

2025 Canada Lynx Mathematical Competition

Official Solutions



A competition of the Canadian Mathematical Society.

Part A: 5 marks each

1. If $S = 2^1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$, what is S ?
(A) 16 (B) 32 (C) 63 (D) 64 (E) 128

Solution: It is not a coincidence that each partial sum doubles the previous total:

$2+2 = 4$, $2+2+4 = 8$, $2+2+4+8 = 16$, $2+2+4+8+16 = 32$,
and $2+2+4+8+16+32 = 64$.

Answer: (D)

2. If $S = \frac{1}{6} + \frac{2}{6} - \frac{3}{6} + \frac{4}{6} + \frac{5}{6} - \frac{6}{6}$, what is S ?
(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) 1 (E) $\frac{4}{3}$

Solution: This expression equals

$$S = \left(\frac{1}{6} + \frac{2}{6} - \frac{3}{6}\right) + \left(\frac{4}{6} + \frac{5}{6} - \frac{6}{6}\right) = \frac{1+2-3}{6} + \frac{4+5-6}{6} = \frac{0}{6} + \frac{3}{6} = \frac{3}{6}.$$

Since $\frac{3}{6} = \frac{1}{2}$, we conclude that $S = \frac{1}{2}$.

Answer: (B)

3. If $1 + 3x = -5 - 7x$, what is the value of $100 + 100x$?
(A) 40 (B) 60 (C) 140 (D) 160 (E) 200

Solution: Since $1 + 3x = -5 - 7x$, we can add $7x$ to both sides to get $1 + 3x + 7x = -5 - 7x + 7x$, which simplifies to $1 + 10x = -5$. We subtract 1 from both sides to get $1 + 10x - 1 = -5 - 1$, which simplifies to $10x = -6$.

We can solve for x here, but because we wish to determine the value of $100 + 100x$, a much faster solution is to notice that $10x = -6$ implies $100x = -60$, and so $100 + 100x$ must equal $100 - 60$, which equals 40.

Answer: (A)

4. Let $\{10, 11, 12, 13, \dots, 97, 98, 99\}$ be the set of two-digit numbers. How many two-digit numbers are divisible by 3?
(A) 30 (B) 31 (C) 32 (D) 33 (E) 34

Solution: Let's consider the set of all positive integers less than 100 that are divisible by 3. This set consists of $\{3, 6, 9, 12, 15, \dots, 99\}$, which we can rewrite as $\{3 \times 1, 3 \times 2, 3 \times 3, 3 \times 4, 3 \times 5, \dots, 3 \times 33\}$.

Clearly there are 33 integers in this set. To obtain the answer to this question, we must remove the three elements $\{3, 6, 9\}$ since we only want to include the two-digit integers that are divisible by 3. So the answer is $33 - 3 = 30$.

Answer: (A)

5. Consider a three-digit number ABC . Write it backwards (CBA), and subtract the smaller number from the larger number. Which of these answers cannot be the final difference?
(A) 297 (B) 396 (C) 595 (D) 693 (E) 792

Solution: Let A, B, C be digits from 0 to 9. Thus, the 3-digit number can be written as $100A + 10B + C$, and so if we write this number backwards, we get $100C + 10B + A$. The difference of these two numbers is $(100A + 10B + C) - (100C + 10B + A) = 99A - 99C = 99(A - C)$.

If $A \geq C$ then this difference is $99(A - C)$, and if $C > A$ this difference is $99(C - A)$. Either way, this difference *must* be a multiple of 99.

Of the five options presented, we see that (c) is not a multiple of 99 since it is equal to $595 = 6 \times 99 + 1$. However, the remaining options are multiples of 99.

Indeed, we can achieve the differences 297, 396, 693, 792 by ensuring $A - C$ equals 3, 4, 7, 8, respectively.

Answer: (C)

Part B: 5 marks each

6. At a grocery store, you can buy 3 avocados and 2 bananas for \$10.50, and buy 4 avocados and 3 bananas for \$14.50. How much does it cost to buy 5 avocados and 4 bananas?

