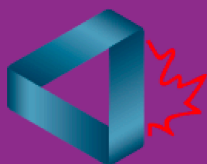




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Crux Mathematicorum with Mathematical Mayhem

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IN MEMORIAM

We are sad to announce the passing of a longtime *CruX* contributor Titu Zvonaru. Titu has shared his love for mathematics with many colleagues, he contributed materials to *CruX*, *The American Mathematical Monthly*, *School Science and Mathematics*, *Octagon Mathematical Magazine*, *Mathematical Reflections* and many other publications. His first contribution to *CruX* came in 2003, and his latest appeared 22 years later as a featured solution to problem 4985 in 51(4).

Titu was married for 48 years to Magdalena, they had no children. He has a sister and a niece who is a mathematics teacher in Bacău County. He is buried at the St. Ilie, Vermecsti, Comănești.



Titu Zvonaru, 27 November 1953 – 11 April 2025

MATHEMATTIC

No. 65

The problems featured in this section are intended for students at the secondary school level.

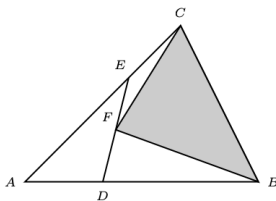
Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 15, 2025**.

MA321. Proposed by Daniel Rasmussen.

Find all pairs of non-zero integers (a, b) that share no common factors other than 1 and such that $a^2 + b^2$ divides $a^2b + ab^2 + ab$.

MA322. In the diagram, the area of the triangle ABC is 1, $AD = \frac{1}{3}AB$, $EC = \frac{1}{3}AC$ and $DF = FE$. Find the area of the shaded triangle.



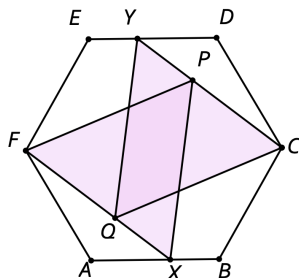
MA323. Five people are trapped on an island. Each person can form an alliance with any of the other people on the island. The alliances are mutual, so that if A is allied with B , then B is allied with A . However, if A is allied with B and B is allied with C , it is not necessarily true that A and C are allied. If every person on the island is allied with every other person on the island, then there are 10 pairs of people on the island who each have the four alliances.

1. Is it possible that no two people on the island have the same number of alliances? Explain.
2. What is the smallest possible number of pairs of people on the island with the same number of alliances?

MA324. Let N be a 3-digit number with three distinct non-zero digits. We say that N is mediocre if it has the property that when all six 3-digit permutations of N are written down, the average is N . For example, $N = 481$ is mediocre since the average of $\{481, 148, 184, 418, 814, 841\}$ is 481. Determine the largest 3-digit mediocre number.

MA325. *Proposed by Arsalan Wares.*

Suppose hexagon $ABCDEF$ is regular with points X and Y on sides AB and DE , respectively. Two congruent overlapping equilateral triangles, FXP and CYQ , partially cover hexagon $ABCDEF$ as shown. If vertices P and Q are on sides YC and FX , respectively, determine the exact value of $AX : AB$.



.....

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

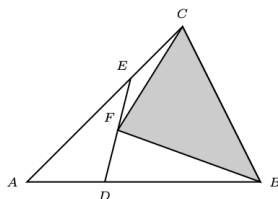
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 juillet 2025.



MA321. *Soumis par Daniel Rasmussen.*

Trouvez toutes les paires d'entiers non nuls (a, b) qui ne partagent aucun facteur commun autre que 1 telles que $a^2 + b^2$ divise $a^2b + ab^2 + ab$.

MA322. Dans le diagramme, l'aire du triangle ABC est 1, $AD = \frac{1}{3}AB$, $EC = \frac{1}{3}AC$ et $DF = FE$. Trouvez l'aire du triangle ombré.



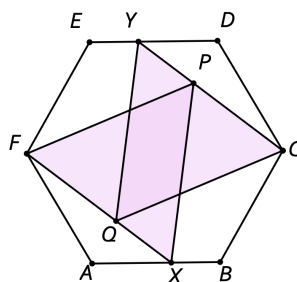
MA323. Cinq personnes sont piégées sur une île. Chaque personne peut former une alliance avec n'importe laquelle des autres personnes présentes sur l'île. Les alliances sont mutuelles, de sorte que si A est allié à B , alors B est allié à A . Cependant, si A est allié à B et que B est allié à C , il n'est pas nécessairement vrai que A et C sont alliés. Si chaque personne sur l'île est alliée avec chaque autre personne sur l'île, alors il y a 10 paires de personnes sur l'île qui ont chacune formé les quatre alliances.

1. Est-il possible qu'aucune personne sur l'île n'ait le même nombre d'alliances? Justifiez.
2. Quel est le plus petit nombre possible de paires de personnes sur l'île avec le même nombre d'alliances?

MA324. Soit N un nombre à 3 chiffres dont les trois chiffres sont distincts et non nuls. On dit que N est médiocre s'il a la propriété que lorsque les six permutations à 3 chiffres de N sont écrites, la moyenne de celles-ci est N . Par exemple, $N = 481$ est médiocre car la moyenne de $\{481, 148, 184, 418, 814, 841\}$ est 481. Déterminez quel est le plus grand nombre médiocre à 3 chiffres.


MA325. *Soumis par Arsalan Wares.*

Supposons que l'hexagone $ABCDEF$ soit régulier avec les points X et Y sur les côtés AB et DE , respectivement. Deux triangles équilatéraux congruents se chevauchant, FXP et CYQ , recouvrent partiellement l'hexagone $ABCDEF$ comme indiqué. Si les sommets P et Q sont respectivement sur les côtés YC et FX , déterminez la valeur exacte de $AX : AB$.



MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2024: 50(10), 493-495.

—————

MA296. Nonzero integers a , b and c satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$. Prove that among a , b , c there are two integers which have a common divisor larger than 1.

Originally Estonian Math Competitions 2012/2013, O1.

We received 5 correct submissions. Presented is the one by Prithwijit De.

We observe that the given condition is equivalent to

$$ab + bc + ca = 0.$$

If at least two of a , b , c are odd, then among the three numbers ab , bc and ca , either exactly one is odd or all three are odd. In either case, $ab + bc + ca$ is odd and cannot be equal to zero. Thus at least two of a , b and c are even.

MA297. The numbers $1, 2, \dots, 2012$ are written on the blackboard in some order, each of them exactly once. Between each two neighbouring numbers the absolute value of their difference is written and the original numbers are erased. This is repeated until only one number is left on the blackboard. What is the largest possible number that can be left on the blackboard?

Originally Estonian Math Competitions 2012/2013, O8.

We received 5 submissions of which 4 were mostly correct. We present the solution by Mihika Bansal, modified by the editor.

It is clear that the largest number cannot increase on any step. After the first step, the largest possible number is 2011 and the smallest possible number is 1. After the second step, the largest possible number is 2010 and the smallest one is 0, so in the end the final number cannot be larger than 2010. Indeed, the strategic approach is to put the two largest numbers on the outside and arrange the rest in consecutive order. We get:

$$\begin{array}{cccccccc}
 2012 & 1 & 2 & 3 & \dots & 2010 & 2011 & \\
 & 2011 & 1 & 1 & 1 & \dots & 1 & \\
 & & 2010 & 0 & 0 & \dots & 0 & \\
 & & & & \dots & & & \\
 & & & & & 2010 & 0 & \\
 & & & & & & 2010 &
 \end{array}$$

Therefore the largest possible number is 2010.

MA298. A game board consists of 10 squares in a row that are numbered 1 to 10. A button is placed on a numbered square. Then on each move, the button can be moved to a square with a number that is either smaller by 2 or double the number. Does there exist an initial location for the button that allows the button to visit all squares of the board? (Note: The button is allowed to visit one square several times.)

Originally Estonian Math Competitions 2013/2014, O4 adapted.

We received 5 submissions, 4 of which were completely correct. We present the solution by Aditi Gupta.

There does not exist an initial location for the button that allows the button to visit all squares of the board.

We have two types of moves – we can either double the number or subtract 2 from the number. If we double the number, the product will be even. If we subtract 2 from an even number, the answer will always be even. Therefore, after reaching an even number we cannot go back to an odd number. This means that we have to cover all the odd numbers before moving to even numbers. Since we have to subtract, we should start from the largest odd number, 9:

$$9 - 2 = 7, 7 - 2 = 5, 5 - 2 = 3 \text{ and } 3 - 2 = 1.$$

Now we start doubling the numbers: $1 \times 2 = 2$, $2 \times 2 = 4$, and $4 \times 2 = 8$. To reach 6, we subtract 2 from 8. We have covered all of the numbers except 10. The only way to reach 10 is to double 5, but that is not possible: If we stop at 5 while subtracting, all of the odd numbers cannot be covered; if we do not stop, we cannot come back to 5. Hence, it is not possible to reach 10.

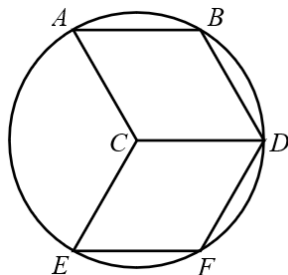
Editor's Comments. Every solver realized that any successful tour would have to visit the odd-numbered squares backwards, starting with the highest. However, every tour starting at 9 misses either 10 or both 3 and 1. As pointed out by the Missouri Problem Solving group, if we replace 10 by N , then a successful tour exists just when $N = 2, 3$ or $N \equiv 0, 1 \pmod{4}$.

MA299. In the plane there are six different points A, B, C, D, E, F such that $ABCD$ and $CDEF$ are parallelograms. What is the maximum number of those points that can be located on one circle?

Originally Estonian Math Competitions 2013/2014, O9.

There were 2 complete and correct submissions proposed by Missouri State University Problem Solving Group and Richard Hess. They are jointly presented below.

The figure on the next page shows the maximum achievable is five points. The two parallelograms must share two points and no parallelogram can have more than three points on the circle, which demonstrates that five is the maximum.



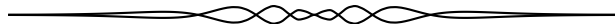
MA300. Proposed by Sicheng Du.

Find the value of

$$T = \frac{\sin 20^\circ}{\sin 100^\circ} + \frac{\sqrt{3} \sin 20^\circ}{4 \sin^2 100^\circ}.$$

We received 8 correct solutions. We present the solution by Catherine Jian.

$$\begin{aligned} T &= \frac{\sin 20^\circ}{\sin 100^\circ} + \frac{\sqrt{3} \sin 20^\circ}{4 \sin^2 100^\circ} \\ &= \frac{4 \sin 20^\circ \sin 100^\circ + 2 \sin 60^\circ \sin 20^\circ}{4 \sin^2 100^\circ} \\ &= \frac{2(\cos 80^\circ - \cos 120^\circ) + (\cos 40^\circ - \cos 80^\circ)}{4 \sin^2 100^\circ} \\ &= \frac{1 + \cos 40^\circ + \cos 80^\circ}{4 \sin^2 100^\circ} \\ &= \frac{1 + \cos 20^\circ}{4 \sin^2 100^\circ} \\ &= \frac{2 \cos^2 10^\circ}{4 \sin^2 100^\circ} \\ &= \frac{1}{2}. \end{aligned}$$



PROBLEM SOLVING VIGNETTES

No. 37

Shawn Godin

Barbeau, Klamkin, Moser, Farey, Pell, Pythagoras, and
the square root of 2: Part II

In the previous column [2025: 51(3), 113–120] we looked at a problem from [1] which led to us exploring rational approximations to $\sqrt{2}$. Along the way we ran into certain *Pythagorean triples* and something called *Pell's equation*, which we will talk a bit more about in this column. Let's recall a few things from the last column that we will need to continue our analysis.

We saw that $\sqrt{2}$ can be expressed as

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}} \quad (1)$$

which led to its *simple continued fraction*, namely

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\ddots}}}}$$

Truncating this infinite continued fraction at various points yields its *convergents*,

$$1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \frac{1393}{985}, \frac{3363}{2378}, \frac{8119}{5741}, \frac{19601}{13860} \dots,$$

which turned out to be increasingly good rational approximations to $\sqrt{2}$, that we labelled:

$$c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, \dots$$

From the sequence of convergents, we defined the sequences of the numerators and denominators, h_n and k_n , respectively, as

$$c_n = \frac{h_n}{k_n}$$

While exploring these sequences we came up with the following useful identities:

$$h_{n+1} = h_n + 2k_n \quad (2)$$

$$h_{n+1} = 2h_n + h_{n-1} \quad (3)$$

$$k_{n+1} = k_n + h_n \quad (4)$$

$$k_{n+1} = 2k_n + k_{n-1} \quad (5)$$

Now we can continue our investigation.

