2024 Canada Jay
Mathematical Competition

Official Solutions

A competition of the Canadian Mathematical Society.

Part A: 4 marks each

A1. Bob asks his friend Mike to type all numbers between 1 and 100, inclusive, with a comma (",") between each pair of consecutive numbers. To type the numbers 1 to 11 Mike presses 23 keys to get:

1,2,3,4,5,6,7,8,9,10,11

How many key presses does Mike need to complete Bob's task?

(A) 192 **(B)** 201 **(C)** 287 **(D)** 291

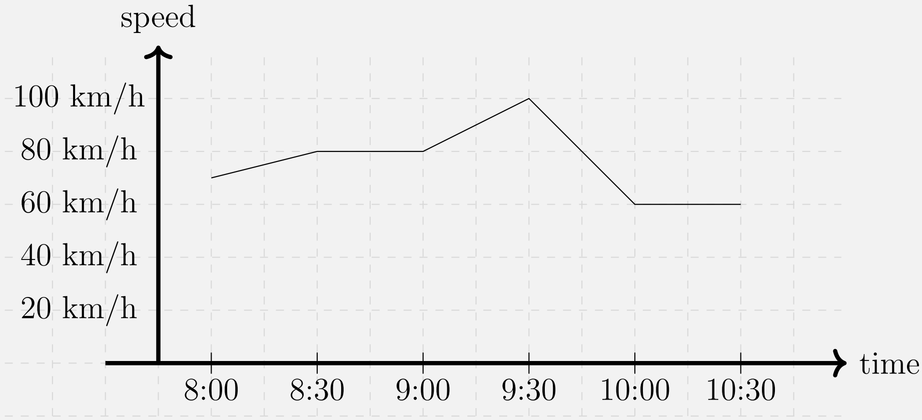
Solution: Mike needs to press 1 key 9 times (1-9), 2 keys 90 times (10-99), 3 keys 1 time (100) and the comma 99 times. Thus, he needs to press

$$9 + 180 + 3 + 99 = 291$$

keys. □

Answer: D

A2. Reem took a trip. The graph below shows the speeds Reem drove during her trip. What percentage of the time did Reem drive at 80 km/hr or faster?



(A) 30 % **(B)** 40 % **(C)** 50 % **(D)** 75 %

Solution: Reem traveled 75 minutes out of 150 minutes at a speed of 80km/h or faster. □

Answer: C

A3. Dana has 100 building blocks. Some are small and some are big. She can combine 3 small blocks into a big one. After combining all her small blocks, she now has 40 big blocks.

How many big blocks did she start with?

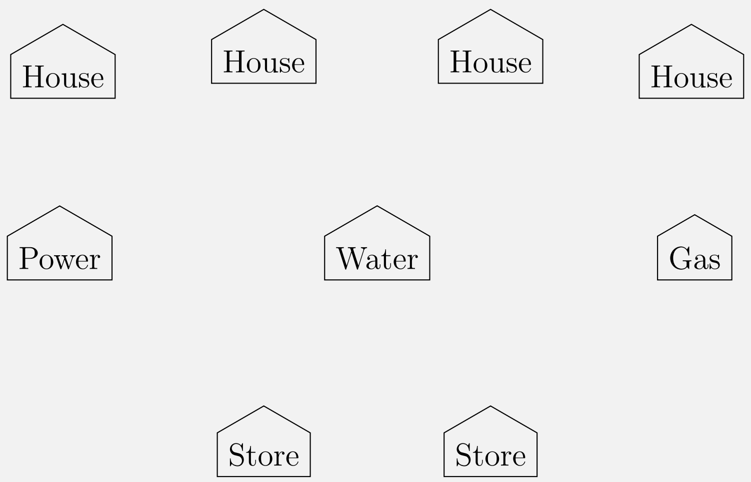
(A) 10 **(B)** 15 **(C)** 20 **(D)** 40

Solution: Every time she combines blocks, the number of blocks goes down by 2. Since the number of blocks decreased by 60, Dana combined blocks 30 times. This means she combined 90 small blocks into 30 big blocks.

