**P1.** The $n$ players of a hockey team gather to select their team captain. Initially, they stand in a circle, and each person votes for the person on their left.

The players will update their votes via a series of rounds. In one round, each player $a$ updates their vote, one at a time, according to the following procedure: At the time of the update, if $a$ is voting for $b$, and $b$ is voting for $c$, then $a$ updates their vote to $c$. (Note that $a$, $b$, and $c$ need not be distinct; if $b = c$, then $a$'s vote does not change for this update.) Every player updates their vote exactly once in each round, in an order determined by the players (possibly different across different rounds).

They repeat this updating procedure for $n$ rounds. Prove that at this time, all $n$ players will unanimously vote for the same person.

***Comment.*** *Unfortunately, the level of difficulty of this problem was the result of an unintentional clerical error. The version of this problem which had been initially slated for the contest was much easier. It had stated, "They repeat the same updating procedure on the second day, the third day, and so on. Prove that eventually, all players will unanimously vote for the same person." In other words, there had been no bound of $n$ rounds.*

## Solution 1

Initially, all players are in a cycle. Note that once a player leaves the cycle, they cannot rejoin. Furthermore, a new cycle cannot be created. Hence, at any point in time, the graph corresponding to the votes will be a functional graph with a single cycle.

We will first prove that after $\lfloor \log_2 n \rfloor$ rounds, the cycle will become a self-loop. Then, we will show that in the next $\lfloor \log_2 n \rfloor$ rounds, all other players vote for the player in the self-loop.

To show the first step, assume the cycle has size $K > 1$ at the beginning of a round. Consider arbitrary player $a$ in the cycle who is updating their vote. Say $a \to b \to c$, all in the cycle. Then $a \to c$ now, bumping $b$ out of the cycle and reducing its size to $K - 1$. Note that $b$ can now update their vote as well without affecting the size of the cycle. If we consider all $K$ original players in the cycle, we see that at least $\lceil \frac{K}{2} \rceil$ of them must still be in the cycle at the time of their update, and hence the cycle's size is reduced to at most $\lfloor \frac{K}{2} \rfloor$. After $\lfloor \log_2 n \rfloor$ rounds, the cycle must be reduced to size 1.

Now that the cycle has been reduced to a single player, say $z$, consider any path from a player $a$ to $z$. No players can be added to this path now. With a similar argument as the cycle, the length of the path must halve each round. In particular, a path of length $L$ to the cycle gets reduced to length $\lceil \frac{L}{2} \rceil$ (note the ceiling, we had the floor for the cycle). After $\lceil \log_2 n \rceil$ rounds, the path must be reduced to length 1.

Thus, after $\lfloor \log_2 n \rfloor + \lceil \log_2 n \rceil$ rounds, the graph has been completely reduced. For $n \geq 5$, $\lfloor \log_2 n \rfloor + \lceil \log_2 n \rceil \leq 2\lfloor \log_2 n \rfloor + 1 \leq n$. For the other $n$, we can manually check that $\lfloor \log_2 n \rfloor + \lceil \log_2 n \rceil \le n$.

**Comment 1.** In this proof, it is important that we consider the cycle size and path sizes in disjoint rounds. In the rounds where the cycle size is decreasing, it's possible for the path sizes to increase by $Cn$ for some constant $C$.

**Comment 2.** This proof can actually be refined to prove that $\log_2 n + O(\log_2 \log_2 n)$ rounds are sufficient (and constructions show that this is necessary). Consider a fixed player $a$ and the path from $a$ to the cycle as it changes across the rounds. In round $i$, let $\ell_i$ be the number of players which were part of the cycle at the beginning of round $i$, and became part of this path during round $i$. (Note that it's possible that a player was part of the path for only a portion of round $i$, and is no longer part of the path at the end. It should still be counted.) We can see that $\ell_i$ is bounded by size of the cycle at the start of round $i$, which we know is $\le \frac{n}{2^{i-1}}$.

Now note that the size of this path after round $k$ is at most

$$1 + \frac{\ell_1}{2^{k-1}} + \frac{\ell_2}{2^{k-2}} + \ldots + \frac{\ell_{k-1}}{2} + \ell_k \le 1 + \frac{kn}{2^{k-1}}.$$

Hence, when $k = \log_2 n + C \log_2 \log_2 n$ for $C$ sufficiently large, we must have that every player is voting for the self-loop.

