = 0,$$

ou montrez qu'aucune suite de ce type n'existe.
(On rappelle qu'une suite est dite dense si tout intervalle ouvert (a, b) contient au moins un terme de la suite.)

5095. *Soumis par Nikolai Osipov.*

(a) Soient x, y, z des entiers strictement positifs tels que $(x^2-1)(y^2-1) = z^2-1$. Montrez que x, y et z sont deux à deux premiers entre eux.

(b) Soient $a \geqslant 2$ et $b \geqslant 1$ des entiers tels que b divise $a^2 - 2$. Montrez que l'équation

$$x^2 - (a^2-1)y^2 = \frac{2-a^2}{b}$$

admet des solutions entières x et y si et seulement si $b = 1$.

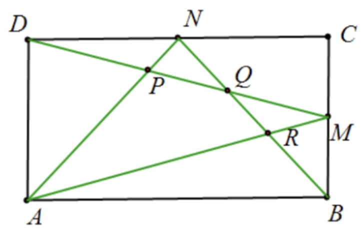
5096. *Soumis par Vasile Cîrtoaje.*

Soient a, b, c et d des nombres réels non négatifs tels que tout au plus un d'entre eux soit supérieur à 1 et que $ab + ac + ad + bc + bd + cd = 6$. Montrez que

$$\frac{1}{(a+b+c)^2} + \frac{1}{(b+c+d)^2} + \frac{1}{(c+d+a)^2} + \frac{1}{(d+a+b)^2} \geq \frac{4}{9}.$$

5097. *Soumis par Xicheng Peng.*

Soit $ABCD$ un parallélogramme. Soient M et N les milieux des côtés BC et CD, respectivement. Soit R le point d'intersection de AM et BN, Q le point d'intersection de DM et BN, et P le point d'intersection de AN et DM. Montrez que les points A, P, Q et R sont concycliques si et seulement si $BA \perp BC$.



5098. *Soumis par Mihaela Berindeanu, modifié par le comité de rédaction.*

Étant donné un quadrilatère cyclique $BCAD$ (où A et B séparent C de D) tel que $CB = CD$, on définit A_1 et B_1 comme les pieds des hauteurs issues de A et de B dans le triangle ABC. Montrez que l'orthocentre H du triangle ABC est le milieu de AA_1 si et seulement si DB_1 est perpendiculaire à DB.

5099. *Soumis par Michel Bataille.*

Soit s un entier positif et soit $a_0 = 1,\ a_k = \prod_{j=1}^{k}(s+j)$ for $k \geq 1$. Évaluez

$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{(2ns)^k}.$$

5100. *Soumis par Huseyin Yigit Emekci.*

Soit $n \geq 2$ un entier et soient $a_i, \ldots, a_n$ des nombres réels strictement positifs tels que $a_1 + \cdots + a_n = 1$. Montrez que

$$\sum_{k=2}^{n}\frac{a_k}{1-a_k}(a_1+a_2+\cdots+a_{k-1})^4 < \frac{1}{5}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2025: 51(5), p. 235–238.

5041. *Proposed by Giuseppe Fera.*

If chords of an ellipse E subtend a constant angle at the focus O, prove that their envelope is an ellipse with axes parallel to those of E and O as focus.

We received 6 solutions all of which were correct and complete. We present the solution by Theo Koupelis.

Let the equation for ellipse E be

$$x^2/a^2 + y^2/b^2 = 1, \tag{1}$$

with foci at $O = (-c, 0)$ and $O' = (c, 0)$, where $c^2 = a^2 - b^2$, and $e = c/a$ is the eccentricity of the ellipse. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points on ellipse E such that $\angle POQ = \theta =$ constant. Let $y = mx + k$ be the equation of the line PQ. We have $PO = a + ex_1$ and $QO = a + ex_2$, and thus, using the law of cosines in triangle POQ we get

$$\cos\theta = \frac{PO^2 + QO^2 - PQ^2}{2PO \cdot QO} = \frac{c^2 + c(x_1 + x_2) + x_1x_2 + y_1y_2}{c^2 + c(x_1 + x_2) + b^2 + e^2x_1x_2}. \tag{2}$$

The coordinates of the points P, Q satisfy (1) and thus

$$\frac{x^2}{a^2} + \frac{(mx + k)^2}{b^2} = 1 \Longrightarrow (b^2 + m^2a^2) \cdot x^2 + 2a^2km \cdot x + a^2(k^2 - b^2) = 0. \tag{3}$$

The solutions of (3) are x_1 and x_2, and therefore $x_1 + x_2 = -2a^2km/(b^2 + m^2a^2)$, and $x_1x_2 = a^2(k^2 - b^2)/(b^2 + m^2a^2)$. Also,

$$y_1y_2 = (mx_1 + k)(mx_2 + k) = m^2x_1x_2 + km(x_1 + x_2) + k^2.$$

Substituting into (2) we get

$$\cos\theta = \frac{a^2(c^2 - b^2)m^2 - 2a^2ckm + k^2(a^2 + b^2) + b^2(c^2 - a^2)}{a^4m^2 - 2a^2ckm + b^4 + k^2c^2}. \tag{4}$$

By construction, the envelope of the chords PQ is a closed curve. Assuming it is an ellipse E' with axes parallel to those of E, and having O as one of its foci, its equation is

$$\frac{(x - \ell)^2}{A^2} + \frac{y^2}{B^2} = 1. \tag{5}$$

The equation of a tangent to E' at a point $T = (x_o, y_o)$ is given by

$$y'\Big|_T = -\frac{B^2}{A^2} \cdot \frac{x_o - \ell}{y_o}. \tag{6}$$

The point T also belongs to the line $y = mx + k$, and thus $y'\Big|_T = m$. Substituting into (6) we get

$$x_o = \frac{\ell B^2 - kmA^2}{B^2 + m^2A^2}, \quad \text{and} \quad y_o = mx_o + k. \tag{7}$$

Substituting (7) into (5) we get

$$(k + m\ell)^2 = B^2 + m^2A^2. \tag{8}$$

To find A, B, ℓ that satisfy (8) for any k, m that satisfy (4), we look at extreme cases. When the tangents to E' are perpendicular to the major axis, substituting into (2) $t := x_1 = x_2$ and $y_2 = -y_1$, where $y_1^2 = b^2(a^2 - x_1^2)/a^2$, we get

$$[a^2 + b^2 - c^2 \cdot \cos\theta] \cdot t^2 + 2ca^2(1 - \cos\theta) \cdot t + a^4 - 2a^2b^2 - a^4 \cdot \cos\theta = 0. \tag{9}$$

