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Editorial Board

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MathemAttic

No. 62

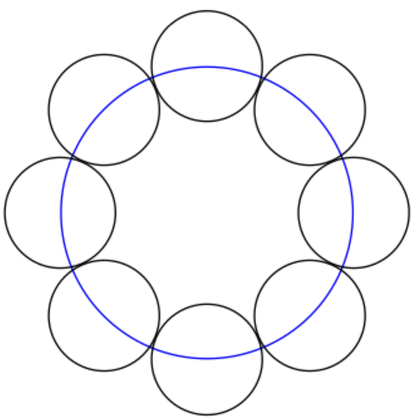
The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2025**.

MA306. The power seven lottery awards prize money in powers of 7. For example, prize categories are $1 = 7^0$, $7 = 7^1$, $49 = 7^2$, $343 = 7^3$, etc. In this lottery there are several awards that amount to 777777 and there are no more than 6 awards for each prize category. How many awards are there?

MA307. Eight circles of radius 1 have centers on a larger common circle and adjacent circles are tangent. Find the area of the common circle. See the illustration below.



MA308. There are 46656 6-digit numbers that can be formed from the digits 1, 2, 3, 4, 5, and 6, with repetition of digits allowed. If these numbers are listed in order what is the 2018th number in the list?

MA309. Alice walks down to the bottom of an escalator that is moving up. Alice counts 150 steps. Alice's friend Bob walks up to the top of the escalator and counts 75 steps. Alice's speed of walking (number of steps per unit time) is 3 times Bob's walking speed. How many steps are visible on the escalator at a given time?

MA310. In a town where any pair of people are either friends or strangers to each other, any two friends do not have common friends, while any two strangers have exactly two common friends. Prove that in this town everyone has exactly the same number of friends.

. . . .

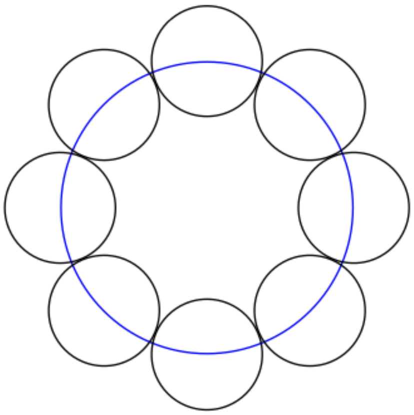
Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2025**.

MA306. La loterie puissance 7 attribue des prix en puissances de 7. Par exemple, les catégories de prix sont les suivantes : $1 = 7^0$, $7 = 7^1$, $49 = 7^2$, $343 = 7^3$, etc. Dans cette loterie, il y a plusieurs prix qui totalisent 777777 et il n'y a pas plus de 6 prix pour chaque catégorie de prix. Combien y a-t-il de prix ?

MA307. Huit cercles de rayon 1 ont leur centre inscrit sur un cercle commun qui est plus grand. Les cercles adjacents sont tangents. Trouvez l'aire du cercle commun. Voir l'illustration ci-dessous.



MA308. Il y a 46656 nombres à 6 chiffres qui peuvent être formés à partir des chiffres 1, 2, 3, 4, 5 et 6 si la répétition des chiffres est autorisée. Si ces nombres sont classés dans l'ordre, quel est le 2018ème nombre de la liste ?

MA309. Alice descend les marches d'un escalier mécanique qui va dans une direction ascendante. Alice compte 150 marches. Bob, l'ami d'Alice, monte les marches de l'escalier mécanique et compte 75 marches. La vitesse de marche d'Alice (le nombre de marches par unité de temps) est 3 fois supérieure à celle de Bob. Combien de marches sont visibles sur l'escalier mécanique à un moment donné ?

MA310. Dans une ville où deux personnes quelconques sont soit amies, soit étrangères l'une pour l'autre, deux amis quelconques n'ont pas d'amis communs, tandis que deux étrangers quelconques ont exactement deux amis communs. Prouvez que dans cette ville, tout le monde a exactement le même nombre d'amis.

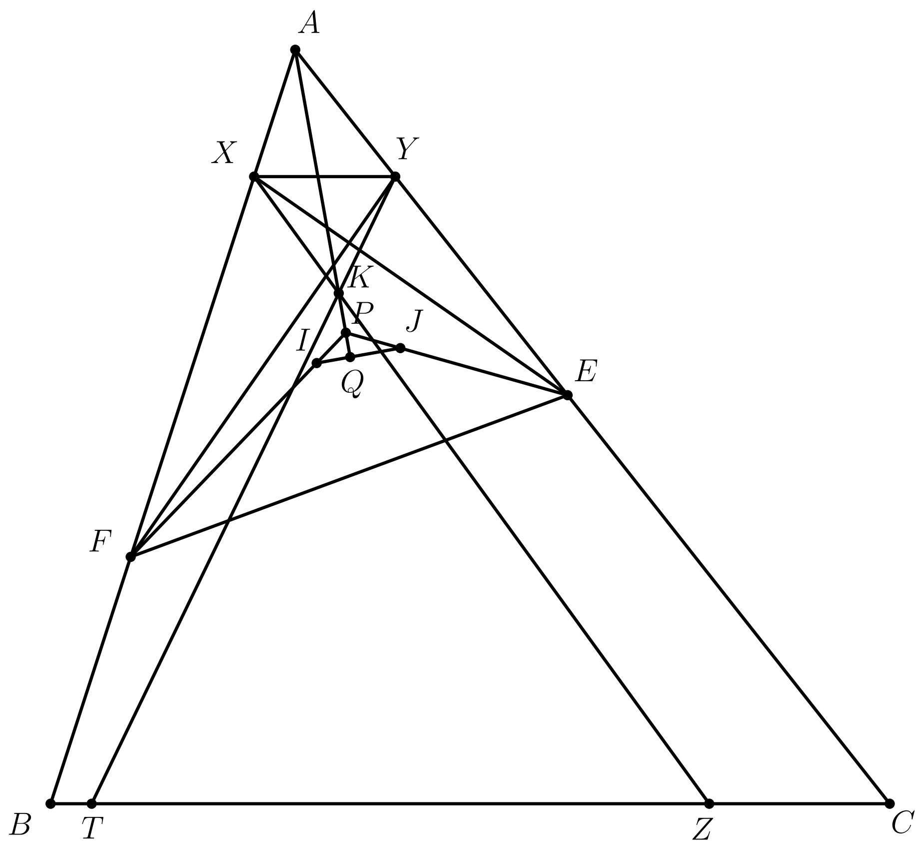
MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2024: 50(7), p. 342-343.

MA281. *Proposed by Trinh Quoc Khanh.*

Given a triangle ABC, let E, F be the feet of altitudes from B, C. The circle centered at B with radius BE intersects segments BA, BC at X, Z. The circle centered at C with radius CF intersects segments CA, CB at Y, T. XZ intersects YT at K. Let I, J be the incenters of $\triangle XEF, \triangle YEF$, respectively. Prove that AK is perpendicular to IJ.

We received no correct submissions to this problem. We present the proposer's solution.



Let the intersection point of EJ, FI be P and let AK meets IJ at Q.

Claim 1. X, Y, E, F are concyclic.

Since $CY = CF$, we have

$$\angle FYC = \frac{180^\circ - \angle YCF}{2} = \frac{180^\circ - (90^\circ - \angle BAC)}{2} = \frac{90^\circ + \angle BAC}{2}.$$

Analogously, we have

$$\angle EXB = \frac{90^\circ + \angle BAC}{2} \implies \angle FYC = \angle EXB \implies \angle FYE = \angle EXF,$$

which implies that X, Y, E, F are concyclic.

Claim 2. AK is the angle bisector of $\angle BAC$.

From Claim 1, $\angle AXY = \angle AEF$, but it is known that $\angle AEF = \angle ABC$. Hence

$$\angle AXY = \angle ABC \implies XY \parallel BC \implies \angle YXZ = \angle XZB.$$

But from the hypothesis, $BX = BZ \implies \angle XZB = \angle BXZ$, so $\angle YXZ = \angle BXZ$. Similarly, $\angle XYT = \angle CYT$. So K is the $A-$excenter of $\triangle AXY$.

We have FI, EJ, AK are the internal angle bisectors of $\triangle AEF$, so they are concurrent at P.

Now, let's do some angle chasing:

$$\angle FIE = 90^\circ + \frac{\angle FXE}{2}$$

and

$$\angle EJF = 90^\circ + \frac{\angle FYE}{2},$$

so $FIJE$ is a cyclic quadrilateral. Furthermore,

$$\angle QPJ = \frac{\angle FAE + \angle FEA}{2} \quad \text{and} \quad \angle PJQ = \angle PFE = \frac{\angle AFE}{2}.$$

This implies that

$$\angle QPJ + \angle PJQ = \frac{\angle FAE + \angle FEA + \angle AFE}{2} = 90^\circ.$$

Therefore $AK \perp IJ$.

MA282. *Proposed by Neculai Stanciu.*

Prove that $2^{2n+5} + 9n^2 + 3n + 4$ is divisible by 18 for any non-negative integer n.

We received 17 submissions, 7 of which were correct and complete. The problem was published with a typo from the original proposal; this was later corrected after input from our readers. As a result, we include in the count of the correct submissions those who submitted counterexamples of the initial problem statement. We *present the solution to the correct problem statement by the proposer and Vasile Teodorovici, done independently.*

We proceed by induction. For the basis step of $n = 0$, we have

$$2^{2(0)+5} + 9(0)^2 + 3(0) + 4 = 36,$$

which is divisible by 18. Next, assume that

$$2^{2(n)+5} + 9(n)^2 + 3(n) + 4$$

is divisible by 18 for some n and consider the case for $n + 1$. We have that

$$\begin{aligned} 2^{2(n+1)+5} &+ 9(n+1)^2 + 3(n+1) + 4 \\ &= 2^{2n+7} + 9n^2 + 21n + 16 \\ &= 2^{2n+7} + 36n^2 + 12n + 16 - 27n^2 + 9n \\ &= 2^2\left(2^{2(n)+5} + 9(n)^2 + 3(n) + 4\right) - 9n(3n-1) \end{aligned}$$

which is divisible by 18 using the induction hypothesis and the fact that n and $3n - 1$ have different parities so their product is even.

Therefore by mathematical induction $2^{2n+5} + 9n^2 + 3n + 4$ is divisible by 18 for any non-negative integer n.

MA283. The integers from 1 to 9 are listed on a blackboard. If an additional m eights and k nines are added to the list, the average of all of the numbers in the list is 7.3. Find the value of $k + m$.

Originally problem 23 from the 2002 Fermat Contest.

We received 10 submissions, 3 of which were correct and complete. We present two solutions.

Solution 1, by Meryem Bourget. Let the average of these numbers be $\bar{x}$. We have

$$\bar{x} = \frac{1+2+3+4+5+6+7+8(m+1)+9(k+1)}{m+k+9} = 7.3$$

That is,

$$1+2+3+4+5+6+7+8+9+8m+9k = 7.3(m+k+9)$$

We do some basic algebra to finally obtain

$$\begin{aligned} \left(\frac{9\cdot 10}{2}\right) + 8m + 9k &= 7.3\cdot 9 + 7.3(m+k) \\ 0.7m + 1.7k &= 20.7 \end{aligned}$$

or more simply,

$$17k + 7m = 207. \tag{1}$$

The last equation is a linear Diophantine equation in two unknowns $k, m \in \mathbb{Z}$. Such an equation can be solved using the division algorithm as described below. First, since the $\gcd(17, 7) = 1$, equation (1) has infinitely many solutions in $\mathbb{Z}$ – we are only interested in the positive solution, i.e. when $k, m \in \mathbb{N}$. We apply the division algorithm successively to obtain

$$\begin{aligned} 17 &= 2 \cdot 7 + 3 \\ 7 &= 2 \cdot 3 + 1 \\ 3 &= 1 \cdot 3 + 0 \end{aligned}$$

By reversing our steps, we deduce

$$1 = 7 - 2 \cdot 3 = 7 - 2(17 - 2 \cdot 7) = 5 \cdot 7 - 2 \cdot 17.$$

We now multiply both sides of last equation by 207 to finally obtain

$$\begin{aligned} 207 &= 17(-2 \cdot 207) + 7(5 \cdot 207) \\ &= 17(-414) + 7(1035) \end{aligned}$$

Therefore, the general solution of (1) is

$$\begin{aligned} k &= -414 + 7n \\ m &= 1035 - 17n, \end{aligned}$$

where $n \in \mathbb{Z}$. The only possible choice of n that makes both k and m positive is $n = 60$. Hence, we obtain

$$k = -414 + 7(60) = 6 \quad \text{and} \quad m = 1035 - 17(60) = 15,$$

and therefore $k + m = 21$.