## Solution 2

We will use induction on $n$.

**Inductive Hypothesis.** Let $G$ be any functional graph with $n$ nodes and a single cycle. Then after $n$ rounds of the given operation, $G$ will become a self-loop with $n-1$ nodes pointing to it.

**Base Case.** The cases $n \le 2$ are clear.

**Inductive Step.** Assume that the hypothesis is proved for $n = k-1$ and $n = k-2$. We will prove it for $n = k$. Consider any initial functional graph with $k$ nodes and a single cycle. Note there is some node $a$ which has in-degree 0 (i.e. no nodes point to it), or all $k$ nodes are in the cycle.

In the first case, consider $G \setminus \{a\}$. Note that all operations except $a$'s own updates are independent of where $a$ is. By the inductive hypothesis, after $k-1$ rounds, $G \setminus \{a\}$ has become a single self-loop and $k-2$ nodes pointing to it. Regardless of where $a$ is, it will point to the self-loop after one more round and we are done.

In the case where all $k$ nodes are in a cycle, consider the very first operation $z \to a \to b \implies z \to b, a \to b$. This creates a zero in-degree node $a$, but $z$'s operation has been used for the first round so the inductive hypothesis cannot be naively applied. Instead, consider $b \to c$ (possibly $c = z$ if $k = 3$). At some point in the first round, $b$ will be updated. Either $c$ will become another zero in-degree node, or $a$ will be the only node that points to $c$. Either way, consider $G \setminus \{a, c\}$. By the induction hypothesis, after rounds 2 through $k-1$, this graph will become a self-loop with $k-3$ nodes pointing to it. It's also easy to see that $a$ and $c$ both have in-degree 0 after round 2. Then in one more round after round $k-1$, we must have $a$ and $c$ pointing to the self-loop. So we are done for $n = k$.

By induction, we are done for all $n$.

**Comment.** If the problem were instead to prove the result after $2n$ rounds, the induction would be much easier.

**P2.** Determine all positive integers $a, b, c, p$ where $p$ and $p+2$ are odd primes and

$$2^a p^b = (p+2)^c - 1.$$

## Solution

The only solution is $(a, b, c, p) = (3, 1, 2, 3)$. First, factor the right hand side. This gives us

$$2^a p^b = (p+1)((p+2)^{c-1} + (p+2)^{c-2} + \cdots + (p+2) + 1).$$

Since $\gcd(p, p+1) = 1$ it must be the case that $p+1 = 2^x$ for some positive integer $x \le a$ and so $p = 2^x - 1$ and $p+2 = 2^x + 1$. Now for $x \ge 3$, $2^x + 1$ is not prime if $x$ is odd (since it is 0 mod 3) and $2^x - 1$ is not prime if $x$ is even (since it is 0 mod 3). This means $x \le 2$, and the only admissible such $x$ is $x = 2$ since otherwise $p$ is not prime. So, the original equation becomes

$$2^a 3^b = 5^c - 1.$$

Now $3|(5^c - 1)$, so evaluating $5^c - 1$ mod 3 gives that $c = 2d$ for some positive integer $d$ and hence

$$2^a 3^b = (5^d - 1)(5^d + 1).$$

Observe $5^d - 1$ and $5^d + 1$ are both even and have greatest common divisor 2 because they are 2 apart. Since $4|(5^d - 1)$ this implies $5^d - 1 = 2^{a-1}$ and $5^d + 1 = 2 \cdot 3^b$. Now, 3 is not a factor of $5^d - 1$ because $5^d - 1 = 2^{a-1}$. Thus, by evaluating mod 3, $d$ must be odd. If $d > 1$, this is impossible as $5^d - 1 = (5-1)(5^{d-1} + 5^{d-2} + \cdots + 5 + 1)$ and the latter factor has an odd prime factor, contradicting $5^d - 1$ is a power of 2. Thus $d = 1$ and so $c = 2$, implying that $2^a 3^b = 24$ so $a = 3$ and $b = 1$. Hence, the only solution is $(a, b, c, p) = (3, 1, 2, 3)$.

