Official 2025 Problem Set

J1. Suppose an infinite non-constant arithmetic progression of integers contains 1 in it. Prove that there are an infinite number of perfect cubes in this progression.

(A perfect cube is an integer of the form k^3, where k is an integer. For example, -8, 0, and 1 are perfect cubes.)

J2. Let $ABCD$ be a trapezoid with parallel sides AB and CD, where $BC \neq DA$. A circle passing through C and D intersects AC, AD, BC, and BD again at W, X, Y, and Z respectively. Prove that WZ, XY, and AB are concurrent.

J3. The n players of a hockey team gather to select their team captain. Initially, they stand in a circle, and each person votes for the person on their left.

The players will update their votes via a series of rounds. In one round, each player a updates their vote, one at a time, according to the following procedure: At the time of the update, if a is voting for b, and b is voting for c, then a updates their vote to c. (Note that a, b, and c need not be distinct; if $b = c$, then a's vote does not change for this update.) Every player updates their vote exactly once in each round, in an order determined by the players (possibly different across different rounds).

They repeat this updating procedure for n rounds. Prove that at this time, all n players will unanimously vote for the same person.

J4. Determine all positive integers a, b, c, p where p and $p + 2$ are odd primes and

$$2^a p^b = (p + 2)^c - 1.$$

J5. A polynomial $c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$ with degree d is *reflexive* if there is an integer $n \geq d$ such that $c_i = c_{n-i}$ for every $0 \leq i \leq n$, where $c_i = 0$ for $i > d$. Let $\ell \geq 2$ be an integer and $p(x)$ be a polynomial with integer coefficients. Prove that there exist reflexive polynomials $q(x), r(x)$ with integer coefficients such that

$$(1 + x + x^2 + \cdots + x^{\ell-1})p(x) = q(x) + x^\ell r(x).$$