Solution 2, by Adam Mawani. Since we are solving for $k + m$, we can introduce the variable $n = k + m$. Since the sum of the integers 1 through 9 is 45, we can determine that:

$$\begin{aligned} &\frac{1+2+3+4+5+6+7+8(m+1)+9(k+1)}{m+k+9} = 7.3 \\ &\Leftrightarrow \frac{45+8m+9k}{9+k+m} = 7.3 \\ &\Leftrightarrow 45+8m+9k = 7.3(9+k+m) \\ &\Leftrightarrow 45+8m+9k = 65.7+7.3k+7.3m \\ &\Leftrightarrow 0.7m+1.7k = 20.7 \\ &\Leftrightarrow 7m+17k = 207 \\ &\Leftrightarrow 7n+10k = 207, \end{aligned}$$

where $n = k + m$.

Since multiples of 10 end in 0, $7n$ must end in 7, meaning n must end in 1. Since k is positive, $n \in \{1, 11, 21\}$ since these are the only numbers ending in 1 that are below $207/7 \approx 29.6$.

If $n = 1$, then $k = 20$, which is impossible because $n = k + m$, where both k and m are positive integers. If $n = 11$, then $k = 13$, which is again impossible since $n < k$. However, if $n = 21$ then $k = 6$, which is possible. Therefore, $n = k + m = 21$.

Editor's Comments. For problems MA282 and MA283, many solvers provided incomplete solutions, skipping intermediate steps and relying on the reader to justify the assertions. We encourage solvers to include solutions showing all the necessary steps.

MA284. The lengths of all six edges of a tetrahedron are integers. The lengths of five of the edges are 14, 20, 40, 52, and 70. How many possible lengths for the sixth edge are there?

Originally problem 25 from the 2002 Fermat Contest.

There were 2 correct and 4 incomplete solutions submitted, along with 1 incorrect solution. We present 2 solutions.

Let (u, v, w) denote a triangle with sidelengths $u < v < w$. Note that $w < u + v$.

Solution 1, by Vasile Teodorovici.

There are two triangles whose sidelengths are among the five given numbers and two triangles with the unknown sidelength x. The only possible triangles with known sidelengths are $(14, 40, 52)$, $(20, 52, 70)$, $(20, 40, 52)$ and $(40, 52, 70)$.

If $(14, 40, 52)$ is one triangle, a second with one of these edges has sidelengths 20 and 70. The only possibility is that it is $(20, 52, 70)$.

If $(20, 52, 70)$ is one triangle, the only possibility for a triangle with sidelengths 14 and 40 is $(14, 40, 52)$.

One of two possibilities for the remaining pair of triangular faces is $(14, 20, x)$ and $(40, 70, x)$, with $6 < x < 34$ and $30 < x < 110$, or $x = 31, 32, 33$. The second possibility for the pair is $(14, 70, x)$ and $(20, 40, x)$ with $56 < x < 84$ and $20 < x < 60$, or $x = 57, 58, 59$.

If $(20, 40, 52)$ is one triangle, then the sidelengths 14 and 70 cannot be combined with any of these sidelengths to constitute a triangle.

If $(40, 52, 70)$ is one triangle, then since $14 + 20$ is less than any of these sidelengths, the situation is again impossible.

Hence there are six possibilities for x, namely 31, 32, 33, 57, 58, 59.

Solution 2, by Catherine Jian.

Let $ABCD$ be the tetrahedon, and suppose that $CD = x$. AB cannot be equal to 14 or 20, since these two must be side lengths of either triangle ABC or ABD, which is impossible because $14 + 20$ is less than the other lengths given. Nor can AB be equal to 70, since one of triangles ABC and ABD must have the other two side lengths from among 14, 20, 40, any pair of which add to less than 70. Therefore $AB = 52$.

As in the previous solution, the only possible choices for triangles ABC and ABD are $(14, 40, 52)$ and $(20, 52, 70)$ with, for example, $(AC, AD) = (14, 20)$ or $(AC, AD) = (14, 70)$. The result follows.

Editor's Comments. The given solutions work if we interpret the problem as determining possible *maps* of tetrahedra, *i.e.*, planar representations where one triangular face ABC has three others ABD, BCD, CAD splayed about it. Then it is just a matter of making sure that the sides of the triangles satisfy the triangular inequality. However, a worrier might ask whether these splayed triangles can be folded up to form a tetrahedron. This cannot be taken for granted, as this simple example due to Karl Wirth and André Dreiding indicates. Suppose that five sides have length 4 and the remaining length 7. Then there are two possible faces ABC and ABD with sides (4, 4, 4) and two ACD and BCD with sides (4, 4, 7). If ABC and ABD are laid flat on opposite sides of AB, then $CD = 4\sqrt{3} = \sqrt{48} < 7$, making it impossible to fold the triangles into a tetrahedron.

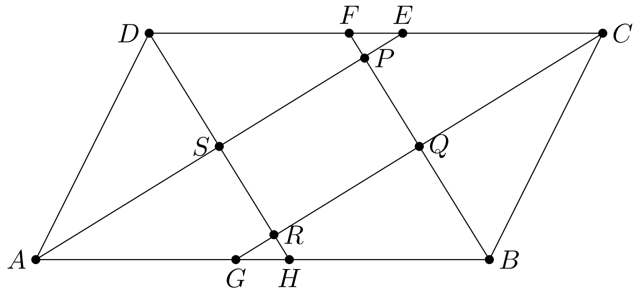
In their paper, *Edge length distances determining tetrahedrons* (Elem. Math. 64 (2009), 160-170 https://ems.press/content/serial-article-files/45383?nt=1, Wirth and Dreiding consider this question in detail. In particular, they show that given six distances, you can form a nondegenerate tetrahedron provided you can find four triples in which each pair appears exactly twice, the triangle inequality is satisfied and the Cayley-Menger determinant is positive. This determinant gives rise to the volume of the tetrahedron.

For the tetrahedra in the problem, we find that each of the values of x gives rise to a legitimate solid. The values of x and the corresponding volumes are $(31, 469.303)$, $(32, 551.000)$, $(33, 483.086)$, $(57, 233.157)$, $(58, 551.236)$, $(59, 219.077)$.

MA285. In parallelogram $ABCD$, $AB = a$ and $BC = b$, $a > b$. Angles A, B, C, D are bisected. The intersection points of these angle bisectors are the vertices of quadrilateral $PQRS$. Prove that $PR = a - b$.

Adapted from problem 8b from the 2000 Euclid Contest.

There were 7 correct solutions. We present the solution by Catherine Jian and Rovsen Pirkuliyev, done independently.



In the diagram above, let AE, BF, CG, DH be the respective bisectors of angles A, B, C, D. Since $\angle DAE = \angle DEA$ and $\angle BGC = \angle BCG$, triangles DAE and CBG are isosceles. Therefore DS right bisects AE and BP right bisects CG. Therefore $PQRS$ is a rectangle.

Since $AE \| GC$, $ASQG$ is a parallelogram, and so

$$PR = SQ = AG = AB - BG = AB - BC = a - b,$$

as desired.

Editor's Comments. An alternative approach after the recognition that $PQRS$ is a rectangle is to note that

$$PS = RQ = RC - QC = a\cos\theta - b\cos\theta = (a-b)\cos\theta$$

and $SR = PQ = (a-b)\cos(90^\circ - \theta) = (a-b)\sin\theta$, where $\theta = \angle DCR = \angle BCQ$, and thence find the length of PR.

TEACHING PROBLEMS

No. 29

John McLoughlin

Considering *Three Problems for Consideration*

The preceding issue of *Teaching Problems* featured *Three Problems for Consideration* (see ***Crux*** 50(10), p. 499 – 500). Those problems are restated here with the intention of elaborating upon merits, extensions, or ideas underlying their inclusion as teaching problems. A common thread is accessibility as each of these problems could be used at a middle school level.

A Special Ten-Digit Number

0's	1's	2's	3's	4's	5's	6's	7's	8's	9's
—	—	—	—	—	—	—	—	—	—

The challenge is to find a special ten-digit number. The first digit gives the number of 0's in the number, the second digit the number of 1's, the third digit the number of 2's and so on. (For example, a number beginning with 2 must have exactly two 0's in it.)

One Dollar on Average

Three items are available to purchase. They cost 50 cents, 2 dollars and 3 dollars respectively. Your challenge is to purchase at least one of each item such that the average price of the items you purchase is exactly 1 dollar.

Pairwise Sums

A list of six positive integers p, q, r, s, t, u satisfies $p < q < r < s < t < u$. There are exactly 15 pairs of numbers that can be formed by choosing two different numbers from this list. The sums of these 15 pairs of numbers are:

$$25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117$$

Determine the value of $(r + s)$.

Discussion of the Problems

The problems are discussed here in reverse order. An elaboration upon a brief commentary of the second problem will follow in a subsequent issue. The first and third problems are discussed in detail here.

Pairwise Sums

The idea of determining the value of a sum or an expression without determining the values of the individual variables seems to be a point of discomfort for many. This particular Gauss Contest question is a thoughtful elaboration of this idea with six variables as it invites insightful thinking along with the usual methods for attacking such a problem. The insight is required as unlike the typical example of such a problem, the answer itself already appears in the list of sums. It is unclear as to which of those values is the desired sum. If one tries to guess, it is likely that the solver will realize it is clearly not the first value of 25 as that must be $(p+q)$, the sum of the smallest two integers, and likewise not 117 as it must represent $(t+u)$. Aha! (We'll come back to this point.)

Consider for a moment the classic example that was likely my first introduction to the idea of not solving for the variables yet finding the sum:

> Given $a+b=20$, $b+c=29$ and $c+d=41$, determine the value of $(a+d)$.

Summing the three equations simplifies to $a+2(b+c)+d=90$ and substituting 29 as the sum of b and c leads to the solution of the problem. That is $(a+d)=32$.

Familiarity with this simpler problem is likely to be found amongst many of those who solved the *Pairwise Sums* question. However, given that the students writing the contest would have been in Grade 8 it is plausible that an attack that began with summing the 15 given values would have been fruitful if followed by that *Aha!* sort of thinking. That is, the total of the values being 980 would represent five times the sum of the six variables, as each variable would be in five pairings. The solution follows as $980/5=196$. Removal of the smallest and largest sums leaves the sum of the middle two values giving $(r+s)=196-(25+117)=54$.

The *Pairwise Sums* problem can be accessible while also yielding an opportunity to incorporate reasoning and move beyond solving for specific values. Any problems that push against immediate tendencies or common comforts potentially develop problem solving abilities. Though it is accessible at a middle school level, my experience suggests it is valuable in teaching secondary math teachers and students. The experience with the question meets up against instincts to begin solving for r and s to get the sum. This alone is valuable.

Before closing discussion on this problem, let us revert to the simpler problem. It is interesting to ask students to pick a value and call it a. Suppose a student selects 6. It would follow then that b must be 14 to give the required sum of 20. Proceeding further would give $c=15$ and $d=26$. Note that $(a+d)$ would be $6+26$, or 32, as found above. Curiously this will be the case regardless of the initial value selected for a. This is a learning moment for many. It calls forth a method of tighter mathematical rigour that ensures this must be the case. Suppose we generalize with expressions in terms of a. The values of the four variables could be expressed as $(a,b,c,d)=(a,20-a,9+a,32-a)$. Again, we find that $a+d=a+(32-a)=32$.

One Dollar on Average

This problem is commonly approached using trial and error. Observe that the smallest possible number of items to be ordered includes at least one 2 dollar and at least one 3 dollar item. Thus, the total is at least 5 dollars making it sensible to add several of the cheaper 50 cent items to bring the average item price down to one dollar. It is quickly noted that six of the 50 cent items bring the overall average to the desired amount of one dollar.