 

**P3.** A polynomial $c_d x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0$ with degree $d$ is *reflexive* if there is an integer $n \geq d$ such that $c_i = c_{n-i}$ for every $0 \leq i \leq n$, where $c_i = 0$ for $i > d$. Let $\ell \geq 2$ be an integer and $p(x)$ be a polynomial with integer coefficients. Prove that there exist reflexive polynomials $q(x), r(x)$ with integer coefficients such that

$$(1 + x + x^2 + \cdots + x^{\ell-1})p(x) = q(x) + x^\ell r(x).$$

## Solution 1

Let $d$ be the degree of $p$ and let $k$ be any non-negative integer. We will choose

$$q(x) = \frac{x^{d+k+\ell}p\left(\frac{1}{x}\right) - p(x)}{x-1},$$
$$r(x) = \frac{p(x) - x^{d+k}p\left(\frac{1}{x}\right)}{x-1}.$$

First, we must show that both $q$ and $r$ are integer polynomials. Consider the numerator in $q$'s definition, $x^{d+k+\ell}p\left(\frac{1}{x}\right) - p(x)$. This is clearly an integer polynomial. As it is equal to 0 when evaluated at $x-1$, $x-1$ divides it. Furthermore, as $x-1$ is monic, the quotient has integer coefficients. The argument for $r$ is similar.

Next, we will show that this choice of $q$ and $r$ satisfies the desired equation. Plugging them into the RHS of the equation gives

$$\begin{aligned} q(x) + x^\ell r(x) &= \frac{x^{d+k+\ell}p\left(\frac{1}{x}\right) - p(x)}{x-1} + x^\ell \left(\frac{p(x) - x^{d+k}p\left(\frac{1}{x}\right)}{x-1}\right) \\ &= \frac{x^{d+k+\ell}p\left(\frac{1}{x}\right) - p(x) + x^\ell p(x) - x^{d+k+\ell}p\left(\frac{1}{x}\right)}{x-1} \\ &= \left(\frac{x^\ell - 1}{x-1}\right)p(x) \\ &= (1 + x + \cdots + x^{\ell-1})p(x) \end{aligned}$$

as desired.

Finally, we will show that $q$ and $r$ are indeed reflexive. We can re-interpret the reflexive condition as such:

Polynomial $a(x)$ is reflexive iff there is an integer $n \geq \deg(a)$ for which

$$a(x) = x^n a\left(\frac{1}{x}\right).$$

We have

$$\begin{aligned} q(x) &= \frac{x^{d+k+\ell}p\left(\frac{1}{x}\right) - p(x)}{x-1} \\ &= x^{d+k+\ell-1} \cdot \frac{p\left(\frac{1}{x}\right) - x^{-(d+k+\ell)}p(x)}{\frac{x-1}{x}} \\ &= x^{d+k+\ell-1} \cdot \frac{x^{-(d+k+\ell)}p(x) - p\left(\frac{1}{x}\right)}{\frac{1}{x}-1} \\ &= x^{d+k+\ell-1} q\left(\frac{1}{x}\right) \end{aligned}$$

as desired. Similarly,

$$\begin{aligned} r(x) &= \frac{p(x) - x^{d+k}p\left(\frac{1}{x}\right)}{x-1} \\ &= x^{d+k-1} \cdot \frac{x^{-(d+k)}p(x) - p\left(\frac{1}{x}\right)}{\frac{x-1}{x}} \\ &= x^{d+k-1} \cdot \frac{p\left(\frac{1}{x}\right) - x^{-(d+k)}p(x)}{\frac{1}{x}-1} \\ &= x^{d+k-1} r\left(\frac{1}{x}\right). \end{aligned}$$

## Solution 2

We write degree $n$ polynomial $p$ as

$$p(x) := \sum_{i=0}^{n} p_i x^i.$$

Define vector $P \in \mathbb{Z}^{n+1}$ as

$$P := \begin{pmatrix} p_0 & p_1 & \cdots & p_n \end{pmatrix}^T.$$

We also denote $X \in \mathbb{Z}[x]^N$ for $N$ some sufficiently high degree (e.g. $N > 2n + \ell$) as the vector of powers of $x$, i.e.

$$X := \begin{pmatrix} 1 & x & x^2 \cdots & x^{N-1} \end{pmatrix}^T.$$

For a matrix $M \in \mathbb{Z}^{(n+1)\times N}$, $P^T M X$ is an integer polynomial of degree $< N$. Note that if the non-zero entries of matrix $M$ are horizontally symmetric, then the resulting polynomial must be reflexive.