- (A) \$18 (B) \$18.50 (C) \$19 (D) \$19.50 (E) \$20

Solution: Let A be the price of an avocado and B be the price of a banana.

From the information given, we have $3A + 2B = 10.50$ and $4A + 3B = 14.50$. We can solve for A and B .

Multiplying the first equation by 3 and multiplying the second equation by 2, we get $9A + 6B = 31.50$ and $8A + 6B = 29.00$.

Subtracting the latter from the former, we get $(9A + 6B) - (8A + 6B) = 31.50 - 29.00$, which implies $A = 2.50$.

Since $3A + 2B = 10.50$, we have $2B = 10.50 - 3A = 10.50 - 3 \times 2.50 = 10.50 - 7.50 = 3.00$. Thus, $B = 1.50$.

Since $A = 2.50$ and $B = 1.50$, we have $5A + 4B = 5 \times 2.50 + 4 \times 1.50 = 12.50 + 6.00 = 18.50$.

A much faster solution exists by taking the difference of the two equations, $(4A + 3B) - (3A + 2B) = 14.50 - 10.50$, and seeing that this simplifies to $A + B = 4$. Since we know that $4A + 3B = 14.50$, if we add $A + B = 4$, we get $5A + 4B = 14.50 + 4 = 18.50$, and we immediately obtain the correct answer without having to solve for A and B .

Another solution is to subtract the first equation from twice the second equation. We then instantly get the correct answer.

$$5A + 4B = 2(4A + 3B) - 1(3A + 2B) = 2 \times 14.50 - 1 \times 10.50 = 29.00 - 10.50 = 18.50.$$

Answer: (B)

7. At the beginning of this year (2025), Xavier has \$100 000 in his bank account and Yvette has \$100 in her bank account. Every year, Xavier's account decreases by 10% and Yvette's account increases by 80%. When will be the first year that Yvette has more money than Xavier?

- (A) 2030 (B) 2035 (C) 2040 (D) 2045 (E) 2050

Solution: Let $x(n)$ be the amount of money in Xavier's account n years after 2025. Similarly define $y(n)$.

Since Xavier's account decreases by 10% each year, we have $x(k) = (1 - 0.1)x(k-1) = x(k-1) \times 0.9$ for all $k \geq 1$. We know that $x(0) = 100,000$ and so we have $x(1) = 100,000 \times 0.9$, $x(2) = 100,000 \times 0.9^2$, $x(3) = 100,000 \times 0.9^3$, and so on. In general, $x(n) = 100,000 \times 0.9^n$.

Since Yvette's account increases by 80% each year, we have $y(k) = (1 + 0.8)y(k-1) = y(k-1) \times 1.8$ for all $k \geq 1$. By the same reasoning, since $y(0) = 100$, we have $y(n) = 100 \times 1.8^n$.

We are looking for the smallest value of n for which Yvette will have more money than Xavier, which is equivalent to the inequality $100 \times 1.8^n > 100,000 \times 0.9^n$. This simplifies to $2^n \cdot (0.9)^n = 1.8^n > 1000 \times 0.9^n$, which implies $2^n > 1000$ since we can divide both sides by 0.9^n .

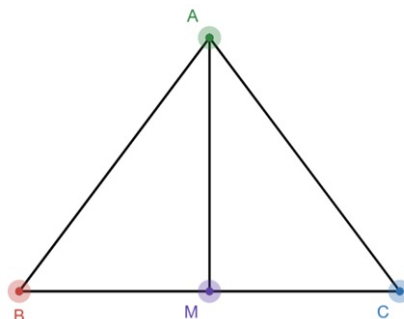
If $n = 9$ then $2^n = 512 < 1000$ but if $n = 10$ then $2^n = 1024 > 1000$. And so 10 years after the year 2025 will be the first time that Yvette will have more money than Xavier. This occurs in the year 2035.

Answer: (B)

8. $\triangle ABC$ is an isosceles triangle with $AB = AC$, with base $BC = 12$. If the area of this triangle is 48 square units, how many units is the perimeter of $\triangle ABC$?