For the purpose of this issue, that is all that will be stated. However, this problem among other related problems will be revisited in future. Extending the discussion at that time will add to the significance of this sort of problem for teaching and learning purposes.

A Special Ten-Digit Number

This problem was introduced to me by Ed Barbeau. What follows here is consideration of the problem from my perspective of working with the problem. Interested readers can see Ed's book [1] for a detailed discussion.

Why do I like this problem? It's playful. On the one hand, numbers with curious properties intrigue me. On the other, it surprised me to find such a number and then not find another. That is, there is the sense of accomplishment in finding the number and then further work established the uniqueness of 6 210 001 000 as that number possessing the required properties.

Playing with the problem initially helps greatly with understanding the task at hand. Suppose the number begins with say a 2. Then that tells us that there are only two appearances of 0 in the number. However, we also know that the third digit must be at least 1 (as it tells how many times the number 2 appears). So now we have a start of 2_1 but the blank space must tell us the number of 1's and so it must be 1. Oops, now it would become 2 as there is already another 1, and we quickly see that this is not as automatic a problem as it may appear using strictly trial and error. Stop and think. What else do we know?

A helpful insight comes with the recognition that the digits in the ten-digit number must total 10 as the digits are telling us the number of appearances of respective digits. Hence, they must total the number of digits in the length of the number. Going back to our example above, beginning with a 2 would leave a total of 8 for the remaining digits with only two 0's. Think about that. This would require seven nonzero digits summing to 8. This would only be possible with six 1's and one 2. Now the number of 1's would have to be large and further each of the 1's would require the appearance of the corresponding digit. This means that possibly the digit 7 say would appear once and 4 would appear once and on. But if we had the digit 1 telling us that the number contains a 7, for example, then it follows that there must be 7 appearances of some other digit. Soon we realize this is impossible and go back to the drawing board.

It is far from evident at the outset that we really need two things to happen.

Algebraically we could imagine $(a, b, c, d, e, f, g, h, i, j)$ representing the number of appearances of the digits 0 through 9 respectively. We require both of the following equations to hold.

$$a + b + c + d + e + f + g + h + i + j = 10$$
$$0a + 1b + 2c + 3d + 4e + 5f + 6g + 7h + 8i + 9j = 10$$

The second equation suggests that a is quite large. Also, we can note that at most one of g, h, i, and j can be positive and cannot exceed 1. It turns out that the only solution to this set of equations is given by

$$(a, b, c, d, e, f, g, h, i, j) = (6, 2, 1, 0, 0, 0, 1, 0, 0, 0).$$

Removing the commas in this set gives the special ten-digit number, namely, 6210001000.

The fact that this problem can be played with in elementary school and explored to a level of conclusiveness in secondary school math adds to its value as a teaching problem. In general, problems that invite messing around while encouraging development of insightful conjectures or observations are among those that attract me as a teacher of mathematics. The mucking around along with the desire to conclusively prove a result are elements of being a mathematician that ought to be more commonplace in the mathematical experiences of students.

References

[1] Ed Barbeau, *After Math: Puzzles and Brainteasers*, Wall and Emerson, Toronto, 1995.

Competition Highlights

The Canada Jay Mathematical Competition

by Nicolae Strungaru

The Canada Jay Mathematical Competition was launched in 2020 under the leadership of Shawn Godin. It started under the name Canadian Mathematical Gray Jay Competition (CMGC), and was renamed to Canada Jay Mathematical Competition(CJMC) in 2022. The competition is open to all students in grades K-8 and is aimed at grades 5-8.

The competition is 90 minutes long, and consists of 15 multiple choice questions, split into three equal parts (A,B,C). Starting from 2024, each question in part A and part B is worth 5 points, while each question in part C is worth 6 points. A point is awarded for each unanswered question, while 0 points are awarded for the wrong answer. The maximum possible score is 80 points.

The first competition took place in October 2020 and had over 2,200 students from Canada and China. This year, there were over 3,500 participants, including 640 participants from 10 countries outside Canada. 75 students obtained a perfect score, and the median score was 36/80.

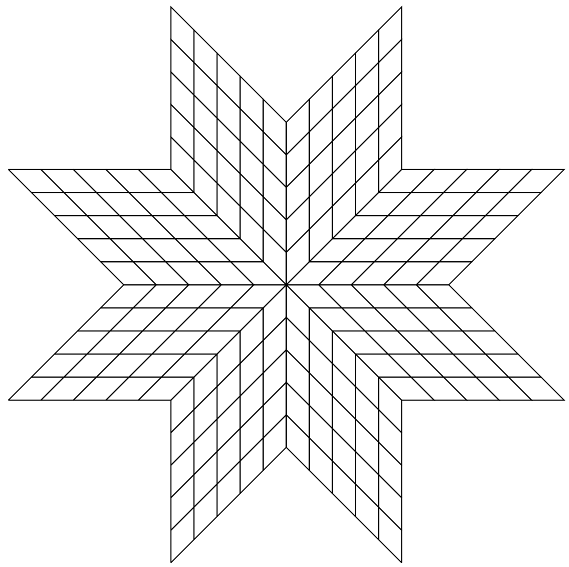
We complete this column by presenting two questions from this year's contest, with their solutions. We start by presenting Problem B05 with two solutions.

Problem B05

Grandma is sewing a traditional Indigenous design typically found on blankets and elsewhere, called a starblanket design, on a tapestry. The pattern is made up of 200 rhombi (see next page).

For each edge, no matter if it is on the outside of the star or between two rhombi, Grandma uses exactly 10cm of thread. How much thread did she use in total?

A. 34 m   B. 40 m   C. 44 m   D. 48 m   e. 80 m



Solution

In total, the 200 rhombi have 800 edges.

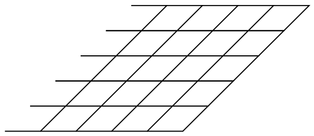
The outside of the star has 16 sides, each consisting of 5 edges. Therefore, there are 80 edges on the outside. For those edges, grandma used 800 cm of thread, or 8m.

Each of the remaining 720 edges is common to two rhombi. Therefore, grandma only sews 360 edges of 10 cm, or 36 m on the inside edges.

In total, she used 44 m of thread.

Second Solution

The starblanket can be made by sewing eight copies of the following shape, and rotating counterclockwise by 45° when moving from one shape to the next.



This shape has six horizontal threads and five vertical threads, each of exactly half meter. Therefore, the length of thread is

$$8 \times .5 \times 11 = 4 \times 11 = 44\text{m} .$$

Answer: (C)

The interested reader can find some more fun problems about the starblanket design in *Crux Mathematicorum*, Vol. 47(1), January 2021 at pages 18-24.

Next, we present Problem C03 with its solution.

Problem C03

A store sells bottles of juice. The bottles come packed in boxes of either 4, 9 or 15 bottles per box and the store only sells full boxes. Sarah is having a party and she needs exactly 50 bottles (as she doesn't want leftovers). What is the smallest number of boxes Sarah needs to buy to get exactly 50 bottles?

A. 4 B. 5 C. 6 D. 7 E. 10

Solution

Note first that buying 4 or more boxes of 15 bottles exceeds 50 bottles. This means that Sarah needs to buy 0, 1, 2 or 3 boxes of 15 bottles.

Case 1: She buys 3 boxes of 15 bottles. Then, she needs to buy an extra 5 bottles, which is not possible.

Case 2: She buys 2 boxes of 15 bottles. Then, she needs to buy an extra 20 bottles. She can buy at most 2 boxes of 9 bottles, and checking the cases she buys 0, 1, 2 boxes of 9 bottles we see that 0 boxes of 9 bottles and 5 boxes of 4 bottles is the only possibility.

In this case, she buys $2 + 0 + 5 = 7$ boxes.

Case 3: She buys 1 box of 15 bottles. Then, she needs to buy an extra 35 bottles. It follows that she can buy at most 3 boxes of 9 bottles. Also, since she needs an odd number of bottles, the number of boxes of 9 bottles must be odd, so 1 or 3.

The only possibility is 3 boxes of 9 bottles and 2 boxes of 4 bottles.

In this case, she buys $1 + 3 + 2 = 6$ boxes.

Case 4: She buys 0 boxes of 15 bottles. Since each box has at most 9 bottles, and she needs to buy 50 bottles, she needs 6 or more boxes.

Regardless, the smallest number of boxes is 6.

Answer: (C)

OLYMPIAD CORNER

No. 430

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **April 15, 2025**.

OC716. Show that for every integer $n \geq 6$, there exists a convex hexagon which can be dissected into exactly n congruent triangles.

OC717. Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let S_k be the sum of the products of $a_1, a_2, \ldots, a_n$ taken k at a time. Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \ldots a_n,$$

for $k = 1, 2, \ldots, n-1$.

OC718. In a triangle ABC, the median and the angle bisector at A meet the side BC at M and N respectively. The perpendicular at N to NA meets MA in Q and BA in P, and the perpendicular at P to BA meets AN produced in O. Prove that QO is perpendicular to BC.

OC719. Let p be a prime number and $n_1, n_2, \ldots, n_p \in \mathbb{N}^*$, at least two of which are distinct. Denote by d the greatest common divisor of the numbers $n_1, n_2, \ldots, n_p$. Prove that the polynomial

$$f = \frac{X^{n_1} + X^{n_2} + \cdots + X^{n_p} - p}{X^d - 1}$$

is irreducible in $\mathbb{Q}[X]$.

OC720. A cube is sub-divided into 27 rectangular prisms by planes parallel to its faces. If exactly two of these prisms are cubes, prove that the two have equal sides.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2025**.

OC716. Montrez que pour tout entier $n \geq 6$, il existe un hexagone convexe qui peut être scindé en exactement n triangles congruents.

OC717. Soient $a_1, a_2, \ldots, a_n$ des nombres réels positifs, et soit S_k la somme des produits de $a_1, a_2, \ldots, a_n$ pris k à la fois. Montrez que

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \ldots a_n,$$

pour $k = 1, 2, \ldots, n-1$.

OC718. Dans un triangle ABC, la médiane et la bissectrice de l'angle A rencontrent le côté BC respectivement en M et N. La perpendiculaire à NA en N rencontre MA en Q et BA en P, et la perpendiculaire à BA en P rencontre AN en O. Montrez que QO est perpendiculaire à BC.

OC719. Soit p un nombre premier et $n_1, n_2, \ldots, n_p \in \mathbb{N}^*$, dont au moins deux sont distincts. On note d le plus grand diviseur commun des nombres $n_1, n_2, \ldots, n_p$. Montrez que le polynôme

$$f = \frac{X^{n_1} + X^{n_2} + \cdots + X^{n_p} - p}{X^d - 1}$$

est irréductible dans $\mathbb{Q}[X]$.

OC720. Un cube est divisé en 27 prismes rectangulaires par des plans parallèles à ses faces. Si exactement deux de ces prismes sont des cubes, montrez que ces deux prismes ont des côtés égaux.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2024: 50(7), p. 356–357.

OC691. Prove that a number written only by zeros and ones, with the number of ones being at least two, cannot be a perfect square.

Originally Saint Petersburg Mathematical Olympiad 1962 - Qualifying Round, Problem 4.

We received 2 submissions. We include the comment by Oliver Geupel.

It is a well-known open problem whether the assertion in the problem statement is true or false. We refer to ***Crux*** Problem 909* proposed by Stan Wagon, USA [1984:20; 1985: 94-95; 2012: 144]: For which positive integers n is it true that, whenever an integer's decimal expansion contains only zeros and ones, with exactly n ones, then the integer is not a perfect square?

Editor's Comment.

The difficulty in this problem arises from the cases where the number ends in 01.

The idea is this: we can assume by contradiction that

$$N = \sum_{i=0}^{n} a_i\, 10^i, \quad a_i \in \{0,1\}, \quad \text{and} \quad \sum_{i=0}^{n} a_i \geq 2,$$

and that N is a perfect square, say $N = m^2$.

Since every digit of N is either 0 or 1, the last two digits of N can only be one of 00, 01, 10, or 11. However, it is well known that a perfect square is congruent to 0 or 1 modulo 4. Since

$$10 \equiv 2 \pmod 4 \quad \text{and} \quad 11 \equiv 3 \pmod 4,$$

the only possibilities for the last two digits of a square are 00 or 01.

