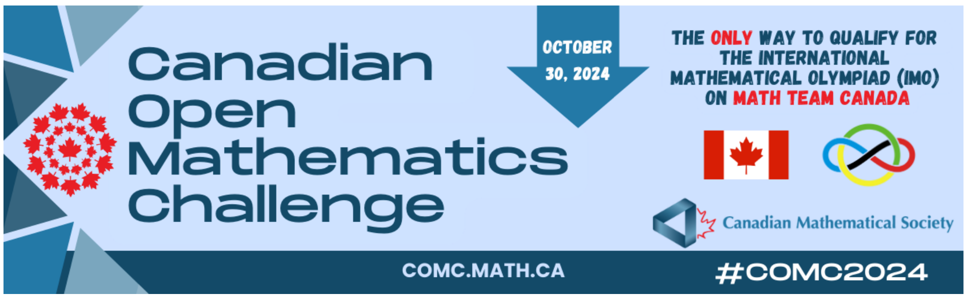
2024 Canadian Open Mathematics Challenge

Official Solutions



A competition of the Canadian Mathematical Society.

The COMC has three sections:

A. Short answer questions worth 4 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

B. Short answer questions worth 6 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

C. Multi-part full solution questions worth 10 marks each. Solutions must be complete and clearly presented to receive full marks.

Section A

A1 Two locations A and B are connected by a 5-mile trail which features a lookout C. A group of 15 hikers started at A and walked along the trail to C. Another group of 10 hikers started at B and walked along the trail to C. The total distance travelled to C by all hikers from the group that started in A was equal to the total distance travelled to C by all hikers from the group that started in B. Find the distance (in miles) from A to C along the trail.

Solution: Let the distance from A to C be x. Then the distance from B to C is $5-x$. The total distance travelled to C by all hikers from the group that started in A is $15x$. The total distance travelled to C by all hikers from the group that started in B is $10(5-x)$. We need $15x = 10(5-x)$. Then $25x = 50$, so $x = 2$.

Answer: $\boxed{2}$.

A2 Alice and Bob are running around a rectangular building measuring 100 by 200 meters. They start at the middle of a 200 meter side and run in the same direction, Alice running twice as fast as Bob. After Bob runs one lap around the building, what fraction of the time were Alice and Bob on the same side of the building?

Solution: The perimeter of the building is 600 m. When Alice completes the first 100 m to the corner, Bob completes his first 50 m. During this time they are running on the same side of the building. A similar situation will occur in the end when Alice runs last 100 m of her second lap and Bob runs his last 50 m. We have $\frac{50+50}{600} = \frac{1}{6}$.

Answer: $\boxed{\frac{1}{6}}$.

A3 Colleen has three shirts: red, green, and blue; three skirts: red, green, and grey; three scarves: red, blue, and grey; and three hats: green, blue, and grey. How many ways are there for her to pick a shirt, a skirt, a scarf, and a hat, so that two of the four clothes are one color and the other two are a different color?

Solution: We arrange the information in a table and observe that each pair of items have two common colors.

shirts	*red*	*green*	*blue*	
skirts	*red*	*green*		*grey*
scarves	*red*		*blue*	*grey*
hats		*green*	*blue*	*grey*

In order to satisfy the rule, for example, a shirt and a skirt could be either both red or both green. At the same time a scarf and a hat could be either both blue or both grey.

There are 3 ways to choose the pairs that are the same color. For example, we can pair a shirt with either a skirt, or a scarf or a hat; then remaining two items form another pair.

Then there are two ways to pick a colour for the first pair and another two ways to pick a color for the remaining pair. In total we have $3 \times 2 \times 2 = 12$.

Answer: $\boxed{12}$.

A4 Consider a sequence of consecutive even numbers starting from 0, arranged in a staggered format, where each row contains one more number than the previous row. The beginning of this arrangement is shown below:

0				
2	4			
6	$\underline{8}$	10		
12	14	16	18	
20	22	24	26	28

The number in the middle of the third row is 8. What is the number in the middle of the 101-st row?

Solution: We can observe the pattern by looking at the numbers in the middle of each row with odd number.

$\underline{0}$						
2	4					
6	$\underline{8}$	10				
12	14	16	18			
20	22	$\underline{24}$	26	28		
30	32	34	36	38	40	
42	44	46	$\underline{48}$	50	52	54

They are 0, 8, 24, 48. The numbers are in the form $n^2 - 1$, where $n = 1, 3, 5, 7$.

Alternatively, we notice the numbers in the beginning and in the end of each "odd" row

row	*beginning*	*end*
1	$0 = 1 \times 0$	$0 = 0 \times 3$
3	$6 = 3 \times 2$	$10 = 2 \times 5$
5	$20 = 5 \times 4$	$28 = 4 \times 7$
7	$42 = 7 \times 6$	$54 = 6 \times 9$
n	$n \times (n-1)$	$(n-1) \times (n+2)$

Thus, in the beginning of each row n we have a number in the form $n(n-1) = n^2 - n$, while in the end of the same row we have a number $(n-1)(n+2) = n^2 + n - 2$. The number is the middle of a row $n = 1, 3, 5, 7, \ldots$ is an average of the two numbers in the ends of this row. This way we obtain $\frac{1}{2}(n^2 - n + n^2 + n - 2) = n^2 - 1$.

Now, if $n = 101$ then $n^2 - 1 = 10200$.