There are two such tangents, at the vertices of ellipse E', where $t_1 = A + \ell$ and $t_2 = -A + \ell$. Thus, from (9) we get

$$\ell = -\frac{ca^2(1 - \cos\theta)}{a^2 + b^2 - c^2 \cdot \cos\theta} \quad \text{and} \quad A^2 = \frac{2a^2b^4(1 + \cos\theta)}{[a^2 + b^2 - c^2 \cdot \cos\theta]^2}. \tag{10}$$

When the tangents to E' are at the co-vertices and thus parallel to the major axis, substituting into (2) $y_1 = y_2 = B$ and $x_2 = -x_1$, where $x_1^2 = a^2(b^2 - y_1^2)/b^2$, we get

$$B^2 = \frac{b^4(1 + \cos\theta)}{a^2 + b^2 - c^2 \cdot \cos\theta}. \tag{11}$$

Substituting the expressions for A, B, ℓ from (10) and (11) into (8), with $\cos\theta$ given by (4), we get that (8) is an identity for k, m. Thus, the envelope of all chords of an ellipse E given by (1), subtending a constant angle θ at the focus O, is an ellipse E', given by (5), with axes parallel to those of E and having O as one of its foci; the expressions for A, B, ℓ are given by (10) and (11).

Note that using the expressions in (7), (10), and (11), we get that OT is the angle bisector of $\angle POQ$.

Editor's Comments. This result in various generality appeared before on many websites and forums. The oldest reference that we could find is Problem 69 on p. 284 in "Conic Sections Treated Geometrically" by W.H. Besant (London, 1895):

> 69. The chords of a conic which subtend the same angle at the focus all touch another conic having the same focus and directrix (`https://archive.org/details/cu31924059322481/page/n305/mode/2up`).

5042. *Proposed by Michel Bataille.*

Let w be such that $w^5 = 1$ and $w \neq 1$. Find the complex roots of the polynomial

$$\sum_{1\le i<j<k\le 4} (x + w^i + w^j + w^k)^3 - \sum_{1\le i<j\le 4} (x + w^i + w^j)^3 + \sum_{1\le i\le 4} (x + w^i)^3.$$

We received 11 solutions, all of which were correct and complete. We present the solution by Catherine Jian.

From $w^5 = 1$ and $w \neq 1$, we have $1 + w + w^2 + w^3 + w^4 = 0$, which gives

$$\begin{aligned}\sum_{1\le i<j<k\le 4} &(x + w^i + w^j + w^k)^3 + \sum_{1\le i\le 4} (x + w^i)^3 \\ &= \sum_{1\le i\le 4} (x - 1 - w^i)^3 + \sum_{1\le i\le 4} (x + w^i)^3.\end{aligned}$$

Then using

$$a^3 + b^3 = (a + b)((a + b)^2 - 3ab),$$

we get

$$(x - 1 - w)^3 + (x + w)^3 = (2x - 1)((2x - 1)^2 - 3(x^2 - x - w - w^2)).$$

Note that $w^2 + w^4 + w^6 + w^8 = w^2 + w^4 + w + w^3 = -1$, we then have

$$\begin{aligned}\sum_{1\le i\le 4} (x - 1 - w^i)^3 + \sum_{1\le i\le 4} (x + w^i)^3 &= (2x - 1)(4(2x - 1)^2 - 3(4x^2 - 4x + 2)) \\ &= (2x - 1)(4x^2 - 4x - 2).\end{aligned}$$

Also note that

$$\begin{aligned}(x + w + w^2)^3 + (x + w^3 + w^4)^3 &= (2x - 1)((2x - 1)^2 - 3(x^2 - x + 2 + w + w^4)),\\ (x + w + w^3)^3 + (x + w^2 + w^4)^3 &= (2x - 1)((2x - 1)^2 - 3(x^2 - x + 2 + w^3 + w^2)),\\ (x + w + w^4)^3 + (x + w^2 + w^3)^3 &= (2x - 1)((2x - 1)^2 - 3(x^2 - x - 1)),\end{aligned}$$

thus

$$\begin{aligned}\sum_{1\le i<j\le 4} (x + w^i + w^j)^3 &= (2x - 1)(3(2x - 1)^2 - 3(3x^2 - 3x + 2)) \\ &= (2x - 1)(3x^2 - 3x - 3).\end{aligned}$$

Therefore, the original polynomial can be simplified to

$$(2x - 1)[(4x^2 - 4x - 2) - (3x^2 - 3x - 3)] = (2x - 1)(x^2 - x + 1),$$

and the three roots are $\frac{1}{2}, \frac{1 + \sqrt{3}i}{2}, \frac{1 - \sqrt{3}i}{2}$.

5043. *Proposed by Daniel Sitaru.*

Let $b \geq a \geq 1$. Prove that

$$\int_a^b \int_a^b \frac{dxdy}{1+\sqrt{xy}} \leq (b-a)\ln\Big(\frac{b+1}{a+1}\Big).$$

We received 7 submissions. All solutions are correct, complete, and use the same approach described below.

Since for any $a, b \geq 1$ we have

$$\begin{aligned} 2(b-a)\ln\left(\frac{b+1}{a+1}\right) &= \int_a^b \frac{dx}{1+x} \cdot \int_a^b dy + \int_a^b dx \cdot \int_a^b \frac{dy}{1+y} \\ &= \int_a^b \int_a^b \left(\frac{1}{1+x} + \frac{1}{1+y}\right) dx\, dy, \end{aligned}$$

it suffices to show that

$$\frac{2}{1+\sqrt{xy}} \leq \frac{1}{1+x} + \frac{1}{1+y}, \qquad x, y \geq 1.$$

The latter inequality is readily seen to be equivalent to

$$(\sqrt{xy}-1)(\sqrt{x}-\sqrt{y})^2 \geq 0, \qquad x, y \geq 1,$$

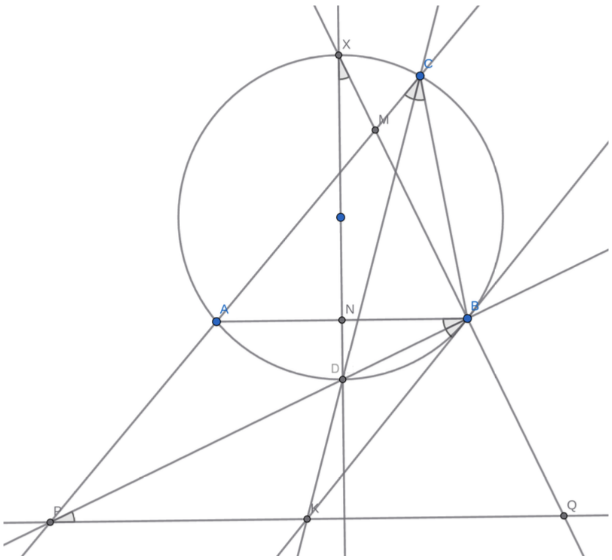
which is trivial. This concludes the proof.

