1. Week 1

We give two entry level problems this week.

Problem A Determine all pairs (x, y) of real numbers which satisfy:

$$\begin{cases} x^3 + y^3 & = 7 \\ xy(x+y) & = -2 \end{cases}$$

Solution:

Problem 1 of the German Mathematical Olympiad, which appeared in Crux Mathematicorum at [2006:279] . We present the solution by Pierre Bornsztein that appeared at [2007:287].

Any such pair must satisfy

$$(x+y)^3 = x^3 + y^3 + 3xy(x+y) = 7 - 6 = 1 \ .$$

Therefore, $x + y = 1$. The second equation implies that $xy = -2$ and hence x, y must be the two roots of

$$Z^2 - Z - 2 = 0 \ .$$

It follows that the only potential solutions to the problem are $(x, y) = (-1, 2)$ and $(x, y) = (2, -1)$. It is easy to see that both of those are indeed solutions.

Therefore

$$\{(-1, 2); (2, -1)\}$$

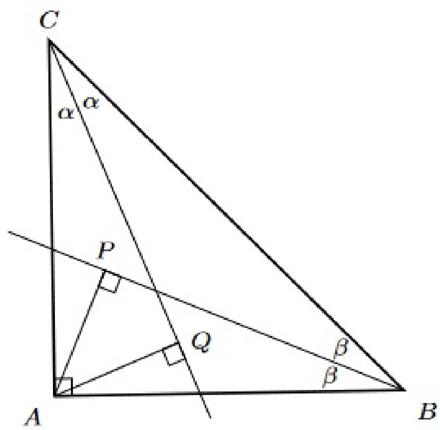
are all the pairs.

Problem B

Triangle ABC has $\angle BAC = 90^\circ$. The feet of the perpendiculars from A to the internal bisectors of $\angle ABC$ and $\angle ACB$ are P and Q, respectively. Determine the measure of $\angle PAQ$.

Solution:

Problem M483 of the Mathematical Mayhem which appeared in Crux Mathematicorum at [2011:136]. We present the solution by Gusnadi Wiyoga that appeared at [2012:45].



$$\begin{aligned}\angle PAQ &= \angle PAB - \angle QAB = \angle PAB - (\angle CAB - \angle CAQ)\\ &= (90^\circ - \beta) - (90^\circ - (90^\circ - \alpha)) = 90^\circ - (\alpha + \beta)\\ &= 90^\circ - 45^\circ = 45^\circ\end{aligned}$$

2. Week 2

Problem Find all triplets of positive integers (x, y, z) such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{5}\ .$$

Solution: Problem 23 of list of problems proposed for the 26th I.M.O. in Finland which appeared in Crux Mathematicorum at [1985:307]. We present the solution by Ed Doolittle, George Evagelopoulos and Bob Prielipp that appeared at [1988:45].

Since the equation is symmetric in (x, y, z) we can find the solutions for which $x \le y \le z$, and by permutations find all solutions.

Let (x, y, z) be a solution with $x \le y \le z$. Then

$$\frac{4}{5} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \le \frac{1}{x} + \frac{1}{x} + \frac{1}{x}\ .$$

Therefore,

$$\frac{3}{x} \ge \frac{4}{5} \Longrightarrow x \le \frac{15}{4}\ .$$

Since x is an integer we get $x \le 3$. Now, $x = 1$ is clearly impossible, so we must have $x = 2$ or $x = 3$.

Case 1: If $x = 2$ then

$$\frac{1}{y} + \frac{1}{z} = \frac{4}{5} - \frac{1}{2} = \frac{3}{10}\ . \tag{1}$$

Similarly to the above,

$$\frac{2}{y} \ge \frac{1}{y} + \frac{1}{z} = \frac{3}{10} \Longrightarrow y \le \frac{20}{3}\ .$$

It follows that $2 = x \le y \le 6$ giving 5 potential choices for y. Only $y = 4$ and $y = 5$ work, giving the solutions

$$(2, 4, 20) \text{ and } (2, 5, 10)\ .$$

Case 2: If $x = 3$ then

$$\frac{1}{y} + \frac{1}{z} = \frac{4}{5} - \frac{1}{3} = \frac{7}{15}\ . \tag{2}$$

Similarly to the above,

$$\frac{2}{y} \ge \frac{1}{y} + \frac{1}{z} = \frac{7}{15} \Longrightarrow y \le \frac{30}{7}\ .$$

It follows that $3 = x \le y \le 4$ giving 2 potential choices for y. None of them work.

This means that all triples are $(2, 4, 20); (2, 5, 10)$ and their permutations.

Editor's Notes:

- It is a standard technique to solve an equation of the form

  $$F(X_1, \dots X_n) = C$$

  in the positive integers by looking at all the possible ways in which the variables are ordered, and bound the expression we want to make equal to a constant from above and/or below by an expression in a single variable. This often leads us to finitely many potential values for that variable, and each potential value leads to a similar equation with one less variable. Repeating the argument leads the solution.