- (A) 16 (B) 24 (C) 32 (D) 40 (E) Impossible to determine

Solution: Let M be the midpoint of BC . Since $\triangle ABC$ is isosceles with $AB = AC$, the line segment AM must be the height of the triangle with base BC .



Since the area of this triangle is 48 square units, we have $\frac{AM \times BC}{2} = 48$, which implies $AM = \frac{2 \times 48}{BC} = \frac{96}{12} = 8$.

We know $BC = 12$. Since M is the midpoint of BC , we have $BM = 6$ and $MC = 6$. By the Pythagorean Theorem, $AB = AC = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$.

It follows that the perimeter of $\triangle ABC$ is $AB + AC + BC = 10 + 10 + 12 = 32$.

Answer: (A) (C) (D) (E)

9. An infinite sequence x_1, x_2, x_3, \dots of positive integers is constructed as follows:

We start with $x_1 = c$ for some positive integer c . For each $n \geq 1$,

If x_n is a perfect square, then $x_{n+1} = \sqrt{x_n}$.

If x_n is not a perfect square, then $x_{n+1} = x_n + 3$.

For example, if $c = 43$, then the first six terms of this infinite sequence are 43, 46, 49, 7, 10, 13.

For which of these values of c will the infinite sequence *never* decrease?

(A) 123 (B) 124 (C) 125 (D) 126 (E) 127

Solution: The rules of this sequence are simple: if x_n is a perfect square, the next term is $\sqrt{x_n}$. Otherwise, the next term increases by 3.

For example, if $c = 123$, then the first nine terms of the sequence are

$$123, 126, 129, 132, 135, 138, 141, 144, 12.$$

The above sequence shows that the answer cannot be 123 or 126 since this sequence contains a perfect square ($144 = 12^2$), and so this sequence decreases when it goes from 144 to 12.

Similarly, the answer cannot be 124 or 127 since the sequence

$$124, 127, 130, 133, 136, 139, 142, 145, 148, 151, 154, 157, 160, 163, 169, 13$$

also contains a perfect square ($169 = 13^2$), and so this sequence decreases when it goes from 169 to 13.

Finally, consider the sequence that starts with $c = 125$. We get

$$125, 128, 131, 134, 137, 140, 143, \dots$$

Notice that each term in this sequence is equal to $3k + 2$ for some integer k . (We say that each term in this sequence is congruent to 2 modulo 3). We claim that this sequence contains no perfect squares.

Suppose, on the contrary, that $3k + 2 = n^2$ for some integers n and k . We consider three cases: $n = 3m$, $n = 3m + 1$, and $n = 3m + 2$ for some integer m .

- (i) If $n = 3m$, then $3k + 2 = n^2 = (3m)^2 = 9m^2$, which simplifies to $2 = 9m^2 - 3k$, or $\frac{2}{3} = 3m^2 - k$.
- (ii) If $n = 3m + 1$, then $3k + 2 = n^2 = (3m + 1)^2 = 9m^2 + 6m + 1$, which simplifies to $1 = 9m^2 + 6m - 3k$, or $\frac{1}{3} = 3m^2 + 2m - k$.
- (iii) If $n = 3m + 2$, then $3k + 2 = n^2 = (3m + 2)^2 = 9m^2 + 12m + 4$, which simplifies to $-2 = 9m^2 + 12m - 3k$, or $-\frac{2}{3} = 3m^2 + 4m - k$.

In all three cases, we derive a contradiction since the left side is a fraction while the right side is an integer.

We have proven that the infinite sequence starting with $c = 125$ contains no perfect squares. And so this sequence will never decrease since every term is 3 more than the previous term.

Answer: (C)

10. Consider a circle with equation $x^2 + y^2 = 100$ and a line with equation $y = \frac{x}{7} + \frac{50}{7}$. The circle and the line intersect at two points, P and Q . What is the length of PQ ?

(A) $7\sqrt{2}$ (B) 10 (C) 14 (D) $10\sqrt{2}$ (E) $14\sqrt{2}$

Solution: Our first solution is algebraic, where we solve a system of two equations in two unknown variables to obtain the coordinates of points P and Q .