If N ends in 00, then $100 \mid N$, so we can write

$$N = 100\, N_1,$$

where N_1 is also a perfect square. Moreover, the decimal digits of N_1 are also only 0's and 1's, and the number of 1's in N_1 is the same as in N (namely, at least two). Repeating this process if necessary, we may assume that our square N does not end in 00, so that in fact N ends in 01.

Since N ends in 01, we have $m^2 \equiv 1 \pmod{100}$. A routine check shows that if $m^2 \equiv 1 \pmod{100}$, then $m \equiv 1, 49, 51, 99 \pmod{100}$. Since none of $1, 49, 51, 99$

satisfies the condition, we assume $m \geq 100$, that is $m = 100k + r$, where k, r are positive integers with $r \in \{1, 49, 51, 99\}$. Now,

$$m^2 = (100k + r)^2 = 10^4 k^2 + 200kr + r^2.$$

For large k one can intuitively guess that there are other digits different from 0 and 1 in the decimal representation of m^2, but no rigorous proof has been found yet.

OC692. Prove that for any positive integer n, the number

$$n(2n + 1)(3n + 1) \cdot \ldots \cdot (1966n + 1)$$

is divisible by every prime number less than 1966.

Originally Saint Petersburg Mathematical Olympiad 1966 - Grade 7, Problem 3.

We received 9 correct solutions. We present the solution by UCLan Cyprus Problem Solving Group.

Let $p < 1966$ be a prime. If $p \mid n$, then the result is immediate. Otherwise, $n, 2n, 3n, \ldots, pn$ are all distinct modulo p. Indeed, if $in \equiv jn \pmod{p}$, then $p \mid (j - i)n$ and since $p \nmid n$, then $p \mid j - i$. But for $j \neq i$ with $i, j \in \{1, 2, \ldots, p\}$ this is impossible. So $n, 2n, \ldots, pn$ form a complete residue system modulo p and therefore there is kn such that $kn \equiv -1 \pmod{p}$. Then $p \mid kn + 1$ and since $k \leq p < 1996$ the result follows.

OC693. On the side AC of triangle ABC, point E is chosen. Bisector AL intersects segment BE at point X. It turns out that $AX = XE$ and $AL = BX$. What is the ratio of angles A and B of the triangle?

Originally Leonhard Euler Competition 2023 - 2nd Round, Day 1, Problem 4.

We received 9 solutions and we present 2 of them.

Solution 1, by Theo Koupelis.

Triangle AXE is isosceles, because $AX = XE$, and thus

$$\angle XEA = \angle XAE = \angle XAB = \angle A/2.$$

Thus, BA is tangent to the circle (AXE) at point A. Therefore,

$$AB^2 = BX \cdot BE = BX \cdot (BX + XE) = AL \cdot (AL + AX).$$

Let point D be on the ray AL such that $LD = AX$ and thus

$$AD = AL + LD = AL + AX.$$

Then $BX = AL = XD$, and thus

$$AX/XE = XD/BX = 1$$

and $BD \parallel AE$. Also, $AB^2 = AL \cdot AD$, and therefore AB is tangent to the circle (BLD). Thus,

$$\angle LAC = \angle LDB = \angle LBA,$$

or $\angle A = 2\angle B$.

Solution 2, by UCLan Cyprus Problem Solving Group.

By two applications of the Sine Law we have

$$\frac{\sin B}{\sin(A/2)} = \frac{AL}{BL} = \frac{BX}{BL} = \frac{\sin(\angle BLX)}{\sin(\angle(BXL)}.$$

Since $AX = XE$, then $\angle AEX = \angle EAX = A/2$ and so $\angle BXL = 180^\circ - A$. Furthermore $\angle BXL = 180^\circ - (B + A/2)$. Thus

$$\frac{\sin B}{\sin(A/2)} = \frac{\sin(B + A/2)}{\sin(A)}.$$

Thus

$$\sin(A)\sin(B) = \sin(A/2)\sin(B + A/2)$$

which gives

$$\frac{\cos(A-B) - \cos(A+B)}{2} = \frac{\cos(B) - \cos(A+B)}{2}.$$

We deduce that $\cos(B) = \cos(A - B)$ which, since $A - B \in (-\pi, \pi)$, occurs only if $A - B \in \{-B, B\}$. Since $A \neq 0^\circ$, then $A = 2B$, i.e. the required ratio is equal to 2.

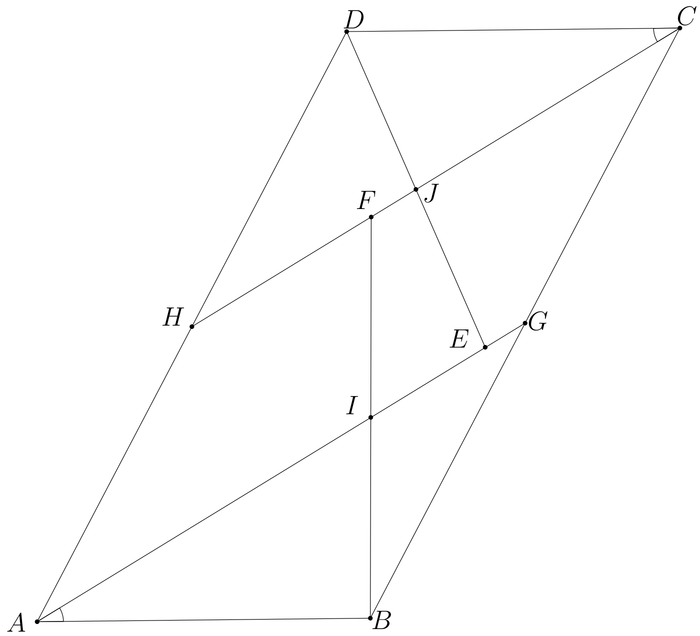
OC694. Inside the parallelogram $ABCD$, a point E is marked, lying on the bisector of angle A, and a point F is marked, lying on the bisector of angle C. It is known that the midpoint of the segment BF lies on the segment AE. Prove that the midpoint of the segment DE lies on the line CF.

Originally Leonhard Euler Competition 2023 - 2nd Round, Day 2, Problem 7.

We received 6 solutions and we present 2 of them.

Solution 1, by Oliver Geupel.

Let G be the point of intersection of the lines AE and BC, and let H be the point of intersection of the lines CF and AD. Let I be the midpoint of the segment BF. Let J be the point of intersection of the lines CF and DE.



The lines AG and CH are parallel since $\angle BAG = \angle DCH$. Hence,

$$\frac{BG}{GC} = \frac{BI}{IF} = 1.$$

By symmetry it follows that

$$\frac{DH}{HA} = \frac{BG}{GC} = 1.$$

Thus,

$$\frac{DJ}{JE} = \frac{DH}{HA} = 1,$$

that is, the line CF meets the segment DE at its midpoint.

Solution 2, by UCLan Cyprus Problem Solving Group.

Let M, N be the intersections of AE and CF with BC and AD respectively. Since

$$\angle MAD = \frac{\angle BAC}{2} = \frac{\angle DCB}{2} = \angle NCB,$$

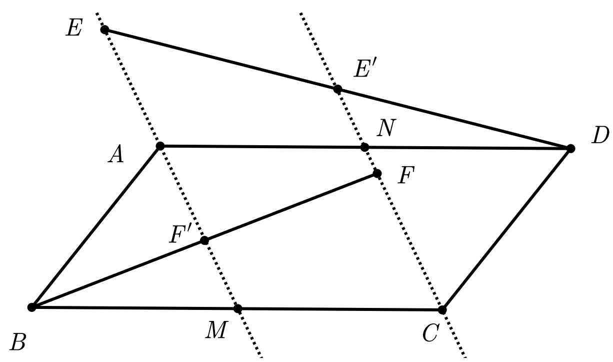
then AM and CN are parallel.

Let F' be the midpoint of BF and E' the midpoint of DE. Then

$$\frac{1}{2} = \frac{BF'}{BF} = \frac{BM}{BC}$$

and therefore M is the midpoint of BC.

The triangles BAM and DCN are congruent (all angles equal and $BA = CD$). Therefore $DN = BM = BC/2 = AD/2$. Thus N is the midpoint of AD and therefore
$$\frac{DE'}{DE} = \frac{DN}{DA} = \frac{1}{2}.$$
It follows that E' is the midpoint DE and therefore it indeed lies on CF.



OC695. Let us call two numbers *almost equal* if they are equal or differ from each other by at most 1. A checkered rectangle with side lengths equal to natural numbers a and b is such that it is impossible to cut out a rectangle along the grid lines whose area is almost equal to half the area of the original rectangle. What is the smallest value that the number $|a - b|$ can take?

Originally Leonhard Euler Competition 2023 - 2nd Round, Day 2, Problem 8.

We received 4 correct solutions. We present the solution by Oliver Geupel.

It is impossible to cut a rectangle with area 22 or 23 out of a 9-by-5 rectangle along the grid lines. Hence, the smallest value of $|a - b|$ is at most 4. We show that the smallest value is in fact equal to 4. The proof is by contradiction. Suppose $|a-b| \leq 3$. If a or b is an even number, say, a is even, then it is possible to cut out a rectangle with side lengths $a/2$ and b, which contradicts the hypothesis of the problem. Hence a and b are odd, so that $|a - b|$ is equal to either 0 or 2. Consider first the case $a = b$. If $a = b = 1$, we can cut out a 1-by-1 rectangle. If $a = b \geq 2$, then we can cut out an $(a - 1)$-by-$(a + 1)/2$ rectangle, where it holds
$$a^2/2 - (a - 1)(a + 1)/2 = 1/2,$$
a contradiction. It remains to consider the case $|a - b| = 2$. We may assume that $b = a + 2$. Since
$$(a + 1)^2/2 - a(a + 2)/2 = 1/2,$$
an $(a + 1)/2$-by-$(a + 1)$ rectangle can be cut out, a contradiction. The proof is complete.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2025**.

5011. *Proposed by Fedor Petrov and Max A. Alekseyev.*

Let a, b, c be the zeros of a cubic polynomial, and let α, β be the zeroes of its derivative. Prove that

$$\frac{(a-b)^4+(a-c)^4+(b-c)^4}{(\alpha-\beta)^4}$$

is a constant that does not depend on the polynomial.

5012. *Proposed by Tran Quang Hung and Nguyen Minh Ha.*

Given a right triangle ABC with $\angle A$ being the right angle. Construct a rectangle $MNPQ$ such that M and N lie on AB and AC, respectively, while P and Q lie on BC. Let E be the intersection of MQ and BN, and F be the intersection of NP and CM. Prove that $AE = AF$ if and only if $AB = AC$ or $MNPQ$ is a square.

5013. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Let AM be the median from the vertex A of a triangle ABC to the midpoint M of BC, and let P be the point of that median (extended, if necessary beyond A) for which $PM = BM = CM$, while Q is the projection of the orthocenter of the triangle on that median. Prove that

$$\frac{AP}{PM} = \frac{PQ}{QM}.$$

5014. *Proposed by Michel Bataille.*

Let n be an integer with $n \geq 2$ and $A \in \mathcal{M}_n(\mathbb{C})$ of rank $r \geq 1$.

a) Show that $A = XY$ for some pair $(X, Y) \in \mathcal{M}_{n,r}(\mathbb{C}) \times \mathcal{M}_{r,n}(\mathbb{C})$ such that $\operatorname{rank}(X) = \operatorname{rank}(Y) = r$.

b) If $A^2 = A$, prove that the product YX is independent of the pair (X, Y).

5015. *Proposed by Mohammad Bakkar.*

Let $\mathcal{C}, \mathcal{P}$ be the sets of circles and points in the plane, respectively. Find all functions $f : \mathcal{C} \to \mathcal{P}$ such that for all nonconcentric circles ω_1, ω_2 from $\mathcal{C}$, we have $f(\omega_1), f(\omega_2), M_{\omega_1,\omega_2}$ are collinear, where M_{ω_1,ω_2} is the intersection of the radical axis of the two circles with the line that passes through their centers.