Answer: $\boxed{10,200.}$

Section B

B1 For any positive integer number k, the factorial $k!$ is defined as a product of all integers between 1 and k inclusive: $k! = k \times (k-1) \times \cdots \times 1$.

Let $s(n)$ denote the sum of the first n factorials, i.e.

$$s(n) = \underbrace{n \times (n-1) \times ... \times 1}_{n!} + \underbrace{(n-1) \times (n-2) \times ... \times 1}_{(n-1)!} + \cdots + \underbrace{2 \times 1}_{2!} + \underbrace{1}_{1!}$$

Find the reminder when $s(2024)$ is divided by 8.

Solution:

We claim that $s(n)$ leaves a remainder 1 when divided by 8 for all $n \neq 2$.
Trying the first few cases, we find that $s(2)$ leaves a remainder 3 when divided by 8.

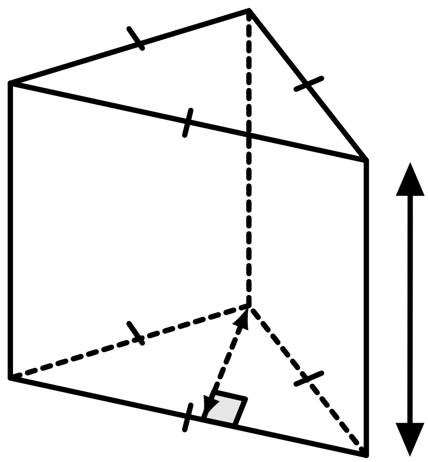
n	$s(n)$	$s(n) \mod 8$
1	1	1
2	3	3
3	9	1
4	9+24	1

We notice that from $n = 4$ onward, n factorial will always have a factor of $4 \times 3 \times 2$ and will always be divisible by 8, adding nothing to the remainder.

Therefore $s(n) \mod 8 = 1$ for all $n \geq 3$. In particular, the reminder when $s(2024)$ is divided by 8 is 1.

Answer: $\boxed{1}$

B2 David wanted to calculate the volume of a prism with an equilateral triangular base. He was given the height of the prism $H = 15$ and the height of the base $h = 6$. He accidentally swapped the values of H and h in his calculations. By what number should he multiply his result to get the correct volume?



Solution: The volume of a prism is $V = H \times A$, where H is the height of the prism and A is the area of the base. The area of the triangular base is $A = \frac{1}{2}ah$, where a is the length of a side of the triangle and h is corresponding height of the triangle. Since the triangle is equilateral, all its sides are equal and all angles are $60°$. Therefore, $a = 2h\tan(30°)$, and so $A = h^2\tan(30°)$.

The correct formula for the volume is $V = Hh^2\tan(30°)$. The mistaken answer is $V' = hH^2\tan(30°)$. And their ratio is

$$\frac{V}{V'} = \frac{Hh^2\tan(30°)}{hH^2\tan(30°)} = \frac{h}{H} = \frac{6}{15} = \frac{2}{5} = 0.4$$

Thus, David should multiply his result by 0.4 to get the correct volume.

Answer: $\boxed{0.4}$.

B3 Let a, b, c, d be four **distinct** integers such that:

$$\min(a, b) = 2$$
$$\min(b, c) = 0$$
$$\max(a, c) = 2$$
$$\max(c, d) = 4$$

Here $\min(a, b)$ and $\max(a, b)$ denote respectively the minimum and the maximum of two numbers a and b.

Determine the fifth smallest possible value for $a + b + c + d$.

Solution:

$$\min(a, b) = 2 \quad ①$$
$$\min(b, c) = 0 \quad ②$$
$$\max(a, c) = 2 \quad ③$$
$$\max(c, d) = 4 \quad ④$$

From statements ① and ②, we can conclude that $c = 0 \leq b$ and furthermore that $2 \leq b$ and $2 \leq a$. From statement ③, we then know that $a = 2$, and from ④, it follows that $d = 4$. $b \geq 2$ remains unknown and since a, b, c, d are distinct, we have $b \neq 0, 2, 4$. Therefore, $a + b + c + d = 2 + b + 0 + 4 = b + 6$, where b could be 3 or $b \geq 5$. In the former case we have $a + b + c + d = 9$ and in the latter case $a + b + c + d \geq 11$. Thus, the five smallest possible value for $a + b + c + d$ are $9, 11, 12, 13, 14$.

Answer: $\boxed{14}$

B4 Initially, the integer 80 is written on a blackboard. At each step, the integer x on the blackboard is replaced with an integer chosen uniformly at random among $[0, x-1]$, unless $x = 0$, in which case it is replaced by an integer chosen uniformly at random among $[0, 2024]$. Let $P(a, b)$ be the probability that after a steps, the integer on the board is b. Determine

$$\lim_{a\to\infty} \frac{P(a, 80)}{P(a, 2024)}$$

(that is, the value that the function $\frac{P(a,80)}{P(a,2024)}$ approaches as a goes to infinity).

Solution 1:

If 0 is written on the board, then we choose one of the integers from the interval $[0, 2024]$ with probability $\frac{1}{2025}$.

If 1 is written on the board, then we choose 0 with probability 1.

If 2 is written on the board, then we choose one of the integers from the interval $[0, 1]$ with probability $\frac{1}{2}$.

If 3 is written on the board, then we choose one of the integers from the interval $[0, 2]$ with probability $\frac{1}{3}$.

If $x > 0$ is written on the board, then we choose one of the integers from the interval $[0, x-1]$ with probability $\frac{1}{x}$.