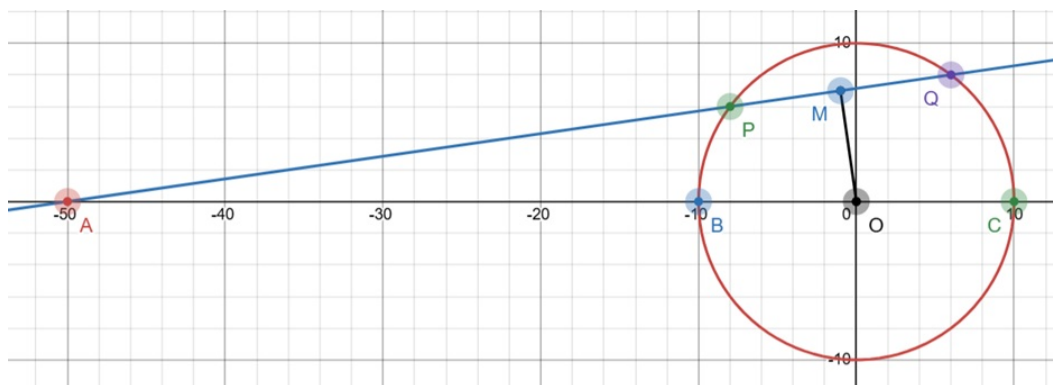
The equation of the line is $7y = x + 50$, which implies $x = 7y - 50$. Substituting into the equation of our circle, we get $100 = x^2 + y^2 = (7y - 50)^2 + y^2 = 49y^2 - 700y + 2500 + y^2 = 50y^2 - 700y + 2500$.

Dividing both sides by 50, we get $2 = y^2 - 14y + 50$, which implies $0 = y^2 - 14y + 48 = (y - 6)(y - 8)$. Hence, $y = 6$ or $y = 8$. Substituting these values back into the equation $7y = x + 50$, we see that $y = 6$ implies $x = -8$ and $y = 8$ implies $x = 6$.

Therefore, the points P and Q must be $(-8, 6)$ and $(6, 8)$. By the Pythagorean Theorem, the length of PQ is $\sqrt{(-8 - 6)^2 + (6 - 8)^2} = \sqrt{14^2 + 2^2} = \sqrt{200} = 10\sqrt{2}$.

Our second solution is geometric. Let M be the midpoint of PQ , and let B and C be the points where the circle intersects with the x -axis.

Letting $A = (-50, 0)$, we see that $AP \cdot AQ = AB \cdot AC$ by the Power-of-a-Point Theorem. Since M is the midpoint of PQ and the origin O is the



midpoint of BC , we have

$$\begin{aligned} AP \cdot AQ &= AB \cdot AC \\ (AM - MP) \cdot (AM + MQ) &= (AO - OB) \cdot (AO + OC) \\ (AM - MP) \cdot (AM + MP) &= (50 - 10) \cdot (50 + 10) \\ AM^2 - MP^2 &= 40 \cdot 60 \\ MP^2 &= AM^2 - 2400 \end{aligned}$$

Since the equation of the line AP is $y = \frac{x}{7} + \frac{50}{7}$, its slope is $\frac{1}{7}$. In other words, $\tan(\angle OAM) = \frac{1}{7}$.

This implies $\sin(\angle OAM) = \frac{1}{\sqrt{1^2+7^2}} = \frac{1}{\sqrt{50}}$ and $\cos(\angle OAM) = \frac{7}{\sqrt{1^2+7^2}} = \frac{7}{\sqrt{50}}$.

Since $\triangle OMA$ is right-angled at point M , we have $\cos(\angle OAM) = \frac{AM}{AO}$, implying $AM = 50 \cdot \frac{7}{\sqrt{50}} = 7\sqrt{50}$, and so $AM^2 = 7^2 \cdot 50 = 49 \cdot 50 = 2450$.

It follows that $MP^2 = AM^2 - 2400 = 2450 - 2400 = 50$, which implies $MP = 5\sqrt{2}$. Therefore, $PQ = MP + MQ = 2MP = 10\sqrt{2}$.