5016. *Proposed by Yagub Aliyev.*

Let $n > 2$ and polynomial $p_n(t)$ be defined by

$$(t-1)^2 p_n(t) = ((n-1) - (n-2)t)(1 - nt^{n-1} + (n-1)t^n).$$

Prove that $p_n(t)$ has exactly one maximum point in interval $[0, +\infty)$.

5017. *Proposed by Michel Bataille.*

Let x, y, and z be non-negative real numbers such that $x + y + z = 1$. Prove that

$$(1-x)(1-y)(1-z) \geq \frac{8\sqrt{3}}{9}\sqrt{xyz}.$$

5018. *Proposed by Nguyen Viet Hung.*

Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{a^2 b}{2a+b} + \frac{b^2 c}{2b+c} + \frac{c^2 a}{2c+a} + \frac{2}{9}M \geq 1,$$

where $M = \max\{(a-b)^2, (b-c)^2, (c-a)^2\}$.

5019. *Proposed by Vasile Cirtoaje.*

Let $a_1 \geq a_2 \geq \cdots \geq a_9 \geq 0$ such that $a_1 a_2 + a_2 a_3 + \cdots + a_9 a_1 = 9$. Prove that

$$a_1 + a_2 + \cdots + a_6 \geq 6.$$

5020. *Proposed by Mihaela Berindeanu.*

Let a, b, c be three numbers greater than zero with the propriety $abc = 1$. Show that

$$\frac{\ln a}{1+a+ab} + \frac{\ln b}{1+b+bc} + \frac{\ln c}{1+c+ca} \leq 0.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2025**.

5011. *Soumis par Fedor Petrov et Max A. Alekseyev.*

Soient a, b et c les zéros d'un polynôme cubique et soient α et β les zéros de sa dérivée. Montrez que

$$\frac{(a-b)^4 + (a-c)^4 + (b-c)^4}{(\alpha-\beta)^4}$$

est une constante qui ne dépend pas du polynôme.

5012. *Soumis par Tran Quang Hung et Nguyen Minh Ha.*

Étant donné un triangle rectangle ABC dont $\angle A$ est l'angle droit. Construisons un rectangle $MNPQ$ tel que M et N se trouvent respectivement sur AB et AC, tandis que P et Q se trouvent sur BC. Soit E l'intersection de MQ et BN, et soit F l'intersection de NP et CM. Prouvez que $AE = AF$ si et seulement si $AB = AC$ ou $MNPQ$ est un carré.

5013. *Soumis par Mihaela Berindeanu, modifié par le comité de rédaction.*

Soit AM la médiane entre le sommet A d'un triangle ABC et le milieu M de BC, et soit P le point de cette médiane (prolongée, si nécessaire, au-delà de A) pour lequel $PM = BM = CM$, tandis que Q est la projection de l'orthocentre du triangle sur cette médiane. Montrez que

$$\frac{AP}{PM} = \frac{PQ}{QM}.$$

5014. *Soumis par Michel Bataille.*

Soit n un entier vérifiant $n \geq 2$ et $A \in \mathcal{M}_n(\mathbb{C})$ de rang $r \geq 1$.

a) Montrez que $A = XY$ pour un certain couple $(X, Y) \in \mathcal{M}_{n,r}(\mathbb{C}) \times \mathcal{M}_{r,n}(\mathbb{C})$ tel que $\text{rang}(X) = \text{rang}(Y) = r$.

b) Si $A^2 = A$, montrez que le produit YX est indépendant du couple (X, Y).

5015. *Soumis par Mohammad Bakkar.*

Soient $\mathcal{C}$ et $\mathcal{P}$, les ensembles des cercles et des points dans le plan, respectivement. Trouvez toutes les fonctions $f : \mathcal{C} \to \mathcal{P}$ telles que pour tous les cercles non concentriques ω_1, ω_2 de $\mathcal{C}$, on a que $f(\omega_1), f(\omega_2)$ et M_{ω_1,ω_2} sont colinéaires, où M_{ω_1,ω_2} désigne l'intersection de l'axe radical des deux cercles avec la droite qui passe par leurs centres.

5016. *Soumis par Yagub Aliyev.*

Étant donné $n > 2$, soit $p_n(t)$ le polynôme défini par

$$(t-1)^2 p_n(t) = ((n-1) - (n-2)t)(1 - nt^{n-1} + (n-1)t^n).$$

Montrez que $p_n(t)$ a exactement un point maximum dans l'intervalle $[0, +\infty)$.

5017. *Soumis par Michel Bataille.*

Soient x, y et z des nombres réels non négatifs tels que $x + y + z = 1$. Montrez que

$$(1-x)(1-y)(1-z) \geq \frac{8\sqrt{3}}{9}\sqrt{xyz}.$$

5018. *Soumis par Nguyen Viet Hung.*

Soient a, b et c des nombres réels positifs tels que $a + b + c = 3$. Montrez que

$$\frac{a^2 b}{2a+b} + \frac{b^2 c}{2b+c} + \frac{c^2 a}{2c+a} + \frac{2}{9}M \geq 1,$$

où $M = \max\{(a-b)^2, (b-c)^2, (c-a)^2\}$.

5019. *Soumis par Vasile Cirtoaje.*

Soient $a_1 \geq a_2 \geq \cdots \geq a_9 \geq 0$ tels que $a_1 a_2 + a_2 a_3 + \cdots + a_9 a_1 = 9$. Montrez que

$$a_1 + a_2 + \cdots + a_6 \geq 6.$$

5020. *Soumis par Mihaela Berindeanu.*

Soient a, b et c trois nombres plus grand que zéro avec la propriété que $abc = 1$. Montrez que

$$\frac{\ln a}{1+a+ab} + \frac{\ln b}{1+b+bc} + \frac{\ln c}{1+c+ca} \leq 0.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2024: 50(7), p. 367–370.

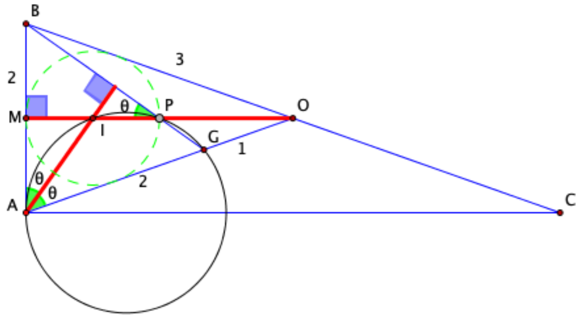
4961. *Proposed by Michel Bataille.*

Let ABC be a triangle with $\angle BAC = 90^\circ$ and $BC = 3AB$. Let O and G be the circumcenter and the centroid of ΔABC and let I be the incenter of ΔOAB. Prove that OI and BG intersect on the circumcircle of ΔAGI and that $IO = \sqrt{3}\,IA$.

All 18 of the submissions that we received were correct, and we will feature two of the approaches that were used.

Solution 1 combines ideas from the solutions of Martin Dimitrov, Chikara Tsugawa, and the group from Logroño that consists of M. Bello, M. Benito, Ó. Ciaurri and E. Fernández.

Without loss of generality we will take $AB = 2$, so that $BC = 3AB = 6$. The circumcenter O of ΔABC is the midpoint of the hypotenuse BC; consequently, ΔOAB is isosceles with $AO = BO = 3$. Moreover, since AO is a median of the given triangle ABC, we have $AG = \frac{2}{3}AO = 2$, so that $AG = AB$ and ΔABG is also isosceles. Denote by P the intersection of OI and BG.



Step 1. P lies on the circumcircle of ΔAGI. In the isosceles triangle BOA, the bisector OI of the angle at O is perpendicular to the base AB. Similarly, in the isosceles triangle AGB, the bisector AI of the angle at A is perpendicular to the base BG. Because corresponding sides of the angles BPI and IAB (marked θ in the accompanying diagram) are perpendicular, they are equal. But AI bisects $\angle GAB$, so that also $\angle GAI = \angle IAB = \theta$. Since the internal angle $\angle GAI$ at A of the quadrilateral $AIPG$ equals the external angle $\angle BPI$ at the opposite vertex P, the quadrilateral is cyclic, as desired. [*Comment.* Tsugawa observes that in an arbitrary right triangle ABC with circumcenter O on the hypotenuse BC, centroid G, I the incenter of triangle OAB, and $P = OI \cap BG$, then the

circle (AIP) intersects the ray $\overrightarrow{AG}$ at the point X for which $AB = AX$. When $BC = 3AB$ as in our problem, that point is G.]

Step 2. $IO = \sqrt{3} \cdot IA$. Denote the midpoint of AB by M. As before, M is the foot of the bisector OI of the angle at O of ΔOAB. In the right triangle AOM,

$$OM = \sqrt{OA^2 - AM^2} = \sqrt{3^2 - 1^2} = 2\sqrt{2}.$$

Since AI is the bisector of ΔOAM, I divides MO in the ratio $1 : 3$ so that

$$MI = \frac{\sqrt{2}}{2} \quad \text{and} \quad IO = \frac{3\sqrt{2}}{2}.$$

Finally, in the right triangle IAM, $IA = \sqrt{AM^2 + MI^2} = \sqrt{1 + \frac{1}{2}} = \frac{\sqrt{3}}{\sqrt{2}}$, whence $IO = \sqrt{3} \cdot IA$, as claimed.

Solution 2, by the Eagle Problem Solvers of Georgia Southern University.

Position A at the origin, B at $(0, 1)$ and C at $(2\sqrt{2}, 0)$, so that $BC = 3$. [Observe that the scale used here is half of that used in Solution 1.] Because the chord BC subtends a right angle $\angle BAC$, BC must be a diameter of the circumcircle of ΔABC, while its midpoint O is the circumcenter; thus, $O = \left(\sqrt{2}, \frac{1}{2}\right)$. Notice that $OA = OB = \frac{3}{2}$, so that ΔOAB is isosceles with a perimeter of 4. Since the coordinates of the incenter I are the weighted averages of the coordinates of the vertices, then

$$I = \frac{3}{8}(0, 0) + \frac{3}{8}(0, 1) + \frac{1}{4}(\sqrt{2}, \frac{1}{2}) = \left(\frac{\sqrt{2}}{4}, \ \frac{1}{2}\right),$$

and OI is the horizontal line $y = \frac{1}{2}$. We compute

$$IA = \sqrt{\frac{1}{8} + \frac{1}{4}} = \frac{\sqrt{3}\sqrt{2}}{4} \text{ and } IO = \sqrt{2} - \frac{\sqrt{2}}{4} = \frac{3\sqrt{2}}{4} = \sqrt{3}\, IA.$$

Meanwhile, the coordinates of the centroid G are the averages of the coordinates of the vertices A, B, and C, so $G = \left(\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$. Thus, the line BG has slope $\frac{-1}{\sqrt{2}}$ and equation $y = -\frac{x}{\sqrt{2}} + 1$. The point P of intersection of OI and BG is therefore $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$. According to Ptolemy's theorem, the points A, I, P, and G are concyclic (in that order) if and only if $AI \cdot PG + IP \cdot AG = AP \cdot IG$. We compute

$$AI \cdot PG + IP \cdot AG = \frac{\sqrt{6}}{4} \cdot \frac{\sqrt{3}}{6} + \frac{\sqrt{2}}{4} \cdot 1 = \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{4} = \frac{3\sqrt{2}}{8} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{6}}{4} = AP \cdot IG.$$

Thus, by Ptolemy's theorem, P lies on the circumcircle of ΔAGI.

Editor's comments. As suggested by the diagram and proved tacitly in the second solution, the incircle of ΔOAB also contains P (the point common to the the lines

OI and BG, as well as the circumcircle of triangle AGI). Surprisingly, although several readers essentially provided a proof and a suggestive diagram, only Didier Pinchon explicitly stated the result. Equally surprisingly, nobody commented that the center of the circle $(AIPG)$ lies on the leg AC of the given triangle.

4962. *Proposed by Leonard Giugiuc and Richdad Phuc.*

Prove that if ABC is an acute angled triangle, then

$$\frac{\cos A}{\cos(B-C)}+\frac{\cos B}{\cos(C-A)}+\frac{\cos C}{\cos(A-B)}\le 2(\cos^2 A+\cos^2 B+\cos^2 C).$$

There were 4 correct and two incomplete solutions. We present 2 solutions.