Then we have

$$P(a, 2024) = \frac{1}{2025}P(a-1, 0),$$

$$P(a, 2023) = \frac{1}{2025}P(a-1, 0) + \frac{1}{2024}P(a-1, 2024),$$

$$P(a, 2022) = \frac{1}{2025}P(a-1, 0) + \frac{1}{2024}P(a-1, 2024) + \frac{1}{2023}P(a-1, 2023),$$

and in general

$$P(a, b) = \frac{1}{2025}P(a-1, 0) + \sum_{i=b+1}^{2024} \frac{1}{i}P(a-1, i).$$

Now let's define $f(x) = \lim_{a\to\infty} P(a, x) = \lim_{a\to\infty} P(a-1, x)$. Then, taking limits of both sides of our equation, we get

$$f(b) = \frac{1}{2025}f(0) + \sum_{i=x+1}^{2024} \frac{1}{i}f(i).$$

Consequently we obtain

$$f(2024) = \frac{f(0)}{2025},$$

$$f(2023) = \frac{f(0)}{2025} + \frac{1}{2024}f(2024) = \frac{f(0)}{2025}(1 + \frac{1}{2024}) = \frac{f(0)}{2024},$$

And then by induction we can show that

$$f(x) = \frac{f(0)}{x+1}, \quad 0 \leq x \leq 2024.$$

Specifically, we assume that $f(i) = \frac{f(0)}{i+1}$ for $x < i \leq 2024$. Then

$$f(x) = \frac{1}{2025}f(0) + \sum_{i=x+1}^{2024} \frac{1}{i}f(i) = f(0)\left(\frac{1}{2025} + \sum_{i=x+1}^{2024} \frac{1}{i(i+1)}\right) = \frac{f(0)}{x+1}$$

Here we used $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$, and by telescoping, $\sum_{i=x+1}^{2024} \frac{1}{i(i+1)} = \frac{1}{x+1} - \frac{1}{2025}$.
So we get

$$\frac{f(80)}{f(2024)} = \frac{2025}{81} = 25.$$

<u>Answer:</u> $\boxed{25}$.

Solution 2:

We first prove the following fact for all $1 \leq b \leq 2024$:

$$P(a, b-1) = P(a, b) + \frac{1}{b}P(a-1, b).$$

Let X_a denote the random variable which equals the integer written on the board after a steps. So, $P(a, b) = \text{Prob}(X_a = b)$. (Here "Prob" stands for "probability".) By the total probability formula, since the integer written on the board after $a-1$ steps is either b or not b, we have

$$P(a, b) = \text{Prob}(X_a = b|X_{a-1} \neq b)\text{Prob}(X_{a-1} \neq b)$$
$$+ \text{Prob}(X_a = b|X_{a-1} = b)\text{Prob}(X_{a-1} = b). \qquad (1)$$

Here $\text{Prob}(A|B)$ is the conditional probability of event A given that event B takes place. Similarly,

$$P(a, b-1) = \text{Prob}(X_a = b-1|X_{a-1} \neq b)\text{Prob}(X_{a-1} \neq b)$$
$$+ \text{Prob}(X_a = b-1|X_{a-1} = b)\text{Prob}(X_{a-1} = b). \quad (2)$$

Notice that conditional probabilities

$$\text{Prob}(X_a = b|X_{a-1} = b) = 0, \qquad \text{Prob}(X_a = b-1|X_{a-1} = b) = \frac{1}{b}.$$

As well,

$$\text{Prob}(X_a = b | X_{a-1} = c) = \text{Prob}(X_a = b-1 | X_{a-1} = c) \quad \text{for all} \quad c \neq b.$$

Then,

$$\text{Prob}(X_a = b | X_{a-1} \neq b) = \text{Prob}(X_a = b-1 | X_{a-1} \neq b).$$

Thus, subtracting equation (1) from equation (2), we obtain

$$P(a, b-1) - P(a, b) = \frac{1}{b}\text{Prob}(X_{a-1} = b),$$

which is equivalent to $P(a, b-1) = P(a, b) + \frac{1}{b}P(a-1, b)$.

Now let's define $f(b) = \lim_{a\to\infty} P(a, b)$. Then, taking limits of both sides of our equation, we get $f(b-1) = \frac{b+1}{b}f(b)$, and multiplying through, we get $bf(b-1) = (b+1)f(b)$. We thus see that

$$f(0) = 2f(1) = 3f(2) = \cdots = (b+1)f(b),$$

for all $b \in [0, 2024]$. In particular, $81f(80) = 2025f(2024)$, so the desired answer is $\frac{f(80)}{f(2024)} = \frac{2025}{81} = 25$.

Remark 1:

A part of the first solution can be expressed in the matrix form.

Consider a (2025×2025)-matrix M, where M_{ij} is the probability to chose $i = 0, 1, \ldots 2024$ given that $j = 0, 1, \ldots 2024$ is written on the board. Then we have:

$$\begin{bmatrix} \frac{1}{2025} & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{2023} & \frac{1}{2024} \\ \frac{1}{2025} & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{2023} & \frac{1}{2024} \\ \frac{1}{2025} & 0 & 0 & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{2023} & \frac{1}{2024} \\ \frac{1}{2025} & 0 & 0 & 0 & \frac{1}{4} & \cdots & \frac{1}{2023} & \frac{1}{2024} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2025} & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2024} \\ \frac{1}{2025} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} P(a-1,0) \\ P(a-1,1) \\ P(a-1,2) \\ P(a-1,3) \\ \cdots \\ P(a-1,2023) \\ P(a-1,2024) \end{bmatrix} = \begin{bmatrix} P(a,0) \\ P(a,1) \\ P(a,2) \\ P(a,3) \\ \cdots \\ P(a,2023) \\ P(a,2024) \end{bmatrix}$$