5044. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Given a circle Γ with center O and a chord AB, let X be the midpoint of the larger arc AB, and C be an arbitrary point of that arc. Define K to be the point where the bisector of $\angle ACB$ intersects the tangent to Γ at B, while M is the intersection of AC and BX. Prove that the line MK contains the midpoint of AB.

There were 10 correct solutions. We present 4 of them.

Solution 1, by Michel Adamaszek. Let N be the midpoint of AB, D be the midpoint of the shorter arc AB, P be the interesection of AC and BD, and Q the intersection of PK and BX.



Let $\alpha = \angle BXD = \angle BCD = \angle ACD = \angle ABD$, all subtended by equal arcs. Since D lies on the bisector of angle ACB, C, D, K are collinear. Moreover, $\angle DBK = \alpha$, as the angle between tangent and chord equals that subtended by the chord.

Since $\angle PBK = \angle PCK = \alpha$, $PKBC$ is concyclic so that $\angle BPK = \angle PBA = \angle PBK = \alpha$. Therefore $PQ \| AB$ and $PK = BK$. Observe that $\angle XBD = 90^\circ$ so that triangle PBQ is right and the point K on its hypotenuse is equidistant from B and P, so it is the midpoint of PQ ($\angle KBQ = \angle KQB = 90^\circ - \alpha$).

Since $PQ \| AB$, the triangles MAB and MPQ are homothetic with centre M, with the midpoint N of AB corresponding to the midpoint K of PQ. Thus, M, N, K are collinear, as desired.

Solution 2, by Michel Bataille.

We use barycentric coordinates relative to (A, B, C) and denote by a, b, c the respective sidelengths of BC, CA, AB. The equation of the bisector CK is $bx - ay = 0$. The equation of Γ being $a^2yz + b^2zx + c^2xy = 0$, the equation of the tangent to Γ at $(x_0 : y_0 : z_0)$ is

$$a^2(zy_0 + yz_0) + b^2(xz_0 + zx_0) + c^2(yx_0 + xy_0) = 0.$$

In particular, the equation of the tangent at $B = (0 : 1 : 0)$ is $c^2x + a^2z = 0$ and so $K = (a^2 : ab : -c^2)$

On the other hand, X is the intersection other than C of Γ and the external bisector of $\angle ACB$ ($bx + ay = 0$), so that $X = (a(b-a) : -b(b-a) : c^2)$ and the equation of BX is $c^2x - a(b-a)z = 0$. We deduce that $M = (a(b-a) : 0 : c^2)$.

Since the midpoint of AB is $(1:1:0)$ and

$$\begin{vmatrix} 1 & a(b-a) & a^2 \\ 1 & 0 & ab \\ 0 & c^2 & -c^2 \end{vmatrix} = \begin{vmatrix} 1 & a(b-a) & a^2 \\ 0 & -a(b-a) & a(b-a) \\ 0 & c^2 & -c^2 \end{vmatrix} = ac^2(b-a) - ac^2(b-a) = 0,$$

this midpoint is on the line MK.

Solution 3, by Sicheng Du.

Recall the external trigonometric form of Ceva's theorem and its converse: Suppose that p, q, r are three lines through the respective vertices B, M, C of a triangle such that p makes angles α_1 and α_2 with BM and BC respectively, q makes angles β_1 and β_2 with MC and MB respectively, and r makes angles γ_1 and γ_2 with CB and CM respectively. Then p, q and r are concurrent if and only if

$$\frac{\sin\alpha_1}{\sin\alpha_2} \cdot \frac{\sin\beta_1}{\sin\beta_2} \cdot \frac{\sin\gamma_1}{\sin\gamma_2} = 1.$$

Apply this result to the present situation with p the tangent to Γ at B, q the line MN and r the bisector of angle ACB, so that $\gamma_1 = \gamma_2$ and the third term of the product is equal to 1. Since the angles between p and the chords BX and BC are equal to the angles subtended by these chords,

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{\sin\angle BAX}{\sin\angle BAC}.$$

Applying the sine law to triangles AMN and BMN and noting the equalities $AX = BX$ and $AN = BN$, we have that

$$\begin{aligned} \frac{\sin\beta_1}{\sin\beta_2} &= \frac{\sin\angle AMN}{\sin\angle BMN} = \frac{(AN/MN)\sin\angle NAM}{(BN/MN)\sin\angle NBM)} \\ &= \frac{\sin\angle BAC}{\sin\angle ABX} = \frac{\sin\angle BAC}{\sin\angle BAX}. \end{aligned}$$

The product of the three ratios is equal to 1, so that p, q and r pass through a common point. This point is K, the intersection of p and q; hence M, N and K must be collinear.

Solution 4, by Theo Koupelis.

Wolog, let $AC > BC$. Let CK intersect AB at F and the circle Γ at D (the midpoint of the smaller arc AB). Let N be the midpoint of the chord AB. Then O, N, D are collinear and $OD \perp AB$. We have $\alpha = \angle KBD = \angle BXD = \angle BCD = \angle DCA = \angle DBA$. Let $\beta = \angle CDB = \angle CAB$.