Answer: (D)

Part C: 6 marks each

11. Let x be a real number with $0 < x < \frac{\pi}{2}$ for which $\frac{\sin x + 1}{\cos x} = \frac{4}{3}$.

What is the value of $\frac{\cos x + 1}{\sin x}$?

- (A) $\frac{3}{4}$ (B) 4 (C) 5 (D) 6 (E) 7

Solution: The equation $\frac{\sin x + 1}{\cos x} = \frac{4}{3}$ is equivalent to $3 \sin x + 3 = 4 \cos x$. Since $\sin^2 x + \cos^2 x = 1$, we get:

$$\begin{aligned} 3 \sin x + 3 &= 4 \cos x \\ (3 \sin x + 3)^2 &= (4 \cos x)^2 \\ 9 \sin^2 x + 18 \sin x + 9 &= 16 \cos^2 x \\ 9 \sin^2 x + 18 \sin x + 9 &= 16(1 - \sin^2 x) \\ 9 \sin^2 x + 18 \sin x + 9 &= 16 - 16 \sin^2 x \\ 25 \sin^2 x + 18 \sin x - 7 &= 0 \\ (25 \sin x - 7)(\sin x + 1) &= 0 \end{aligned}$$

Since we are given that x satisfies the inequality $0 < x < \frac{\pi}{2}$, we conclude that $\sin x$ must equal $\frac{7}{25}$ since $\sin x$ cannot equal -1 .

Because $\sin^2 x + \cos^2 x = 1$, we get $\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{7^2}{25^2}} = \sqrt{\frac{25^2 - 7^2}{25^2}} = \sqrt{\frac{625 - 49}{625}} = \sqrt{\frac{576}{625}} = \frac{24}{25}$.

We conclude that $\frac{\cos x + 1}{\sin x} = \frac{\frac{24}{25} + 1}{\frac{7}{25}} = \frac{\frac{49}{25}}{\frac{7}{25}} = 7$.

Answer: (E)

12. For how many positive integers n is $\frac{n^2}{n+45}$ an integer?

- (A) 7 (B) 8 (C) 9 (D) 14 (E) 15

Solution: We have

$$\frac{n^2}{n+45} = \frac{n^2 - 2025}{n+45} + \frac{2025}{n+45} = \frac{(n+45)(n-45)}{n+45} + \frac{2025}{n+45} = (n-45) + \frac{2025}{n+45}.$$

Thus, we want to find the number of positive integers n for which $\frac{2025}{n+45}$ is an integer, i.e., $n+45$ is a divisor of 2025.

Notice that $2025 = 45^2 = (3 \cdot 3 \cdot 5)^2 = 3^4 \cdot 5^2$. Each divisor of 2025 is of the form $3^a 5^b$, where $0 \leq a \leq 4$ and $0 \leq b \leq 2$. There are 5 options for a and 3 options for b .

Listing these $5 \times 3 = 15$ divisors from smallest to largest, we get:

$$1, 3, 5, 9, 15, 25, 27, 45, 75, 81, 135, 225, 405, 675, 2025.$$

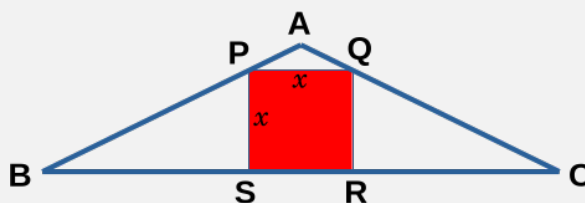
For each of these 15 divisors, we set it equal to $n+45$ and solve for n . Remembering that n must be a positive integer, we only look for divisors that are greater than 45. Therefore, we get:

$$n+45 \in [75, 81, 135, 225, 405, 675, 2025] \rightarrow n \in [30, 36, 90, 180, 360, 630, 1980].$$

We see that there are exactly 7 positive integers n for which $n+45$ is a divisor of 2025.