Equation (1) suggests it might be interesting to consider the function

$$f(x) = 1 + \frac{1}{1+x}.$$

For one thing, (1) implies that $\sqrt{2}$ is a *fixed point* for this function, that is

$$f(\sqrt{2}) = \sqrt{2}.$$

Also, if we start with $x = 1$ and *iterate* the function, we get

$$\begin{aligned} f(1) &= 1 + \frac{1}{1+1} = \frac{3}{2} \\ f\left(\frac{3}{2}\right) &= 1 + \frac{1}{1+\frac{3}{2}} = \frac{7}{5} \\ f\left(\frac{7}{5}\right) &= 1 + \frac{1}{1+\frac{7}{5}} = \frac{17}{12} \\ f\left(\frac{17}{12}\right) &= 1 + \frac{1}{1+\frac{17}{12}} = \frac{41}{29} \\ \vdots & \qquad \qquad \qquad \vdots \end{aligned}$$

It seems that our new function can be used to generate the convergents to the continued fraction of $\sqrt{2}$. This makes sense if we rewrite the convergents as we have done for c_4 below

$$c_4 = \frac{17}{12} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = 1 + \frac{1}{1 + 1 + \frac{1}{1 + 1 + \frac{1}{1+1}}} = 1 + \frac{1}{1 + \left(1 + \frac{1}{1 + \left(1 + \frac{1}{1+1}\right)}\right)}$$

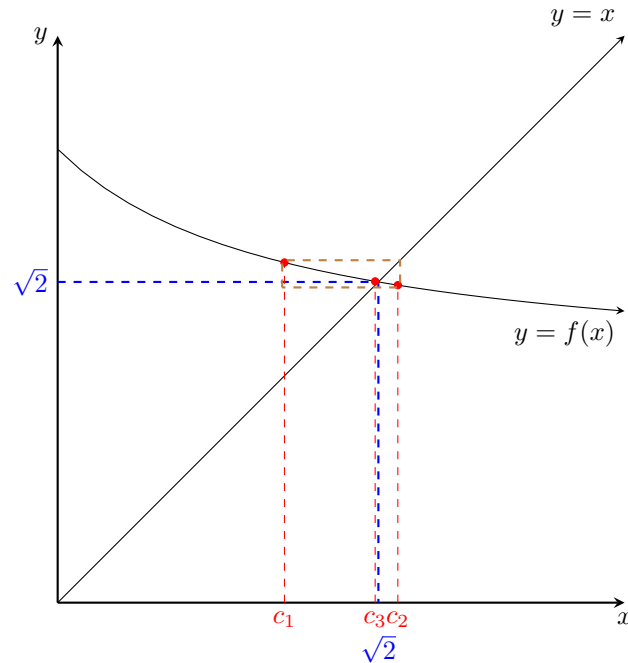
Here we see at the bottom, the lone brown one, 1 , is c_1 . Next, the expression in the blue brackets would be $f(c_1) = c_2$. Similarly in red we have

$$f(c_2) = f(f(c_1)) = c_3$$

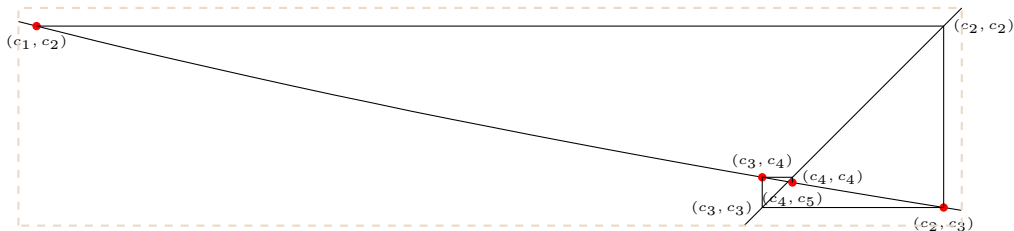
and, finally, in black,

$$f(c_3) = f(f(f(c_1))) = c_4.$$

Plotting $y = f(x)$ and $y = x$ on the grid on the next page along with the points (c_1, c_2) , (c_2, c_3) , and (c_3, c_4) , we can see how quickly the convergents converge on the square root of 2.



If we go a step further we can draw the *cobweb plot* for function f . Starting at $x = c_1$ plot the point $(c_1, f(c_1)) = (c_1, c_2)$. Draw a line segment from this point horizontally until it intersects the line $y = x$, which will occur, since we are keeping the y values constant, at the point (c_2, c_2) . From there we draw a line segment from this new point vertically until we intersect the graph $y = f(x)$ which occurs at the point $(c_2, f(c_2)) = (c_2, c_3)$. The process then continues producing a “cobweb” that converges on the fixed point at $(\sqrt{2}, \sqrt{2})$. The graph below is of the region in the brown box in the graph above magnified eight times. We can see one more point, but because of the rate of convergence, we would have to zoom in more to discern the next point.



Let's see what else f can tell us about the convergents. Given two real numbers

$x, y \geq 1$, then

$$\begin{aligned} f(x) - f(y) &= 1 + \frac{1}{1+x} - \left(1 + \frac{1}{1+y}\right) \\ &= \frac{1}{1+x} - \frac{1}{1+y} \\ &= \frac{1+y - (1+x)}{(1+x)(1+y)} \\ &= \frac{-(x-y)}{(1+x)(1+y)} \\ &\leq \frac{-(x-y)}{4}. \end{aligned}$$

Hence, if $x > y$ then $f(x) < f(y)$ and vice versa. Also, $f(x)$ and $f(y)$ are closer together than x and y . Specifically, since $f(\sqrt{2}) = \sqrt{2}$, then

$$f(x) - \sqrt{2} \leq \frac{-(x - \sqrt{2})}{4},$$

so we can deduce that our convergents alternate between greater than and less than $\sqrt{2}$ and they get closer at each step, verifying

$$c_1 < c_3 < c_5 < c_7 < \cdots < \sqrt{2} < \cdots < c_8 < c_6 < c_4 < c_2. \quad (6)$$

which we saw in *Part I*.

Let's examine what our function f can tell us about our sequences h_n and k_n that we recalled earlier. We have already seen that $f(c_n) = c_{n+1}$. Combining this with $c_n = \frac{h_n}{k_n}$, we get

$$\begin{aligned} c_{n+1} &= f(c_n), \\ \frac{h_{n+1}}{k_{n+1}} &= 1 + \frac{1}{1 + \frac{h_n}{k_n}} = 1 + \frac{1}{\frac{k_n + h_n}{k_n}} = 1 + \frac{k_n}{k_n + h_n} = \frac{k_n + h_n + k_n}{k_n + h_n} = \frac{h_n + 2k_n}{k_n + h_n}. \end{aligned}$$

Since we assumed that h_n and k_n were defined so that the fraction was in lowest terms, this means that $\gcd(h_n, k_n) = 1$. Hence

$$\begin{aligned} \gcd(h_n + 2k_n, h_n + k_n) &= \gcd(h_n + 2k_n - (h_n + k_n), h_n + k_n) \\ &= \gcd(k_n, h_n + k_n) \\ &= \gcd(k_n, h_n + k_n - k_n) \\ &= \gcd(k_n, h_n) = 1, \end{aligned}$$

from which we can conclude

$$\begin{aligned} h_{n+1} &= h_n + 2k_n, \\ k_{n+1} &= h_n + k_n. \end{aligned}$$

our equations (2) and (4), developed in part I.

Since the convergents alternate below and above as in (6), then the difference between successive terms should give us a rough idea of the error in the approximation. That is, for example, if

$$c_n < \sqrt{2} < c_{n+1}$$

then, as c_{n+1} is closer to $\sqrt{2}$ than c_n ,

$$\left| \sqrt{2} - c_{n+1} \right| < \frac{c_{n+1} - c_n}{2}.$$

Examining the difference between consecutive convergents we get

$$\begin{aligned} c_{n+1} - c_n &= \frac{h_{n+1}}{k_{n+1}} - \frac{h_n}{k_n} \\ &= \frac{h_{n+1}k_n - h_nk_{n+1}}{k_nk_{n+1}} \end{aligned}$$

Focusing on the numerator and using (3) and (5) yields

$$\begin{aligned} h_{n+1}k_n - h_nk_{n+1} &= (2h_n + h_{n-1})k_n - h_n(2k_n + k_{n-1}) \\ &= 2h_nk_n + h_{n-1}k_n - 2h_nk_n - h_nk_{n-1} \\ &= h_{n-1}k_n - h_nk_{n-1} \\ &= -(h_nk_{n-1} - h_{n-1}k_n) \end{aligned} \tag{7}$$

which would be the negative of the numerator of $c_n - c_{n-1}$. If we go back to the first pair of approximations, we get

$$\frac{3}{2} - \frac{1}{1} = \frac{1}{2} \Rightarrow h_2k_1 - h_1k_2 = 1$$

which, with (7), implies inductively that

$$h_{n+1}k_n - h_nk_{n+1} = (-1)^{n+1}. \tag{8}$$

Hence,

$$\left| \sqrt{2} - \frac{h_{n+1}}{k_{n+1}} \right| < \frac{1}{2k_nk_{n+1}}$$

gives a nice approximation to the “error” of using the convergent c_{k+1} as an approximation to $\sqrt{2}$.

Also hidden in (8) is a connection to what are called *Farey sequences*. The Farey sequence F_n is the list of all fractions, reduced to lowest terms, with maximum denominator n in the interval $[0, 1]$. Equation (8) tells us specifically that $(c_n - 1)$ and $(c_{n+1} - 1)$ are consecutive terms in the Farey sequence $F_{k_{n+1}}$.

Returning to (8) and using (2) and (4) yields

$$\begin{aligned} (-1)^{n+1} &= h_{n+1}k_n - h_nk_{n+1} \\ &= (h_n + 2k_n)k_n - h_n(k_n + h_n) \\ &= h_nk_n + 2k_n^2 - h_nk_n - h_n^2 \\ &= 2k_n^2 - h_n^2 \end{aligned}$$

which can be rewritten as

$$h_n^2 - 2k_n^2 = (-1)^n \quad (9)$$

which we met in passing in part I.

The equation

$$x^2 - dy^2 = N \quad (10)$$

for given integers d and N and unknown integers x and y is called *Pell's equation*. The solutions to this nonlinear Diophantine equation is closely related to the simple continued fraction expansion of \sqrt{d} , as is hinted in (9).

I will most likely return to the topics of continued fractions, Pell's equation, and Farey sequences in later columns. Impatient readers can likely find some information about continued fractions, Pell's equation, Pythagorean triples, and Farey sequences in an introductory textbook on number theory. For example [4] talks about these topics in chapter 7, section 7.8, section 5.3, and section 6.1, respectively. To delve a bit deeper I would suggest [3] to learn more about continued fractions and [2] to learn about Pell's equation. I will leave you with a few problems related to the topics from this column.

1. Given $\alpha = \sqrt{5}$,
 - (a) Generate the continued fraction for α .
 - (b) Compute the first few convergents to the continued fraction in (a).
 - (c) Writing the convergents as $c_n = \frac{h_n}{k_n}$, as in the column, generate equations similar to (2), (4), (3), and (5).
 - (d) Show that the sequences in (b) satisfy the Pell equation

$$h_n^2 - 5k_n^2 = \pm 1$$

where the sign on the right is positive or negative depending on the parity of n .

2. Show that if $n = a^2 + 1$ for some positive integer a and if the function

$$f(x) = a + \frac{1}{a+x}$$

is iterated over and over, starting with $x = 1$, then the resultant sequence of values converges to \sqrt{n} . In particular, we saw earlier that iterating the function with $a = 1$ will yield $\sqrt{2}$ while using $a = 2$ will yield $\alpha = \sqrt{5}$ (see problem 1).

3. Given two fractions in lowest terms $\frac{a}{b}$ and $\frac{c}{d}$ with $\frac{a}{b} < \frac{c}{d}$, the quantity

$$\frac{a+c}{b+d}$$

is called the *mediant* of the two fractions.

- (a) Show that the mediant of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ with $\frac{a}{b} < \frac{c}{d}$ satisfies

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

- (b) Notice if we take the mediant of two consecutive convergents we get twice the reciprocal of the latter convergent. For example, the mediant of $c_3 = \frac{7}{5}$ and $c_4 = \frac{17}{12}$ is

$$\frac{7+17}{5+12} = \frac{24}{17} = 2 \left(\frac{12}{17} \right).$$

Assuming this is true for all consecutive pairs of convergents, derive some more relationships between the h_n s and the k_n s and use (2), (4), (3), and (5) to verify them.

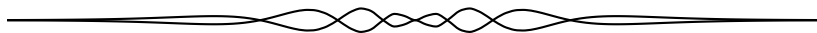
The author would like to thank, once again, Ed Barbeau for his comments and suggestions. As usual, Ed's input greatly improved this article and its prequel.

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The online journal allows for use of colours and visuals. For those who print the journal in black and white, there are times that references in articles such as those concerning colours in the preceding piece may not flow naturally. Readers are encouraged to refer to the online presentation in such instances.



OLYMPIAD CORNER

No. 433

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

*To facilitate their consideration, solutions should be received by **July 15, 2025**.*

OC731. 12 children came to a physical education lesson, all of different strengths. The teacher divided them into two teams of 6 people 10 times, each time in a new way, and held a tug-of-war competition. Could it be that all 10 times the competition ended in a draw (that is, the sum of the strengths of the children in the teams was equal)?

OC732. Prove that among the vertices of a convex nonagon, three can be found that form an obtuse triangle, none of whose sides coincide with the sides of the nonagon.

OC733. In an acute triangle ABC , the altitude AH is drawn. Points M and N are the midpoints of segments BH and CH . Prove that the intersection point of the perpendiculars dropped from points M and N to lines AB and AC , respectively, is equidistant from points B and C .

OC734. A natural number has exactly 50 divisors. Is it possible that no difference between two of its distinct divisors is divisible by 100?

OC735. An odd number $n \geq 3$ is given. In a $2n \times 2n$ grid, $2(n-1)^2$ cells are painted. What is the greatest number of L -trominos that can be guaranteed to be cut out of an unpainted grid?

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2025**.

OC731. 12 enfants sont venus à un cours d'éducation physique. Chacun d'eux dispose de forces différentes. Le professeur les a divisés en deux équipes de 6 personnes à 10 reprises, chaque fois d'une nouvelle manière, et a organisé une compétition de tir à la corde. Se peut-il que chacune des 10 reprises la compétition se soit terminée par un match nul (c'est-à-dire que la somme des forces des enfants dans les équipes était égale) ?

OC732. Prouvez que parmi les sommets d'un nonagone convexe, on peut en trouver trois qui forment un triangle obtus dont aucun des côtés ne coïncide avec les côtés du nonagone.

OC733. Dans un triangle aigu ABC , on trace la hauteur AH . Les points M et N sont les milieux des segments BH et CH . Prouvez que le point d'intersection des perpendiculaires, respectivement abaissées des points M et N sur les droites AB et AC , est équidistant des points B et C .