 

Let $A$ be the matrix corresponding to $(1 + x + \cdots + x^{\ell-1})p(x)$. The non-zero entries of the matrix form a parallelogram. In particular, $A$ is a sparse matrix with 1s across the upper $\ell$ diagonals. For example, for $\ell = 3, n = 4$,

$$A = \begin{pmatrix} 1 & 1 & 1 & & & & & \cdots \\ & 1 & 1 & 1 & & & & \cdots \\ & & 1 & 1 & 1 & & & \cdots \\ & & & 1 & 1 & 1 & & \cdots \\ & & & & 1 & 1 & 1 & \cdots \end{pmatrix}.$$

We will now construct matrices $Q, R \in \mathbb{Z}^{(n+1)\times N}$ such that $Q$ and $R$ will correspond to $q(x)$ and $x^\ell r(x)$, respectively. Thus, we require

$$\begin{aligned} & (1 + x + x^2 + \cdots + x^{\ell-1})p(x) = q(x) + x^\ell r(x) \\ \iff \quad & P^T AX = P^T QX + P^T RX \\ \iff \quad & P^T (A - Q - R)X = 0. \end{aligned}$$

So it suffices to find $Q$ and $R$ horizontally symmetric and such that $Q + R = A$. Note that the first $\ell$ columns of $R$ must also be zero.

As it turns out, many constructions exist. For example, consider $Q$ with 1 entries forming a isosceles triangle with base from 0 to $2n$:

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \cdots \\ & & 1 & 1 & 1 & 1 & 1 & & & \cdots \\ & & & 1 & 1 & 1 & & & & \cdots \\ & & & & 1 & & & & & \cdots \end{pmatrix}.$$

Then the difference, $R = A - Q$ is

$$R = \begin{pmatrix} & -1 & -1 & -1 & -1 & -1 & -1 & \cdots \\ & & -1 & -1 & -1 & -1 & & \cdots \\ & & & -1 & -1 & & & \cdots \\ & & & & & & & \cdots \\ & & & 1 & 1 & & & \cdots \end{pmatrix}.$$

Extending this triangle structure to an isosceles trapezoid (even a self-intersecting one) works as well. More rigorously, define $Q$ to be the matrix whose entries are 1 at the isosceles trapezoid from (zero-indexed) entries at

$$(0, 0), (0, n + k), (n, k), (n, n)$$

in that order, for any non-negative integer $k$ with $k \geq \ell$. (If we choose $k < n$, the entries may be $-1$ to account for the self-intersection.) Define $R$ to be the matrix whose entries are $-1$ at the isosceles trapezoid from entries at

$$(0, \ell), (0, n + k), (n, k), (n, n + \ell).$$

Then their total is the matrix whose entries are 1 at the parallelogram formed by

$$(0,0),(0,\ell-1),(n,n+\ell-1),(n,n),$$

which is precisely $A$.

 

**P4.** Let $ABC$ be a triangle with circumcircle $\Gamma$ and $AB \neq AC$. Let $D$ and $E$ lie on the arc $BC$ of $\Gamma$ not containing $A$ such that $\angle BAE = \angle DAC$. Let the incenters of $BAE$ and $CAD$ be $X$ and $Y$ respectively, and let the external tangents of the incircles of $BAE$ and $CAD$ intersect at $Z$. Prove that $Z$ lies on the common chord of $\Gamma$ and the circumcircle of $AXY$.

## Solution

Let $AX$ and $AY$ intersect $(ABC)$ again at $P$ and $Q$, let the inradii of $ABE$ and $ACD$ be $r_B$ and $r_C$, and let $(AXY)$ intersect $(ABC)$ again at $N$.