Answer: (A)

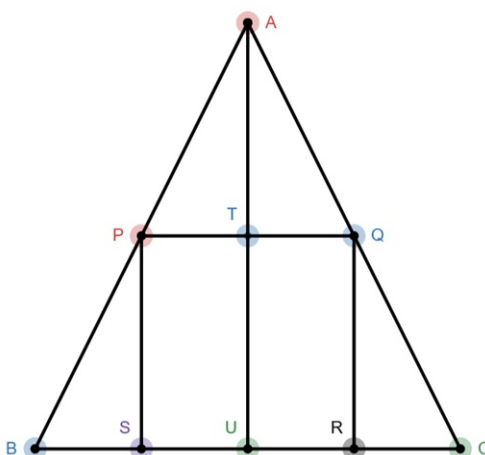
13. Let ABC be an isosceles triangle with $AB = AC$, whose area is 8 square units. Construct a square $PQRS$ so that P is on AB , Q is on AC , R is on BC , and S is on BC .



Let x be the side length of the square. What is the maximum possible value of x ?

- (A) $\frac{4}{3}$ (B) $\sqrt{2}$ (C) 2 (D) $\sqrt{3}$ (E) $2\sqrt{3}$

Solution: Let U be the midpoint of BC , and let T be the intersection of height AU with the side PQ .



Let $AU = h$ be the height of the triangle and let $BC = b$ be the base of the triangle. We are given that x is the side length of the square $PQRS$.

Because $\triangle ABC$ is isosceles with $AB = AC$, we have $BU = \frac{b}{2}$ and $BS = BU - SU = \frac{b}{2} - \frac{x}{2}$.

Notice that $\triangle ABU$ is similar to $\triangle PBS$. It follows that $\frac{AU}{BU} = \frac{PS}{BS}$, which implies $AU \times BS = BU \times PS$. From above, we get $h \left(\frac{b}{2} - \frac{x}{2} \right) = \frac{b}{2} \cdot x$.

This simplifies to $h(b - x) = bx$, or $hb = hx + bx = x(h + b)$. And we conclude that $x = \frac{hb}{h+b}$.

We know that h and b are positive numbers. By the Arithmetic Mean - Geometric Mean Inequality, we have $\frac{h+b}{2} \geq \sqrt{hb}$, with equality if and only if $h = b$. We can prove the above inequality by showing that it is equivalent to $(\sqrt{h} - \sqrt{b})^2 \geq 0$, where equality holds if and only if $h = b$.

Therefore, the side length x satisfies $x = \frac{hb}{h+b} = \frac{hb}{2} \cdot \frac{2}{h+b} \leq \frac{hb}{2} \cdot \frac{1}{\sqrt{hb}} = \frac{\sqrt{hb}}{2}$.

Given that the area of $\triangle ABC$ is 8 square units, we have $8 = \frac{AU \cdot BC}{2} = \frac{hb}{2}$, which implies $hb = 16$. From above, if $hb = 16$, then $x \leq \frac{\sqrt{hb}}{2} = \frac{\sqrt{16}}{2} = \frac{4}{2} = 2$.

The maximum value of $x = 2$ occurs when h and b are equal, i.e., $h = 4$ and $b = 4$.

Answer: (C)

14. Consider all 9-letter strings, where each letter is an A or a B. We say that a string is *diverse* if it does not contain two consecutive As nor three consecutive Bs. For example, ABABABABA and BBABABBAB are diverse but AAABABBBB and BBABBAABB are not.

How many 9-letter strings are diverse?

- (A) 18 (B) 19 (C) 20 (D) 21 (E) 22

Solution: Let $f(n)$ be the number of n -letter diverse strings. For example, we can easily check that $f(1) = 2$ and $f(2) = 3$. We have $f(3) = 4$ since the only diverse strings are $\{ABA, ABB, BBA, BAB\}$, and $f(4) = 5$ since the only diverse strings are $\{ABAB, ABBA, BABA, BABB, BBAB\}$.

Let us prove that $f(n) = f(n - 2) + f(n - 3)$.

To count the number of n -letter diverse strings, we consider two cases: when the final letter is A and when the final letter is B.