If F is symmetric in $X_1, \ldots, X_n$ we can always assume without loss of generality that $X_1 \leq X_2 \leq \ldots \leq X_n$. Otherwise, multiple (but finitely many!) cases need to be considered.

- (1) can also be solved the following way:

$$\begin{aligned} \frac{1}{y} + \frac{1}{z} = \frac{4}{5} - \frac{1}{2} = \frac{3}{10} &\Longrightarrow \\ \frac{y+z}{yz} = \frac{3}{10} &\Longrightarrow \\ 3xy - 10x - 10y = 0 &\Longrightarrow \\ 9xy - 30x - 30y = 0 &\Longrightarrow \\ (3x-10)(3y-10) = 100 & \end{aligned}$$

and considering all possible factorisations of 100 into product of integers. Note here that $3x - 10$ and $3y - 10$ could be negative!

(2) can be solved similarly.

3. Week 3

Problem Let a, b, c, d be real numbers, not all zero. Prove that the polynomial

$$P(X) = X^6 + aX^3 + bX^2 + cX + d$$

cannot have six real roots.

Solution: Problem 2 of the 1989 Indian Mathematical Olympiad, which appeared in Crux Mathematicorum at [1990:133] . We present the solution by J. Lou and Michael Selby which appeared at [1992:6] .

Let us assume by contradiction that $P(X)$ has six real roots, call them $r_1, r_2, r_3, r_4, r_5, r_6$. Then, $P(X)$ is divisible by $(X-r_1)(X-r_2)(X-r_3)(X-r_4)(X-r_5)(X-r_6)$ and hence

$$P(X) = Q(X)(X-r_1)(X-r_2)(X-r_3)(X-r_4)(X-r_5)(X-r_6)$$

for some polynomial Q. Comparing the degrees, we get that Q must be a constant polynomial. Comparing the coefficients of X^6 we get that the constant must be 1. Therefore,

(3)

$$X^6 + aX^3 + bX^2 + cX + d = P(X) = (X-r_1)(X-r_2)(X-r_3)(X-r_4)(X-r_5)(X-r_6)\,.$$

Now, opening the brackets on the right hand side, and comparing the coefficients of X^5 and X^4, respectively, we get

$$r_1 + r_2 + r_3 + r_4 + r_5 + r_6 = 0$$
$$r_1r_2 + r_1r_3 + \ldots + r_5r_6 = 0$$

It follows that

$$r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2 = (r_1 + r_2 + r_3 + r_4 + r_5 + r_6)^2 - 2\,(r_1r_2 + r_1r_3 + \ldots + r_5r_6) = 0$$

Since $r_1, \ldots, r_6$ are real numbers, we get $r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = 0$. But then, (3) gives

$$a = b = c = d = 0$$

which is a contradiction.

Since we got a contradiction, our assumption that $P(X)$ has six real roots is wrong.

Editor's note: The relations we used above are a particular case of the so called **Vieta's formulas**. Since these form a very useful tool when dealing with polynomials, let us briefly review them below.

Consider a polynomial

$$P(X) = a_nX^n + \ldots + a_1X + a_0$$

of degree n (meaning $a_n \neq 0$) with real coefficients.

Assume that the polynomial has n distinct roots $r_1, \ldots, r_n$ (which by the Fundamental Theorem of Algebra always happens in complex numbers!).

Then, similarly to the above we can argue that

$$P(X) = a_n(X - r_1)(X - r_2)\dots(X - r_n)\ .$$

Opening the brackets and equating the coefficients we get the Vieta formulas, relating $a_0, \dots, a_n$ to $r_1, \dots, r_n$:

$$\begin{cases} r_1 + r_2 + \dots + r_n & = -\frac{a_{n-1}}{a_n} \\ r_1r_2 + r_1r_3 + \dots + r_{n-1}r_n & = \frac{a_{n-2}}{a_n} \\ r_1r_2r_3 + r_1r_2r_4 + \dots + r_{n-2}r_{n-1}r_n & = -\frac{a_{n-3}}{a_n} \\ \dots & \dots \\ r_1r_2\dots r_n & = (-1)^n\frac{a_0}{a_n}\ . \end{cases}$$

4. Week 4

This week we look at a functional equation.

Problem

Find all the functions f defined for all non-zero real numbers satisfying the following two conditions:

(a) For all non-zero x we have

$$f(x) = xf(\frac{1}{x})$$

(b) For all pairs (x, y) of non-zero real numbers with $x + y \neq 0$ we have

$$f(x) + f(y) = 1 + f(x + y)$$

Solution:

Problem 4 of the 1991 Australian Mathematical Olympiad, which appeared in Crux Mathematicorum at [1992:129] . We present a solution by the editor .