First note that $\angle BAP = \frac{1}{2}\angle BAE = \frac{1}{2}\angle CAD = \angle QAC$, so $XP = BP = CQ = CY$. This thus implies that, since $NXP$ and $NYQ$ are spirally similar, $NX = NY$ and $NP = NQ$ (in particular, $N$ is the midpoint of arc $XAY$ on $(AXY)$). Now, note that

$$\begin{aligned}\frac{ZX}{ZY} &= \frac{r_b}{r_c}\\ &= \frac{AX\sin(\angle PAE)}{AY\sin(\angle QAD)}\\ &= \frac{AX}{AY}.\end{aligned}$$

Since $Z$ lies on the ray $YX$, we have that $Z$ lies on the external angle bisector of $\angle XAY$. But so does $N$, hence $Z$, $A$, $N$ are collinear, as desired.

**P5.** A rectangle $R$ is divided into a set $S$ of finitely many smaller rectangles with sides parallel to the sides of $R$ such that no three rectangles in $S$ share a common corner. An ant is initially located at the bottom-left corner of $R$. In one operation, we can choose a rectangle $r \in S$ such that the ant is currently located at one of the corners of $r$, say $c$, and move the ant to one of the two corners of $r$ adjacent to $c$.

Suppose that after a finite number of operations, the ant ends up at the top-right corner of $R$. Prove that some rectangle $r \in S$ was chosen in at least two operations.

## Solution 1

Consider the following version of the problem:

> A rectangle $R$ is divided into a set $S$ of finitely many smaller rectangles such that no three rectangles in $S$ share a common corner. For each $r \in S$, draw two non-intersecting arcs inside $r$, connecting the pairs of adjacent corners of $r$ (there are two ways to do this, by connecting either the horizontally or vertically adjacent corners). Prove that there does not exist a path from the bottom-left corner of $R$ to the top-right corner of $R$ by walking only along these arcs.

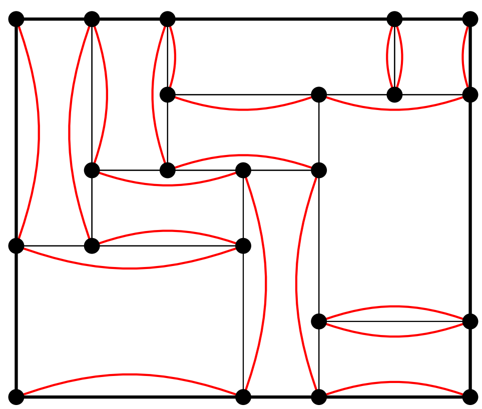

Figure 1: A possible diagram of all the arcs.

In this problem, consider an undirected graph where nodes correspond to corners of rectangles in $S$ and edges correspond to the arcs, connecting the two nodes that the arc connects. The degrees of the nodes corresponding to the corners of $R$ are all exactly 1. Since no three rectangles in $S$ share a common corner, all intersection points have a pattern like $\vdash$, $\dashv$, $\perp$, or $\top$, so the degree of all other nodes is exactly 2. Therefore, this graph can be decomposed into several paths and cycles. The only possible endpoints of paths are the degree 1 nodes, which are the corners of $R$. It follows that if there exists a path from the bottom-left corner to the top-right corner of $R$, then there also exists a path from the bottom-right corner to the top-left corner of

 

$R$. However, this is impossible because these two paths (viewed as planar curves inside $R$) must intersect, which cannot occur. Therefore, this claim is proved.

Returning to the original problem, suppose some operations were performed while choosing each rectangle at most once. Draw two arcs inside every rectangle, either both horizontal if the ant used this rectangle to move horizontally or both vertical if the ant used this rectangle to move vertically (or pick one arbitrarily if this rectangle was not used). By the new version of the problem, there does not exist a path from the bottom-left corner to the top-right corner of $R$, and it follows that it is impossible for the ant to have reached the top-right corner of $R$, finishing the proof.

## Solution 2

Suppose that no rectangle was chosen in at least two operations. In particular, a rectangle cannot be selected in two consecutive operations.