Solution 1, by the proposers. Since

$$\cos(B-C)=\cos A+\cos(B+C)+\cos(B-C)=\cos A+2\cos B\cos C,$$

then

$$\frac{\cos A}{\cos(B-C)}=\frac{\cos^2 A}{\cos^2 A+2\cos A\cos B\cos C}=\frac{1}{1+x},$$

where $x=(2\cos B\cos C)/\cos A$. Similarly

$$\frac{\cos B}{\cos(C-A)}=\frac{1}{1+y}\quad\text{and}\quad\frac{\cos C}{\cos(A-B)}=\frac{1}{1+z},$$

where

$$y=\frac{2\cos C\cos A}{\cos B}\quad\text{and}\quad z=\frac{2\cos A\cos B}{\cos C}.$$

The inequality to be established is

$$2\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right)\le xy+yz+zx.$$

Now

$$\begin{aligned}\cos^2 A&+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C\\&=\cos^2(B+C)+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C\\&=\cos^2 B\cos^2 C-2\sin B\sin C\cos B\cos C+(1-\cos^2 B)(1-\cos^2 C)\\&\quad+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C\\&=1+2\cos^2 B\cos^2 C+2\cos B\cos C(-\sin B\sin C+\cos A)\\&=1+2\cos B\cos C(\cos B\cos C-\sin B\sin C-\cos(B+C))\\&=1.\end{aligned}$$

Since $\cos^2 A=yz/4$, $\cos^2 B=zx/4$ and $\cos^2 C=xy/4$, we find that

$$xy+yz+zx+xyz=4.$$

Let $p = x + y + z$, $q = xy + yz + zx$ and $r = xyz$, so that $q + r = 4$. By the arithmetic-geometric means inequality, $q \geq 3r^{2/3}$, so that $4 \geq r^{2/3}(3 + r^{1/3})$, whence $0 \leq r \leq 1$. We have that $3 \leq 4 - r = q \leq 4$. Observe that

$$\begin{aligned} q^2 - 3pr &= x^2y^2 + y^2z^2 + z^2x^2 + 2pr - 3pr \\ &= \tfrac{1}{2}((xy - yz)^2 + (yz - zx)^2 + (zx - xy)^2) \geq 0, \end{aligned}$$

so that $q^2 \geq 3pr$.

In terms of p, q, r, the desired inequality is

$$2\left(\frac{q + 2p + 3}{5 + p}\right) \leq q,$$

or $pr = p(4-q) \leq 3q-6$. This is now clear. Since $0 \geq (q-3)(q-6) = q^2-3(3q-6)$, it follows that

$$pr \leq \frac{q^2}{3} \leq 3q - 6.$$

Equality occurs if and only if the triangle is equilateral.

Solution 2, by Sicheng Du.

Let $x = \cot B \cot C$, $y = \cot A \cot C$, $z = \cot A \cot B$ and also $p = x + y + z$, $q = xy + yz + zx$ and $r = xyz$. Since the triangle is acute, all these quantities are positive.

We begin with some preliminary observations.

$$\begin{aligned} 0 = \sin(A + B + C) &= \sin(A + B)\cos C + \cos(A + B)\sin C \\ &= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C \\ &= \sin A \sin B \sin C(\cot B \cot C + \cot A \cot C + \cot A \cot B - 1). \end{aligned}$$

Then $p = x + y + z = 1$.

We verify that $q > r$. With $K = (\cot A \cot B \cot C)(\sin A \sin B \sin C)^{-1}$, we have

$$\begin{aligned} q - r = xy + yz + zx - xyz &= \cot A \cot B \cot C(\cot A + \cot B + \cot C - \cot A \cot B \cot C) \\ &= K(\cos A \sin B \sin C + \sin A \cos B \sin C + \sin A \sin B \cos C - \cos A \cos B \cos C) \\ &= K(-\cos A \cos(B + C) + \sin A \sin(B + C)) = K(\cos^2 A + \sin^2 A) = K > 0. \end{aligned}$$

As in Solution 1, we can show that $q^2 \geq 3pr = 3r$. Finally,

$$1 - 3q = (x + y + z)^2 - 3(xy + yz + zx) = \tfrac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] \geq 0.$$

We have that

$$\frac{\cos A}{\cos(B - C)} = \frac{-\cos(B + C)}{\cos(B - C)} = \frac{1 - x}{1 + x}$$

and

$$\cos^2 A = \frac{\cot^2 A}{1+\cot^2 A} = \frac{yz}{x+yz} = \frac{yz(y+z)}{(x+y)(x+z)(y+z)}.$$

The inequality to be established is

$$\frac{1-x}{1+x} + \frac{1-y}{1+y} + \frac{1-z}{1+z} \le 2 \cdot \frac{xy(x+y)+yz(y+z)+zx(x+z)}{(x+y)(y+z)(z+x)}.$$

The left side of this inequality is equal to

$$\frac{3+p-q-3r}{1+p+q+r} = \frac{4-q-3r}{2+q+r}$$

and the right side is equal to

$$2 \cdot \frac{xy(1-z)+yz(1-x)+zx(1-y)}{(1-z)(1-x)(1-y)} = \frac{2q-6r}{q-r}.$$

Since

$$\begin{aligned}(2q-6r)(2+q+r)-(4-q-3r)(q-r) &= 3q^2-8r-2qr-9r^2 \\ &\ge 3q^2-(8q^2/3)-(2q^3/3)-q^4 \\ &= \tfrac{1}{3}q^2(1+q)(1-3q) \\ &\ge 0,\end{aligned}$$

the desired inequality follows.

Comments by the editor. Michel Bataille considered a triangle with sides d, e, f, angles $D = 180^\circ - 2A$, $E = 180^\circ - 2B$, $F = 180^\circ - 2C$, inradius r and circumradius R. Then

$$\frac{\cos A}{\cos(B-C)} = \frac{2\sin(D/2)\cos(D/2)}{2\cos(D/2)\cos((F-E)/2} = \frac{\sin D}{\sin E + \sin F} = \frac{d}{e+f}$$

and

$$\begin{aligned}2(\cos^2 A + \cos^2 B + \cos^2 C) &= 2(\sin^2(D/2) + \sin^2(E/2) + \sin^2(F/2)) \\ &= 3 - (\cos D + \cos E + \cos F) \\ &= 2 - \frac{r}{R}.\end{aligned}$$

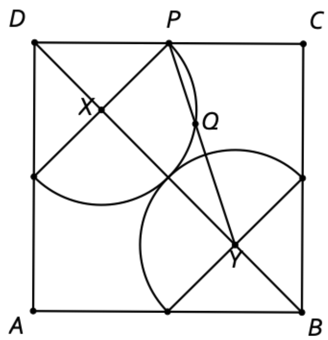
The required inequality is equivalent to

$$\frac{d}{e+f} + \frac{e}{f+d} + \frac{f}{d+e} \le 2 - \frac{r}{R}.$$

This was posed earlier in *Crux* as Problem 4212, with the solution appearing in *Crux* 44:2 (February, 2018), 74.

4963. *Proposed by Arsalan Wares.*

Two congruent semicircular arcs touch at the midpoint of diagonal BD of square $ABCD$ as shown. The diameters of the congruent semicircular arcs are perpendicular to diagonal BD. The arcs have terminal points on the sides of square $ABCD$. Points X and Y are the centers of the arcs. Point P is a terminal point, as shown. Point Q is the point of intersection of PY and one of the arcs, as shown. Determine the exact value of PQ/QY.



We received 29 submissions, all of which were correct. We present 7 solutions.

Solution 1, by Michali Adamaszek.

PXY is a right triangle with $PX : XY = 1 : 2$ and PXQ is isosceles. Let M be the midpoint of PQ, which is also the base of the altitude from vertex X in triangle PXY. One easily checks that in a right triangle with legs a, b the base of altitude divides the hypotenuse in ratio $a^2 : b^2$, in this case $PM : MY = 1 : 4$, therefore PQ is $2/5$ of the hypotenuse and $PQ : QY = 2 : 3$

Solution 2, by Brian D. Beasley.

In right triangle PXY, let $r = PX$ and $\theta = \angle XPY$. Since $XY = 2r$, we obtain

$$\cos\theta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \sin\theta = \frac{2}{\sqrt{5}}.$$

This in turn implies

$$\cos(2\theta) = \left(\frac{1}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2 = -\frac{3}{5} \quad \text{and} \quad \sin(2\theta) = 2\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{4}{5}.$$

Next, in triangle PXQ, since $PX = QX$, we have $\angle PQX = \theta$ and therefore $\angle PXQ = \pi - 2\theta$. Then the Law of Cosines yields

$$PQ^2 = 2r^2 - 2r^2\cos(\pi - 2\theta) = r^2[2 + 2\cos(2\theta)].$$

Similarly, in triangle YXQ, we have $\angle QXY = 2\theta - \frac{\pi}{2}$ and hence

$$QY^2 = r^2 + 4r^2 - 4r^2\cos\left(2\theta - \frac{\pi}{2}\right) = r^2[5 - 4\sin(2\theta)].$$

Thus

$$\frac{PQ}{QY} = \sqrt{\frac{2+2\cos(2\theta)}{5-4\sin(2\theta)}} = \sqrt{\frac{2+2(-3/5)}{5-4(4/5)}} = \frac{2}{3}.$$

Solution 3, by Brian Bradie.

We have:

$$\begin{aligned}\angle PXQ &= \pi - 2\tan^{-1} 2 = 2\left(\frac{\pi}{2} - \tan^{-1} 2\right) = 2\tan^{-1}\frac{1}{2}\\ &= \tan^{-1}\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{2}\cdot\frac{1}{2}} = \tan^{-1}\frac{4}{3}.\end{aligned}$$

It then follows from the Law of Cosines that

$$PQ^2 = r^2 + r^2 - 2r^2\cos\angle PXQ = 2r^2\left(1-\frac{3}{5}\right) = \frac{4r^2}{5},$$

so

$$PQ = \frac{2r\sqrt{5}}{5}.$$

Next,

$$QY = r\sqrt{5} - \frac{2r\sqrt{5}}{5} = \frac{3r\sqrt{5}}{5},$$

and

$$\frac{PQ}{QY} = \frac{2}{3}.$$

Solution 4, by UCLan Cyprus Problem Solving Group.

Use coordinates $A=(0,0), B=(4,0), C=(4,4)$ and $D=(0,4)$.

Then we have $X=(t,4-t)$ for some $t \in (0,2)$. Since PX is perpendicular to BD, its gradient is 1 and so its equation is $y=x+c$. Since X belongs to it we get $c=4-2t$. Since the $y_P=4$, then $P=(2t,4)$. We have $PX^2=2t^2$.

Letting M be the midpoint of BD, we have $M=(2,2)$ and so $PM^2=2(2-t)^2$. Since $PX^2=PM^2$ we get $t=1$. Thus $X=(1,3)$ and $P=(2,4)$. Similarly we have $Y=(3,1)$. The equation of the circle centred at X of radius XM is

$$(x-1)^2+(y-3)^2=2$$

and the equation of PY is $y=-3x+10$. So at $Q=(x,y)$ we have

$$2=(x-1)^2+(-3x+7)^2=10x^2-44x+50$$

giving $x=2$ or $x=12/5$. [We know that $x=2$ is one solution since P belongs on both the circle and the line. Since the product of the solutions (by Vieta) is $48/10$, then the other solution is $x=12/5$.]

Therefore $Q = (12/5, 14/5)$. Then

$$\frac{PQ^2}{QY^2} = \frac{(2/5)^2 + (6/5)^2}{(3/5)^2 + (9/5)^2} = \frac{4+36}{9+81} = \frac{4}{9}.$$

Thus $PQ/QY = 2/3$.

Solution 5, by M. Bello, M. Benito, Ó. Ciaurri and E. Fernández.

Let M denotes the midpoint of the diagonal BD. Suppose wlog that $AB = 2$, and so $DM = \sqrt{2}$. Let P' be the other terminal point of the semicircle with center X. We have $\angle PMP' = \pi/2$ and $\angle XPM = \pi/4$, so MP is perpendicular to CD, and P is the midpoint of CD. Then, $XD = XP = XM = \sqrt{2}/2$. In the same way $YM = \sqrt{2}/2$.