Because CF is the angle bisector of $\angle BCA$,

$$\frac{AB}{BF} = \frac{AF}{BF} + 1 = \frac{AC}{BC} + 1 = \frac{AC+BC}{BC}.$$

Also

$$\frac{AB}{2} \cdot \left[1 - \frac{AC - BC}{AC + BC}\right] = \frac{AB \cdot BC}{AC + BC} = BF.$$

Hence

$$\begin{aligned} BF = \frac{AB \cdot BC}{BC + AC} &\Longrightarrow FN = \frac{AB}{2} - BF = \frac{AB}{2} \cdot \frac{AC - BC}{AC + BC} \\ &\Longrightarrow \frac{FN}{NA} = \frac{AC - BC}{AC + BC} = \frac{\sin(\beta + 2\alpha) - \sin\beta}{\sin(\beta + 2\alpha) + \sin\beta} = \frac{\sin\alpha \cdot \cos(\alpha + \beta)}{\cos\alpha \cdot \sin(\alpha + \beta)}, \end{aligned}$$

using the sine law on triangle ABC.

An application of the sine law on triangles ABM and CBM yields

$$\frac{AM}{\cos\alpha} = \frac{BM}{\sin\beta} \qquad \text{and} \qquad \frac{MC}{\cos(\alpha + \beta)} = \frac{BM}{\sin 2\alpha},$$

whence

$$\frac{AM}{MC} = \frac{\cos\alpha \cdot \sin 2\alpha}{\cos(\alpha + \beta) \cdot \sin\beta}.$$

Finally, applying the sine law to triangles CBK and BFK yields

$$\frac{CK}{\sin\beta} = \frac{KB}{\sin\alpha} \qquad \text{and} \qquad \frac{KF}{\sin 2\alpha} = \frac{KB}{\sin(\alpha + \beta)},$$

whence

$$\frac{CK}{KF} = \frac{\sin(\alpha + \beta) \cdot \sin\beta}{\sin\alpha \cdot \sin 2\alpha}.$$

Since

$$\frac{FN}{NA} \cdot \frac{AM}{MC} \cdot \frac{CK}{KF} = 1,$$

the converse of Menelaus' theorem applied to triangle FAC assures us that M, N and K are collinear.

Comment by the editor. Three solvers hammered out the result using analytic geometry through a straightforward but not easily digestible computation.

5045. *Proposed by Nguyen Tuan Anh.*

Let $n \geq 2$ be a positive integer. Prove that

$$\sum_{k=0}^{n-2} \binom{n-k}{2} F_{k+1} = F_{n+5} - (n+4) - \binom{n+2}{2},$$

where F_n is the n-th Fibonacci number defined by

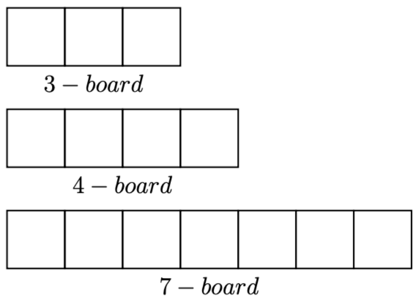
$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n,\ \forall n \in \mathbb{N}.$$

We received 18 solutions of which 16 were correct and complete. We present the solution by Jaimin Patel. A similar solution was submitted by the proposer.

Solution: We begin by rewriting the desired equality as

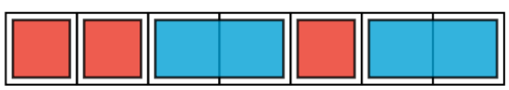
$$F_{n+5} = (n+4) + \binom{n+2}{2} + \sum_{k=0}^{n-2} \binom{n-k}{2} F_{k+1}.$$

Consider the following definitions. *Definition.* An *n-board* is a $1 \times n$ rectangle made up of 1×1 squares called *cells*.



A *square* is a 1×1 tile, and a *domino* is a 1×2 tile.

A *tiling* of an n-board uses squares and dominos to completely cover the $1 \times n$ area. For example, a tiling of a 7-board using squares and dominos:



For each positive integer n, let t_n denote the number of possible tilings of an n-board using squares and dominos. In particular, $t_1 = 1, t_2 = 2$. For every $n \geq 3$, consider an n-board, and two cases based on whether the last tile is a square or a domino.

- If the last tile is a square, the first $n-1$ cells can be tiled in t_{n-1} ways.
- Otherwise if the last tile is a domino, it occupies cells $n-1$ and n, so the first $n-2$ cells can be tiled in t_{n-2} ways.

Thus for $n \geq 3$, $t_n = t_{n-1} + t_{n-2}$. Note that with the initial conditions $t_1 = 1$ and $t_2 = 2$, this corresponds (by shifting the index) to the Fibonacci sequence F_{n+1}. In other words, if we define $t_{-1} = 0$ and $t_0 = 1$, then $F_n = t_{n-1}$ for all $n \geq 0$.

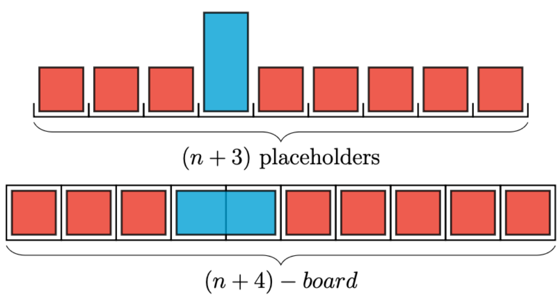
If we have a board of size $n+4$, there are F_{n+5} ways to tile this board using squares and dominos, as above. We will now count the same thing in a different way, considering four cases based on the number of dominos used in a given tiling.

Case 1: No dominos.

In this case only squares are used, and so this can only occur one way.

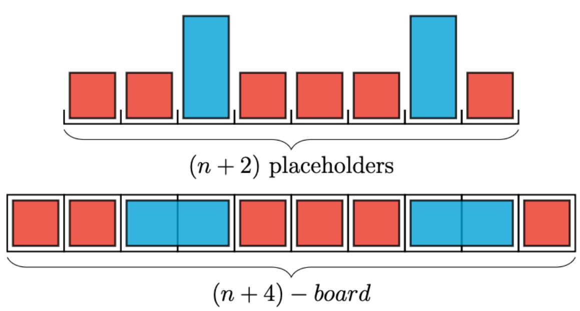
Case 2: One domino.

To count the number of tilings with exactly one domino, consider $n+3$ placeholders. These placeholders do not correspond to the actual squares of the board; rather, they indicate possible positions where we can place a single vertical domino, with the remaining positions filled by squares, as illustrated in the figure below. The vertical domino can be placed in any one of the $n+3$ placeholders. After placing it, we rotate the vertical domino to integrate it into the tiling. This rotation requires one additional square, resulting in a tiling of an $(n+4)$-board. Thus there are $n+3$ distinct tilings with exactly one domino.