Dana started with 10 big blocks. □

Answer: A

A4. In a small town there are two stores, three utility companies (gas, power, and water) and four houses. A map of the town is drawn below.



Each house and each store needs to be connected via a pipe to each utility. How many pipes in total are needed?

(A) 18 **(B)** 20 **(C)** 26 **(D)** 36

Solution: Each of the 3 utility companies must be connected to 4 houses and 2 stores, and hence needs 6 pipes. In total, we need 18 pipes. □

Answer: A

A5. Together, Ana, Brenda, and Carla, have 67 candies. Aunt Mary gives more candies to each one of them. If they each received the same amount of candy from Aunt Mary and Ana now has 27 candies, Brenda has 39 candies, and Carla has 31 candies, then how many candies did Aunt Mary give to each of them?

(A) 8 **(B)** 10 **(C)** 15 **(D)** 30

Solution: After receiving the candies, the kids have

$$27 + 39 + 31 = 97$$

candies. Therefore they received 30 candies in total. Each kid received 10 candies. □

Answer: B

Part B: 5 marks each

B1. The factorial of a positive integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. For example, 3! (pronounced "3 factorial") is

$$3! = 3 \times 2 \times 1$$

while

$$5! = 5 \times 4 \times 3 \times 2 \times 1 \ .$$

What is the last digit of

$$1! + 2! + 3! + 4! + 5! + \ldots + 2024! \qquad ?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: We notice that from 5! onwards, every factorial will have $\times 2$ and $\times 5$ in it, and will end in 0.

Since

$$1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33$$

ends in 3, the last digit of our sum is 3. □

Answer: D

B2. At Canada Jay School, two thirds of the students are in math club, and two fifths of the students are in physics club. If there are 105 students in the school, what is the smallest number of students that must belong to both clubs?

(A) 0 **(B)** 5 **(C)** 7 **(D)** 42 **(E)** 105

Solution: There are 70 students in the math club and 42 in the physics club. Since $70 + 42 = 112$ and there are 105 students in the school, at least 7 students must be in both clubs.

7 is possible if the first 70 students are in the math club and the last 42 students are in the physics club. □

Answer: C

B3. A prime number is an integer, greater than and not including 1, that is divisible by only 1 and itself.

Example: 2 and 3 are prime, but 4 and 6 are not (4 is divisible by 2, and 6 is divisible by 2 and 3).

The faces of two six sided dice are labelled with the first six prime numbers. If the two dice are rolled, what is the probability that the sum of the two faces is even?

(A) $\frac{1}{4}$ **(B)** $\frac{5}{18}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{13}{18}$ **(E)** $\frac{3}{4}$

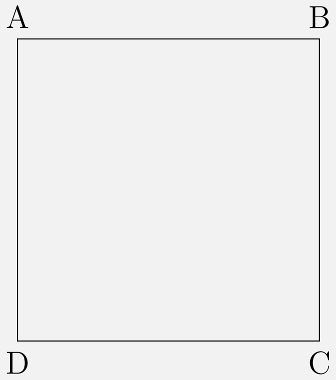
Solution: The first 6 prime numbers are $2, 3, 5, 7, 11, 13$. There is a $\frac{5}{6} \cdot \frac{5}{6}$ chance that both faces are odd and a $\frac{1}{6} \cdot \frac{1}{6}$ chance that both faces are even. Therefore, the probability is

$$\frac{26}{36} = \frac{13}{18}.$$

□

Answer: D

B4. Two runners, Joy and Hope, run around a square (ABCD) with side lengths of 200m.