OC734. Un nombre naturel a exactement 50 diviseurs. Est-il possible qu'aucune différence entre deux de ses diviseurs distincts ne soit divisible par 100 ?

OC735. Soit un nombre impair $n \geq 3$. Dans une grille de dimension de $2n \times 2n$, il y a $2(n-1)^2$ cases peintes. Quel est le plus grand nombre de L -triominos que l'on peut garantir de découper dans la grille non peinte ?

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2024: 50(10), 505-507.

OC706. Let a and b be distinct positive integers such that $3^a + 2$ is divisible by $3^b + 2$. Prove that $a > b^2$.

Originally from the 2024 41st Balkan Mathematical Olympiad.

We received a single submission from Vivek Mehra, presented below.

We have that $a > b$. Let $a = bk + r$ with r, k nonnegative integers that satisfy $0 \leq r \leq b - 1$ and $1 \leq k$.

Then

$$3^a + 2 = 3^{kb+r} + 2 \equiv (-2)^k 3^r + 2 \pmod{3^b + 2}.$$

Hence $3^b + 2$ divides

$$(-2)^k 3^r + 2 - 3^b - 2 = 3^r((-2)^k - 3^{b-r}).$$

Since $\gcd(3^r, 3^b + 2) = 1$, we must have that $3^b + 2$ divides $(-2)^k - 3^{b-r}$.

Assume k is even. Then $3^b + 2$ divides $2^k - 3^{b-r}$, which is a non-zero number. Hence, either

$$3^{b-r} - 2^k \geq 3^b + 2$$

or

$$2^k - 3^{b-r} \geq 3^b + 2.$$

The first possibility is not possible. The second implies $k > b$ and $a = bk + r > b^2$.

Assume k is odd and $r = 0$. Then $a > b$ implies $k > 1$. We find that $3^b + 2$ divides

$$2^k + 3^b - 3^b - 2 = 2^k - 2.$$

Since $2^k - 2 > 0$, we must have $2^k - 2 \geq 3^b + 2$ and $k > b$. This implies $a > b^2$.

Assume k is odd and $r \geq 1$. Then

$$3^r - 1 \geq 2 \times 3^{r-1}.$$

Since $3^b + 2$ divides $2^k + 3^{b-r}$, we have

$$2^k \geq 3^{b-r}(3^r - 1) + 2 \geq \frac{2}{3} \times 3^b + 2.$$

So $2^{k-1} \geq 3^{b-1} + 1$. This implies $k > b$ and $a > b^2$.

OC707. For any positive real number x , we define $[x]$ to be the greatest integer that does not exceed x and $\{x\}$ to be the fractional part of x such that $\{x\} = x - [x]$. Let a , b and c be positive real numbers that satisfy the following system of equations:

$$\begin{cases} a + 2[b] + 2\{c\} = 4.6 \\ [a] + \{b\} - c = 1 \\ 2\{a\} + 2b + [c] = 4.5 \end{cases}$$

What is the largest possible value of $a + b + c$?

Originally from the 2024 InIMC, Lucknow, Individual Contest.

We received 4 submissions, 3 of which were correct and complete. We present the solution by Theo Koupelis.

Let $a = n + r_1$, $b = m + r_2$, and $c = k + r_3$, where n, m, k are nonnegative integers,

$$0 \leq r_1 < 1, 0 \leq r_2 < 1 \text{ and } 0 \leq r_3 < 1.$$

From the second of the given system of equations we get $n + r_2 - k - r_3 = 1$, and thus $n - k = 1 + r_3 - r_2$. But $1 + r_3 - r_2 \in (0, 2)$ and thus $n - k = 1$, with $n \geq 1$, and $r_2 = r_3$. Substituting these expressions into the first and third of the given equations we get

$$n + r_1 + 2m + 2r_2 = 4.6,$$

and

$$2r_1 + 2m + 2r_2 + (n - 1) = 4.5.$$

Subtracting the above two equations we get $r_1 = 0.9$. Finally, substituting this value of r_1 into the first equation we get $n + 2m = 3.7 - 2r_2$. Clearly $m = \{0, 1\}$. For $m = 0$ we get $(n, r_2) = (2, 0.85)$ or $(3, 0.35)$. For $m = 1$, we get $(n, r_2) = (1, 0.35)$.

Thus, the three solutions to the given system of equations are

$$(a, b, c) = (2.9, 0.85, 1.85), (3.9, 0.35, 2.35) \text{ and } (1.9, 1.35, 0.35).$$

Therefore, the largest possible value of $a + b + c$ is 6.6.

OC708. Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

Originally from the 2020 Australian Mathematical Olympiad.

We received 17 submissions, 12 of which were correct and complete. We present a typical solution.

The initial equation $\frac{a+b}{2} - \sqrt{ab} = 1$ is equivalent to $(\sqrt{a} - \sqrt{b})^2 = 2$ and to

$$(\sqrt{a} - \sqrt{b} - \sqrt{2})(\sqrt{a} - \sqrt{b} + \sqrt{2}) = 0.$$

It follows that either $\sqrt{a} = \sqrt{b} + \sqrt{2}$, or $\sqrt{b} = \sqrt{a} + \sqrt{2}$. After squaring each side, we obtain $a = b + 2 + 2\sqrt{2b}$, or $b = a + 2 + 2\sqrt{2a}$. Since $\sqrt{2b}$, or $\sqrt{2a}$ need to be integers, we obtain the solutions

$$(a, b) \in \{(2(k+1)^2, 2k^2), (2k^2, 2(k+1)^2) \mid k \in \mathbb{N}\}.$$

OC709. Amy and Bec play the following game. Initially, there are three piles, each containing 2020 stones. The players take turns to make a move, with Amy going first. Each move consists of choosing one of the piles available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to make a move. Prove that Bec can always win the game, no matter how Amy plays.

Originally from the 2020 Australian Mathematical Olympiad.

We received 3 submissions, all of which were correct and complete. We present the solution by Roy Barbara.

Given that the three initial piles are of equal size (2020 stones), the game's outcome remains unchanged if we consider it starting with a single pile. We will prove the following:

Let $n \geq 1$. If the game begins with a single pile of n stones:

- (a) The first player has a winning strategy if $n \equiv 0 \pmod{3}$ or $n \equiv 2 \pmod{3}$.
- (b) The second player has a winning strategy if $n \equiv 1 \pmod{3}$.

Once this general result is established, we can conclude that Bec has a winning strategy when starting with 2020 stones, as $2020 \equiv 1 \pmod{3}$.

We will prove this by induction on n . The result holds for $n = 1, 2, 3$. Let $n \geq 4$ and assume that the result holds for any m with $1 \leq m < n$. We will refer to the first player as X and the second player as Y .

Case 1: $n \equiv 0 \pmod{3}$

X starts and divides the pile containing n stones into one pile of $n - 2$ stones and two piles, each containing 1 stone. Player Y then has only one valid move: to select the pile with $n - 2$ stones. Since $n - 2 < n$ and $n - 2 \equiv 1 \pmod{3}$, by the induction hypothesis, X has a winning strategy.

Case 2: $n \equiv 2 \pmod{3}$

X starts and divides the pile containing n stones into one pile of $n - 1$ stones and one pile of 1 stone. Player Y then has only one valid move: to select the pile with $n - 1$ stones. Since $n - 1 < n$ and $n - 1 \equiv 1 \pmod{3}$, by the induction hypothesis, X has a winning strategy.

Case 3: $n \equiv 1 \pmod{3}$

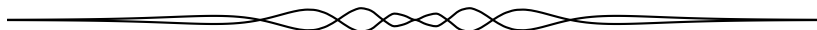
Suppose player X starts and divides the pile containing n stones into two piles: one containing $p \geq 1$ stones and another containing $q \geq 1$ stones. If $p \equiv 1 \pmod{3}$ and $q \equiv 1 \pmod{3}$ then $n \equiv 2 \pmod{3}$, which contradicts our assumption that $n \equiv 1 \pmod{3}$. Then at least one of p and q , assume p , must satisfy $p \equiv 0 \pmod{3}$ or $p \equiv 2 \pmod{3}$. Player Y can then select the pile with p stones, removing the pile with q stones. Since $p < n$, and by the induction hypothesis, Y has a winning strategy.

Now suppose player X starts and divides the pile containing n stones into three piles: one containing $p \geq 1$ stones, one containing $q \geq 1$ stones, and another containing $r \geq 1$ stones. If $p \equiv 1 \pmod{3}$, $q \equiv 1 \pmod{3}$, $r \equiv 1 \pmod{3}$ then $n \equiv 0 \pmod{3}$, which contradicts our assumption that $n \equiv 1 \pmod{3}$. Then at least one of p , q , and r , assume p , must satisfy $p \equiv 0 \pmod{3}$ or $p \equiv 2 \pmod{3}$. Player Y can then select the pile with p stones, removing the piles with q or r stones. Since $p < n$, by the induction hypothesis Y has a winning strategy.

OC710. A subset S of size n of a plane consisting of points with both coordinates integer is given, where n is an odd number. The injective function $f: S \rightarrow S$ satisfies the following: for each pair of points $A, B \in S$, the distance between points $f(A)$ and $f(B)$ is not smaller than the distance between points A and B . Prove there exists a point X such that $f(X) = X$.

Originally from the 2018 Polish Mathematical Olympiad.

This question is identical to OC 579. All 4 submissions we received were correct and complete. Therefore, we will not present any of the received solutions and instead direct the reader to the featured solution of OC 579.



Exploring The Linearity of Power Differences in Olympiad Geometry

Sicheng Du

1 Introduction

The linearity of power differences is a lemma that provides a computational approach to solving Olympiad geometry problems. It is the purpose of this article to familiarize the readers with the lemma so as to expand their problem-solving toolbox.

It will be assumed that the reader has already learned about the power of a point, radical axes, and radical centers. Note also, that it is recommended to append a proof of the linearity of power differences whenever using it in a proof-writing competition.

2 Linearity of Power Differences

Without further delay, we will introduce the main lemma. Later on, the notation $\text{Pow}_\omega(X)$ refers to the power of point X with respect to circle ω ; bold lowercase letters represent the position vectors of the points labeled by their respective uppercase letters, e.g. $\mathbf{a} = \overrightarrow{OA}$. The selection of the origin O will turn out to be irrelevant.

Lemma 1 (Linearity of Power Differences). *Let ω_1 and ω_2 be two circles. Then the function*

$$f(X) = \text{Pow}_{\omega_1}(X) - \text{Pow}_{\omega_2}(X)$$

defined for all points X is linear. In particular, if points A, B, C are such that $\mathbf{c} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$, then $f(C) = \lambda f(A) + (1 - \lambda)f(B)$.

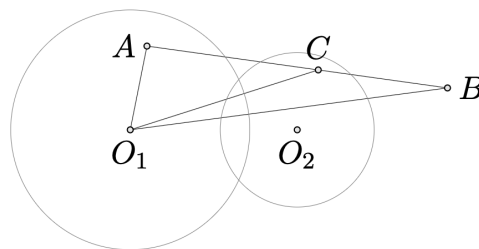


Figure 1: Linearity of power differences

The *linearity* refers to the fact that $f(C)$ can be expressed as a linear combination of $f(A)$ and $f(B)$, in the same way \mathbf{c} can be expressed in terms of \mathbf{a} and \mathbf{b} .

This lemma can be proven using vectors as follows.

Proof. Let the centers and radii of ω_1, ω_2 be O_1, O_2 and r_1, r_2 respectively. By definition, $f(X)$ can be transformed into

$$\begin{aligned} f(X) &= [|\mathbf{x} - \mathbf{o}_1|^2 - r_1^2] - [|\mathbf{x} - \mathbf{o}_2|^2 - r_2^2] \\ &= (\mathbf{x} - \mathbf{o}_1) \cdot (\mathbf{x} - \mathbf{o}_1) - (\mathbf{x} - \mathbf{o}_2) \cdot (\mathbf{x} - \mathbf{o}_2) - r_1^2 + r_2^2 \\ &= 2\mathbf{x} \cdot (\mathbf{o}_2 - \mathbf{o}_1) + \mathbf{o}_1 \cdot \mathbf{o}_1 - \mathbf{o}_2 \cdot \mathbf{o}_2 - r_1^2 + r_2^2, \end{aligned}$$

which is linear in \mathbf{x} . □

Let us reconsider **Lemma 1** in a Cartesian coordinate system. It can be shown that there exist real numbers u, v , and w such that for any point C with coordinates (x_C, y_C) , we have $f(C) = ux_C + vy_C + w$. With this property, the following generalization can be deduced.

Lemma 2. Let ω_1 and ω_2 be two circles. Define $f(X) = \text{Pow}_{\omega_1}(X) - \text{Pow}_{\omega_2}(X)$ for all points X . If points A_1, A_2, \dots, A_n , and B are such that

$$\mathbf{b} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \dots + \lambda_n \mathbf{a}_n,$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are real numbers that sum to 1, then

$$f(B) = \lambda_1 f(A_1) + \lambda_2 f(A_2) + \dots + \lambda_n f(A_n).$$

Proof. Set up an arbitrary Cartesian coordinate system. Let ω_1, ω_2 have centers and radii $(X_1, Y_1), (X_2, Y_2)$ and r_1, r_2 respectively. For any point C with coordinates (x_C, y_C) ,

$$\begin{aligned} f(C) &= [(x_C - X_1)^2 + (y_C - Y_1)^2 - r_1^2] - [(x_C - X_2)^2 + (y_C - Y_2)^2 - r_2^2] \\ &= 2(X_2 - X_1)x_C + 2(Y_2 - Y_1)y_C + (X_1^2 - X_2^2 + Y_1^2 - Y_2^2 - r_1^2 + r_2^2). \end{aligned}$$

So $f(C) = ux_C + vy_C + w$, where

$$u = 2(X_2 - X_1), \quad v = 2(Y_2 - Y_1), \quad w = X_1^2 - X_2^2 + Y_1^2 - Y_2^2 - r_1^2 + r_2^2$$

are constants regardless of C .