Case 3: Two dominos.

In this case, we proceed in a similar manner by considering $n+2$ placeholders. From these, we select two positions for the vertical dominos and fill the remaining n placeholders with squares. The number of ways to choose two positions from $n+2$ is $\binom{n+2}{2}$. As before, each vertical domino is rotated to become part of the tiling. Thus, there are $\binom{n+2}{2}$ distinct tilings with exactly two dominos.



Case 4: More than two dominos.

Since we are considering more than two dominos, we divide this case into subcases based on the position of the third domino in the tiling (counting from left to right).

- The third domino could be placed in the 5^{th} and 6^{th} cells. Since this is the third domino, the first two dominos must occupy the first four cells. This leaves $n+4-6=n-2$ cells to tile freely. The number of ways to tile these remaining cells is F_{n-1}. Thus, there are $\binom{2}{2}\cdot F_{n-1}$ such tilings.

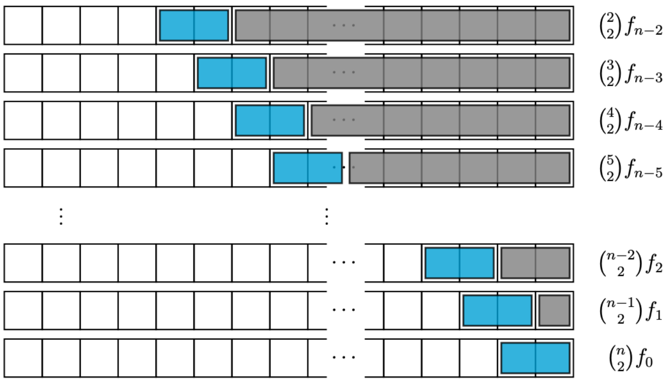
- The third domino could be placed in the 6^{th} and 7^{th} cells. In this case, the first five cells must contain exactly two dominos and one square. Using the same technique as in Case 3, we choose two positions for vertical dominos among three placeholders, and fill the remaining one with a square. The number of such arrangements is $\binom{3}{2}$. After placing the third domino, we are left with $n+4-7=n-3$ cells to tile freely, which can be done in F_{n-2} ways. Thus, there are $\binom{3}{2}\cdot F_{n-2}$ such tilings.

- Continuing in this manner, we can systematically consider each possible position for the third domino, case by case.

- In the final case, the third domino is placed in the $(n+3)^{\text{rd}}$ and $(n+4)^{\text{th}}$ cells. The number of ways to arrange two dominos and $n-2$ squares in the first n positions is $\binom{n}{2}$. Since there are no cells left after the third domino, the number of ways to tile the rest is $t_0=F_1=1$. Therefore, there are $\binom{n}{2}\cdot F_1$ such tilings.

In total, the number of tilings with at least three dominos is given by:

$$\binom{2}{2}F_{n-1}+\binom{3}{2}F_{n-2}+\binom{4}{2}F_{n-3}+\cdots+\binom{n}{2}F_1=\sum_{k=0}^{n-2}\binom{n-k}{2}F_{k+1}$$



Note: any arrangement of an $(n+4)$-board falls into one of cases 1 through 4.

Also note that these cases are pairwise exclusive. Hence,

$$F_{n+5} = 1 + (n+3) + \binom{n+2}{2} + \sum_{k=0}^{n-2} \binom{n-k}{2} F_{k+1}$$
$$F_{n+5} = (n+4) + \binom{n+2}{2} + \sum_{k=0}^{n-2} \binom{n-k}{2} F_{k+1}$$

5046. *Proposed by Nguyen Viet Hung.*

Prove that for all positive real numbers x, y, z

$$\frac{x^3+y^3}{x^2+xy+y^2} + \frac{y^3+z^3}{y^2+yz+z^2} + \frac{z^3+x^3}{z^2+zx+x^2} \geq \frac{2(x^2+y^2+z^2)}{x+y+z}.$$

There were 17 correct solutions submitted by 13 solvers. In addition, there were two solutions that relied to an undue extent on electronic computation. We present 4 solutions.

Let $S = \sum x^3(x^2+xy+y^2)^{-1}$ and $T = \sum y^3(x^2+xy+y^2)^{-1}$, with the sums cyclic with three terms. Then $S - T = \sum(x-y) = 0$, so that the left side of the inequality is equal to $S + T = 2S$.

Solution 1, by Sicheng Du.

The difference between the two sides of the inequality is

$$\begin{aligned}
&\left[\sum\left(\frac{x^3+y^3}{x^2+xy+y^2} - \frac{x+y}{3}\right)\right] - \left[\frac{2(x^2+y^2+z^2)}{x+y+z} - \frac{2(x+y+z)}{3}\right] \\
&\quad = \left[\sum \frac{2(x+y)(x-y)^2}{3(x^2+xy+y^2)}\right] - \frac{2[(x-y)^2+(y-z)^2+(z-x)^2]}{3(x+y+z)} \\
&\quad = \frac{2}{3}\sum\left(\frac{x+y}{x^2+xy+y^2} - \frac{1}{x+y+z}\right)(x-y)^2 \\
&\quad = \frac{2(xy+yz+zx)}{x+y+z}\sum \frac{(x-y)^2}{x^2+xy+y^2} \geq 0,
\end{aligned}$$

with equality if and only if $x = y = z$.

Solution 2, by Michel Bataille, Huseyin Yigit Emekci, Arkan Manva and Cao Minh Quang (independently).

Applying the Cauchy-Schwarz inequality to the vectors $((x^3+x^2y+xy^2)^{1/2}, \dots)$ and $(x^2(x^3+x^2y+xy^2)^{-1/2}, \dots)$ and noting that

$$\sum(x^3+x^2y+xy^2) = (x^2+y^2+z^2)(x+y+z),$$

we find that

$$\begin{aligned} S &= \sum\left(\frac{x^4}{x^3+x^2y+xy^2}\right) \\ &\geq \left[\sum(x^3+x^2y+xy^2)\right]^{-1}(x^2+y^2+z^2)^2 = (x^2+y^2+z^2)(x+y+z)^{-1}. \end{aligned}$$

Equality occurs when the two vectors are proportional: $x^2 = \lambda(x^3+x^2y+xy^2)$, *etc.*. Adding these equations yields

$$x^2+y^2+z^2 = \lambda(x+y+z)(x^2+y^2+z^2)$$

from which

$$x(x^2+xy+xz) = x^2(x+y+z) = \lambda(x+y+z)(x^3+x^2y+xy^2) = x(x^2+xy+y^2).$$

Hence $y^2 = xz$; likewise $x^2 = yz$ and $z^2 = xy$. Therefore $x = y = z$.