Let z be any non-zero number. Setting $x = y = \frac{1}{2z}$ in (b) we get

$$2f(\frac{1}{2z}) = 1 + f(\frac{1}{z}) .$$

Next, by (a) we have

$$2zf(\frac{1}{2z}) = f(2z)$$

$$zf(\frac{1}{z}) = f(z)$$

Therefore

$$f(2z) = 2zf(\frac{1}{2z}) = z + zf(\frac{1}{z}) = z + f(z) .$$

One another hand, by (b) we also have

$$2f(z) = 1 + f(2z) .$$

This gives

$$1 + z + f(z) = 1 + f(2z) = 2f(z)$$

and hence, the only potential solution is

$$f(z) = z + 1 .$$

It is easy to check that the function $f(z) = z+1$ does satisfy (a) and (b). Therefore, the only function satisfying (a), (b) is $f(z) = z + 1$.

5. Week 5

Problem

Six musicians participate in a music festival. At each concert, some of them play music, and the others listen. What is the minimal number of concerts so that each musician listens to all the others?

Solution:

Problem 3 of the XXXIII Spanish Mathematical Olympiad, which appeared in Crux Mathematicorum [2000:196]. We present the solution by Pierre Bornsztein which appeared at [2002:296-297], slightly modified.

Let us call the musicians A, B, C, D, E, F. Whenever one musician listens to another musician, we will refer to this as an "interaction". Since each of the six musicians must listen to each of the other five musicians, there must be at least 30 different interactions.

Let us now look at the possible number of interactions in each concert.

Number of musicians playing	Number of musicians listening	Number of interactions
1	5	5
2	4	8
3	3	9
4	2	8
5	1	5
6	0	0

Therefore, for each concert there are at most 9 different interactions happening. Since we need 30 different interactions, the number of concerts should be at least 4.

Now, 4 is possible, as the table below shows a way to achieve it.

	Playing	Listening
Concert 1	ABC	DEF
Concert 2	AEF	BCD
Concert 3	BDF	ACE
Concert 4	CDE	ABF

Therefore, the minimal number of concerts is 4.

6. Week 6

Problem Let a, b and c be positive integers such that

$$\frac{a\sqrt{3}+b}{b\sqrt{3}+c}$$

is a rational number. Show that

$$\frac{a^2+b^2+c^2}{a+b+c}$$

is an integer.

Solution:

Problem 2 from the Finnish High School Math Contest 2004, final round, that appeared in Crux Mathematicorum [2007:85], whose solution by Geoffrey A. Khandall appeared at [2008:31]. We give an alternate solution by the editor.

Let

$$r := \frac{a\sqrt{3}+b}{b\sqrt{3}+c} \ .$$

Then

$$a\sqrt{3}+b = br\sqrt{3}+cr \Longrightarrow$$
$$(a-br)\sqrt{3} = cr-b \ .$$

We must have $a-br=0$, as otherwise

$$\sqrt{3} = \frac{cr-b}{a-br}$$

would be a rational number.

But if $a-br=0$ then

$$cr-b=(a-br)\sqrt{3}=0 \ .$$

Therefore

$$b = cr$$
$$a = br = cr^2 \ .$$

It follows that

$$\begin{aligned}\frac{a^2+b^2+c^2}{a+b+c} &= \frac{c^2r^4+c^2r^2+c^2}{cr^2+cr+c} = c\frac{r^4+r^2+1}{r^2+r+1} \\ &= c\frac{(r^4+2r^2+1)-r^2}{r^2+r+1} = c\frac{(r^2+1)^2-r^2}{r^2+r+1} = c\frac{(r^2+1+r)(r^2+1-r)}{r^2+r+1} \\ &= c(r^2+1-r) = cr^2-cr+c = a-b+c\end{aligned}$$

which is an integer.

7. Week 7

Problem

Let n be a positive integer. Prove that the equation

$$\sqrt{x}+\sqrt{y}=\sqrt{n}$$

has a solution (x, y) with x, y positive integers if and only if there exists some integer $m>1$ such that m^2 divides n.

Solution:

Problem 3 from Day 2 of the 2012 Indonesia National Science Olympiad, that appeared in Crux Mathematicorum [2013:166-167], whose solution by Michel Bataille appeared at [2014:244]. We give an alternate solution by the editor.

If there exists some integer $m>1$ such that m^2 divides n, then, we can write $n=km^2$ for positive integer k. Then, $x=k(m-1)^2, y=k$ satisfy

$$\sqrt{x}+\sqrt{y}=(m-1)\sqrt{k}+\sqrt{k}=m\sqrt{k}=\sqrt{km^2}=\sqrt{n}\ .$$

Conversely, suppose that there exists positive integers x, y such that

$$\sqrt{x}+\sqrt{y}=\sqrt{n}\ .$$

Squaring both sides we get

$$n=x+y+2\sqrt{xy}$$

which implies that $2\sqrt{xy}$ is an integer. Since x, y are integers, it follows that xy is the square of an integer, lets call it k.