Let $f(x) = \lim_{a\to\infty} P(a,x) = \lim_{a\to\infty} P(a-1,x)$, $x = 0,1,\ldots,2024$. Then we obtain

$$\begin{bmatrix} \frac{1}{2025} & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{2023} & \frac{1}{2024} \\ \frac{1}{2025} & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{2023} & \frac{1}{2024} \\ \frac{1}{2025} & 0 & 0 & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{2023} & \frac{1}{2024} \\ \frac{1}{2025} & 0 & 0 & 0 & \frac{1}{4} & \cdots & \frac{1}{2023} & \frac{1}{2024} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2025} & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2024} \\ \frac{1}{2025} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \cdots \\ f(2023) \\ f(2024) \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \cdots \\ f(2023) \\ f(2024) \end{bmatrix}$$

which is equivalent to $f(b) = \frac{1}{2025} f(0) + \sum_{i=x+1}^{2024} \frac{1}{i} f(i)$ for $0 \le b \le 2024$.

Remark 2: In our definition of $f(b)$, we implicitly assumed that $f(b)$ exists. We also used the fact that the limit of the ratio is the ratio of the limits, if they exist.

For completeness, we present a proof of the existence of $f(b)$ below. We start with the recursion

$$P(a,b) = \frac{1}{2025} P(a-1,0) + \sum_{i=b+1}^{2024} \frac{1}{i} P(a-1,i),$$

and let $Q(a,b) = (b+1)P(a,b)$. Then, the recursion becomes

$$Q(a,b) = (b+1)\left(\frac{Q(a-1,0)}{2025} + \sum_{i=b+1}^{2024} \frac{Q(a-1,i)}{i(i+1)} \right).$$

Let $M(a) = \max_{0\le i\le 2024} Q(a,i)$. Then the following inequality holds for any b:

$$Q(a,b) \le (b+1)\left(\frac{1}{2025} + \sum_{i=b+1}^{2024} \frac{1}{i(i+1)} \right) M(a-1) = M(a-1)$$

and so

$$M(a) \le M(a-1).$$

Thus, $M(a)$ is non-increasing as $a \to \infty$, and hence approaches a limiting value M. We now claim that for *every* i, we have $\lim_{a\to\infty} Q(a,i) = M$. Assume there is some minimal index j without this property; then, there is some ε such that $Q(a,j) < M - \varepsilon$ for arbitarily large a. Also, assume that there is some N for which $\max_{0\le i\le 2024} Q(a,i) < M + \varepsilon/2^{100}$ for $a > N$. Then, for an index $a > N$ with $Q(a,i) < M - \varepsilon$, we have

$$Q(a+1,j-1) = j\left(\frac{Q(a,0)}{2025} + \sum_{i=j}^{2024} \frac{Q(a,i)}{i(i+1)} \right) < M - \frac{\varepsilon}{2j(j+1)}.$$

This is true for arbitrary choice of a, so this contradicts the minimality of j, unless the minimal index is 0. Finally, in the case that $\lim_{a\to\infty} Q(a,0) \neq M$, we again say that there is some ε such that $Q(a,0) < M - \varepsilon$ for arbitrarily large a, and for some N, we have $\max_{0\le i\le 2024} Q(a,i) < M + \varepsilon/2^{100}$ for $a > N$. Then, for any index b, we get

$$Q(a+1,b) < M - \frac{\varepsilon}{10000},$$

which contradicts our assumption that $\max_{0\le i\le 2024} Q(a+1,i) \ge M$.

Section C

C1 Let the function $f(x,y,t) = \frac{x^2-y^2}{2} - \frac{(x-yt)^2}{1-t^2}$ for all real values x, y and $t \neq \pm 1$.

(a) Evaluate $f(2,0,3)$ and $f(0,2,3)$.

Solution:
$f(2,0,3) = 2 + 1/2 = 5/2$ and $f(0,2,3) = -2 + 9/2 = 5/2$.

(b) Show that $f(x,y,0) = f(y,x,0)$ for any values of (x,y).

Solution:
$f(x,y,0) = \frac{x^2-y^2}{2} - x^2 = -\frac{x^2+y^2}{2}$.
$f(y,x,0) = -\frac{y^2+x^2}{2} = f(x,y;0)$.

(c) Show that $f(x,y,t) = f(y,x,t)$ for any values of (x,y) and $t \neq \pm 1$.

Solution:
$f(x,y,t) = \frac{x^2-y^2}{2} - \frac{(x-yt)^2}{1-t^2} = \frac{(x^2-y^2)(1-t^2)-2(x-yt)^2}{2(1-t^2)} = \frac{4xyt-(x^2+y^2)(1+t^2)}{2(1-t^2)}$.

Here we used $(x^2-y^2)(1-t^2)-2(x-yt)^2 = x^2-y^2-t^2x^2+t^2y^2-2x^2+4xyt-2t^2y^2 = -x^2-y^2-t^2x^2-t^2y^2+4xyt = 4xyt-(x^2+y^2)(1+t^2)$.

This expression is invariant under interchanging of x and y, and so does $f(x,y,t)$ for all $t \neq \pm 1$.