Comment by the editor. A variant of this is due to Prithwijit De and the proposer, who expressed the left side as

$$\frac{(x+y)(x^3+y^3)}{(x+y)(x^2+xy+y^2)}.$$

The Cauchy-Schwarz inequality was applied to replace the numerator of each summand by $(x^2+y^2)^2$ and then applied again on the whole sum.

Solution 3, by Didier Pinchon.

Let

$$(u,v,w) = ((x^2+xy+y^2)^{-1}, (y^2+yz+z^2)^{-1}, (z^2+zx+x^2)^{-1})$$

and $r = (x^2+y^2+z^2)^{-1}$. Then $rx^2+ry^2+rz^2 = 1$.

Applying the inequality of the weighted arithmetic and harmonic means,

$$\begin{aligned} rS &= (rx^2)(xu)+(ry^2)(yv)+(rz^2)(zw) \\ &\geq \left(\frac{rx^2}{xu}+\frac{ry^2}{yv}+\frac{rz^2}{zw}\right)^{-1} \\ &= \left[r\left(\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right)\right]^{-1} = [r(x+y+z)(x^2+y^2+z^2)]^{-1} = (x+y+z)^{-1}. \end{aligned}$$

Hence

$$S+T = 2S \geq \frac{2(x^2+y^2+z^2)}{(x+y+z)},$$

as desired.

Equality occurs if and only if $x^2+xy+y^2 = y^2+yz+z^2 = z^2+zx+x^2 \Leftrightarrow (x+y+z)(y-z) = 0$, *etc.* $\Leftrightarrow x = y = z$.

Solution 4, by Kee-Wai Lau and Angel Plaza (independently).

Take the difference between the two sides; the numerator of the fraction obtained is

$$2\left(\sum xy\right)\left(\sum x^4(y^2+z^2) - xyz\sum x^2(y+z)\right).$$

It remains to establish that the second factor is non-negative.

Lau applied the arithmetic-geometric means inequality, for two variables, then to three, to obtain

$$\sum x^4(y^2+z^2) \geq 2xyz\sum x^3 \geq xyz(3xyz + \sum x^3).$$

Finally,

$$3xyz + \sum x^3 - \sum x^2(y+z) = \sum x(x-y)(x-z) \geq 0,$$

by the Schur inequality, so that

$$\sum x^4(y^2+z^2) \geq xyz\sum x^2(y+z),$$

and the desired inequality follows.

Plaza noted that the second factor is non-negative because of the Muirhead inequality with $[4,2,0] \succ [3,2,1]$.

Equality holds if and only if $x = y = z$.

Comment by the editor. Michal Adamaszak and Walther Janous expressed the numerator in the form

$$2(\sum x^5y^3 + \sum x^5y^2z - \sum x^4y^2y^2 - \sum x^3y^3z^2)$$

and used the Muirhead inequality.

5047. *Proposed by Sicheng Du.*

Let real numbers a and b be such that $a^2+b^2 \leq 1$. Find the minimum of

$$\sqrt{2-2a} + \sqrt{2+a+\sqrt{3}b} + \sqrt{2+a-\sqrt{3}b}.$$

We received 12 submissions and 11 of them were complete and correct. We feature the following two solutions, slightly modified by the editor.

First, we show that the minimum is only achieved when $a^2+b^2=1$. Indeed, we have

$$\begin{aligned}\left(\sqrt{2+a+\sqrt{3}b} + \sqrt{2+a-\sqrt{3}b}\right)^2 &= 4+2a+2\sqrt{(2+a)^2-3b^2}\\ &= 4+2a+2\sqrt{4+4a+4a^2-3(a^2+b^2)}.\end{aligned}$$

Next, we present two different ways to show the minimum is $2\sqrt{3}$.

Solution 1, by Michel Bataille.

By the above discussion, it suffices to show that the function $g : [-1, 1] \to \mathbb{R}$ defined by

$$g(a) = \sqrt{2-2a} + \sqrt{4+2a+2|2a+1|},$$

has minimum $2\sqrt{3}$. Observe that

$$g(a) = \begin{cases} 2\sqrt{2-2a} & \text{if } -1 \le a \le -\frac{1}{2}, \\ \sqrt{2-2a} + \sqrt{6+6a} & \text{if } -\frac{1}{2} \le a \le 1. \end{cases}$$

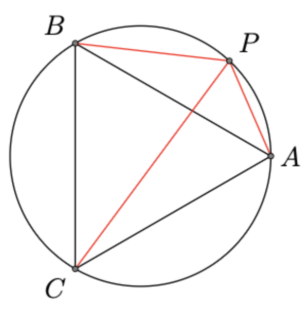
It is easy to verify that g is decreasing on the interval $[-1, -\frac{1}{2}]$, increasing on $[-\frac{1}{2}, \frac{1}{2}]$ and decreasing on $[\frac{1}{2}, 1]$. Since $g\left(-\frac{1}{2}\right) = g(1) = 2\sqrt{3}$, we conclude that $g(a) \ge 2\sqrt{3}$ for all $a \in [-1, 1]$, as required.

Solution 2, by Michal Adamaszek, C.R. Pranesachar, and the proposer (independently).

Since $a^2 + b^2 = 1$, we have

$$\begin{aligned} &\sqrt{2-2a} + \sqrt{2+a+\sqrt{3}b} + \sqrt{2+a-\sqrt{3}b} \\ &= \sqrt{(1-a)^2+b^2} + \sqrt{\left(a+\frac{1}{2}\right)^2 + \left(b-\frac{\sqrt{3}}{2}\right)^2} + \sqrt{\left(a+\frac{1}{2}\right)^2 + \left(b+\frac{\sqrt{3}}{2}\right)^2}. \end{aligned}$$

Let $P = (a, b)$ be a point on the unit circle and let $A = (1, 0)$, $B = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $C = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ be three vertices of an equilateral triangle inscribed in that circle. Then the above equation shows that it is equivalent to minimizing $|PA| + |PB| + |PC|$. By symmetry, we may assume that P is on the shorter arc AB.