They both start running at 9:00. Joy starts at point A and runs around the square at a speed of 75m/min counterclockwise ($A \to D, D \to C, C \to B$, and $B \to A$). Hope starts at point B and runs around the square at a speed of 125m/min clockwise ($B \to C, C \to D, D \to A$ and $A \to B$). At what time do they meet for the second time?

(A) 9:06 **(B)** 9:07 **(C)** 9:08 **(D)** 9:09 **(E)** 9:10

Solution: Let us note that the first time they met they run together distance $A - D - C - B$, which is 600 m.

Between the first and second meeting they need to run together another full lap. Therefore, up to the second meeting they ran together

$$600 + 800 = 1400 \text{ m} \quad .$$

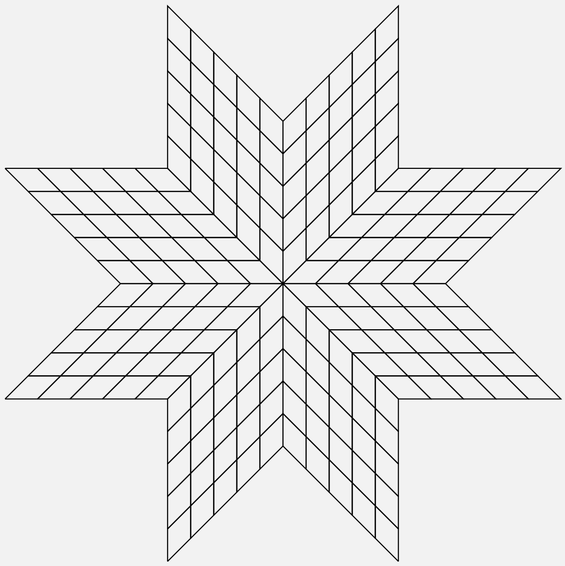
If t is the time, in minutes, from $9:00$ until their second meeting, we have

$$75t + 125t = 1400 \Longrightarrow t = \frac{1400}{200} = 7 \text{ min} \quad .$$

□

Answer: **B**

B5. Grandma is sewing a traditional Indigenous design typically found on blankets and elsewhere, called a starblanket design, on a tapestry. The pattern is made up of 200 rhombi.



For each edge, no matter if it is on the outside of star or between two rhombi, Grandma uses exactly 10cm of thread. How much thread did she use in total?

(A) 34 m **(B)** 40 m **(C)** 44 m **(D)** 48 m **(E)** 80 m

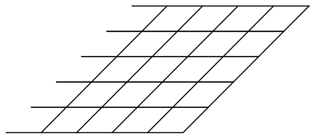
Solution: In total, the 200 rhombi have 800 edges.

The outside of the star has 16 sides, each consisting of 5 edges. Therefore, there are 80 edges on the outside. For those edges, grandma used 800 cm of thread, or 8m.

Each of the remaining 720 edges is common to two rhombi. Therefore, grandma only sews 360 edges of 10 cm, or 36 m on the inside edges.

In total, she used 44 m of thread.

Second solution: The starblanket can be made by making, sewing eight copies of the following shape, and rotating counterclockwise by 45° when moving from one shape to the next.



This shape has six horizontal threads and five vertical threads, each of exactly half meter. Therefore, the length of thread is

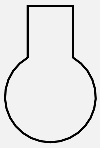
$$8 \times .5 \times 11 = 4 \times 11 = 44\text{m} .$$