The power of point Y with respect the circle with center X gives

$$PY \cdot QY = YM \cdot YD = \frac{\sqrt{2}}{2} \cdot \frac{3\sqrt{2}}{2} = \frac{3}{2}.$$

But $PY = \sqrt{PX^2 + XY^2} = \sqrt{5/2}$, so

$$QY = \frac{3/2}{\sqrt{5/2}} = \frac{3}{\sqrt{10}}.$$

On the other hand,

$$PQ = PY - QY = \sqrt{\frac{5}{2}} - \frac{3}{\sqrt{10}} = \sqrt{\frac{2}{5}}$$

and, consequently,

$$\frac{PQ}{QY} = \frac{\sqrt{2}/\sqrt{5}}{3/\sqrt{10}} = \frac{2}{3}.$$

Solution 6, by Ángel Plaza.

Note that triangle PXB is right with right angle at X and $XY = 2PX$, angle $\angle XYP = \arctan\frac{1}{2}$, angle $\angle YPX = \frac{\pi}{2} - \arctan\frac{1}{2}$. Since $\angle PQX = \angle XYP$, then

$$\angle PXQ = \pi - 2\angle QPX = 2\arctan\frac{1}{2} \quad \text{and} \quad \angle QXY = \frac{\pi}{2} - 2\arctan\frac{1}{2}.$$

Now in order to determine the exact value of PQ/QY, it is enough to apply the cosine law to triangles QPX and YPX. Let $PX = a$, and $XY = 2a$, then

$$PQ^2 = 2a^2 - 2a^2\cos\left(2\arctan\frac{1}{2}\right) = a^2\frac{4}{5}$$

and

$$QY^2 = 5a^2 - 4a^2\sin\left(2\arctan\frac{1}{2}\right) = a^2\frac{9}{5}.$$

Therefore, $PQ/QY = 2/3$.

Solution 7, by Theo Koupelis.

Let K be the midpoint of BD. We have $AC \perp BD$ and the diagonals AC, BD bisect the right angles at the verices of the square. Because the arcs are congruent and $PX \perp BD$, we have $\angle DPX = \angle DCA = 45°$, and thus

$$DX = PX = XK = KY = YB := x.$$

From the right triangle PXY we get $PY = \sqrt{5}x$. Using the power of point Y with respect to circle (X, x) we get

$$YQ \cdot YP = YX^2 - XP^2 \quad \text{or} \quad YQ \cdot \sqrt{5}x = 4x^2 - x^2,$$

and thus $YQ = 3\sqrt{5}x/5$. Therefore,

$$PQ = YP - YQ = 2\sqrt{5}x/5$$

and $PQ/QY = 2/3$.

4964. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(-x) = 1 - \int_0^x \mathrm{e}^{-t} f(x-t)\mathrm{d}t, \quad \forall x \in \mathbb{R}.$$

We received 18 submissions, 14 of which found $f(x) = x + 1$ correctly, but only 8 showed it actually works. We present the solution by Michel Bataille.

We show that the function f_0 defined by $f_0(x) = x + 1$ is the unique solution.

First, note that for any continuous function f, we have

$$\int_0^x e^{-t} f(x-t)\,dt = \int_0^x e^{u-x} f(u)\,du = e^{-x}F(x)$$

where F is the differentiable function defined by $F(x) = \int_0^x e^u f(u)\,du$ (so that $F'(x) = e^x f(x)$). In particular, we obtain

$$F_0(x) := \int_0^x e^u f_0(u)\,du = \int_0^x (u+1)e^u\,du = ue^u\Big|_0^x = xe^x$$

so that $1 - e^{-x}F_0(x) = 1 - x = f_0(-x)$ for all x, proving that f_0 is a solution.

Conversely, let f be any solution. Then we have $f(x) = 1 - e^x F(-x)$ for all x; hence f is differentiable on $\mathbb{R}$ (as F is) and

$$f'(x) = -e^x F(-x) + e^x F'(-x) = f(x) - 1 + f(-x).$$

Therefore f' is differentiable and, for all x,

$$f''(x) = f'(x) - f'(-x) = 0.$$

It follows that f' is constant. Since $f(0) = 1 = f'(0)$, we see that $f'(x) = 1$ and $f(x) = x + 1$ for all x. This completes the proof.

Editor's Comments. Every correct submission used the substitution $u = x - t$ and the factorization $e^{u-x} = e^{-x}e^u$ to eliminate the dependence of the integrand on x. Ultimately this led most solvers to conclude that $f'' = 0$. Letting $f(x) = ax + b$ shows that the original integral equation is equivalent to the global vanishing of

$$g(x) := (2b - a - 1) + (a - b)e^{-x}.$$

Since 1 and the exponential are linearly independent (as functions on $\mathbb{R}$), g is constantly zero if and only if

$$2b - a - 1 = 0 \quad \text{and} \quad a - b = 0$$

i.e., if and only if $a = b = 1$.

Kernels other than $\theta(t) = e^{-t}$ are possible. P. Perfetti took $\theta(t) = ke^{-t}$ and showed that

$$f(x) = \frac{k}{\sqrt{k^2-1}} \sin\sqrt{k^2-1}x + \frac{k}{k+1}\cos\sqrt{k^2-1}x + \frac{1}{k+1}$$

for $k \neq 1$. E. J. Ionaşcu pointed out that if $\theta(t) = -\cos t$ then

$$f(x) = 1 + \frac{\cos\phi x - \cos\phi^{-1}x}{\sqrt{5}} - \frac{\sin\phi x + \sin\phi^{-1}x}{\sqrt{5}}$$

where ϕ is the golden mean. Is there a general formula for f in terms of θ?

4965. *Proposed by Ángel Plaza.*

Prove that the following identities hold:

a) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+4} - \cdots\right) = \frac{7\pi^2}{96} - \frac{\ln^2 2}{8},$$

b) $$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+4} - \cdots\right) = \frac{\ln^2 2}{8} + \frac{11\pi^2}{96},$$

We received 18 submissions, of which 16 were correct. We present the solution by Gordon Russ.

For each positive integer n, let

$$c_n = \frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+4} - \frac{1}{n+6} + \cdots.$$

Our goal is to find the sums of the series $\sum_{n=1}^{\infty}(-1)^{n+1}c_n/n$ and $\sum_{n=1}^{\infty}c_n/n$. We first look at the even and odd terms of the sequence $\{c_n\}$ separately by defining

$$a_n = 2c_{2n} = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \cdots;$$
$$b_n = c_{2n-1} = \frac{1}{2n-1} - \frac{1}{2n+1} + \frac{1}{2n+3} - \frac{1}{2n+5} + \cdots.$$

Recalling two familiar sums, we note that $a_1 = \ln 2$ and $b_1 = \pi/4$. It is easy to verify that

$$0 < a_n \le \frac{1}{n} \qquad \text{and} \qquad 0 < b_n \le \frac{1}{2n-1}$$

for each positive integer n. It follows that the series $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ both converge; we denote these sums by A and B, respectively. Using the fact that $a_n + a_{n+1} = 1/n$ for each n, we find that

$$\begin{aligned}\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} &= \sum_{n=1}^{\infty}(a_n + a_{n+1})^2 = \sum_{n=1}^{\infty}(a_n^2 + 2a_na_{n+1} + a_{n+1}^2)\\ &= A + \sum_{n=1}^{\infty} 2a_n\left(\frac{1}{n} - a_n\right) + A - a_1^2 = 2\sum_{n=1}^{\infty}\frac{a_n}{n} - a_1^2;\end{aligned}$$

and thus

$$\sum_{n=1}^{\infty}\frac{a_n}{n} = \frac{\pi^2}{12} + \frac{a_1^2}{2} = \frac{\pi^2}{12} + \frac{(\ln 2)^2}{2}.$$

Similarly, using the fact that $b_n + b_{n+1} = 1/(2n-1)$ for each n, we obtain

$$\begin{aligned}\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} &= \sum_{n=1}^{\infty}(b_n + b_{n+1})^2 = \sum_{n=1}^{\infty}(b_n^2 + 2b_nb_{n+1} + b_{n+1}^2)\\ &= B + \sum_{n=1}^{\infty} 2b_n\left(\frac{1}{2n-1} - b_n\right) + B - b_1^2 = 2\sum_{n=1}^{\infty}\frac{b_n}{2n-1} - b_1^2;\end{aligned}$$

and thus

$$\sum_{n=1}^{\infty}\frac{b_n}{2n-1} = \frac{\pi^2}{16} + \frac{b_1^2}{2} = \frac{3\pi^2}{32}.$$

Given the values for these two sums, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{c_n}{n} &= \sum_{n=1}^{\infty}\frac{c_{2n-1}}{2n-1} + \sum_{n=1}^{\infty}\frac{c_{2n}}{2n}\\ &= \sum_{n=1}^{\infty}\frac{b_n}{2n-1} + \sum_{n=1}^{\infty}\frac{a_n}{4n}\\ &= \frac{3\pi^2}{32} + \left(\frac{\pi^2}{48} + \frac{(\ln 2)^2}{8}\right)\\ &= \frac{11\pi^2}{96} + \frac{(\ln 2)^2}{8},\end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} c_n}{n} = \sum_{n=1}^{\infty} \frac{c_{2n-1}}{2n-1} - \sum_{n=1}^{\infty} \frac{c_{2n}}{2n}$$
$$= \sum_{n=1}^{\infty} \frac{b_n}{2n-1} - \sum_{n=1}^{\infty} \frac{a_n}{4n}$$
$$= \frac{3\pi^2}{32} - \left(\frac{\pi^2}{48} + \frac{(\ln 2)^2}{8}\right)$$
$$= \frac{7\pi^2}{96} - \frac{(\ln 2)^2}{8}.$$

This completes the proof.

Editor's Comments. We chose the featured solution for its simplicity. Some solvers showed that

$$(a) = \int_0^1 \frac{\ln(1+x)}{x(1+x^2)}\,dx \quad \text{and} \quad (b) = -\int_0^1 \frac{\ln(1-x)}{x(1+x^2)}\,dx$$

and then evaluated the integrals.

4966. *Proposed by Vasile Cîrtoaje.*

For given $n \geq 2$, find the largest integer k such that

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_k^2}{k}$$

for all nonnegative numbers $a_1, a_2, \ldots, a_n$ satisfying $a_1 \leq a_2 \leq \cdots \leq a_n$.

We received 6 submissions and 5 of them were correct. We present the following solution by the majority of solvers.

Note that if $a_1 = 0$ and $a_2 = a_3 = \ldots = a_n = 1$, then we get

$$\frac{(n-1)^2}{n^2} \geq \frac{k-1}{k} \Longrightarrow k \leq \frac{n^2}{2n-1} = \frac{n}{2} + \frac{n}{2(2n-1)} < \frac{n+1}{2}.$$

It follows that the largest k satisfying the given inequality and condition is at most $\lfloor n/2 \rfloor$. We show that we can indeed take $k = \lfloor n/2 \rfloor$.

First consider the case n is even. Write $n = 2m$, we need to show

$$\left(\frac{a_1 + a_2 + \cdots + a_{2m}}{2m}\right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_m^2}{m}. \tag{1}$$

Equivalently, we need to show

$$m \cdot (a_1 + a_2 + \cdots + a_{2m})^2 \geq 4m^2(a_1^2 + a_2^2 + \cdots + a_m^2)$$
$$\Longleftrightarrow \sum_{j=1}^{2m} a_j^2 + \sum_{1 \leq j < \ell \leq 2m} 2a_j a_\ell \geq 4m \cdot (a_1^2 + a_2^2 + \cdots + a_m^2).$$

From the given condition $a_1 \leq a_2 \leq \cdots \leq a_n$ we get $\sum_{j=1}^{2m} a_j^2 \geq 2\sum_{j=1}^{m} a_j^2$, and thus it suffices to show that

$$\sum_{1\leq j<\ell\leq 2m} a_j a_\ell \geq (2m-1)(a_1^2+a_2^2+\cdots+a_m^2). \tag{2}$$

But using the given condition we get

$$\begin{aligned} a_1(a_2+a_3+\cdots+a_{2m}) &\geq (2m-1)a_1^2, \\ a_2(a_3+a_4+\cdots+a_{2m}) + a_{2m-1}a_{2m} &\geq (2m-1)a_2^2, \\ a_3(a_4+a_5+\cdots+a_{2m}) + a_{2m-2}(a_{2m-1}+a_{2m}) &\geq (2m-1)a_3^2, \\ \cdots\cdots \\ a_m(a_{m+1}+a_{m+2}+\cdots+a_{2m}) + a_{m+1}(a_{m+2}+\cdots+a_{2m}) &\geq (2m-1)a_m^2. \end{aligned}$$

Adding the above we get inequality (2), as required.