By Ptolemy's theorem, we have $|PC||AB| = |PA||BC| + |PB||AC|$. It follows that $|PA| + |PB| = |PC|$ and thus $|PA| + |PB| + |PC| = 2|PC| \ge 2\sqrt{3}$, where the equality holds when $P \in \{A, B\}$.

5048. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\lim_{n\to\infty}\int_0^1 \sqrt[n]{\frac{1}{2}\ln\frac{1+x}{1-x}-x-\frac{x^3}{3}-\cdots-\frac{x^{2n-1}}{2n-1}}\,\mathrm{d}x.$$

We received 6 solutions, all of which were correct. We present the solution by Giuseppe Fera.

We prove that the given limit is $\frac{1}{3}$. Since

$$\frac{1}{2}\ln\frac{1+x}{1-x}=\sum_{k=1}^{\infty}\frac{x^{2k-1}}{2k-1} \qquad \text{for} \qquad 0\le x<1,$$

then

$$\frac{1}{2}\ln\frac{1+x}{1-x}-x-\frac{x^3}{3}-\ldots-\frac{x^{2n-1}}{2n-1}=\sum_{k=n}^{\infty}\frac{x^{2k+1}}{2k+1}=\sum_{k=0}^{\infty}\frac{x^{2n+2k+1}}{2n+2k+1}.$$

Hence

$$\sqrt[n]{\frac{1}{2}\ln\frac{1+x}{1-x}-x-\frac{x^3}{3}-\ldots-\frac{x^{2n-1}}{2n-1}}=x^2\sqrt[n]{\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2n+2k+1}}.$$

Now, we have

$$\frac{x}{2n+1}\le\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2n+2k+1}\le\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2n+1}=\frac{x}{(2n+1)(1-x^2)}.$$

Consequently,

$$\int_0^1 x^2\sqrt[n]{\frac{x}{2n+1}}\,dx\le\int_0^1 x^2\sqrt[n]{\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2n+2k+1}}\,dx\le\int_0^1 x^2\sqrt[n]{\frac{x}{(2n+1)(1-x^2)}}\,dx.$$

The lower bound is

$$\int_0^1 x^2\sqrt[n]{\frac{x}{2n+1}}\,dx=\frac{1}{\sqrt[n]{2n+1}}\frac{n}{3n+1}$$

and so

$$\lim_{n\to\infty}\int_0^1 x^2\sqrt[n]{\frac{x}{2n+1}}\,dx=\frac{1}{3}.$$

As to the upper bound, note that $\frac{x}{1-x^2}<1$ for $0\le x<\frac{\sqrt{5}-1}{2}$ and $1\le\frac{x}{1-x^2}$ for $\frac{\sqrt{5}-1}{2}\le x<1$. Let

$$F(x)=\begin{cases}x^2 & \text{if } 0\le x<\frac{\sqrt{5}-1}{2},\\ x^2\sqrt{\frac{x}{1-x^2}} & \text{if } \frac{\sqrt{5}-1}{2}\le x<1.\end{cases}$$

For $n \geq 2$,

$$x^2 \sqrt[n]{\frac{x}{(2n+1)(1-x^2)}} \leq F(x)$$

holds in the interval $0 \leq x < 1$, and clearly $\int_0^1 F(x)\,dx < \infty$. Therefore, the dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_0^1 x^2 \sqrt[n]{\frac{x}{(2n+1)(1-x^2)}}\,dx = \int_0^1 \lim_{n\to\infty} x^2 \sqrt[n]{\frac{x}{(2n+1)(1-x^2)}}\,dx = \int_0^1 x^2\,dx = \frac{1}{3}.$$

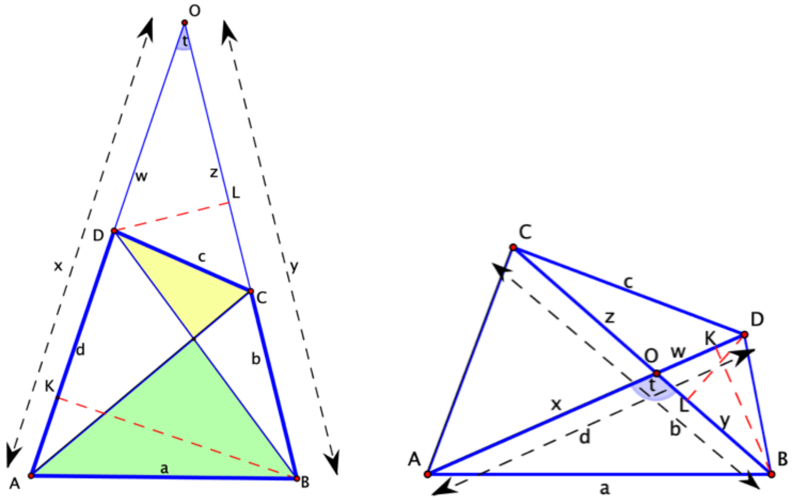
Finally, the squeeze theorem implies the result.

5049. *Proposed by Xicheng Peng.*

In quadrilateral $ABCD$, line AD intersects line BC at point O. Letting $|XYZ|$ denote the area of ΔXYZ, prove that

$$AB\cdot BC\cdot CD\cdot DA\cdot\sin(\angle BAD + \angle BCD) = 2(|ABC|-|DBC|)(OA\cdot OD - OB\cdot OC)$$

We received 6 submissions, which were all correct under the additional assumption that the given quadrilateral is convex. In fact, the three solutions that used coordinates were valid without the convexity assumption, as were the others if modified to allow for positive and negative quantities. We feature the solution by C.R. Pranesachar that has been so modified by the editor.



Notation. The vertices of the given quadrilateral $ABCD$ are any four points in the plane, no three collinear, such the lines AD and BC intersect in a finite point O. We direct the line OA from O to A, and OB from O to B. Segments on those lines, therefore, have *signed lengths*

$$x = OA > 0, \quad y = OB > 0, \quad z = OC \neq 0, \quad w = OD \neq 0.$$

Thus (as shown by both examples in the accompanying figure), the sides of the quadrilateral have signed lengths $a = AB > 0$ and $c = CD > 0$, while

$$b = BC = BO{+}OC = OC{-}OB = z{-}y, \quad \text{and} \quad d = DA = DO{+}OA = OA{-}OD = x{-}w.$$

Finally, we let $t = \angle AOB$ be the *directed angle* whose measure equals the angle through which the line OA must be rotated about O in the positive direction to coincide with OB.