Let $d=\text{gdc}(x, y)$. Then there exists relatively prime integers x', y' such that

$$\begin{cases} x & =dx' \\ y & =dy' \end{cases} .$$

It follows that

$$k^2=xy=d^2x'y'\ .$$

Since $\gcd(x', y')=1$, x' and y' must have distinct prime factors. Then, any prime divisor of x' (or y' respectively) must appear in x'(or y' respectively) at an even power.

This implies that $x'=m^2$ and $y'=l^2$ for some positive integers l, r. Therefore, $x=dl^2, y=dr^2$.

Then,

$$\sqrt{n}=\sqrt{x}+\sqrt{y}=l\sqrt{d}+r\sqrt{d}=\sqrt{d(l+r)^2}\ .$$

Setting $m=l+r$ we get $m\geq 1+1=2$ and $m^2|n$.

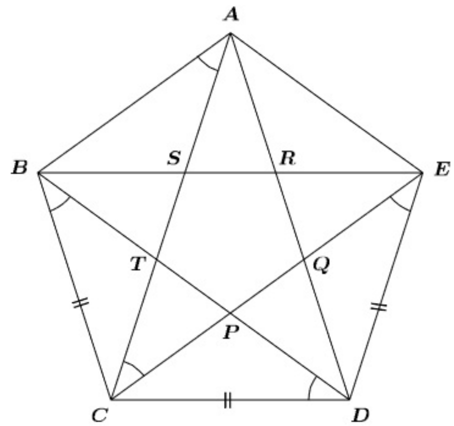
8. Week 8

Problem Let $ABCDE$ be a convex pentagon such that $BC = CD = DE$ and each diagonal of the pentagon is parallel to one of its sides. Prove that all the angles in the pentagon are equal, and that all sides are equal.

Solution:

Problem 1 of the Australian Mathematical Olympiad 1996, that appeared in Crux Mathematicorum [1999:74]. We give the solution by Toshio Seimiya which appeared at [2000:458-459], slightly modified.

Label the intersections of the diagonal P, Q, R, S, T, as in the picture



Since $BE \parallel CD$ and $AC \parallel DE$, the quadrilateral $SCDE$ is a parallelogram, and hence $SC = DE = CB$. Therefore

$$\angle CBE = \angle CBS = \angle CSB = \angle DEB \ .$$

Next, since $\angle CBE = \angle DEB, CB = DE$ and $CD \parallel DE$ it follows that $CDEB$ is an isosceles trapezoid. It follows that $\angle CBD = \angle CED$, and that $BCDE$ are on the same circle.

Using $AB \parallel CE, AC \parallel DE$ and $BC = CD$ we get

$$\angle BAC = \angle ACE = \angle CED = \angle CBD = \angle BDC \ . \tag{4}$$

This implies that $ABCD$ are on the same circle. Since ABC determine an unique circle, it follows that both D, E must be on this circle. Therefore, $ABCDE$ are on the same circle.

Now, since $ABCD$ are on the same circle, we have $\angle ABC + \angle ADC = 180°$. Since $AB \parallel CD$ we also have $\angle ABC + \angle BAD = 180°$. Therefore, $ABCD$ is an isosceles trapezoid and hence $AB = CD$.

Exactly the same argument shows that $AEDC$ is an isosceles trapezoid and hence $AE = CD$. It follows that

$$AB = BC = CD = DE = EA \ .$$

Next, by (4) we have

$$\angle BAC = \angle ACE = \angle CED = \angle CBD = \angle BDC$$

Let us call the value of this angle α.

Since CDE and ABC are isosceles triangles we also get

$$\angle ACB = \angle DCE = \alpha \ .$$

Using $BE \parallel CD$ we also get

$$\angle EBD = \angle BDC = \alpha$$

$$\angle BEC \angle ECD = \alpha$$

Next, ABE isosceles and $AB \parallel CD$ give

$$\angle AEB = \angle ABE = \angle BEC = \alpha \ .$$

$BC \parallel AD, AE \parallel BD$ and EAD isosceles give

$$\angle EDA = \angle EAD = \angle ABD = \angle DBC = \alpha \ .$$

Finally, $AC \parallel DE$ gives

$$\angle CAD = \angle ADE = \alpha \ .$$

It follows that all five angle in the pentagon are equal to 3α.

Editor's note: As in the original solution, once we get that $ABCDE$ is inscribed in a circle, and $AB = BC = CD = DE = EA$, we can use the following property of the circle to deduce that all angles are equal:

Property: Let A, B, C, D be on a circle. If $AB = CD$ then, the corresponding arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are equal.