(d) Given

$$g(x,y,s) = \frac{(x^2-y^2)(1+\sin s)}{2} - \frac{(x-y\sin s)^2}{1-\sin s}$$

for all real values x, y and $s \neq \frac{\pi}{2} + 2\pi k$, where k is an integer number, show that $g(x,y,s) = g(y,x,s)$ for any values of (x,y) and s in the domain of $g(x,y,s)$.

Solution
We have $g(x,y,s) = \frac{(x^2-y^2)(1-\sin^2 s)-2(x-y\sin s)^2}{2(1-\sin s)}$.

Observe that letting $\sin s = t$ and repeating the calculations in part (c), we obtain $g(x,y) = \frac{4xyt-(x^2+y^2)(1+t^2)}{2(1-t)}$. This expression in invariant under interchanging of x and y for all $t \neq 1$ or, equivalently, $\sin s \neq 1$.

That is, for any value $s \neq \frac{\pi}{2} + 2\pi k$, where k is an integer number.

Alternative solution:

Observe that

$$g(x, y, s) = (1 + \sin(s))f(x, y, \sin(s))\,.$$

When $1 + \sin(s) = 0$ we have $g(x, y, s) = 0 = g(y, x, s)$ and when $1 + \sin(s) \neq 0$ the claim follows from (c).

C2 (a) How many ways are there to pair up the elements of 1, 2, ..., 14 into seven pairs so that each pair has sum at least 15?

Solution:

One way. 1 has to be paired with 14, then 2 with 13, etc.

(b) How many ways are there to pair up the elements of 1, 2, ..., 14 into seven pairs so that each pair has sum at least 13?

Solution:

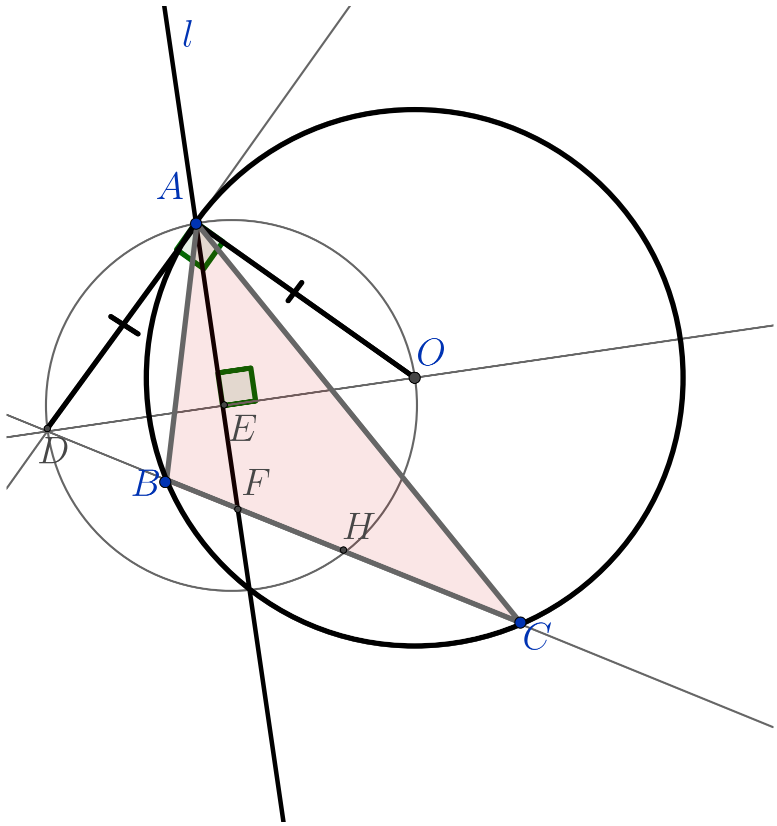
3^6 ways. 1 can be paired with any of 12, 13, 14. Then there are 3 candidate pairs for 2: the three numbers of 11, 12, 13, 14 that 1 is not paired with. Then there are 3 candidate pairs for 3: the three numbers of 10, 11, 12, 13, 14 that 1 and 2 are not paired with.We repeat this process to form six pairs, and then the last two numbers will be paired automatically.

(c) How many ways are there to pair up the elements of 1, 2, ..., 2024 into 1012 pairs so that each pair has sum at least 2001?

Solution: We can pair up each of 1, 2, ... 1000 in 25 ways. That is, 1 can be paired with any of 2000, 2001, ... , 2024; then there are 25 candidate pairs for 2: the 25 numbers of 1999, ... , 2024 that 1 is not paired with, then there are 25 candidate pairs for 3: the 25 numbers of 1998, ... , 2024 that 1 and 2 are not paired with, etc; finally there are 25 candidate pairs for 1000: the 25 numbers of 1001, ... , 2024 that 1, ... , 999 are not paired with.

The other 24 numbers (excluding 1, ... , 1000 and their partners) are all greater or equal than 1001. Therefore, we can pair the smallest of them with any of 23 remaining numbers. This leaves 22 numbers and again we can pair the smallest of them with any of 21 remaining numbers, etc. until only one pair is left. Thus, the 24 numbers can be paired in $23 \times 21 \times \cdots \times 3 \times 1$ ways. So the answer is $25^{1000} \times 23!!$

C3 Let ABC be a triangle for which the tangent from A to the circumcircle intersects line BC at D, and let O be the circumcenter. Construct the line ℓ that passes through A and is perpendicular to OD. ℓ intersects OD at E and BC at F. Let the circle passing through ADO intersect BC again at H. It is given that $AD = AO = 1$.



(a) Find OE.