□

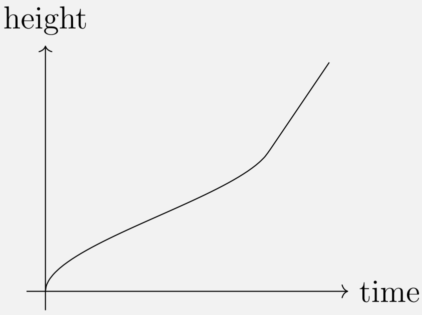
Answer: C

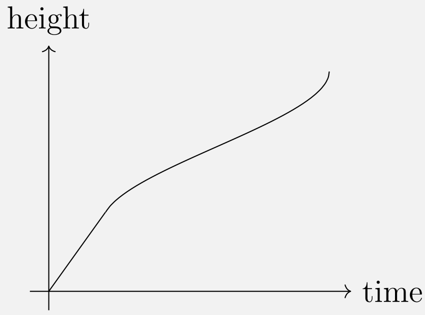
Part C: 5 marks each

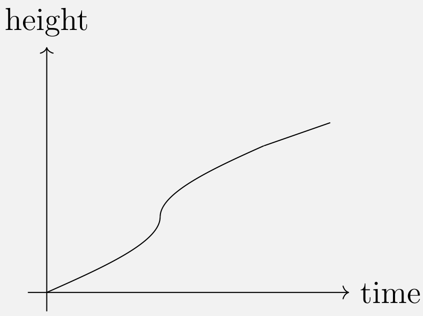
C1. Chemists use round-bottom flasks in their labs and one is shown below.

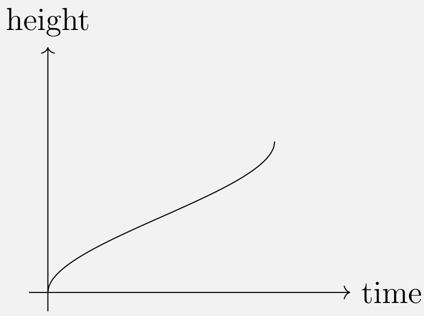


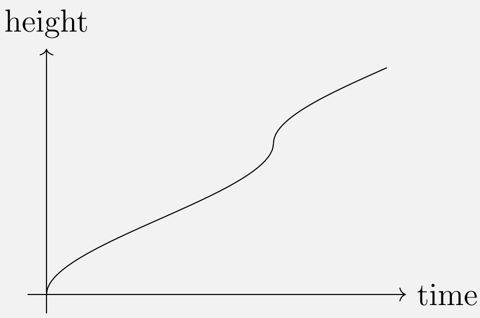
If water flows into the flask at a constant rate and a scientist makes continuous measurements of the height of water in the flask over time, then which graph shows the measurements taken by the scientist?

A.


B.


C.


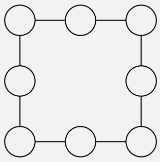
D.


E.


Solution: The height will rise quickly at the beginning, as the flask is very narrow at the bottom. Then, it will rise slower and slower until the water level reaches the widest section of the flask. After this point the height will start rising faster and faster, until it reaches the cylindrical section, where it will rise at a constant rate, giving a straight line segment. The only graph fitting this description is A. □

Answer: A

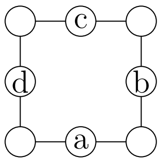
C2. The numbers $1, 2, 3, 4, 5, 6, 7, 8$ can be placed in the circles below with each number used exactly once and such that the sum of the three numbers in the circles on each side of the square is the same for all sides.



What is the largest possible value that sum can take?

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Solution: Denote by a, b, c, d the following circles and s the common sum.



The left and right side contain all numbers excepting a, c. Therefore

$$2s + a + c = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$$

The top and both side contain all numbers excepting b, d. Therefore

$$2s + b + d = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$$

This shows that

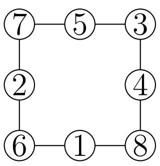
$$a + c = b + d = 36 - 2s\,.$$

In order for s to be as large as possible, $a + c = b + d$ need to be as small as possible. Note that $a + c = b + d = 36 - 2s$ are both even numbers, therefore $a + b + c + d$ needs to be a multiple of 4. On another hand $a + b + c + d \geq 1 + 2 + 3 + 4 = 10$ and hence $a + b + c + d$ needs to be at least 12.