Let the coordinates of A_i be (x_i, y_i) for $i = 1, 2, \dots, n$. Then the coordinates of B are

$$(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n, \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n).$$

So

$$\sum_{i=1}^n \lambda_i f(A_i) = \sum_{i=1}^n \lambda_i (ux_i + vy_i + w) = u \sum_{i=1}^n \lambda_i x_i + v \sum_{i=1}^n \lambda_i y_i + w = f(B).$$

We are done. □

This generalization can be effectively combined with barycentric coordinates. For example, if the centroid of $\triangle ABC$ is G , then $G = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which means that $\mathbf{g} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Consequently, $f(G) = \frac{1}{3}[f(A) + f(B) + f(C)]$.

For the final part of the theoretical section, here are some tips.

- The power of a point to a single circle does *not* have linearity[†]. Instead, it is only related to the distance from the point to the circle's center and the circle's radius.
- If a geometry problem can reduce to something related to the power of a point, but this power cannot be directly evaluated, the linearity of power differences may be considered.
- If only one circle is present and constructing a second is hard, try considering degenerated circles—points can be regarded as circles with radius 0.
- Usually, it simplifies the problem if points that were previously defined can be used to characterize points that were later defined. This idea can guide you when selecting points and circles to apply the linearity of power differences on.

3 Example Problems

Problem 1 (Beijing High School League Preliminaries 2023 Problem 13). *Let $\triangle ABC$ be an acute triangle, with its incircle ω tangent to sides AB , AC at K , L respectively. Let altitude AH intersect the bisectors of $\angle ABC$, $\angle ACB$ at P , Q respectively. Let ω_1 , ω_2 be the circumcircles of $\triangle KPB$, $\triangle LQC$ respectively. If the midpoint of AH is outside ω_1 and ω_2 , show that the tangent lines of ω_1 and ω_2 from the midpoint of AH are equal in length.*

Proof. Let M be the midpoint of AH . By the power of a point theorem, it suffices to prove that the power of M with respect to ω_1 and ω_2 are the same. If we define $f(X) = \text{Pow}_{\omega_1}(X) - \text{Pow}_{\omega_2}(X)$ for all points X , then it suffices to show that $f(M) = 0$. By the linearity of power differences, since $\mathbf{m} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{h}$, then $f(M) = \frac{1}{2}f(A) + \frac{1}{2}f(H)$.

To find $f(A)$, let $a = BC$, $b = CA$, $c = AB$. By the power of a point theorem,

$$\begin{aligned} f(A) &= AK \cdot AB - AL \cdot AC = \frac{b+c-a}{2} \cdot c - \frac{b+c-a}{2} \cdot b \\ &= \frac{(c-b)(b+c-a)}{2}. \end{aligned}$$

To find $f(H)$, let ω_2 intersect BC at points D and C . It seems that ω_1 goes through D as well, and we are going to prove this. Let the incircle be tangent to

[†]The lemma is sometimes referred to as the linearity of power of a point. While both this name and the one used in this article are non-official, the latter seems to be more accurate.

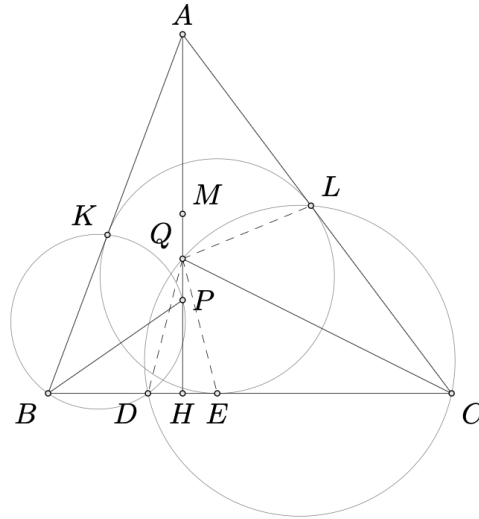


Figure 2: Problem 1(Beijing High School League Preliminaries 2023 Problem 13)

BC at E , then L and E are symmetric with respect to line CQ . Thus,

$$\angle QDE = \angle ALQ = \pi - \angle QLC = \pi - \angle QEC = \angle QED.$$

Moreover, $QH \perp DE$, so D is the reflection of E over line AH . Similarly, ω_1 can be proven to intersect BC at B and the reflection of E over line AH , so D indeed lies on ω_1 . Now,

$$\begin{aligned} f(H) &= HD \cdot HB - (-HD \cdot HC) = HD \cdot BC \\ &= HE \cdot a = a(HC - EC) = a(AC \cos \angle ACB - EC) \\ &= a \left(b \cdot \frac{a^2 + b^2 - c^2}{2ab} - \frac{a + b - c}{2} \right) = \frac{(b - c)(b + c - a)}{2}. \end{aligned}$$

Therefore, $f(M) = \frac{f(A)+f(H)}{2} = 0$ is true, so we are done. □

The following problem is more difficult as it poses a challenge in selecting geometric objects to use the lemma on.

Problem 2 (Proposed by user 0% on the Chinese Math Olympiad forum in 2022). *In $\triangle ABC$, let point H be the orthocenter and BE, CF be altitudes. Let K and M be the midpoints of segments AH and BC respectively. Show that the radical axis of $\odot AHM$ and $\odot BCK$ bisects segment EF .*

Proof. Let N be the midpoint of segment EF , and ω_1, ω_2 be the circumcircles of $\triangle AHM, \triangle BCK$ respectively. Then we need to prove that $f(N) = 0$, where $f(X) = \text{Pow}_{\omega_1}(X) - \text{Pow}_{\omega_2}(X)$ for all points X .

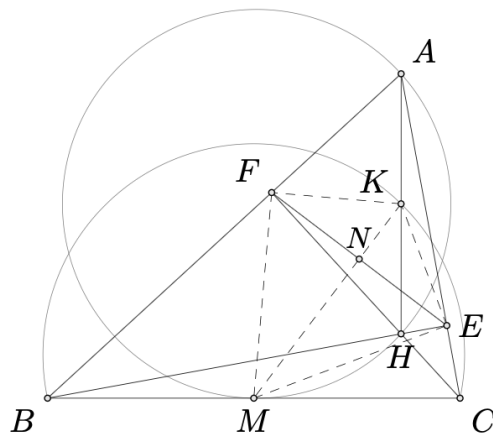


Figure 3: Problem 2 (Problem Proposed by 0% on the Chinese Math Olympiad forum in 2022)

If we write $f(N) = \frac{f(F)+f(E)}{2}$, then it is difficult to proceed because the quantities such as $\text{Pow}_{\omega_2}(E)$ are difficult to express. Alternatively, by Newton's theorem in the complete quadrilateral $ABHECF$, we get that N is on the line KM . Hence,

$$\begin{aligned} f(N) &= \frac{NK}{MK}f(M) + \frac{MN}{MK}f(K) \\ &= \frac{NK[0 - (-BM \cdot MC)] + MN(-AK \cdot KH - 0)}{MK} \\ &= \frac{NK \cdot BM \cdot MC - MN \cdot AK \cdot KH}{MN}. \end{aligned}$$

By properties of the median of right triangles' hypotenuse, we have

$$FK = KE = AK = KH, \quad FM = ME = BM - MC,$$

and

$$\begin{aligned} \angle FME &= \angle FMC - \angle EMC = 2B - (\pi - 2C) = \pi - 2A, \\ \angle FKE &= \angle FKH + \angle HKE = 2\angle FAH + 2\angle HAE = 2A. \end{aligned}$$

So

$$\begin{aligned} \frac{NK}{MN} &= \frac{[FKE]}{[FME]} = \frac{\frac{1}{2}FK \cdot KE \cdot \sin \angle FKE}{\frac{1}{2}FM \cdot ME \cdot \sin \angle FME} \\ &= \frac{AK \cdot KH \cdot \sin 2A}{BM \cdot MC \cdot \sin(\pi - 2A)} = \frac{AK \cdot KH}{BM \cdot MC}. \end{aligned}$$

Thus, $NK \cdot BM \cdot MC = MN \cdot AK \cdot KH$, implying that $f(N) = 0$. We are done. \square

4 Exercise Problems

Problem 3 (China Second Round Olympiad 2021 A1 Problem 1). In triangle ABC where $AB > AC$, internal points X and Y are on the angle bisector of $\angle BAC$ such that $\angle ABX = \angle ACY$. Line BX intersects line CY at point P and the circumcircles of $\triangle BPY$ and $\triangle CPX$, ω_1 and ω_2 , intersect at points Q and P . Show that A , P , and Q are collinear.

Problem 4 (China Girl's 2023 Problem 5). Let $\triangle ABC$ be an acute triangle with $AB < AC$, AH be an altitude, and G be the centroid. Let P, Q be the points of tangency of $\odot I$, the incircle of $\triangle ABC$, with AB, AC , respectively. Let M, N be the midpoints of PB, QC respectively. Let D, E be points on the incircle such that $\angle BDH + \angle ABC = 180^\circ$, $\angle CEH + \angle ACB = 180^\circ$. Prove that the lines MD, NE, HG are concurrent.

Problem 5 (Beijing High School League Preliminaries 2024 Problem 13). The three altitudes AD, BE , and CF of $\triangle ABC$ intersect at H . Construct the line $FG \parallel AC$ through F intersecting BC at G . Let the circumcircle of $\triangle CFG$ be $\odot O$, which intersects line AC at another point P . Construct line $PQ \parallel DE$ through P intersecting AD at Q . Prove that $OD = OQ$.

Problem 6 (Geometry Planet Olympiad 2024.2 Problem 6). Let $\triangle ABC$ be a triangle with I as its incenter. The incircle $\odot I$ touches sides BC, CA, AB at points D, E, F , respectively. The midpoints of segments DE, DF are points X, Y , respectively. Lines BX and CY intersect at point Z . Show that $\odot(XYZ)$ is tangent to $\odot I$.

5 Hints and Diagrams for the Exercise Problems

Problem 3. Prove that the power of A with respect to the two circles are equal.

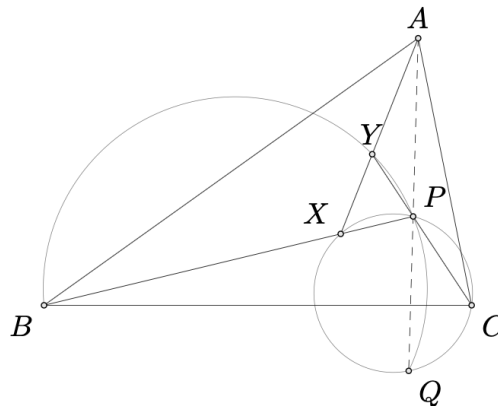


Figure 4: Problem 3 (China Second Round Olympiad 2021 A1 Problem 1)

Problem 4. Consider the radical center of the three circles shown below.

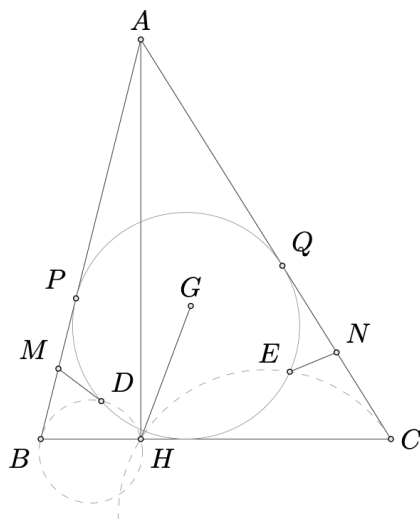


Figure 5: Problem 4 (China Girl's 2023 Problem 5)

Problem 5. Prove that $CHPQ$ is a cyclic quadrilateral, and denote its circumcircle by ω . Next, consider $f(X) = \text{Pow}_{\odot O}(X) - \text{Pow}_{\omega}(X)$.

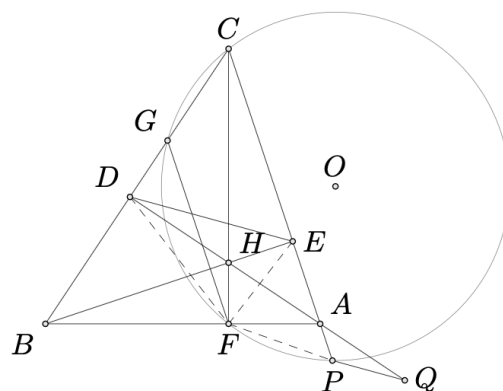


Figure 6: Problem 5 (Beijing High School League Preliminaries 2024 Problem 130)

Problem 6. For full solutions, refer to

<https://artofproblemsolving.com/community/c6h3256045p29901608>.

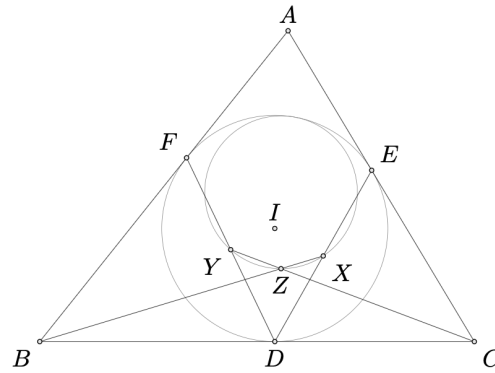
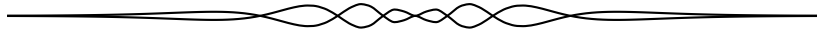


Figure 7: Problem 6 (GPO 2024.2 Problem 6)



PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 15, 2025**.

5041. *Proposed by Giuseppe Fera.*

If chords of an ellipse E subtend a constant angle at the focus O , prove that their envelope is an ellipse with axes parallel to those of E and O as focus.

