2024 Canada Lynx Mathematical Competition

Official Solutions

A competition of the Canadian Mathematical Society.

Part A: 4 marks each

1. Let $S = 1 - 2 + 4 - 8 + 16 - 32 + 64$. What is the value of S?
   (A) 31 **(B)** 33 **(C)** 41 **(D)** 43

Solution: We rewrite the expression as $S = 1 + (-2 + 4) + (-8 + 16) + (-32 + 64)$, upon which we see that the desired answer is $1 + 2 + 8 + 32 = 43$.

Answer: **(D)**

2. If December 1 occurs on a Sunday, on what day of the week will December 14 occur?

   (A) Fri. **(B)** Sat. **(C)** Sun. **(D)** Mon.

Solution: We know that there are seven days in a week. Thus, if December 1 occurs on a Sunday (as it does in the year 2024), then December 8 must also occur on Sunday, and so must December 15.

December 14 is the day before December 15, and Saturday is the day before Sunday.

Hence, if December 1 occurs on a Sunday, then December 14 must occur on a Saturday.

Answer: **(B)**

3. If x and y satisfy $3x - y = 16$ and $3y - x = 24$, what is the value of $5x + 5y$?

   (A) 40 **(B)** 50 **(C)** 80 **(D)** 100

Solution: One way to answer this question is to solve the system of two equations, to determine the values of x and y.

For example, if we multiply the first equation by 3 and add it to the second equation, we get

$$3(3x - y) + 1(3y - x) = 3 \times 16 + 1 \times 24$$

And this simplifies to $9x - 3y + 3y - x = 48 + 24$, or $8x = 72$, and so we get $x = 9$.

Since $3x - y = 16$, we know that $y = 3x - 16$, and so $x = 9$ implies that $y = 3 \times 9 - 16 = 27 - 16 = 11$.

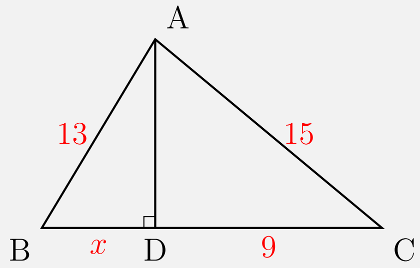
Since $x = 9$ and $y = 11$, we conclude that $5x + 5y = 5 \times 9 + 5 \times 11 = 45 + 55 = 100$.

While the above solution is correct, a far more insightful and elegant solution is to exploit the symmetry in the two equations. If we add the two equations, we see that the left side sums to $(3x - y) + (3y - x) = 2x + 2y$ and the right side sums to $16 + 24 = 40$.

Since $2x + 2y = 40$, we know that $x + y = 20$. And so we immediately see that $5x + 5y = 5(x + y) = 5 \times 20 = 100$, and we answer the question without needing to solve for the variables x and y.

Answer: **(D)**

4. In $\triangle ABC$, D is on side BC so that AD is perpendicular to BC.



As seen in the diagram, $AB = 13$, $AC = 15$, $CD = 9$, and $BD = x$. What is the value of x?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8

Solution: By the Pythagorean Theorem, we know that $AD^2 + CD^2 = AC^2$, which implies $AD^2 = 15^2 - 9^2 = 225 - 81 = 144$, and so $AD = \sqrt{144} = 12$.

Once again, by the Pythagorean Theorem, we know that $AD^2 + BD^2 = AB^2$, and so $12^2 + x^2 = 13^2$. Therefore, $x^2 = 13^2 - 12^2 = 169 - 144 = 25$, and so $x = \sqrt{25} = 5$.

We can also get the answer without solving for AD. From the two equations $AD^2 + CD^2 = AC^2$ and $AD^2 + BD^2 = AB^2$, we know that $AD^2 = AC^2 - CD^2 = AB^2 - BD^2$, which implies $15^2 - 9^2 = 13^2 - x^2$.

It follows that $x^2 = 13^2 + 9^2 - 15^2 = 169 + 81 - 225 = 25$, and so we conclude that $x = \sqrt{25} = 5$.