If $a + b + c + d = 12$ then $a + c = b + d = 6$ in which case

$$6 = 36 - 2s \Longrightarrow s = 15$$

Therefor the largest value is 15. This is indeed achievable, as the table below shows:



□

Answer: D

C3. A store sells bottles of juice. The bottles come packed in boxes of either 4, 9 or 15 bottles per box and the store only sells full boxes. Sarah is having a party and she needs exactly 50 bottles (as she doesn't want leftovers). What is the smallest number of boxes Sarah needs to buy to get exactly 50 bottles?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 10

Solution: Note first that buying 4 or more boxes of 15 bottles exceeds 50 bottles. This means that Sarah needs to buy 0, 1, 2 or 3 boxes of 15 bottles.

Case 1: She buys 3 boxes of 15 bottles. Then, she needs to buy an extra 5 bottles, which is not possible.

Case 2: She buys 2 boxes of 15 bottles. Then, she needs to buy an extra 20 bottles. She can buy at most 2 boxes of 9 bottles, and checking the cases she buys 0,1,2 boxes of 9 bottles we see that 0 boxes of 9 bottles and 5 boxes of 4 bottles is the only possibility.

In this case, she buys $2 + 0 + 5 = 7$ boxes.

Case 3: She buys 1 box of 15 bottles. Then, she needs to buy an extra 35 bottles. It follows that she can buy at most 3 boxes of 9 bottles. Also, since she needs an odd number of bottles, the number of boxes of 9 bottles must be odd, hence 1 or 3.

The only possibility is 3 boxes of 9 bottles and 2 boxes of 4 bottles.

In this case, she buys $1 + 3 + 2 = 6$ boxes.

Case 4: She buys 0 boxes of 15 bottles. Since each box has at most 9 bottles, and she needs to buy 50 bottles, she needs 6 or more boxes.

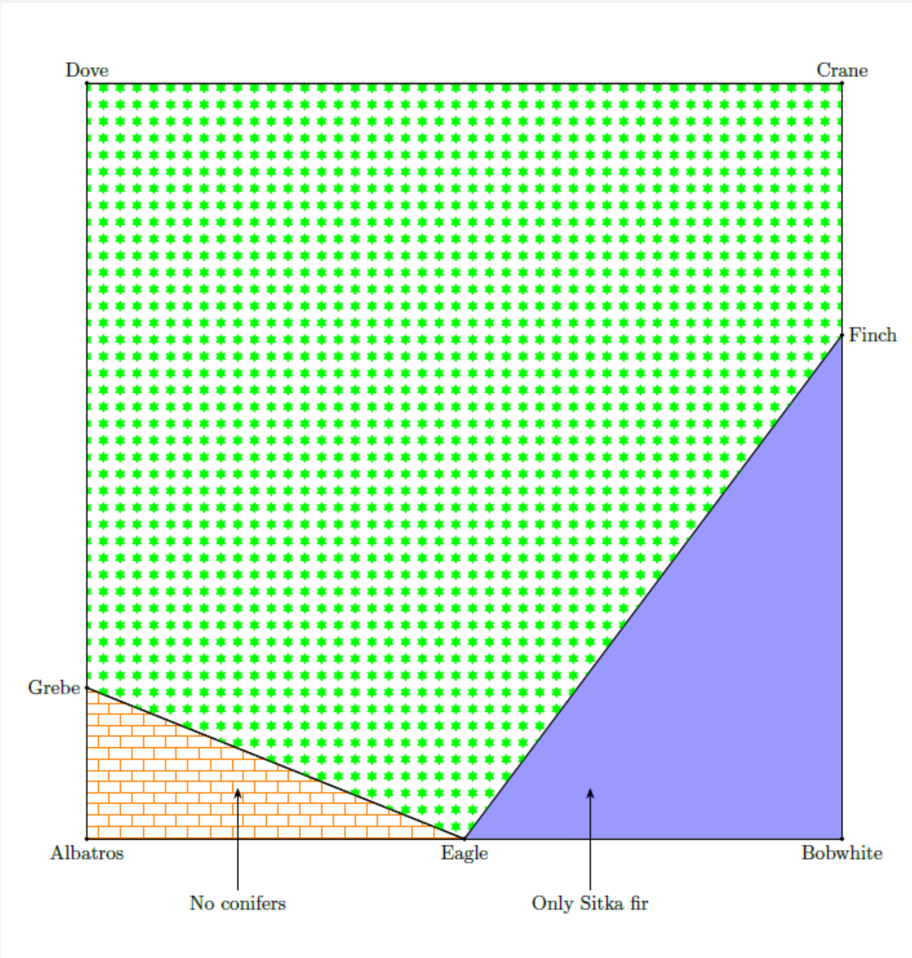
Regardless, the smallest number of boxes is 6. □