At any point in the process, consider whether the last move by the ant was horizontal or vertical, and whether the most recently chosen rectangle was to the left or the right of the ant's path. In the first move, either the ant moved horizontally and the rectangle was to the left, or the ant moved vertically and the rectangle was to the right. We claim that this invariant is preserved throughout the entire process (see Figure 2 for a sample path). Assuming this claim, the final move to the top right corner must select the top right rectangle. If it is vertical, this is to the left of the path, and if it is horizontal, it is to the right of the path, both of which are impossible, providing a contradiction.

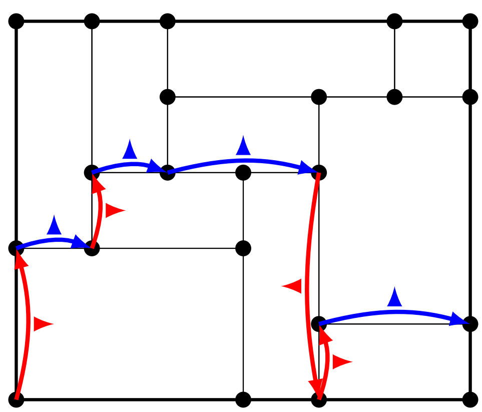

Figure 2: A possible path by the ant. The red arrows are all vertical, with the corresponding rectangle to the right. The blue arrows are all horizontal, with the corresponding rectangle to the left.

It remains to show that the invariant is preserved. Since four rectangles cannot intersect at a corner, each intersection has a pattern like $\vdash$, $\dashv$, $\perp$, or $\top$.

First, assume the ant moves up, hence the chosen rectangle $r$ is on the right. The possible configurations are depicted in the first two diagrams in Figure 3.

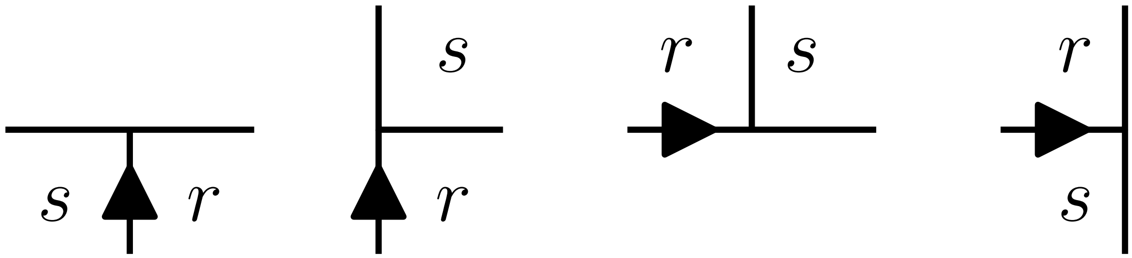


Figure 3: Possible ant moves going up or right.

In each case, the ant must choose rectangle $s$ next (to avoid repeating $r$ twice), and we see that both side choices preserve a horizontal move with $s$ left, or a vertical move with $s$ right. By rotating the picture by 180°, we cover the two possibilities for the ant moving downward.

If the ant moves right, then $r$ must occur on the left, and the possible configurations are the last two diagrams of Figure 3. Once again, rectangle $s$ must be chosen next, and the invariant is similarly preserved. The case of the ant moving left is again handled by a 180° rotation, completing the proof.

## Solution 3

This is a variant of Solution 2. As in that proof, assume that no rectangle was chosen in two consecutive operations. We claim that for every move, the ant is in either the bottom-left or top-right corner of the chosen rectangle, and moves to the bottom-right or top-left corner.

This is clearly true of the first move. If the ant starts at the bottom-left or top-right corner on a move (choosing rectangle $r$), it is clear that they must move to the bottom-right or top-left of $r$. Assume they moved to the bottom-right corner of $r$, and chose rectangle $s$ in the next move. If they are at the bottom-right corner of $s$, then $r$ and $s$ are either equal or overlap, a contradiction. If they are at the top-left of $s$, then $r$ and $s$ intersect at a corner and no sides, and we must have 4 rectangles intersecting at a corner, again a contradiction.

Therefore they must be at the bottom-left or top-right corner of $s$, as desired. The case where the ant is at the top-left corner of $r$ is analogous.

If the ant is able to make it to the top right corner, their final move must select $r$ as the top-right rectangle, and they move to the top-right corner, which is therefore impossible.

 