Answer: **(A)**

5. If $2^7 + 2^7 + 2^8 + 2^7 + 2^7 + 2^8 = 4^x$, what is x?

(A) 4 **(B)** 5 **(C)** 8 **(D)** 10

Solution: Since $2^7 + 2^7 = 2 \times 2^7 = 2^1 \times 2^7 = 2^{1+7} = 2^8$, we can rewrite the left side as $2^8 + 2^8 + 2^8 + 2^8$.

We see that $2^8 + 2^8 + 2^8 + 2^8 = 4 \times 2^8 = 2^2 \times 2^8 = 2^{2+8} = 2^{10}$.

Our goal is to determine the value of x for which $2^{10} = 4^x$. The easiest way to do this is to observe that $4^x = (2^2)^x = 2^{2x}$, and so $2^{10} = 2^{2x}$. We conclude that $10 = 2x$, or $x = 5$.

Answer: **(B)**

Part B: 5 marks each

6. Define $f(x) = x^2 - 3x + 5$ for all real numbers x. What is the smallest possible value of $f(x)$?

(A) 2 **(B)** $\frac{9}{4}$ **(C)** $\frac{5}{2}$ **(D)** $\frac{11}{4}$ **(E)** 3

Solution: Note that $f(x) = x^2 - 3x + 5 = x(x-3) + 5$, and so $f(x)$ is a parabola passing through the two points $(0, 5)$ and $(3, 5)$. The key observation is that the smallest value of $f(x)$ occurs when $x = \frac{3}{2}$, the midpoint of $x = 0$ and $x = 3$.

To prove this, we can multiply both sides by 4, to get $4f(x) = 4x^2-12x+20 = (4x^2 - 12x) + 20$. We then *complete the square* to get $4f(x) = (4x^2 - 12x + 9) + 11 = (2x-3)(2x-3) + 11 = (2x-3)^2 + 11$.

Dividing both sides by 4, we get $f(x) = \frac{(2x-3)^2}{4} + \frac{11}{4}$. Since every perfect square is non-negative, we see immediately that the smallest value of $f(x)$ must occur when $2x - 3 = 0$, which implies $x = \frac{3}{2}$.

For this value of x, we see that $f(x) = 0 + \frac{11}{4} = \frac{11}{4}$. For all other values of x, we have $f(x) > 0 + \frac{11}{4} = \frac{11}{4}$. And so the smallest possible value of $f(x)$ must be $\frac{11}{4}$.

Answer: **(D)**

7. The parabola $y = -x^2 + 10x - 16$ has vertex at point A, and intersects the x-axis at points B and C. What is the area of $\triangle ABC$?

(A) 15 **(B)** 27 **(C)** 30 **(D)** 36 **(E)** 54

Solution: Like we did in the previous question, we can *complete the square* to find the x-coordinate of the vertex. We observe that $y = -x^2+10x-16 = -(x^2 - 10x) - 16 = -(x^2 - 10x + 25) + 25 - 16 = -(x-5)^2 + 9$.

Thus, the vertex of the parabola must occur at $x = 5$, which implies $y = 9$. Thus, point A has coordinates $(5, 9)$.

We know that the parabola intersects the x-axis, which has equation $y = 0$, at points B and C. To obtain these coordinates, we solve for $0 = -x^2+10x-16$.

We see that $0 = -(x-2)(x-8)$, implying that $x = 2$ and $x = 8$. Thus, point B and C have coordinates $(2, 0)$ and $(8, 0)$.

From here we observe that $\triangle ABC$ has length $BC = 8 - 2 = 6$ and height 9. And so the area of this triangle must be $\frac{1}{2} \times 6 \times 9 = 27$.

Answer: **(B)**

8. We say that a positive integer is *happy* if its digits add up to a multiple of 5. For example, 64 is happy since $6 + 4 = 10$, and 10 is a multiple of 5.

   How many positive integers between 10 and 99 are happy?