Answer: C

C4. Campers arrive to Algonquin park to find a perfectly square forest with an area of 6000 km^2 whose map is given below. They are looking to find a Canada Jay bird. They know the Canada Jay is not found where no conifers grow (marked with orange bricks on the map) or where there is only the Sitka spruce (marked solid blue on the map).

Two campers, Ali and Bria, decide to map the areas where a Canada Jay cannot be found. They walk with a steady pace, measuring their times between points of interest. When they return, they make the following comments:

- The walk from Albatross to Eagle took as long as the walk from Eagle to Bobwhite.
- The walk from Dove to Grebe took 4 times as long as the walk from Grebe to Albatross.
- The walk from Bobwhite to Finch took twice as long as the walk from Finch to Crane.



How big is the area (marked with green stars) where a Canada Jay might be spotted?

(A) 3400 **(B)** 4700 **(C)** 5000 **(D)** 5700 **(E)** 6000

Solution: Let x be the side of square. We have $x^2 = 6000$.

Now, the distance from A to G is $\frac{x}{5}$, from A to E and from E to B are $\frac{x}{2}$ and from B to F is $\frac{2x}{3}$.

Therefore

$$\text{orange bricks area} = \frac{x^2}{20} = 300$$
$$\text{solid blue area} = \frac{x^2}{6} = 1000\,.$$

Therefore, Blue jays can be seen in $6000 - 300 - 1000 = 4700\text{km}^2$. □

Answer: **B**

C5. A 6-digit number is said to be "exciting" if it satisfies each of the following properties.

- Each of the digits 4,5,6,7,8,9 occur exactly once;
- The number formed by the first n digits is divisible by n, for each $n = 1, 2, 3, 4, 5, 6$.

For example, 987654 is exciting because 9 is divisible by 1, 98 is divisible by 2, 987 is divisible by 3, 9876 is divisible by 4, 98765 is divisible by 5, and 987654 is divisible by 6.
Not including 987654, how many exciting numbers exist?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: Let us note that the second, fourth and sixth digits must be even. Therefore, they must be $8, 6, 4$ in some order.

The fifth digit must be 5. It follows that the first and third digit are 9 and 7, in some order.

The sum of the first three digits must be divisible by 3, and hence 16 + second digit must be divisible by 3. Since the second digit is $8, 6$ or 4 must be 8.

We therefore concluded that the first and third digits are 9 and 7, in some order, and the second digit must be 8.

We split now the problem into two cases:

Case 1: The number starts with 987. We know that the fourth digit must be 6 or 4, thence the first four digits are either 9876 or 9874. Since this number must be divisible by 4, the fourth digit must be 6. As the fifth digit is 5, we get that in Case 1 we only have the already mentioned exciting number 987654.

<u>Case 2:</u> The number starts with 789. We know that the fourth digit must be 6 or 4, thence the first four digits are either 7896 or 7894. Since this number must be divisible by 4, the fourth digit must be 6. As the fifth digit is 5, we get that in Case 2 we only have one exciting number 789654.

Therefore, the answer is one more exciting number. □

Answer: **B**