Solution:
Triangle ADO is an isosceles right triangle, $AD = AO = 1$, so $OD = \sqrt{2}$ and $OE = \frac{1}{2}OD = \frac{\sqrt{2}}{2}$.

(b) Suppose for this part only that $FH = \frac{1}{\sqrt{12}}$, determine the area of triangle OEF.

Solution:
Because H lies on the circle (ADO) and from $\angle DAO = 90^\circ$ it follows that $\angle OHD = 90^\circ$. Consequently, $\angle OHF = 90^\circ$. So we know that $EOHF$ is cyclic. Now, from power of a point we obtain $DF \cdot DH = DE \cdot DO$. Denote $y = DF$, then $DH = DF + FH = y + \frac{1}{\sqrt{12}}$. We also know that $DE = OE = \frac{1}{\sqrt{2}}$ and $DO = \sqrt{2}$. Thus, we see have $y(y + \frac{1}{\sqrt{12}}) = 1$. We solve for y to obtain

$$y = \frac{-\frac{1}{\sqrt{12}} + \sqrt{\frac{1}{12} + 4}}{2} = \frac{6}{2\sqrt{12}} = \frac{\sqrt{3}}{2}$$

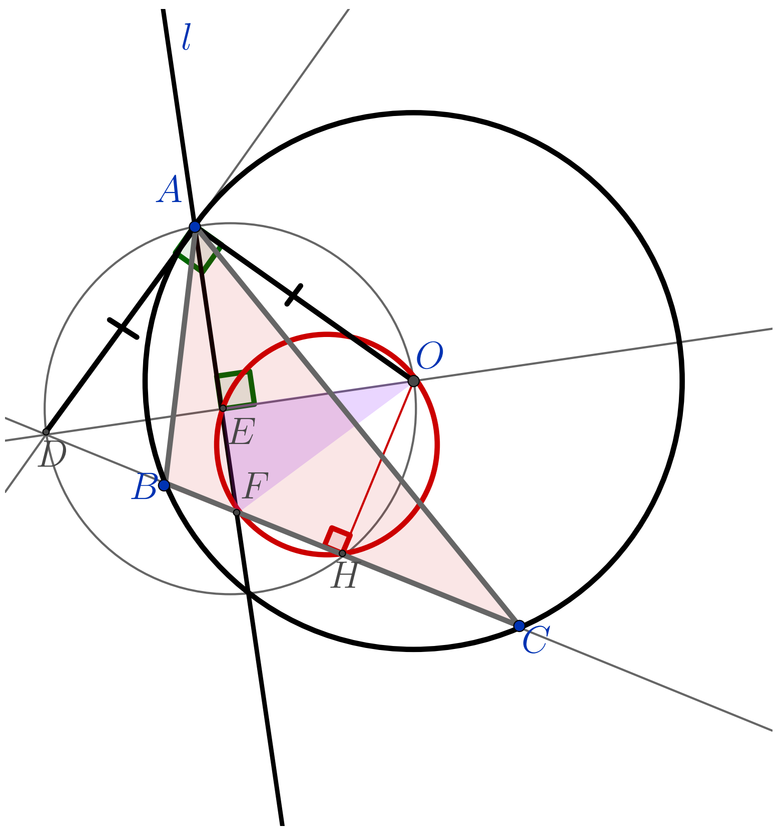
As such, we see that

$$EF = \sqrt{DF^2 - DE^2} = \sqrt{\frac{3}{4} - \frac{1}{2}} = \frac{1}{2}$$

It follows that the area of the right triangle $[\triangle OEF] = \frac{1}{2}OE \cdot EF = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}}{8}$.

(c) Suppose for this part only that $BC = \sqrt{3}$, determine the area of the triangle OEF.

Solution:



From power of a point (or from similar triangles DAB and DCA) we know that $DA^2 = DB \cdot DC$. Let $x = DB$, then $DC = DB + BC = x + \sqrt{3}$, and $DA = 1$. Thus, $x(x + \sqrt{3}) = 1$ implies $x = \frac{-\sqrt{3}+\sqrt{7}}{2}$.

Observe that H is the midpoint of BC since $OH \perp BC$, so $BH = \frac{\sqrt{3}}{2}$. Then $DH = DB + BH = x + \frac{\sqrt{3}}{2} = \frac{\sqrt{7}}{2}$.

As in (b), we have $DF \cdot DH = DE \cdot DO$, and $DE \cdot DO = 1$, so $DF = \frac{2}{\sqrt{7}}$.