Observe that the projection of B on OA, call it K, satisfies

$$\sin\angle BAD = \frac{BK}{a} = \frac{y\sin t}{a} \quad \text{and} \quad \cos\angle BAD = \frac{KA}{a} = \frac{OA + KO}{a} = \frac{x - y\cos t}{a}.$$

Similarly, the projection of D on OB, call it L, satisfies

$$\sin\angle DCB = \sin\angle OCD = \frac{DL}{c} = \frac{w\sin t}{c}$$

and

$$\cos\angle DCB = -\cos\angle OCD = -\frac{LC}{c} = -\frac{OC + LO}{c} = -\frac{z - w\cos t}{c}.$$

Finally, in a similar manner we derive

$$\sin\angle ADC = \frac{z\sin t}{c}.$$

Substituting these values in the left-hand side of the given equation, we have

$$\begin{aligned} & AB\cdot BC\cdot CD\cdot DA\cdot\sin(\angle BAD + \angle DCB) \\ = \;& abcd(\sin\angle BAD\cos\angle DCB + \cos\angle BAD\sin\angle DCB) \\ = \;& abcd\left(-\frac{y\sin t}{a}\cdot\frac{(z - w\cos t)}{c} + \frac{x - y\cos t)}{a}\cdot\frac{w\sin t}{c}\right) \\ = \;& bd(xw - yz)\sin t. \end{aligned}$$

Next, we use square brackets to denote signed area and derive

$$\begin{aligned} [ABC] - [DBC] &= [ABD] - [ACD] \\ &= \frac{1}{2}AB\cdot AD\cdot\sin\angle BAD - \frac{1}{2}DA\cdot DC\cdot\sin ADC \\ &= -\frac{1}{2}ad\cdot\frac{y\sin t}{a} + \frac{1}{2}cd\cdot\frac{z\sin t}{c} \\ &= \frac{1}{2}d(z - y)\sin t = \frac{1}{2}bd\sin t. \end{aligned}$$

We have, finally, the right-hand side of the given equation equals

$$2([ABC] - [DBC])(OA\cdot OD - OB\cdot OC) = 2\cdot\frac{1}{2}bd\sin t\cdot(xw - yz),$$

which equals our expression for the left-hand side.

Editor's comments. The proposer wrote that he had been led to his problem by comparing the two familiar formulas for determining whether four points are concyclic, namely, $\angle BAD + \angle DCB$ and $OA \cdot OD - OB \cdot OC$. The former equals 0 if and only if the points A, B, C, D lie on a circle in the given order (with A and C separating B from D), while the latter equals 0 if and only if the points lie on a circle in any order. It is clear, therefore, that the statement of the problem required further conditions such as "the quadrangle $ABCD$ is convex." The featured solution modified the problem so that all quantities were signed (in which case the angle $\angle BCD$ had to be replaced by $\angle DCB$).

5050. *Proposed by Vasile Cîrtoaje.*

Let a, b, c, d be positive real numbers such that $a \geq b \geq 1 \geq c \geq d$ and $abcd = 1$. Prove that

$$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} + \frac{1}{2d+1} \geq \frac{4}{3}.$$

We received 17 submissions, of which 10 were correct and complete. We present the solution by Michal Adamaszek.

First, we prove by direct calculation that the following inequalities hold for $x, y > 0$:

$$\frac{1}{2x^2+1} + \frac{1}{2y^2+1} \geq \begin{cases} \frac{2}{2xy+1} & \text{when } x, y \geq 1, \\ \frac{2}{3} \cdot \frac{4-xy}{2+xy} & \text{when } x, y \leq 1. \end{cases}$$

We consider the first inequality and find for which values of $x, y > 0$ this inequality holds.

$$\begin{aligned} \Rightarrow \quad & \frac{1}{2x^2+1} + \frac{1}{2y^2+1} \geq \frac{2}{2xy+1} \\ \Rightarrow \quad & \frac{(2y^2+1)+(2x^2+1)}{(2x^2+1)(2y^2+1)} \geq \frac{2}{2xy+1} \\ \Rightarrow \quad & 2(x^2+y^2+1)(2xy+1) \geq 2(2x^2+1)(2y^2+1) \\ \Rightarrow \quad & 2x^3y + 2y^3x + 2xy + x^2 + y^2 + 1 \geq 4x^2y^2 + 2x^2 + 2y^2 + 1 \\ \Rightarrow \quad & (2xy-1)(x-y)^2 \geq 0 \end{aligned}$$

This inequality holds when $x, y \geq 1$ (and, in fact, whenever $xy \geq \frac{1}{2}$). Now, we consider the second inequality and find for which values of $x, y > 0$ this inequality holds.

$$\Rightarrow \quad \frac{1}{2x^2+1}+\frac{1}{2y^2+1}\geq \frac{2}{3}\frac{4-xy}{2+xy}$$

$$\Rightarrow \quad 6(x^2+y^2+1)(2+xy)\geq 2(2x^2+1)(2y^2+1)(4-xy)$$

$$\Rightarrow \quad 4x^3y^3+5x^3y+5xy^3+4xy-16x^2y^2-2x^2-2y^2+2\geq 0$$

$$\Rightarrow \quad 4xy(xy-1)^2+5xy(x-y)^2+2(1-x^2)(1-y^2)\geq 0$$

This inequality holds when $0\leq x,y\leq 1$. We now show how these inequalities imply the solution.

Let $p=\sqrt{cd}$, then:

$$\begin{aligned}\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}+\frac{1}{2d+1} &\geq \frac{2}{2\sqrt{ab}+1}+\frac{2}{3}\cdot\frac{4-\sqrt{cd}}{2+\sqrt{cd}}\\ &=\frac{2}{\frac{2}{p}+1}+\frac{2}{3}\cdot\frac{4-p}{2+p}\\ &=\frac{2p}{p+2}+\frac{2}{3}\cdot\frac{4-p}{2+p}\\ &=\frac{2}{p+2}\left(p+\frac{4}{3}-\frac{1}{3}p\right)=\frac{2}{p+2}\cdot\frac{2}{3}(2+p)=\frac{4}{3}.\end{aligned}$$