Finally, we consider the case that n is odd. Write $n = 2m+1$. Then we have

$$\left(\frac{a_1+a_2+\cdots+a_{2m+1}}{2m+1}\right)^2 \geq \left(\frac{a_1+a_2+\cdots+a_{2m}}{2m}\right)^2 \geq \frac{a_1^2+a_2^2+\cdots+a_m^2}{m}$$

by inequality (1) and the assumption $a_1 \leq a_2 \leq \cdots \leq a_{2m} \leq a_{2m+1}$, as required.

4967. *Proposed by Marian Ursărescu.*

Consider triangle ABC, where the a, b, c are lengths of the sides BC, AC, AB, respectively. Suppose that $a = \dfrac{b+c}{2}$, r is the inradius and R is the circumradius. Prove that:

$$\sin^2\frac{A}{2} \geq \frac{r}{2R}.$$

We received 23 correct solutions for this problem. Many solvers pointed out that under the conditions of the problem strict equality holds true. We present 2 solutions.

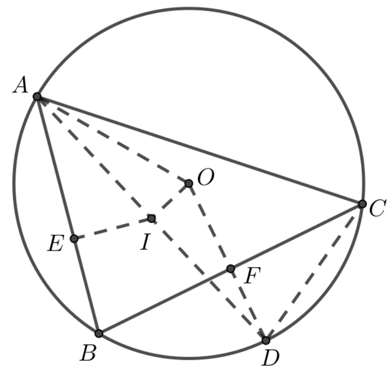
Solution 1, by Michal Adamaszek.

Since $R = \dfrac{a}{2\sin A}$ and $\tan\dfrac{A}{2} = \dfrac{r}{\frac{b+c-a}{2}} = \dfrac{2r}{a}$ we have

$$\frac{r}{2R} = \frac{1}{2}\sin A \tan\frac{A}{2} = \sin^2\frac{A}{2}.$$

Solution 2, by Sicheng Du.

Let I and O be the incenter and circumcenter of $\triangle ABC$. Extend AI intersecting the circumcircle at D, then D is the midpoint of arc BC. Let OD intersect BC at F, then $BF = FC$, and $OF \perp BC$.



Let the incircle be tangent to AB at E, then $IE \perp AB$, and $AE = \dfrac{b+c-a}{2}$. Since $b+c=2a$, then $AE = \frac{a}{2} = FC$. Moreover,

$$\angle BAD = \angle BCD \quad \text{and} \quad \angle AEI = \frac{\pi}{2} = \angle CFD.$$

So $\triangle AEI \cong \triangle CFD$, then $DC = IA$. By the incenter lemma, $DC = DI$, so $IA = DI$.

Hence, $AI \perp IO$. By Euler's theorem, $IO^2 = R^2 - 2Rr$, then by the Pythagorean theorem, $AI^2 = AO^2 - IO^2 = 2Rr$. Therefore,

$$\sin^2 \frac{A}{2} = \sin^2 \angle EAI = \frac{EI^2}{AI^2} = \frac{r^2}{2Rr} = \frac{r}{2R}.$$

Editor's Comments. Chikara Tsugawa found that for any triangle

$$\sin^2 \frac{A}{2} = \frac{r}{2R} \cdot \frac{a}{b+c-a},$$

from which a more general result follows:

- Suppose that $2a > b+c$. Then, $b+c-a < a$. Hence, $\frac{a}{b+c-a} > 1$. Therefore,

$$\sin^2 \frac{A}{2} = \frac{r}{2R} \cdot \frac{a}{b+c-a} > \frac{r}{2R}$$

- Suppose that $2a = b+c$. Then, $b+c-a = a$. Hence, $\frac{a}{b+c-a} = 1$. Therefore,

$$\sin^2 \frac{A}{2} = \frac{r}{2R} \cdot \frac{a}{b+c-a} = \frac{r}{2R}$$

- Suppose that $2a < b+c$. Then, $b+c-a > a$. Hence, $\frac{a}{b+c-a} < 1$.Therefore,

$$\sin^2 \frac{A}{2} = \frac{r}{2R} \cdot \frac{a}{b+c-a} < \frac{r}{2R}$$

4968. *Proposed by Dacian-Dumitru Robu.*

Find all monotonic functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$(x+y)f(xf(y-1)+yf(x-1)) = f(2xy)f(x+y-1)$$

for all $x, y \in \mathbb{Z}$.

We received 18 submissions and 13 of them were correct. We present the following solution by the majority of solvers.

We will show that either f is identically 0 or $f(x) = x+1$ for every $x \in \mathbb{Z}$. It is easy to check that both of these functions satisfy the conditions.

For $y = -x$, we get $f(-2x^2)f(-1) = 0$. So $f(-1) = 0$ or $f(-2x^2) = 0$ for every $x \in \mathbb{Z}$. In the second case we have $f(0) = f(-2) = 0$ and by monotonicity $f(-1) = 0$. So in both cases, we have $f(-1) = 0$. For $y = 0$ we now get

$$xf(0) = f(0)f(x-1).$$

If $f(0) \neq 0$, we deduce that $f(x) = x+1$ for every $x \in \mathbb{Z}$ and we are done. Next assume that $f(0) = 0$.

We first show that $f(n) = 0$ for every positive integer n. Suppose otherwise that this is not the case; let k be the minimum positive integer such that $f(k) \neq 0$. Then for $x = k, y = 1$ we get

$$(k+1)f(kf(0)+f(k-1)) = f(2k)f(k).$$

Since $f(k-1) = f(0) = 0$, it follows that $f(2k)f(k) = 0$ and thus $f(2k) = 0$. However, the monotonicity gives $f(k) = 0$, violating the choice of k.

Finally, we show that $f(-n) = 0$ for every positive integer n using a similar approach. Suppose otherwise that this is not the case; let k be the minimum positive integer such that $f(-k) \neq 0$. Since $f(-1) = 0$, we have $k \geq 2$. For $x = 1$ and $y = -k$ we get

$$(1-k)f(f(-k-1)) = f(-2k)f(-k).$$

Next we consider the following two cases:

- Assume $f(-k) > 0$. Then $f(-k-1) > 0$ by monotonicity and thus we have $f(f(-k-1)) = 0$. It follows that $f(-2k) = 0$, contradicting the monotonicity of f.
- Assume $f(-k) < 0$. For $x = 2k$ and $y = 1-k$, we get

  $$(k+1)f\big(2kf(-k)+(1-k)f(2k-1)\big) = f(2k(1-k))f(k+1) = 0.$$

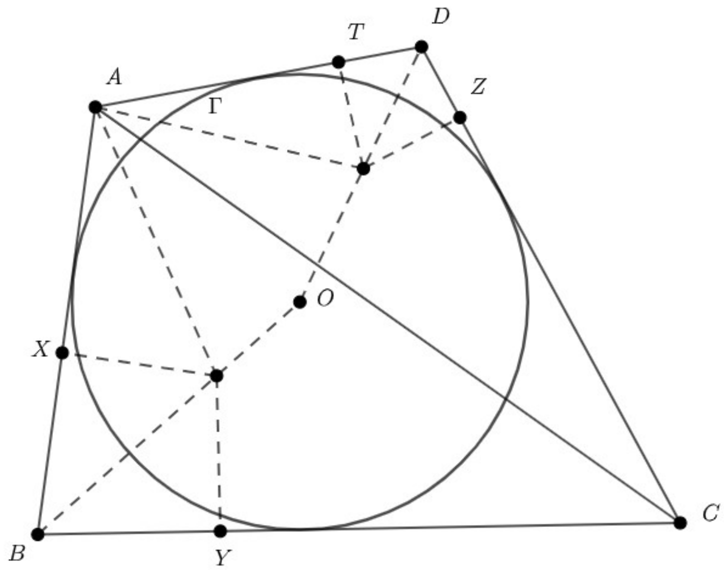
  It follows that $f(2kf(-k)) = 0$. However, $2kf(-k) \leq -2k$, contradicting the monotonicity of f.

Thus, we have proved that f is identically zero.

4969. *Proposed by Mihaela Berindeanu.*

Let $ABCD$ be a circumscribed quadrilateral to a circle Γ and let Ω_1, Ω_2 be the incircles of $\triangle ABC$, respectively $\triangle ACD$. Finally, let $\Omega_1 \cap AB = \{X\}$, $\Omega_1 \cap BC = \{Y\}$, $\Omega_2 \cap CD = \{Z\}$ and $\Omega_2 \cap AD = \{T\}$. If $XY \equiv ZT$, show that $YZ \parallel XT$.

We received 12 submissions, all of which were correct, and we feature the solution by Ivan Hadinata, with some details added by the editor.



Let O be the centre of the incircle Γ. Because O is equidistant from the sides, the bisectors of $\angle A$, $\angle B$, $\angle C$, $\angle D$ of quadrilateral $ABCD$ pass through O. The key to what follows is the Euclidean theorem: The two tangents to a circle from an external point have the same length. The theorem applied to Γ gives us

$$AB + CD = AD + BC, \tag{1}$$

and applied to the incircle Ω_1 of triangle ACD yields

$$AC = AT + CZ, \quad AD = AT + DT, \quad \text{and} \quad CD = DT + CZ. \tag{2}$$

The equations in (2) imply that

$$AT = \frac{1}{2}(AC + AD - CD).$$

Similarly (using triangle ABC and its incircle) we have

$$AX = \frac{1}{2}(AC + AB - BC).$$

But from equation (1) we have $AD - CD = AB - BC$, so that

$$AX = AT.$$

Similarly,

$$CY = \frac{1}{2}(AC + BC - AB) = \frac{1}{2}(AC + CD - AD) = CZ.$$

Therefore, the angle bisectors AO, BO, CO, DO are, respectively, the perpendicular bisectors of segments TX, XY, YZ, ZT. It follows that

$$XO = YO = ZO = TO$$

and, hence, $XYZT$ is a cyclic quadrilateral inscribed in a circle with centre O and radius XO.

Everything up to here holds for any quadrilateral that has an incircle. If we further assume that $XY = ZT$, we obtain $\angle XTY = \angle TYZ$ (because equal chords of a circle subtend equal angles), which is equivalent to $XT \parallel YZ$ (because the transversal TY makes equal alternate interior angles with the lines XT and YZ). This concludes the proof.

Editor's comments. The above equality $AX = AT$ implies immediately that the incircles Ω_1 and Ω_2 are tangent at the same point of AC. Several correspondents observed that this is a known theorem; see, for example, M. Josefsson, More characterizations of Tangential Quadrilaterals, *Forum Geometricorum*, Vol. 11 (2011) 65-82. If, moreover, $XY = ZT$ as in our problem, then the resulting figure is symmetric about the diagonal AC.

4970. *Proposed by George Apostolopoulos.*

Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right)^2 \geq 2(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right).$$

We received 14 solutions. Many solvers noted that the condition "a, b, c are the lengths of the sides of a triangle" is not necessary, as the inequality hold for all positive real numbers a, b, c. The following is the solution by Nguyen Viet Hung.

By the Cauchy-Schwarz inequality we have

$$\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right)(ca + ab + bc) \geq (a+b+c)^2,$$

$$\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right)\left(\frac{1}{ca} + \frac{1}{ab} + \frac{1}{bc}\right) \geq \left(\frac{1}{c} + \frac{1}{a} + \frac{1}{b}\right)^2.$$

Multiplying these inequalities and noting that

$$(ab + bc + ca)\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

we get

$$\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)^2 \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

It remains to show that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}.$$

This is follows from summing up the following inequalities

$$\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b},$$

$$\frac{1}{b}+\frac{1}{c} \geq \frac{4}{b+c},$$

$$\frac{1}{c}+\frac{1}{a} \geq \frac{4}{c+a}.$$

The proof is completed.