If the final letter is B, there are $f(n - 2)$ options for the first $n - 2$ letters, by definition. Thus, each of these n -letter strings is of the form $X_1, X_2, \dots, X_{n-2}, X_{n-1}, B$. If X_{n-2} is A then X_{n-1} must be B since we cannot have two consecutive As. If X_{n-2} is B then X_{n-1} must be A since we cannot have three consecutive Bs. For each of these $f(n - 2)$ options, there is exactly one n -letter diverse string whose final letter is B, since X_{n-1} can be A or B but not both. Therefore, there are exactly $f(n - 2)$ diverse strings in this case.

If the final letter is A, there are $f(n - 3)$ options for the first $n - 3$ letters, by definition. Thus, each of these n -letter strings is of the form $X_1, X_2, \dots, X_{n-3}, X_{n-2}, X_{n-1}, A$. Note that X_{n-1} must be B since we cannot have two consecutive As. If X_{n-3} is A then X_{n-2} must be B since we cannot have two consecutive As. If X_{n-3} is B then X_{n-2} must be A since we cannot have three consecutive Bs. For each of these $f(n - 3)$ options, there is exactly one n -letter diverse string whose final letter is A. Therefore, there are exactly $f(n - 3)$ diverse strings in this case.

We have shown that $f(n) = f(n - 2) + f(n - 3)$. Using this recurrence relation, we can calculate $f(9)$ since we already know the values for the

cases $n = 2, 3, 4$. We have

$$\begin{aligned}
 f(9) &= f(7) + f(6) \\
 &= [f(5) + f(4)] + [f(4) + f(3)] \\
 &= f(5) + f(4) + f(4) + f(3) \\
 &= [f(3) + f(2)] + f(4) + f(4) + f(3) \\
 &= 4 + 3 + 5 + 5 + 4 \\
 &= 21
 \end{aligned}$$

Here is another solution that uses $f(4) = 5$ to prove that $f(9) = 21$.

Suppose the 9-letter string $X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9$ is diverse. Then there are $f(4) = 5$ options for the substring X_1, X_2, X_3, X_4 and $f(4) = 5$ options for the substring X_6, X_7, X_8, X_9 .

Consider all $5 \times 5 = 25$ ways we can form a 9-letter string where the first four letters and the last four letters are diverse substrings.

If both X_4 and X_6 are B, then X_5 must be A, and this 9-letter string is guaranteed to be diverse. Similarly, if both X_4 and X_6 are A, then X_5 must be B, and this 9-letter string is guaranteed to be diverse.

If exactly one of X_4 and X_6 is A, then X_5 cannot be A, but X_5 could be B as long as the 9-letter string does not contain BBB. We show there are only four such cases where the resulting string is not diverse.

- (i) If X_4 is A (which occurs in the strings **ABBA**, **BABA**), then X_5 must be B, so the second 4-letter substring cannot be **BBAB** as otherwise X_5, X_6, X_7 are all B.
- (ii) If X_6 is A (which occurs in the strings **ABAB**, **ABBA**), then X_5 must be B, so the first 4-letter substring cannot be **BABB** as otherwise X_3, X_4, X_5 are all B.

Of the $5 \times 5 = 25$ ways we can form a 9-letter string where the first four letters and the last four letters are diverse substrings, we have identified exactly four scenarios where the result will not be diverse. In the remaining 21 cases, there is exactly one choice of A or B for X_5 that ensures the 9-letter

string is diverse.

Therefore, we conclude that $f(9) = 25 - 4 = 21$.

Answer: (D)

15. Let $f(x) = px^4 + 2025x^3 + qx^2 + 2025x + p$, where p is a positive integer. If $f(x) = 0$ has exactly three solutions, all of which are rational numbers, determine the minimum possible value of p .

(A) 1 (B) 25 (C) 44 (D) 45 (E) 46

Solution: First notice that $f(0) = 0 + 0 + 0 + 0 + p = p$. Since p is a positive integer, $f(0)$ cannot be 0 and so $x = 0$ cannot be a root of $f(x)$.

Let $g(x) = \frac{f(x)}{x^2}$. Since $x = 0$ is not a root of $f(x)$, we see that $x = r$ is a root of $f(x)$ if and only if $x = r$ is a root of $g(x)$.

