Canadian Mathematical Olympiad Qualifying Repêchage 2025

A competition of the Canadian Mathematical Society.

Official Solutions

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1. [**10 points**] Solve the following equation, where A, B, and C are digits and A and C are non-zero: $\overline{ABCB} + 1434 = \overline{CABA}$.

Solution: We start our analysis from the last digit, where we have two cases:

- $B + 4 = A$. Here, we see that $C + 3$ must equal to B. Otherwise, we must have $C + 3 = B + 10$, and the plus 10 carries into the next digit, where we have $B + 4 + 1 \equiv A \pmod{10}$ which contradicts with $B + 4 = A$. So we have that $C + 3 = B$. Finally, from the thousandth digits, we have that $A + 1 = C$. This is again impossible since $A = C - 1 = B - 4 = A - 8$. There are no solutions in this case.

- $B + 4 = 10 + A$. Here we must have $C + 4 = B$, since otherwise $C + 4 = B + 10$, which means that the addition on the tenth digit will carry into the hundredth digit, implying that $B + 4 + 1 \equiv A \pmod{10}$ which contradicts with the assumption that $B + 4 = 10 + A$. Finally, on the thousandth digit, we have $A + 2 = C$. So overall, we have that $(A, B, C) = (A, A + 6, A + 2)$. Here, we see that A can be $1, 2, 3$.

The only possible triples of (A, B, C) are $(1, 7, 3)$, $(2, 8, 4)$, and $(3, 9, 5)$. □

2. [**10 points**] Let triangle ABC be a right triangle with $\angle BAC = 90°$. Let I and O be the incentre and the circumcentre of triangle ABC, respectively. It is given that $\angle IOB = 45°$. Determine all possible angles of $\angle CBA$.

Solution: First, note the circumcenter O of any right triangle ABC is the midpoint of the line segment BC. Moreover, we denote $\angle B = \angle ABC$ and $\angle C = \angle BCA$. So we see that $\angle IOC = 180° - 45° = 135°$. Also, $\angle IAC = \frac{1}{2} \cdot 90° = 45°$. Therefore, we see that points A, I, O, C are concyclic.

Thus, we see that

$$\angle OCI = \angle OAI \implies \angle B = \angle OAB = \angle OAI + \angle IAB = \frac{1}{2}\angle C + 45°$$

Because $\angle B + \angle C = 90°$, we may solve that $\angle C = 30°$ and $\angle B = 60°$.

Finally, we verify that this triangle works: we first know that $\triangle ABO$ is an equilateral triangle, and we may compute that $\angle IAO = \angle BAO - \angle BAI = 60° - 45° = 15° = \angle IAC$, which implies that A, I, O, C are concyclic, so $\angle BOI = \angle IAC = 45°$, which means that this triangle indeed works. $\square$

3. [10 points] Initially, there are 2024 green balls and 1 red ball in a box. Every minute, Kate chooses a random ball from the box. If it is green, she paints it blue and puts it back into the box. If it is blue, she paints it green and puts it back into the box. Finally, if it is red, then she stops the process. What is the expected number of green balls at the end of her process?

Solution: Label the green balls as $G_1, \ldots, G_{2024}$. Define the indicator random variable I_n to be 1 if the ball is green when the red ball is drawn, and 0 if the ball is blue when the red ball is drawn. The expected number of green balls at the end of the process is $E(I_1 + I_2 + \cdots + I_{2024})$. From the linearity of expectation and the symmetry of all the green balls, we see that this expected value is equal to $2024E(I_1) = 2024 \cdot P(\text{ball 1 is green})$.

In order to compute this probability, we consider the process of drawing balls. At each step, either ball G_1 is drawn; the red ball is drawn; or $G_2, \ldots, G_{2024}$ is drawn. Note that in the last case, this does not affect the status of the ball G_1 at all, so we may discard this possibility. As such, we see that at each step, either G_1 is drawn or the red ball is drawn and they happen with equal probability. So we see that for the ball to be green in the end, the ball must be chosen exactly an even number of times before the red ball is chosen. So the probability is

$$P(\text{ball 1 is green}) = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots = \frac{2}{3}$$

As such, we see that the expected number of balls that are green at the end is $\frac{4048}{3}$. $\square$

4. **[10 points]** Assume that $\{a_n\}_{n\geq 1}$ is an infinite arithmetic sequence and $\{b_n\}_{n\geq 1}$ is an infinite geometric sequence. If it is given that $a_1 < a_2$, $b_i = a_i^2$ for $i = 1, 2, 3$, and

$$\lim_{n\to\infty}(b_1 + b_2 + \cdots + b_n) = \sqrt{2} + 1,$$

determine, with proof, all possible sequences $\{a_n\}$.