5042. *Proposed by Michel Bataille.*

Let w be such that $w^5 = 1$ and $w \neq 1$. Find the complex roots of the polynomial

$$\sum_{1 \leq i < j < k \leq 4} (x + w^i + w^j + w^k)^3 - \sum_{1 \leq i < j \leq 4} (x + w^i + w^j)^3 + \sum_{1 \leq i \leq 4} (x + w^i)^3.$$

5043. *Proposed by Daniel Sitaru.*

Let $b \geq a \geq 1$. Prove that

$$\int_a^b \int_a^b \frac{dx dy}{1 + \sqrt{xy}} \leq (b - a) \ln\left(\frac{b+1}{a+1}\right).$$

5044. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Given a circle Γ with center O and a chord AB , let X be the midpoint of the larger arc AB , and C be an arbitrary point of that arc. Define K to be the point where the bisector of $\angle ACB$ intersects the tangent to Γ at B , while M is the intersection of AC and BX . Prove that the line MK contains the midpoint of AB .

5045. *Proposed by Nguyen Tuan Anh.*

Let $n \geq 2$ be a positive integer. Prove that

$$\sum_{k=0}^{n-2} \binom{n-k}{2} F_{k+1} = F_{n+5} - (n+4) - \binom{n+2}{2},$$

where F_n is the n -th Fibonacci number defined by

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}.$$

5046. *Proposed by Nguyen Viet Hung.*

Prove that for all positive real numbers x, y, z

$$\frac{x^3 + y^3}{x^2 + xy + y^2} + \frac{y^3 + z^3}{y^2 + yz + z^2} + \frac{z^3 + x^3}{z^2 + zx + x^2} \geq \frac{2(x^2 + y^2 + z^2)}{x + y + z}.$$

5047. *Proposed by Sicheng Du.*

Let real numbers a and b be such that $a^2 + b^2 \leq 1$. Find the minimum of

$$\sqrt{2 - 2a} + \sqrt{2 + a + \sqrt{3}b} + \sqrt{2 + a - \sqrt{3}b}.$$

5048. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\lim_{n \rightarrow \infty} \int_0^1 \sqrt[n]{\frac{1}{2} \ln \frac{1+x}{1-x} - x - \frac{x^3}{3} - \dots - \frac{x^{2n-1}}{2n-1}} dx.$$

5049. *Proposed by Xicheng Peng.*

In quadrilateral $ABCD$, line AD intersects line BC at point O . Letting $|XYZ|$ denote the area of $\triangle XYZ$, prove that

$$AB \cdot BC \cdot CD \cdot DA \cdot \sin(\angle BAD + \angle BCD) = 2(|ABC| - |DBC|)(OA \cdot OD - OB \cdot OC)$$

5050. *Proposed by Vasile Cîrtoaje.*

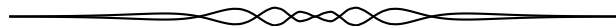
Let a, b, c, d be positive real numbers such that $a \geq b \geq 1 \geq c \geq d$ and $abcd = 1$. Prove that

$$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} + \frac{1}{2d+1} \geq \frac{4}{3}.$$

.....

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2025**.



5041. *Soumis par Giuseppe Fera.*

Si les cordes d'une ellipse E sous-tendent un angle constant au foyer O , montrez que leur enveloppe est une ellipse dont les axes sont parallèles à ceux de E et dont le foyer est O .

5042. *Soumis par Michel Bataille.*

Soit w tel que $w^5 = 1$ et $w \neq 1$. Trouvez les racines complexes du polynôme

$$\sum_{1 \leq i < j < k \leq 4} (x + w^i + w^j + w^k)^3 - \sum_{1 \leq i < j \leq 4} (x + w^i + w^j)^3 + \sum_{1 \leq i \leq 4} (x + w^i)^3.$$

5043. *Soumis par Daniel Sitaru.*

Soit $b \geq a \geq 1$. Montrez que

$$\int_a^b \int_a^b \frac{dx dy}{1 + \sqrt{xy}} \leq (b - a) \ln \left(\frac{b+1}{a+1} \right).$$

5044. *Soumis par Mihaela Berindeanu, modifié par le comité de rédaction.*

Étant donné un cercle Γ de centre O et de corde AB , notons X le milieu de l'arc le plus grand AB , et C un point arbitraire de cet arc. Définissons K comme le point où la bissectrice de $\angle ACB$ coupe la tangente à Γ en B , tandis que M est l'intersection de AC et BX . Montrez que la droite MK contient le milieu de AB .

5045. *Soumis par Nguyen Tuan Anh.*

Soit $n \geq 2$ un entier positif. Montrez que

$$\sum_{k=0}^{n-2} \binom{n-k}{2} F_{k+1} = F_{n+5} - (n+4) - \binom{n+2}{2},$$

où F_n est le n -ième nombre de Fibonacci défini par

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}.$$

5046. *Soumis par Nguyen Viet Hung.*

Montrez que pour tous les nombres réels positifs x, y et z ,

$$\frac{x^3 + y^3}{x^2 + xy + y^2} + \frac{y^3 + z^3}{y^2 + yz + z^2} + \frac{z^3 + x^3}{z^2 + zx + x^2} \geq \frac{2(x^2 + y^2 + z^2)}{x + y + z}.$$

5047. *Soumis par Sicheng Du.*

Soient des nombres réels a et b tels que $a^2 + b^2 \leq 1$. Trouvez le minimum de

$$\sqrt{2 - 2a} + \sqrt{2 + a + \sqrt{3}b} + \sqrt{2 + a - \sqrt{3}b}.$$

5048. *Soumis par Ovidiu Furdui et Alina Sîntămărian.*

Calculez

$$\lim_{n \rightarrow \infty} \int_0^1 \sqrt[n]{\frac{1}{2} \ln \frac{1+x}{1-x} - x - \frac{x^3}{3} - \dots - \frac{x^{2n-1}}{2n-1}} dx.$$

5049. *Soumis par Xicheng Peng.*

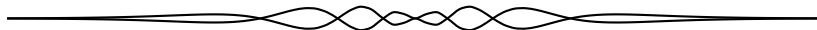
Dans le quadrilatère $ABCD$, la droite AD coupe la droite BC au point O . Sachant que $|XYZ|$ représente l'aire de $\triangle XYZ$, montrez que

$$AB \cdot BC \cdot CD \cdot DA \cdot \sin(\angle BAD + \angle BCD) = 2(|ABC| - |DBC|)(OA \cdot OD - OB \cdot OC)$$

5050. *Soumis par Vasile Cîrtoaje.*

Soient a, b, c et d des nombres réels positifs tels que $a \geq b \geq 1 \geq c \geq d$ et $abcd = 1$. Montrez que

$$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} + \frac{1}{2d+1} \geq \frac{4}{3}.$$



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2024: 50(10), 519-523.

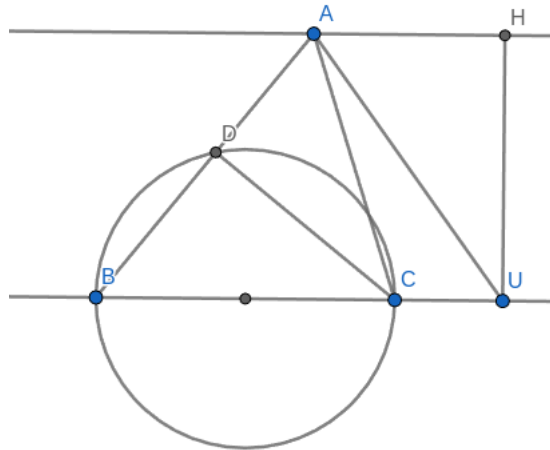
4991. Proposed by Michel Bataille.

Let ABC be a triangle with no right angle and let the line AB intersect the circle γ with diameter BC at D ($D \neq B$). Let U be the point of the line BC such that $\angle BAU = \angle BCA$, H the orthogonal projection of A onto the perpendicular to BC through U and E the point diametrically opposite to D on γ . Prove that AE and BH intersect in a point of γ and that $AB \cdot AD = AH \cdot BC$.

All 6 submissions were correct, and we feature the solution by Michal Adamaszek with some details added by the editor.

We begin by proving the second claim, namely

$$AB \cdot AD = AH \cdot BC. \quad (1)$$



Since CD is the altitude from vertex C of $\triangle ABC$ while HU equals the altitude from A , by computing its area in two ways we have

$$AB \cdot CD = BC \cdot HU. \quad (2)$$

From $AH \parallel BC$ we have

$$\angle BAU = \angle ACB = \angle CAH,$$

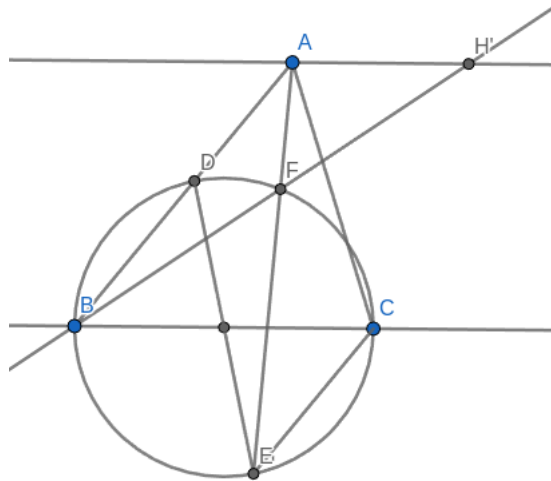
which implies $\angle UAH = \angle DAC$. Consequently, the right-angled triangles AHU and ADC are similar, and

$$\frac{AD}{CD} = \frac{AH}{UH}. \tag{3}$$

Multiplying together the equalities in (2) and (3) we get equation (1), as desired.

Later we will need to know that the vectors \overrightarrow{AH} and \overrightarrow{BC} are parallel when A is outside γ , while $\overrightarrow{AH} \parallel \overrightarrow{CB}$ when A is inside. To see this, fix the point D on γ and let A move along the line BD while the foot of the altitude from A , call it A' , moves along the line BC . Note that $\overrightarrow{AH} = \overrightarrow{A'U}$. When $A \in \gamma$, $D = A = H$ and $A' = U$, while the angle at C (namely $\angle BCA$) equals the complement of the angle at B . When A moves outside γ along BD , the angle at C increases beyond the complement of B , and therefore the $\angle BAU$ likewise increases, and $\overrightarrow{A'U}$ points in the same direction as \overrightarrow{BC} ; when A moves inside, the angle at C decreases, and, therefore, $\angle BAU$ likewise decreases, so that $\overrightarrow{A'U}$ points in the same direction as \overrightarrow{CB} , as claimed.

For the initial claim, let AE intersect γ again at F and let BF intersect the line AH (parallel to BC) at H' . We are to prove that H' coincides with H .



We have

$$\angle H'BA = \angle FBD = \angle FED = \angle AED.$$

Moreover, because both $AH' \parallel BC$ and $CE \parallel AB$,

$$\angle AH'B = \angle CBH' = \angle CBF = \angle CEF = \angle BAE,$$

so the triangles $AH'B$ and DAE are similar; therefore

$$\frac{AH'}{AB} = \frac{AD}{ED} = \frac{AD}{BC}.$$

It follows that

$$AB \cdot AD = AH' \cdot BC.$$

Comparing this equation with equation (1), we deduce that $AH' = AH$. It remains to show that the vectors \overrightarrow{AH} and $\overrightarrow{AH'}$ point in the same direction for us to conclude that, in fact, H' is the point H (so that BH and AE intersect at $F \in \gamma$, as desired). When A is outside γ , so that F lies between E and A , it therefore lies in the interior of the trapezoid $BCH'A$, whence the vectors $\overrightarrow{AH'}$ and \overrightarrow{BC} are parallel. When A is inside γ , it lies between E and F so that F is outside the trapezoid $BCAH'$ and therefore $\overrightarrow{AH'}$ and \overrightarrow{CB} are now parallel. This agrees with the direction of the vector \overrightarrow{AH} , thereby completing the proof.

Editor's comments. None of the submitted solutions mentioned the condition that $\triangle ABC$ contain no right angle; the requirement seems to be unneeded.

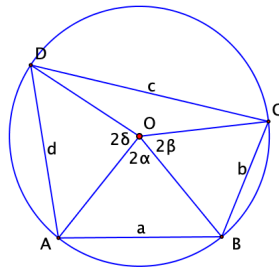
4992. Proposed by Michael Friday, modified by the Editorial Board.

Let $ABCD$ be a convex cyclic quadrilateral with side-lengths $a = AB$, $b = BC$, $c = CD$, $d = DA$ and circumradius R . If the circumcenter is within the quadrilateral, prove that

$$\sqrt{(4R^2 - a^2)(4R^2 - b^2)} + \sqrt{(4R^2 - c^2)(4R^2 - d^2)} = ab + cd.$$

All 16 submissions were correct, and we feature two of them. The first, by Prithwjit De, is typical of 10 of the submissions, while the second, by Bing Jian, is similar to another three.

Solution 1, by Prithwjit De.