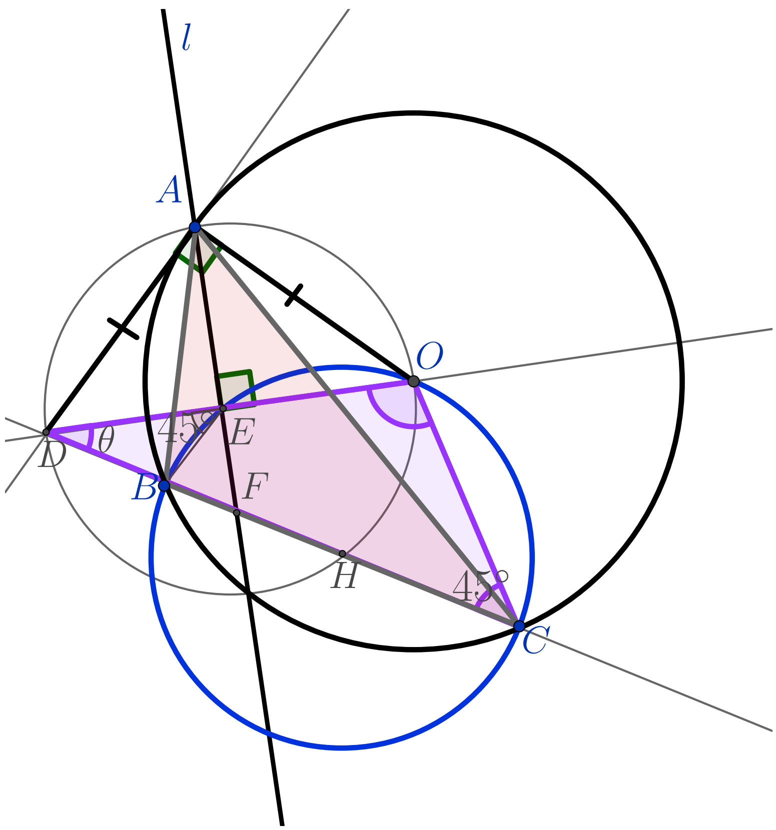
From the right triangle DEF, by Pythagorean's theorem, we obtain

$$EF = \sqrt{DF^2 - DE^2} = \sqrt{\frac{4}{7} - \frac{1}{2}} = \frac{1}{\sqrt{14}}$$

It follows that the area of the right triangle $[\triangle OEF] = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{\sqrt{14}} = \frac{1}{4\sqrt{7}}$.

(d) Suppose that B lies on the angle bisector of DEF, find the area of the triangle OEF.

Solution:



First, we see that $DE \cdot DO = DA^2$ (both sides equal 1) and $DB \cdot DC = DA^2$ as observed in (b). So $DE \cdot DO = DB \cdot DC$, which means that E, O, C, B are concyclic.

Thus, we see that $\angle BEO + \angle OCB = 180°$. Since $\angle BEO + \angle DEB = 180°$ as well, $\angle OCB = \angle DEB = \frac{1}{2}\angle DEF = 45°$.

We have $\angle OCD = \angle OCB = 45°$. Denote $\angle ODC = \theta$. From Sine Law in $\triangle DOC$, we see that

$$\frac{\sqrt{2}}{\sin 45°} = \frac{1}{\sin\theta} \implies \sin\theta = \frac{1}{2} \implies \theta = 30°$$

Therefore, from right triangle DEF, $EF = DE \cdot \tan 30^\circ = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{6}}$.

It follows that the area of the right triangle $[\triangle OEF] = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{\sqrt{6}} = \frac{1}{4\sqrt{3}}$.

C4 Call a polynomial $f(x)$ *excellent* if its coefficients are all in $[0,1)$ and $f(x)$ is an integer for all integers x.

(a) Compute the number of excellent polynomials with degree at most 3.

Solution:

Let $f(x) = ax^3 + bx^2 + cx + d$ be an excellent polynomial. Then:

i. $f(0) = d$ must be an integer, so $d = 0$.

ii. $f(1) = a + b + c$ must be an integer, and so must $f(-1) = -a + b - c$, so by adding these we get that $2b$ must be an integer, so $b = 0$ or $b = \frac{1}{2}$. Moreover, from $f(1) - f(-1)$ we get that $2(a + c)$ must be an integer, so $a + c$ must be one of $0, \frac{1}{2}, 1$, or $\frac{3}{2}$.

iii. $f(2) = 8a + 4b + 2c = 6a + 4b + 2(a + c)$ must be an integer, so since both $4b$ and $2(a + c)$ are integers, $6a$ must be an integer. Thus a must be one of $0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, or $\frac{5}{6}$.

iv. Putting all this together, we obtain the following:
If $b = 0$, then $a + c = 0$ or 1, and thus we have six possibilities:
if $a = 0$ then $c = 0$ and we get $f(x) = 0$;
if $a = \frac{1}{6}$ then $c = \frac{5}{6}$
(and we get $f(x) = \frac{x^3+5x}{6} = \frac{x(x^2+5)}{6}$ which is an integer for every integer x);
if $a = \frac{1}{3}$ then $c = \frac{2}{3}$;
if $a = \frac{1}{2}$ then $c = \frac{1}{2}$;
if $a = \frac{2}{3}$ then $c = \frac{1}{3}$;
if $a = \frac{5}{6}$ then $c = \frac{1}{6}$.

If $b = \frac{1}{2}$, then $a + c = \frac{1}{2}$ or $\frac{3}{2}$, and thus we have another six possibilities:
if $a = 0$ then $c = \frac{1}{2}$;
if $a = \frac{1}{6}$ then $c = \frac{1}{3}$;
if $a = \frac{1}{3}$ then $c = \frac{1}{6}$;
if $a = \frac{1}{2}$ then $c = 0$;
if $a = \frac{2}{3}$ then $c = \frac{5}{6}$;
if $a = \frac{5}{6}$ then $c = \frac{2}{3}$.

Therefore 12 excellent polynomials altogether.

Alternatively, the same result could be found based on part (b) as shown below.

(b) Compute the number of excellent polynomials with degree at most n, in terms of n.

Solution:

We require the following **Lemma**:

Every polynomial of degree at most n that maps integers to integers is an integer combination of the polynomials $1, x, \binom{x}{2}, \ldots, \binom{x}{n}$.