Solution: First, we may let $a_1 = x - d, a_2 = x, a_3 = x + d$. So we see that $b_1 = (x-d)^2, b_2 = x^2, b_3 = (x+d)^2$. Because $b_\bullet$ forms an geometric sequence, we have that

$$(x-d)^2(x+d)^2 = b_1b_3 = b_2^2 = x^4 \implies (x-d)(x+d) = x^2 - d^2 = \pm x^2$$

Note that it is not possible for $x^2 - d^2 = x^2$, as this implies that $d = 0$, meaning both sequences are constant, so the sum $\lim_{n\to\infty} b_1 + b_2 + \cdots + b_n$ diverges. So we must have that $x^2 - d^2 = -x^2$. So we see that $2x^2 = d^2$. As such, we see that $d = \pm\sqrt{2}x$. This means that the first three terms of the b sequence is either $\{(1+\sqrt{2})^2x^2, x^2, (1-\sqrt{2})^2x^2\}$ or $\{(1-\sqrt{2})^2x^2, x^2, (1+\sqrt{2})^2x^2\}$. In the second case, we see that the sequence is increasing, so once again, the sum diverges. In the first case, we see that the sequence is given by $a_n = (3-2\sqrt{2})^{n-1}(3+2\sqrt{2})x^2$.

Finally, we just solve the equation here:

$$\begin{aligned}
\sqrt{2} + 1 &= \lim_{n\to\infty} b_1 + \cdots + b_n \\
&= (3+2\sqrt{2})x^2 \cdot \frac{1}{2\sqrt{2}-2} \\
1 &= \frac{1}{2}(3+2\sqrt{2})x^2 \\
x^2 &= 6 - 4\sqrt{2}
\end{aligned}$$

So we have $x = 2 - \sqrt{2}$ or $x = \sqrt{2} - 2$. Therefore, the first term will be either $\sqrt{2}$ or $-\sqrt{2}$, respectively. We can verify that when $a_1 = \sqrt{2}$, we have $a_2 = 2 - \sqrt{2}$, and so $a_1 < a_2$, and this does not work. So the only solution is the sequence $\{-\sqrt{2}, \sqrt{2} - 2, \dots\}$. $\square$

5. **[10 points]** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $xy = f(x)f(y) - f(x+y)$ for all real numbers x and y.

Solution: We claim that the solutions are $f(x) = 1 \pm x$. It is easy to verify that these satisfy the functional equation.

To show that these are the only solutions, we first plug in $y = 0$ to the original equation to get $f(0)f(x) = f(x)$ for all x. It is also evident that $f(x)$ cannot be 0 for all x, so we get that $f(0) = 1$.

Next, plug in $x = 1$ and $y = -1$ to get that $f(1)f(-1) = 0$. We now split into two cases based on which $f(1)$ and $f(-1)$ is zero.

Case 1: $f(1) = 0$. In this case, plug in $y = 1$ to get $f(x+1) = -x$, and by a change of variables, we get $f(x) = 1 - x$ for all x.

Case 2: $f(-1) = 0$. In this case, plug in $y = -1$ to get $f(x-1) = x$. Again, by a change of variables, we get $f(x) = 1 + x$ for all x. □

6. [**15 points**] In scalene triangle ABC, the circumcentre and incentre are respectively O and I. Let AD be the altitude to line BC, with D lying on line BC. Given that the radius of the circumcircle and A-excircle are equal, prove that the points O, I, and D are collinear.

Solution: Let I_A be the A-excentre of triangle ABC, T be the midpoint of II_A, M the midpoint of BC, X the foot of the perpendicular from I to BC, Y the reflection of X across M, and Z the reflection of X across I.

By standard lemmas (see points 2 and 4 here: https://yufeizhao.com/olympiad/geolemmas.pdf), we have that A, Z, Y are collinear, O, M, T are collinear, T lies on the circumcircle of triangle ABC, and Y is the foot of the perpendicular from I_A to BC. Let U be the midpoint of ZY; this is on line OMT.