Let O be the centre of the circle. Let $\angle AOB = 2\alpha$, $\angle BOC = 2\beta$, $\angle COD = 2\gamma$, $\angle DOA = 2\delta$. Then $a = 2R \sin \alpha$, $b = 2R \sin \beta$, $c = 2R \sin \gamma$, $d = 2R \sin \delta$ and

$$ab + cd = 4R^2(\sin \alpha \sin \beta + \sin \gamma \sin \delta).$$

Also,

$$4R^2 - a^2 = 4R^2(1 - \sin^2 \alpha) = 4R^2 \cos^2 \alpha, \text{ etc.}$$

so that

$$\sqrt{(4R^2 - a^2)(4R^2 - b^2)} + \sqrt{(4R^2 - c^2)(4R^2 - d^2)} = 4R^2(\cos \alpha \cos \beta + \cos \gamma \cos \delta).$$

Now, observe that

$$2(\alpha + \beta + \gamma + \delta) = 2\pi,$$

whence

$$\alpha + \beta = \pi - (\gamma + \delta);$$

This implies

$$\cos(\alpha + \beta) + \cos(\gamma + \delta) = 0,$$

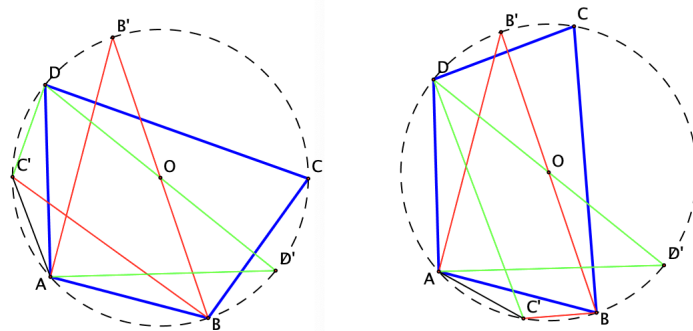
whence

$$\cos \alpha \cos \beta + \cos \gamma \cos \delta = \sin \alpha \sin \beta + \sin \gamma \sin \delta.$$

Multiplying both sides by $4R^2$ yields

$$\sqrt{(4R^2 - a^2)(4R^2 - b^2)} + \sqrt{(4R^2 - c^2)(4R^2 - d^2)} = ab + cd.$$

Solution 2 by Bing Jian.



Draw diameters BB', CC', DD' . Then in the resulting right triangles we have

$$\begin{aligned} AB' &= \sqrt{(4R^2 - a^2)}, & BC' &= \sqrt{(4R^2 - b^2)}, \\ DC' &= \sqrt{(4R^2 - c^2)}, & AD' &= \sqrt{(4R^2 - d^2)}. \end{aligned}$$

Because we have assumed that O is in the interior of $ABCD$, the vertex C must lie on the circumcircle opposite A strictly between B' and D' . Thus C' lies on the arc AB as in the diagram on the right, so that the quadrilateral $AB'BC'$ is convex while $AD'DC'$ is not, or it lies on DA (as on the left) so that $AD'DC'$ is convex while $AB'BC'$ is not. By Ptolemy's theorem applied to $AB'BC'$, we have

$$\sqrt{(4R^2 - a^2)(4R^2 - b^2)} = AB' \cdot BC' = AB \cdot B'C' \pm BB' \cdot AC' = ab \pm 2R \cdot AC',$$

with the top sign when C' is on the arc DA , and the bottom sign when on the arc AB . Similarly for the cyclic quadrilateral $AD'DC'$ (with the top sign when C' is on the arc DA , and the bottom sign when on the arc AB):

$$\sqrt{(4R^2 - c^2)(4R^2 - d^2)} = DC' \cdot AD' = DA \cdot C'D' \mp DD' \cdot AC' = cd \mp 2R \cdot AC'.$$

Adding the two equations, we see that in any case,

$$\sqrt{(4R^2 - a^2)(4R^2 - b^2)} + \sqrt{(4R^2 - c^2)(4R^2 - d^2)} = ab + cd.$$

Editor's comments. Walther Janous observed that by cyclically permuting the edge lengths in the equation we have twice established (that is, substituting b for a , c for b , ...) you get a companion equation,

$$\sqrt{(4R^2 - b^2)(4R^2 - c^2)} + \sqrt{(4R^2 - d^2)(4R^2 - a^2)} = bc + da.$$

By adding these two equations, after a bit of algebra we arrive at the “amazing relation”

$$\frac{\sqrt{(4R^2 - a^2)} + \sqrt{(4R^2 - c^2)}}{a + c} \cdot \frac{\sqrt{(4R^2 - b^2)} + \sqrt{(4R^2 - d^2)}}{b + d} = 1.$$

4993. *Proposed by Tatsunori Irie.*

Let n be an integer such that $n \geq 2$. Show that

$$\sum_{k=1}^n \sec^{2n} \left(\frac{k\pi}{2n+1} \right) \geq \frac{n^n}{(n-1)^{n-1}} \binom{2n+1}{2}.$$

There were 7 correct solutions submitted. Solution 1 was similar to two other solutions. The second is a stronger result.

Solution 1, by Sicheng Du.

Let $x_k = k\pi/(2n+1)$. Then

$$(\cos x_k + i \sin x_k)^{2n+1} = \cos(2n+1)x_k + i \sin(2n+1)x_k = \cos k\pi$$

and

$$\begin{aligned} (\cos x_k + i \sin x_k)^{2n+1} &= \sum_{j=0}^n \binom{2n+1}{2j} \cos^{2n-2j} x_k \sin^{2j} x_k \\ &\quad + i \sum_{j=0}^n (-1)^j \binom{2n+1}{2j+1} \cos^{2n-2j-1} x_k \sin^{2j+1} x_k. \end{aligned}$$

Equating the imaginary parts of these expressions yields

$$0 = \tan x_k \cos^{2n} x_k \sum_{j=0}^n (-1)^j \binom{2n+1}{2j+1} \tan^{2j} x_k.$$

Hence $\tan^2 x_k$ ($1 \leq k \leq n$) are the roots of the polynomial

$$\sum_{j=0}^n (-1)^j \binom{2n+1}{2j+1} t^j$$

and the sum $\sum_{k=1}^n \tan^{2k} x_k$ of these roots is

$$\binom{2n+1}{2n-1} = \binom{2n+1}{2} = n(2n+1).$$

Applying the arithmetic-geometric means inequality, we find that

$$\begin{aligned} \sum_{k=1}^n \sec^{2n} x_k &= \sum_{k=1}^n \left(\tan^2 x_k + (n-1) \frac{1}{n-1} \right)^n \\ &\geq \sum_{k=1}^n \left[n \left(\tan^2 x_k (n-1)^{-(n-1)} \right)^{1/n} \right]^n \\ &= n^n (n-1)^{-(n-1)} \sum_{k=1}^n \tan^2 x_k \\ &\geq n^n (n-1)^{-(n-1)} \binom{2n+1}{2}, \end{aligned}$$

as desired.

Solution 2, by Walther Janous.

By the power means inequality for $n \geq 2$, we have that

$$\begin{aligned} \left(\frac{1}{n} \sum_{k=1}^n (\sec^2 x_k)^n \right)^{1/n} &\geq \frac{1}{n} \sum_{k=1}^n \sec^2 x_k = \frac{1}{n} \left(\sum_{k=1}^n (\tan^2 x_k + 1) \right) \\ &= \frac{1}{2} (n(2n+1) + n) = 2(n+1), \end{aligned}$$

whence

$$\sum_{k=1}^n \sec^{2n} x_k \geq 2^n n (n+1)^n.$$

Note that

$$\frac{2^n n (n+1)^n}{n^n (n-1)^{-(n-1)} n (2n+1)} = \frac{2^n (n^2-1)^{n-1} (n+1)}{n^n (2n+1)}$$

exceeds 1 when $n \geq 2$. The desired result follows.

Comments by the editor. There is a reason that the lower bound found by Janous is much more generous than the one proposed. Three solvers pointed out that, if we replace the left side by the single term $\sec^{2n} x_n$, the inequality is still valid. From the fact that $(n/(n-1))^{-(n-1)} < e < 3$, the right side is less than $3n^2(2n+1)$. On the other hand, since $\sec(n\pi/(2n+1))$ increases with n , for $n \geq 2$, the left side is greater than

$$\sec^{2n} 2\pi/5 = (1 + \sqrt{5})^{2n} > 3^{2n}.$$

4994. Proposed by Vasile Córtoaje.

Let a, b, c, d be nonnegative real numbers such that $ab + bc + cd + da \geq 4$. Prove that

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) \geq (a + b + c + d)^2.$$

We received 10 submissions, all correct. We present the solution by Sicheng Du.

Let $a + c = x$ and $b + d = y$, then we have $xy \geq 4$. Without loss of generality, let $x \geq y$, then $x \geq 2$. The desired inequality can be written as

$$[(a + c)^2 + (1 - ac)^2] [(b + d)^2 + (1 - bd)^2] \geq (x + y)^2.$$

If $y \geq 2$, then

$$\text{LHS} \geq (a + c)^2(b + d)^2 = (xy)^2 \geq (x \cdot 2)^2 \stackrel{x \geq y}{\geq} (x + y)^2.$$

If $y < 2$, then by AM-GM inequality, $bd \leq \frac{1}{4}y^2 < 1$, so

$$\begin{aligned} \text{LHS} &\geq x^2 \left[y^2 + \left(1 - \frac{y^2}{4}\right)^2 \right] \\ &= x^2 \left(1 + \frac{y^2}{4}\right)^2 = \left(x + \frac{xy^2}{4}\right)^2 \\ &= \left(x + \frac{xy}{4} \cdot y\right)^2 \stackrel{xy \geq 4}{\geq} (x + y)^2. \end{aligned}$$

Editor's comments. The majority of solvers found some variant of the featured elementary solution, which was by far the most succinct. It is also possible to use Lagrange multipliers, but that approach is very tedious. A few solvers remarked that equality holds just if

$$a = c = \frac{2}{b + d} \text{ and } bd = 1$$

or vice versa (swapping $a \leftrightarrow b$ and $c \leftrightarrow d$ simultaneously); note this condition contains the special case $a = b = c = d = 1$. The key inequality

$$(p^2 + 1)(q^2 + 1) \geq (p + q)^2$$

is immediate from the identity

$$(p^2 + 1)(q^2 + 1) = (p + q)^2 + (1 - pq)^2$$

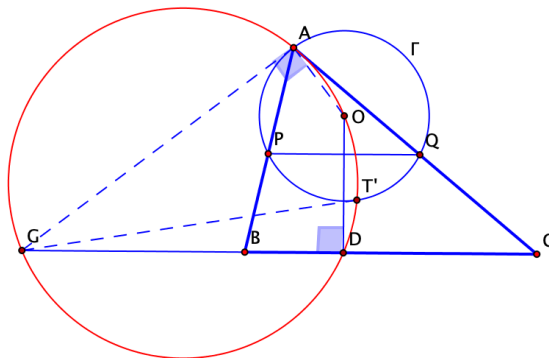
but also follows from Cauchy–Schwarz applied to the vectors $(p, 1)$ and $(1, q)$.

4995. *Proposed by Boriu Wang, Yaozhong Xu and Ruiqi Yan, modified by the Editorial Board.*

Let $\triangle ABC$ be a triangle such that $AB \neq AC$. Let P be a point on the line AB , with Q on AC such that $PQ \parallel BC$; let Γ be the circumcircle of $\triangle APQ$ and denote its center by O . The circumcircles of $\triangle ABQ$ and $\triangle ACP$ intersect at $S \neq A$. The point D lies on BC , and $OD \perp BC$. Prove that the circumcircle of $\triangle OSD$ intersects Γ at A and a second point, call it T , for which both circles (BAC) and (BTC) are tangent to Γ .

The final sentence of the problem was garbled; it should read, Prove that the circle (OSD) intersects Γ at A and at a second point, call it T , the points at which Γ is tangent to (ABC) and (BTC) , respectively. The four submissions we received were all correct; we feature the proposer's solution with some details added by the editor.

We assume without loss of generality that $AB < AC$ so that the tangent to Γ at A must intersect BC at some point, call it G , on the same side of D as B ; denote by T' the point where the second tangent from G touches Γ .



Because

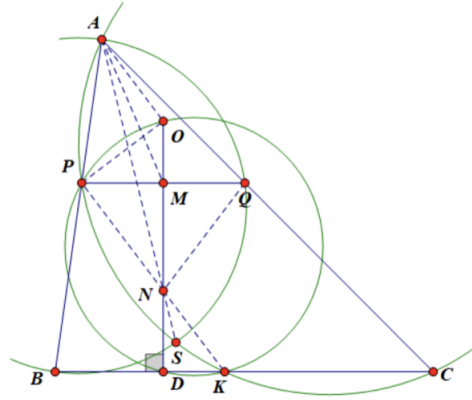
$$\angle OAG = \angle OT'G = \angle ODG = 90^\circ,$$

the points A, O, T', D, G all lie on the circle whose diameter is OG . We next prove that S also lies on this circle, which implies that $T' = T$ (because T was defined to be the point other than A where the circles OSD and Γ intersect).

Proof that S lies on the circle (AOD) .

As in the accompanying figure, let M be the midpoint of PQ , N the intersection of the tangents to Γ at P and Q , and K be the intersection of PN with BC . Denote by S' the second point where AN intersects the circle (ABQ) ; we first show that S' is also on the circle (ACP) so that in fact, $S' = S$. Because OD is the perpendicular bisector of PQ , $N \in OD$; furthermore, because AN must be a symmedian of $\triangle APQ$, we have

$$\angle MAQ = \angle PAN = \angle PAS'.$$



Consequently,

$$\begin{aligned}
 \frac{BS'}{S'Q} &= \frac{\sin \angle BQS'}{\sin \angle S'BQ} && \text{(Law of Sines applied to } \triangle BS'Q) \\
 &= \frac{\sin \angle BAS'}{\sin \angle S'AQ} && (BS'QA \text{ cyclic}) \\
 &= \frac{\sin \angle MAQ}{\sin \angle PAM} && (\angle MAQ = \angle PAN = \angle PAS' = \angle BAS') \\
 &= \frac{AP}{AQ} && \text{(Law of Sines applied to triangles } PAM, MAQ).
 \end{aligned}$$

Because $PQ \parallel BC$, it follows that

$$\frac{BS'}{S'Q} = \frac{AP}{AQ} = \frac{BP}{CQ}.$$

Moreover, because $\angle S'BP = \angle S'BA = \angle S'QC$, we have the triangles $S'PB$ and $S'CQ$ are similar (by side-angle-side), which means that $\angle S'PB = \angle S'CQ = \angle S'CA$. Consequently A, P, S', C are concyclic so that $S' = S$ as claimed. Observe that as a further consequence, S is on the line AN .