Notice that $g(x) = \frac{f(x)}{x^2} = px^2 + \frac{p}{x^2} + 2025x + \frac{2025}{x} + q$. For all $r \neq 0$, we see that $g(r) = g(\frac{1}{r})$. In other words, r is a root if and only if $\frac{1}{r}$ is a root as well.

For example, if 3 and -7 are roots of $g(x)$, then that would imply that $g(x)$ has four roots: $\{3, -7, \frac{1}{3}, -\frac{1}{7}\}$. The only way $g(x)$ can have *three* roots is if one of them is a double root, i.e., $r = \frac{1}{r}$.

Since $f(x)$ has exactly three roots, this means that $g(x)$ has exactly three roots as well. This implies that one of these roots must satisfy $r^2 = 1$, i.e., $r = 1$ or $r = -1$. Let us consider both of these cases.

In our first case, we assume $r = 1$ is a root of $f(x)$. Then $0 = f(1) = p + 2025 + q + 2025 + p = 2p + q + 4050$, and so $q = -2p - 4050$. We can rewrite the polynomial $f(x)$ as

$$\begin{aligned} f(x) &= px^4 + 2025x^3 + qx^2 + 2025x + p \\ &= px^4 + 2025x^3 + (-2p - 4050)x^2 + 2025x + p \\ &= (x^2 - 2x + 1)(px^2 + (2025 + 2p)x + p). \\ &= (x - 1)^2(px^2 + (2025 + 2p)x + p). \end{aligned}$$

If $f(x)$ has exactly three *rational* roots, then the two solutions of $px^2 + (2025 + 2p)x + p = 0$ must both be rational. This occurs precisely when the

discriminant is a perfect square. The discriminant is equal to

$$(2025+2p)^2 - 4 \cdot p \cdot p = 2025^2 + 2025 \cdot 4p + 4p^2 - 4p^2 = 2025(2025+4p) = 45^2(2025+4p)$$

From above, we see that $f(x)$ has exactly three rational roots when $2025+4p$ is a perfect square. We know that $2025+4p$ is an odd integer larger than $2025 = 45^2$, and so we set $2025+4p = 47^2$ to find the *smallest* positive integer p in this case. We find that $2025+4p = 2209$ implies $p = 46$.

In our second case, we assume $r = -1$ is a root of $f(x)$. Then $0 = f(-1) = p - 2025 + q - 2025 + p = 2p + q - 4050$, and so $q = -2p + 4050$. We can rewrite the polynomial $f(x)$ as

$$\begin{aligned} f(x) &= px^4 + 2025x^3 + qx^2 + 2025x + p \\ &= px^4 + 2025x^3 + (-2p + 4050)x^2 + 2025x + p \\ &= (x^2 + 2x + 1)(px^2 + (2025 - 2p)x + p). \\ &= (x + 1)^2(px^2 + (2025 - 2p)x + p). \end{aligned}$$

If $f(x)$ has exactly three *rational* roots, then the two solutions of $px^2 + (2025 - 2p)x + p = 0$ must both be rational. This occurs precisely when the discriminant is a perfect square. The discriminant is equal to

$$(2025-2p)^2 - 4 \cdot p \cdot p = 2025^2 - 2025 \cdot 4p + 4p^2 - 4p^2 = 2025(2025-4p) = 45^2(2025-4p)$$

From above, we see that $f(x)$ has exactly three rational roots when $2025-4p$ is a perfect square. We know that $2025-4p$ is an odd integer less than $2025 = 45^2$, and so we set $2025-4p = 43^2$ to find the *smallest* positive integer p in this case. We find that $2025-4p = 1849$ implies $p = 44$.

Our question asks for the smallest positive integer p for which there exists a polynomial $f(x)$ with exactly three rational roots. From our analysis, we see that $p = 44$ is the correct answer, and in this case $q = -2 \cdot 44 + 4050 = 3962$.

Indeed we can check that $f(x) = 44x^4 + 2025x^3 + 3962x^2 + 2025x + 44 = (x+1)^2(x+44)(44x+1)$, so this polynomial has exactly three rational roots: $x = -1$, $x = -44$, and $x = -\frac{1}{44}$.

Answer: (C)