Proof: We use induction. This is clearly true when $n = 0$. Now, assume it is true for some $n = k$, and let f be an excellent polynomial of degree $k+1$. Then, the polynomial $f(x+1) - f(x)$ is a polynomial of degree at most k and also maps integers to integers, so we can write

$$f(x+1) - f(x) = \sum_{i=0}^{k} a_i \binom{x}{i},$$

for some integers a_i. Then, we have

$$f(x) = f(0) + \sum_{i=0}^{k} a_i \sum_{y=0}^{x-1} \binom{y}{i} = f(0) + \sum_{i=0}^{k} a_i \binom{x}{i+1}.$$

Since $f(0)$ and the a_i are all integers, this proves the lemma.

In particular, all excellent polynomials are of this form.

Given an excellent polynomial f of degree at most n, the lemma implies there exists an integer c such that $g(x) = f(x) - c\binom{x}{n}$ is a polynomial of degree at most $n-1$ that maps integers to integers. Note since $\binom{x}{n} = \frac{1}{n!}x^n + p(x)$, where $p(x)$ is a polynomial of degree at most $n-1$, the coefficient of x^n in $f(x)$ is $\frac{c}{n!}$. Therefore, since f is excellent, c is an integer such that $\frac{c}{n!} \in [0, 1)$ and thus there are $n!$ options for c.

For any such c, $g(x)$ is a polynomial of degree at most $n-1$ that maps integers to integers. Since adding a polynomial with integer coefficients to g will still map integers to integers, there is a unique polynomial h with integer coefficients and degree at most $n-1$ such that $g - h$ is excellent. Conversely, for any excellent polynomial g of degree at most $n-1$, there are $n!$ integers c such that coefficient of x^n in $g(x) + c\binom{x}{n}$ is in $[0, 1)$ and a unique polynomial h with integer coefficients and degree at most $n-1$ such that $f(x) = h(x) + g(x) + c\binom{x}{n}$ is excellent.

Since the above processes are inverses of each other, thus the number of excellent polynomials of degree at most n is $n!$ times the number of excellent polynomials of degree at most $n-1$. So the answer for (a) is $3!2!1! = 12$ and for (b) is $n!(n-1)! \cdots 1! = n(n-1)^2 \cdots 2^{n-1}1^n$.

(c) Find the minimum $n \geq 3$ for which there exists an excellent polynomial of the form $\frac{1}{n!}x^n + g(x)$, where $g(x)$ is a polynomial of degree at most $n-3$.

Solution:

We must find some n such that there are integers c and d such that the polynomial

$$\binom{x}{n} + c\binom{x}{n-1} + d\binom{x}{n-2} - \frac{1}{n!}x^n$$

is of degree at most $n-3$.

Observe that

$$\binom{x}{n} = \frac{x!}{n!(x-n)!} = \frac{x(x-1)(x-2)\cdots(x-(n-1))}{n!}.$$

Now,

$$(x-1)(x-2)\cdots(x-k) = x^k + c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + q(x),$$

where $q(x)$ is a polynomial of degree $k-3$, $c_{k-1} = -\sum_{j=1}^{k} j = -\frac{(k+1)k}{2}$, and

$$c_{k-2} = \sum_{j=1}^{k-1}\sum_{i=j+1}^{k} ij = \frac{1}{2}\left(\frac{(k+1)^2k^2}{4} - \frac{k(k+1)(2k+1)}{6}\right) = \frac{1}{24}(k+1)k(k-1)(3k+2).$$

Taking $k = n-1$, we see that

$$\binom{x}{n} = \frac{1}{n!}\left(x^n - \frac{(n-1)(n)}{2}x^{n-1} + \frac{n(n-1)(n-2)(3n-1)}{24}x^{n-2}\right) + P(x),$$

where $P(x)$ is a polynomial of degree at most $n-3$.

Similarly,

$$\binom{x}{n-1} = \frac{1}{(n-1)!}\left(x^{n-1} - \frac{(n-2)(n-1)}{2}x^{n-2}\right) + Q(x),$$

and

$$\binom{x}{n-2} = \frac{1}{(n-2)!}x^{n-2} + R(x),$$

where Q and R have degree at most $n-3$.
We are forced to choose $c = \frac{n-1}{2}$ and obtain

$$\frac{(n-2)(3n-1)}{24} - \frac{(n-2)(n-1)}{4} + d = 0.$$

Thus, $d = \frac{3n^2-11n+10}{24} = \frac{(3n-5)(n-2)}{24}$. To have integer coefficients c and d, we require $2|(n-1)$ and $24|(3n^2-11n+10)$. The minimum such value for n is 23.

Remark

Here are some examples of non-zero excellent polynomials of degree 2 and 3:

$$\frac{x+x^2}{2}, \quad \frac{x+x^3}{2}, \quad \frac{2x+x^3}{3}, \quad \frac{5x+x^3}{6}, \quad \frac{4x+3x^2+5x^3}{6}, \quad \frac{2x+3x^2+x^3}{6}, \quad \frac{x+3x^2+2x^3}{6}, \quad \frac{5x+3x^2+4x^3}{6}.$$

Note that for a polynomial $P(x)$ of degree n it is sufficient to verify that $P(0), \dots, P(n)$ are integers in order to conclude that $P(k)$ is integer for all integer k. This fact is proven by induction because it is true for $n = 0$ and degree of the polynomial $Q(x) = P(x+1) - P(x)$ is at most $n - 1$. Thus, if $n \geq 1$ then for all $k \geq n$, $P(k+1) = P(k) + Q(k)$ is integer given that $P(k)$ is integer.