We also have

- O, P, D, K are concyclic (because of the right angles at D and P , these points are on the circle whose diameter is OK).
- The circle $(APCS)$ also contains the point K because $\angle PKB = \angle PAC$ as follows: because $\angle POQ$ at the center of Γ equals twice the inscribed angle $\angle PAQ = \angle PAC$, while $\angle POM$ (in the right triangle PON) equals $\angle NPM$ and (because the lines MP and KB are parallel) we have that $\angle PKB = \angle KPM = \angle NPM$; in summary,

$$\angle PKB = \angle KPM = \angle NPM = \angle POM = \frac{1}{2} \angle POQ = \angle PAC.$$

The powers of N with respect to the circles $(OPDK)$ and $(APSK)$ equal

$$ND \cdot NO = NK \cdot NP = NA \cdot NS;$$

thus, we conclude that S is on the circle (AOD) , as claimed.

It remains to investigate the tangency claims. Because the triangles APQ and ABC are homothetic, their circumcircles Γ and (ABC) must be tangent at their common point A . Finally, observe that G is the radical center of the three circles $\Gamma(= (APQT))$, (ABC) , and (BTC) ; in other words, the power of G with respect to the last pair of circles, namely $GB \cdot GC$, equals its power with respect to the first two, namely GA^2 , which must then equal its power with respect to the first and third. But the power of G with respect to the first is GT^2 . We conclude that $GT^2 = GB \cdot GC$, which implies that GT is a common tangent to the circles (BTC) and Γ , as desired.

4996. *Proposed by Yagub Aliyev.*

Let P_α be power mean of non-negative real numbers x_1, \dots, x_n :

$$P_\alpha = \left(\frac{\sum_{i=1}^n x_i^\alpha}{n} \right)^{\frac{1}{\alpha}},$$

where $\alpha \neq 0$. If $\alpha = 0$, then we assume that $P_\alpha = G_n$. Find

$$\lim_{(x_1, x_2, \dots, x_n) \rightarrow (x, x, \dots, x)} \frac{P_\alpha - P_\beta}{P_\gamma - P_\delta},$$

where $x > 0$, α , β , γ , and δ are real numbers and $\gamma \neq \delta$.

We received 2 correct and 1 incorrect solutions for this problem. The following is based on the solution by Theo Koupelis and was revised by The Editorial Board to contain the remainder terms in Peano's form and the homogeneity argument at the end.

Let $x_i = x + \omega_i$, where x is a non-negative real number and ω_i are real numbers for all $i = 1, \dots, n$. As $x_i \rightarrow x$ we get $\omega_i \rightarrow 0$. Thus, as $x_i \rightarrow x$, using the expansion $(1+t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2} \cdot t^2 + o(t^2)$ for $t \rightarrow 0$, we get

$$\begin{aligned} \sum_{i=1}^n x_i^\alpha &= \sum_{i=1}^n (x + \omega_i)^\alpha = x^\alpha \sum_{i=1}^n \left(1 + \frac{\omega_i}{x} \right)^\alpha \\ &= x^\alpha \sum_{i=1}^n \left(1 + \alpha \cdot \frac{\omega_i}{x} + \frac{\alpha(\alpha-1)}{2} \cdot \frac{\omega_i^2}{x^2} + o(\omega_i^2) \right) \\ &= x^\alpha \left(n + \alpha \cdot \frac{\sum_i \omega_i}{x} + \frac{\alpha(\alpha-1)}{2} \cdot \frac{\sum_i \omega_i^2}{x^2} + o\left(\sum_i \omega_i^2 \right) \right), \end{aligned}$$

and therefore

$$\begin{aligned} P_\alpha &= x \left[1 + \alpha \left(\frac{\sum_i \omega_i}{nx} + \frac{\alpha - 1}{2} \cdot \frac{\sum_i \omega_i^2}{nx^2} + o \left(\sum_i \omega_i^2 \right) \right) \right]^{\frac{1}{\alpha}} \\ &= x \left[1 + \frac{\sum_i \omega_i}{nx} + \frac{\alpha - 1}{2n^2 x^2} \cdot \left((n-1) \sum_i \omega_i^2 - 2 \sum_{i,j,i < j} \omega_i \omega_j \right) + o \left(\sum_i \omega_i^2 \right) \right], \end{aligned}$$

with similar expressions for $P_\beta, P_\gamma, P_\delta$. For the case $\alpha = 0$ we get

$$\begin{aligned} G_n^\alpha &= x^n \left(1 + \frac{\omega_1}{x} \right) \left(1 + \frac{\omega_2}{x} \right) \cdots \left(1 + \frac{\omega_n}{x} \right) \\ &= x^n \left(1 + \frac{\sum_i \omega_i}{x} + \frac{\sum_{i,j,i < j} \omega_i \omega_j}{x^2} + o \left(\sum_i \omega_i^2 \right) \right), \end{aligned}$$

and thus

$$\begin{aligned} G_n &= x \left[1 + \left(\frac{\sum_i \omega_i}{x} + \frac{\sum_{i,j,i < j} \omega_i \omega_j}{x^2} + o \left(\sum_i \omega_i^2 \right) \right) \right]^{\frac{1}{n}} \\ &= x \left[1 + \frac{\sum_i \omega_i}{nx} - \frac{1}{2n^2 x^2} \cdot \left((n-1) \sum_i \omega_i^2 - 2 \sum_{i,j,i < j} \omega_i \omega_j \right) + o \left(\sum_i \omega_i^2 \right) \right]. \end{aligned}$$

Since the function $\frac{P_\alpha - P_\beta}{P_\gamma - P_\delta}$ is homogeneous with respect to x_i , its values are constant on rays through the origin. Therefore it suffices to consider its behavior on the portion of the plane $\sum_i x_i = nx$, where $x_i \geq 0$. Then $\sum_i \omega_i = 0$ and consequently

$$(n-1) \sum_i \omega_i^2 - 2 \sum_{i,j,i < j} \omega_i \omega_j = n \sum_i \omega_i^2 - \left(\sum_i \omega_i \right)^2 = n \sum_i \omega_i^2.$$

Thus, in all cases we get

$$\lim_{(x_1, x_2, \dots, x_n) \rightarrow (x, x, \dots, x)} \frac{P_\alpha - P_\beta}{P_\gamma - P_\delta} = \frac{\alpha - \beta}{\gamma - \delta}.$$

Editor's note. Another proof based on Multivariable Taylor's formula appear as Lemma 2.1 in preprint paper [1], where one can find many references with related questions. The homogeneity argument above is similar to the one used in [2], where the case $\alpha = 1, \beta = 0, \gamma = 1$, and $\delta = -1$ was discussed. Note also that we can not replace the limit with one variable limit $\lim_{t \rightarrow 0} \frac{P_\alpha - P_\beta}{P_\gamma - P_\delta}$, by taking $x_i = x + \phi_i t$, where not all ϕ_i ($i = 1, 2, \dots, n$) are equal. It is well known that there are multivariable functions for which the limits at a point in all possible directions are equal but the multivariable limit itself does not exist. Consider for example the limit of the function $f(x, y) = \frac{2x^2 y}{x^4 + y^2}$ at the point $(0, 0)$. Its limit on

any ray through the origin is 0, but on the path $y = x^2$ the limit is 1, and therefore the multivariable limit does not exist [3, Ch. 14, p. 781-782]. Walther Janous mentioned Joel Lee Brenner (1912 – 1997) who was interested in the differences and ratios of power means (see e.g. [4]). J. L. Brenner was also one of the contributors of *CruX* with proposed problems 495 and 843.

[1] Yagub Aliyev, The extremal values of the ratio of differences of power mean, arithmetic mean, and geometric mean, Arxiv, 2024. <https://arxiv.org/abs/2405.11947>

[2] N. Lord, 85.17 More on the Relative Location of Means II, *The Mathematical Gazette*, Vol. 85, No. 502 (Mar., 2001), 114-116. <https://doi.org/10.2307/3620489>

[3] M. D. Weir, J. Hass, G. B. Thomas, *Calculus: early transcendentals*, 12th ed., Addison-Wesley, 2010.

[4] J. L. Brenner, Limits of means for large values of the variables, *Pi Mu Epsilon Journal*, 8 (1985), 160-163. <https://www.pme-math.org/journal/issues/PMEJ.Vol.8.No.3.pdf>

4997. Proposed by Richdad Phuc and Leonard Giugiuc.

Let a, b, c be nonnegative real numbers such that $a \geq 1 \geq b \geq c$ and $a + b + c = 3$. Prove the inequality

$$a^4 + b^4 + c^4 + 14\sqrt{abc} \geq 17.$$

When does the equality hold?

We received 9 submissions and 8 of them were complete and correct. We feature the following two solutions.

We shall prove the stronger inequality that

$$a^4 + b^4 + c^4 + 14(abc)^{6/7} \geq 17, \quad (1)$$

where the equality holds if and only if $(a, b, c) = (1, 1, 1)$ or $(a, b, c) = (2, 1, 0)$. To justify that this is indeed stronger, note that the AM-GM inequality implies $abc \leq 1$ and thus $(abc)^{6/7} \leq \sqrt{abc}$. Thus, we deduce immediately that

$$a^4 + b^4 + c^4 + 14\sqrt{abc} \geq 17,$$

where the equality holds if and only if $(a, b, c) = (1, 1, 1)$ or $(a, b, c) = (2, 1, 0)$. Next, we present two proofs of inequality (1).

Solution 1, by Adamaszek Michal and Borche Joshevski (independently).

Set $b = 1 - x, c = 1 - y$, and $a = 1 + x + y$ for some $0 \leq x \leq y \leq 1$. Then

$$\begin{aligned} a^4 &= 1 + 4(x + y) + 6(x + y)^2 + 4(x + y)^3 + (x + y)^4, \\ b^4 &= 1 - 4x + 6x^2 - 4x^3 + x^4, \quad c^4 = 1 - 4y + 6y^2 - 4y^3 + y^4, \\ abc &= (1 + x + y)(1 - x)(1 - y) = 1 + x^2y + xy^2 - x^2 - xy - y^2. \end{aligned}$$

It follows that

$$\begin{aligned} a^4 + b^4 + c^4 &= 3 + 6(x+y)^2 + 6x^2 + 6y^2 + 12(x^2y + xy^2) + (x+y)^4 + x^4 + y^4 \\ &= 3 + 12(x^2 + xy + y^2) + 12(x^2y + xy^2) + 2(x^2 + xy + y^2)^2. \end{aligned}$$

Let $u = x^2 + xy + y^2$ and $v = x^2y + y^2x$. We have $u, v \geq 0$ and $abc = 1 + v - u \geq 0$. It suffices to show

$$2u^2 + 12(u+v) + 3 + 14(1+v-u)^{6/7} \geq 17.$$

If $v > u$, then we are done. Next we assume that $u \geq v$ and write $u = v + t$ for some $t \in [0, 1]$. Then the inequality becomes

$$v^2 + 2tv + 12v + t^2 + 6t + 7(1-t)^{6/7} \geq 7. \quad (2)$$

Since $t, v \geq 0$, it suffices to show that

$$t^2 + 6t + 7\sqrt[6]{1-t} \geq 7 \iff (t+7)(t-1) + 7(1-t)^{6/7} \geq 0,$$

equivalently, the function $g(t) = (t+7)(1-t)^{1/7}$ satisfies $g(t) \leq 7$ for $t \in [0, 1]$. Indeed, g is decreasing in $[0, 1]$ since

$$g'(t) = (1-t)^{1/7} - \frac{(t+7)}{7(1-t)^{6/7}} = (1-t)^{1/7} \left(1 - \frac{t+7}{7(1-t)} \right) = \frac{8t(1-t)^{1/7}}{7(t-1)}.$$

In particular, $g(t) \leq g(0) = 7$. Finally, note that inequality (2) is an equation precisely when $v = 0$ and $t \in \{0, 1\}$, that is, $x = 0$ and $y \in \{0, 1\}$, equivalently, when $(a, b, c) = (1, 1, 1)$ and $(a, b, c) = (2, 1, 0)$.

Solution 2, by Vivek Mehra and the proposer (independently).

Let $p = abc$ and $q = ab + bc + ca$. It is given that $a + b + c = 3$. Observe that a, b, c are the roots of the polynomial

$$f(x) = (x-a)(x-b)(x-c) = x^3 - 3x^2 + qx - p,$$

with $f(1) \leq 0$ since $a \geq 1 \geq b \geq c$. It follows that $q \leq p + 2 \leq 3$. Also, we have

$$\begin{aligned} a^4 + b^4 + c^4 &= (a^2 + b^2 + c^2)^2 - 2(a^2b^2 + b^2c^2 + c^2a^2) \\ &= (9 - 2q)^2 - 2(q^2 - 6p) = 2q^2 - 36q + 81 + 12p. \end{aligned}$$

If $q < 2$, then $a^4 + b^4 + c^4 \geq 2q^2 - 36q + 81 > 17$ and we are done.

Next, assume that $2 \leq q \leq 3$. Since $p \geq q - 2$, it suffices to show

$$2q^2 - 36q + 81 + 12(q-2) + 14(q-2)^{6/7} - 17 \geq 0,$$

equivalently,

$$(q-2)(q-10) + 7(q-2)^{6/7} \geq 0 \iff 7 \geq (q-2)^{1/7}(10-q).$$

Now we can proceed as in Solution 1.