We now use the main condition in the problem, that is, that $OT = YI_A$. Since these two lines are both perpendicular to BC, they are parallel, and so OYI_AT is a parallelogram. Since $IT = TI_A$, we see that $OITY$ is also a parallelogram. Note that I and U are respectively the midpoints of ZX and ZY, so IU is parallel to BC. Now, since U is the intersection of the diagonals of trapezoid $ATYO$, we get that $AU/UY = OU/UT$. Adding 1 to both sides, we see that $AY/UY = OT/UT$. Now, by projecting the ratio AY/UY to a line perpendicular to BC, we see that $AY/UY = AD/UM = AD/IZ$. Further, from similar triangles AZI and AUT, we have $UT/IZ = TI/AI$. Combining all of this together, we see that $TI/AI = UT/IZ = OT/AD$, and so by SAS similarity, the triangles ADI and TOI are similar, whence O, I, D are collinear. □

7. [15 points] Is it possible to arrange the numbers $1, 2, \ldots, 54^2$ in a 54×54 grid such that any two vertically or horizontally adjacent cells are relatively prime?

Solution: Yes, it is possible. The following construction uses the fact that $54^2 + 1 = 2917$ and $54^2 + 54 + 1 = 2971$ are prime numbers. Define

$$a_{2i+1,j} = 54i + j,$$

where $i \in [0, 26]$ and $j \in [1, 54]$, and

$$a_{2i,j} = 2971 - 54i - j,$$

where $i \in [1, 27]$ and $j \in [1, 54]$. We can check that this uses each of the numbers from 1 to 54^2 exactly once; the numbers from 1 to $27 \cdot 54$ are used in the odd rows, and the remainder are used in the even rows.

Then, it is clear that two cells that differ in the second coordinate by 1 are relatively prime, as the contents of the cells differ by 1. As for two cells that differ in the first coordinate, we have

$$a_{2i,j} + a_{2i+1,j} = 2971$$

and

$$a_{2i+1,j} + a_{2i+2,j} = 2917,$$

and any two positive numbers adding up to a prime number are relatively prime. □

8. [20 points] Let n be a positive integer, and let $2n$ points be equally spaced on a circle. Prove that for any integer $0 \le k \le \frac{n(n-1)}{2}$, there exists a way to draw n line segments, each connecting two distinct points, such that exactly k pairs of these line segments intersect.

Solution: We first note that if we connect diametrically opposite points, then there will be $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of lines that intersect, as every pair of lines will intersect. We will now proceed by induction in the following manner:

Main claim: Given any n line segments that each connect two distinct points, with at least one pair of lines that intersect, there exist points A, B, C, D in that order on the circle, such that AC and BD are originally connected, and upon replacing these segments with AB and CD, there will be one fewer pair of lines that intersect.

It is clear that this claim will solve the problem, as we may iteratively decrease the number of pairs of lines that intersect by 1 until no pairs of lines intersect.

Now, let $\mathcal{W}$ be the set points in arc AB not including C and D, and define $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ for arcs BC, CD, and DA, respectively. Note that the other $n-2$ lines remain the same, so the number of intersections within them remain the same; AC and BD intersect, and they were replaced with AB and CD, which do not. To this end, in order to compare the change in the number of intersections, we only need to consider lines where one is not from the set of points $\{A, B, C, D\}$, and one is. We casework on where the first of the above lines is.

First, note that if the line contains two points in the same set among $\mathcal{W}$, $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$, then it intersects none of AC, BD, AB, CD, so the number of intersections does not change. If one point is in $\mathcal{W}$ or $\mathcal{Y}$, and the other is in $\mathcal{X}$ or $\mathcal{Z}$, then it intersects exactly one of AC and BD, and exactly one of AB and CD, so again, the number of intersections does not change. If one point is in $\mathcal{W}$ and the other is in $\mathcal{Y}$, then it intersects all of AC, BD, AB, CD, so again, the number of intersections does not change. Finally, if one point is in $\mathcal{X}$ and the other is in $\mathcal{Z}$, then it intersects both AC and BD but neither AB nor CD, so the number of intersections decreases by 2.

To this end, we only require a choice of points A, B, C, D such that AC and BD are lines in the original configuration and there are no lines connecting a point in $\mathcal{X}$ to $\mathcal{Z}$. In order to do this, we consider the pair of lines AC and BD for which the number of points in $\mathcal{X}$ (that is, between B and C on the arc not containing A and D) is minimal. Assume there is some $E \in \mathcal{X}$ and $F \in \mathcal{Z}$. Then the lines FE and BD intersect, and the arc BE not including F and D has strictly fewer points than $\mathcal{X}$, which is a contradiction. This gives us the desired quadruplet to prove the main claim. $\square$