Editor's Comment. Borche Joshevski pointed out that that the exponent $6/7$ is optimal. More precisely, for each $\theta > 6/7$, there are a, b, c with $a \geq 1 \geq b \geq c \geq 0$ and $a + b + c = 3$ such that $a^4 + b^4 + c^4 + 14(abc)^\theta < 17$. This can be seen by choosing $a = 1 + x$, $b = 1$, and $c = 1 - x$ with $x > 0$ sufficiently small; in which case we have

$$a^4 + b^4 + c^4 + 14(abc)^\theta - 17 = 12x^2 + 2x^4 + 14(1 - x^2)^\theta - 14.$$

Indeed, when $\theta \in (6/7, 1)$, Bernoulli's inequality implies that

$$12x^2 + 2x^4 + 14(1 - x^2)^\theta - 14 < 12x^2 + 2x^4 + 14(1 - \theta x^2) - 14 = (12 - 14\theta)x^2 + 2x^4 < 0$$

for $x > 0$ sufficiently small; when $\theta \geq 1$, we have

$$12x^2 + 2x^4 + 14(1 - x^2)^\theta - 14 \leq 12x^2 + 2x^4 + 14(1 - x^2) - 14 = 2x^4 - 2x^2 < 0$$

for $x > 0$ sufficiently small.

4998. *Proposed by George Apostolopoulos.*

The diagonals of a convex quadrilateral $ABCD$ meet at point O . The distances from O on the lines AB, BC, CD and DA are equal to x, y, z and t , respectively. If $AB + CD > BC + DA$, prove that

$$\frac{1}{x} + \frac{1}{z} > \frac{1}{y} + \frac{1}{t}.$$

There were 5 solutions submitted, three of which were correct and two were incomplete. We present the solution from Giuseppe Fera.

Let $a = AB$, $b = BC$, $c = CD$, $d = DA$, $p = AO$, $q = BO$, $r = CO$, $s = DO$, $\theta = \angle AOB = \angle COD$, $\alpha = [AOB]$, $\beta = [BOC]$, $\gamma = [COD]$ and $\delta = [DOA]$, where $[\dots]$ denotes the area. The hypothesis is that $a + c > b + d$.

We have $2\alpha = ax$, $2\beta = by$, $2\gamma = cz$, $2\delta = dt$, $\alpha/\delta = q/s = \beta/\gamma$, $\alpha/\beta = p/r = \delta/\gamma$, whereupon $\alpha\gamma = \beta\delta$. Also

$$\begin{aligned} a^2 &= p^2 + q^2 - 4\alpha \cot \theta; & b^2 &= q^2 + r^2 + 4\beta \cot \theta; \\ c^2 &= r^2 + s^2 - 4\gamma \cot \theta; & d^2 &= s^2 + p^2 + 4\delta \cot \theta. \end{aligned}$$

Note that

$$\left(\frac{1}{x} + \frac{1}{z}\right)^2 = \left(\frac{1}{4\alpha\gamma}\right) \alpha\gamma \left(\frac{a}{\alpha} + \frac{c}{\gamma}\right)^2$$

with an analogous expression for $((1/y) + (1/t))^2$. Then

$$\begin{aligned} & \alpha\gamma \left(\frac{a}{\alpha} + \frac{c}{\gamma} \right)^2 \\ &= \frac{\gamma a^2}{\alpha} + \frac{\alpha c^2}{\gamma} + 2ac \\ &= \left(\frac{p^2 + q^2 - 4\alpha \cot \theta}{\alpha} \right) \gamma + \left(\frac{r^2 + s^2 - 4\gamma \cot \theta}{\gamma} \right) \alpha + 2ac \\ &= \frac{(p^2 + q^2)\gamma^2 + (r^2 + s^2)\alpha^2}{\alpha\gamma} - 4(\alpha + \gamma) \cot \theta + 2ac \\ &= \left(\frac{1}{\alpha\gamma} \right) (p^2\gamma^2 + q^2\gamma^2 + r^2\alpha^2 + s^2\alpha^2) + (a^2 + c^2) - (p^2 + q^2 + r^2 + s^2) + 2ac. \end{aligned}$$

Likewise

$$\beta\delta \left(\frac{b}{\beta} + \frac{d}{\delta} \right)^2 = \left(\frac{1}{\beta\delta} \right) (q^2\delta^2 + r^2\delta^2 + s^2\beta^2 + p^2\beta^2) + (b^2 + d^2) - (p^2 + q^2 + r^2 + s^2) + 2bc.$$

Noting that $\alpha\gamma = \beta\delta$, $p\gamma = r\delta$, $q\gamma = s\beta$, $r\alpha = p\beta$ and $s\alpha = q\delta$, we find that

$$\begin{aligned} & \left(\frac{1}{x} + \frac{1}{z} \right)^2 - \left(\frac{1}{y} + \frac{1}{t} \right)^2 \\ &= \frac{1}{4} \left(\left(\frac{a}{\alpha} + \frac{c}{\gamma} \right)^2 - \left(\frac{b}{\beta} + \frac{d}{\delta} \right)^2 \right) \\ &= (4\alpha\gamma)^{-1} (a^2 + c^2 + 2ac - b^2 - d^2 - 2bd) = (4\alpha\gamma)^{-1} ((a + c)^2 - (b + d)^2). \end{aligned}$$

Since $a + c > b + d$, the desired result follows.

Comments by the editor. A quadrilateral is said to be *tangential* if and only if it has an incircle. Vivek Mehra points out that, by Pitot's theorem, $ABCD$ is tangential if and only if $a + c = b + d$. Mehra also notes Problem 3 from Day 1 of the German National Mathematical Olympiad in 2009 that required candidates to prove that $ABCD$ is tangential if and only if $(1/x) + (1/z) = (1/y) + (1/t)$. Thus

$$a + c = b + d \iff \frac{1}{x} + \frac{1}{z} = \frac{1}{y} + \frac{1}{t}.$$

4999. *Proposed by Hung Nguyen Viet.*

Let a, b, c be pairwise distinct real numbers such that

$$\frac{(1 - a^2)(1 - b^2)}{(a - b)^2} + \frac{(1 - b^2)(1 - c^2)}{(b - c)^2} + \frac{(1 - c^2)(1 - a^2)}{(c - a)^2} = 24.$$

Evaluate

$$\frac{1 - ab}{a - b} + \frac{1 - bc}{b - c} + \frac{1 - ca}{c - a}.$$

There were 10 correct solutions, all essentially the same.

Let

$$(x, y, z) = \left(\frac{1-ab}{a-b}, \frac{1-bc}{b-c}, \frac{1-ca}{c-a} \right).$$

Since the left side of the inequality can be written as $(x^2 - 1) + (y^2 - 1) + (z^2 - 1)$, we have $x^2 + y^2 + z^2 = 27$. Since $xy + yz + zx$ is equal to

$$\begin{aligned} & (1-ab)(1-bc)(c-a) + (1-bc)(1-ca)(a-b) + (a-ca)(1-ab)(b-c) \\ &= a^2b - a^2c + b^2c - b^2a + c^2a - c^2b \\ &= -(a-b)(b-c)(c-a) \end{aligned}$$

divided by $(a-b)(b-c)(c-a)$, we have $xy + yz + zx = -1$. It follows that $(x + y + z)^2 = 27 - 2 = 25$, so that $x + y + z = \pm 5$.

We show that both values are possible. Let $c = 1$. Then, for the condition to be satisfied, $y^2 = z^2 = 1$ and $x^2 = 25$. We can achieve this when either $(a, b, c) = (0, 1/5, 1)$ and $(x, y, z) = (-5, -1, 1)$, or $(a, b, c) = (0, -1/5, 1)$ and $(x, y, z) = (5, -1, 1)$.

Comments from the editor. Two solvers did not provide an example to indicate that the situation was possible. Others gave the example in the solution, and one supplied $(a, b, c) = (1, 4, 19)$ for a value of 5 and $(a, b, c) = (4, 1, 19)$ for -5 .

5000. *Proposed by Bill Sands. Dedicated in memoriam to Andy Liu.*

There is a straight row of vertical cylinders stretching in both directions, each of radius 1 metre, and equally spaced at a distance of $s > 2$ metres apart (centre to centre). You are standing at a point $d > 1$ metres from the line through the centres of the cylinders. From your position, you can see a number of the cylinders completely, but eventually the cylinders in both directions become partly covered by cylinders closer to you.

- Show that you can see at most $2 \lceil \frac{d+1}{2} \rceil$ complete cylinders, that is, not partly obscured by other cylinders.
- Suppose that d is an integer and that $s > \frac{2d}{\sqrt{2d-1}}$. Show that you can see at least $2 \lceil \frac{d-1}{2} \rceil$ complete cylinders.
- Suppose in addition that the perpendicular drawn from you to the line of centres hits that line at a point exactly halfway between two neighbouring cylinders. How many cylinders can you see completely?

We only received the proposer's solution. The solution presented is based on the ideas from the proposer's.

First we observe that we can restrict ourselves to two dimensions such that the cylinders become evenly spaced unit circles. Let l be the line through the centres of the circles and let O be the point on l closest to you.

Lemma. A circle whose centre is at distance $w > s$ from O can be seen completely by you if and only if

$$d \geq \frac{2w - s}{\sqrt{s^2 - 4}}.$$

Proof. Let P be the centre of a unit circle on l with $OP = w$ and let Q be the centre of the next circle from P in the direction of O . Let R be the midpoint of PQ . Consider a tangent through R and the unit circles around P and Q . Let S be its intersection with the line through O and your location and T its intersection with the unit circle around Q . See the figure below.

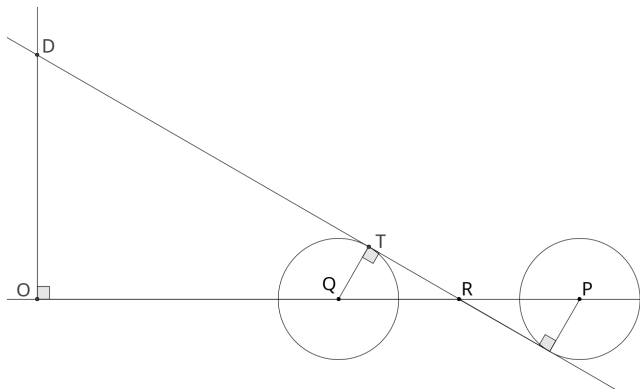
From the figure on the next page it is apparent that we can see the unit circle around P completely if and only if $d \geq DO$. We know that $PR = QR = \frac{s}{2}$ and $QT = 1$. Therefore $RT = \sqrt{\frac{s^2}{4} - 1}$. Since $\triangle RQT$ and $\triangle RSO$ are similar we obtain

$$\frac{DO}{OR} = \frac{QT}{RT}$$

or

$$DO = \frac{OR \cdot QT}{RT} = \frac{w - \frac{s}{2}}{\sqrt{\frac{s^2}{4} - 1}} = \frac{2w - s}{\sqrt{s^2 - 4}},$$

proving the lemma.



a) Suppose to the contrary that we were able to see $2\lceil \frac{d+1}{2} \rceil + 1$ circles. Then in one direction from O there would be at least $\lceil \frac{d+1}{2} \rceil + 1$ centres of circles that we could see completely. Let P be the centre of the furthest of those circles from O . Then $w := OP \geq \lceil \frac{d+1}{2} \rceil s$. Thus

$$\frac{2w - s}{\sqrt{s^2 - 4}} \geq \frac{2\lceil \frac{d+1}{2} \rceil s - s}{\sqrt{s^2 - 4}} > \frac{(2\lceil \frac{d+1}{2} \rceil - 1)s}{s} = 2\left\lceil \frac{d+1}{2} \right\rceil - 1 \geq d.$$

By the lemma, we cannot see the circle around P completely.

b) We need to show that we can see at least $\lceil \frac{d-1}{2} \rceil$ circles in each direction of l from O . Let P be the centre of the last such circle. Then $w := OP \leq \lceil \frac{d-1}{2} \rceil s$ and

$$\begin{aligned} \frac{2w - s}{\sqrt{s^2 - 4}} &\leq \frac{(2\lceil \frac{d-1}{2} \rceil - 1)s}{\sqrt{s^2 - 4}} = \frac{2\lceil \frac{d-1}{2} \rceil - 1}{\sqrt{1 - \frac{4}{s^2}}} \\ &< \frac{2\lceil \frac{d-1}{2} \rceil - 1}{\sqrt{1 - \frac{4(2d-1)}{4d^2}}} = \frac{2\lceil \frac{d-1}{2} \rceil - 1}{\sqrt{\frac{d^2 - 2d + 1}{d^2}}} \\ &= \frac{2\lceil \frac{d-1}{2} \rceil - 1}{\frac{d-1}{d}} \leq \frac{d-1}{d} = d, \end{aligned}$$

where the last inequality uses the fact that d is an integer and thus $2\lceil \frac{d-1}{2} \rceil \leq d$. By the lemma, we are done.

c) We will show that we can see exactly $2\lceil \frac{d}{2} \rceil$ circles completely. The centre of the k -th circle on l from O is at distance $w = ks - \frac{s}{2}$ from O . If $k = \lceil \frac{d}{2} \rceil$, then similar to the calculation in part b)

$$\frac{2w - s}{\sqrt{s^2 - 4}} = \frac{(2\lceil \frac{d}{2} \rceil - 2)s}{\sqrt{s^2 - 4}} \leq \dots \leq \frac{2\lceil \frac{d}{2} \rceil - 2}{\frac{d-1}{d}} \leq \frac{d+1-2}{\frac{d-1}{d}} = d,$$

so we can see the circle completely. If on the other hand $k = \lceil \frac{d}{2} \rceil + 1$ then

$$\frac{2w - s}{\sqrt{s^2 - 4}} = \frac{2\lceil \frac{d}{2} \rceil s}{\sqrt{s^2 - 4}} > \frac{2\lceil \frac{d}{2} \rceil s}{s} \geq d,$$

so we cannot see the circle completely, proving that we can see exactly $2\lceil \frac{d}{2} \rceil$ circles completely.

Proposer's Comments. This is a generalization of a problem appearing in the 1984–1985 Alberta High School Mathematics Competition, Part II. The problem there was to show that at most 100 circles are visible when $d = 99$, which follows from part (a) above. See *The Alberta High School Math Competitions 1957-2006: A Canadian Problem Book*, compiled by Andy Liu, MAA in collaboration with the CMS, 2008. Our proof of the upper bound in (a) is derived from the proof in that book.