   (A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

Solution: Suppose the two digits are x and y, where x is the tens digit and y is the units digit.

We know that $x + y$ is at least $1 + 0 = 1$ and at most $9 + 9 = 18$. Thus, a two-digit integer is happy provided that $x + y$ is equal to 5 or 10 or 15.

We know that $1 \le x \le 9$ and $0 \le y \le 9$. We consider all three cases.

If $x+y = 5$, then (x, y) can be any of the following: $(1, 4), (2, 3), (3, 2), (4, 1), (5, 0)$. These correspond to the happy numbers 14, 23, 32, 41, 50.

If $x+y = 10$, then (x, y) can be any of the following: $(1, 9)$, $(2, 8)$, $(3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4)$, $(7, 3)$, $(8, 2)$, $(9, 1)$. These correspond to the happy numbers 19, 28, 37, 46, 55, 64, 73, 82, 91.

If $x+y = 15$, then (x, y) can be any of the following: $(6, 9), (7, 8), (8, 7), (9, 6)$. These correspond to the happy numbers 69, 78, 87, 96.

In total, there are $5 + 9 + 4 = 18$ happy numbers.

Answer: **(E)**

9. In this 3×3 grid of integers, the sum of each row is 0, the sum of each column is 0, and the sum of each diagonal is 0.

	-4	
		2
x		

What is the value of x?

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Solution: Label the nine cells of the grid as follows:

A	B	C
D	E	F
G	H	I

We first prove that the centre cell *must* be 0.

Since each row, each column, and each diagonal add up to 0, we know that $A + E + I = 0$, $B + E + H = 0$, and $C + E + G = 0$.

Adding the three equations, we get $A + E + I + B + E + H + C + E + G = 0$, which we rewrite as $(A + B + C) + 3E + (G + H + I) = 0$. Since each row sums to 0, we know that $0 + 3E + 0 = 0$, which implies $3E = 0$, or $E = 0$.

Thus, we have proven that the centre cell is $E = 0$, and we also are given that $G = x$, $B = -4$, and $F = 2$. Therefore, $C = 0 - E - G = -x$, and $H = 0 - B - E = 4$.

A	-4	-x
D	0	2
x	4	I

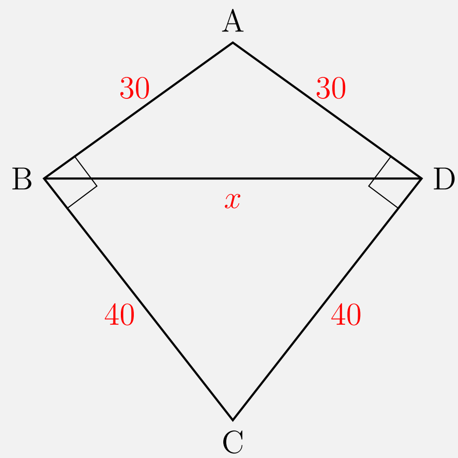
Since the bottom row adds up to 0, we must have $I = -x - 4$. Since the last column adds up to 0, we must have $I = x - 2$. It follows that $-x - 4 = x - 2$, and so $2x = -2$, or $x = -1$.

Note that we can also get the correct answer without realizing that $E = 0$. From the equations $C + E + G = 0$, $G + H + I = 0$, $C + F + I = 0$, $B + E + H = 0$, we know that $0 = (C + E + G) + (G + H + I) - (C + F + I) - (B + E + H) = 2G - F - B$. Therefore, $G = \frac{B+F}{2}$.

Since $G = x$, $B = -4$, and $F = 2$, we conclude that $x = \frac{-4+2}{2} = -1$.

Answer: **(B)**

10. In the diagram, $ABCD$ is a quadrilateral with $AD = AB = 30$, $CD = CB = 40$, and $\angle ADC = \angle ABC = 90°$.



What is the length of the line segment BD?

(A) 24 **(B)** 36 **(C)** 48 **(D)** 50 **(E)** 72

Solution: We determine the length of BD by calculating the area of the quadrilateral in two different ways. Let this area be S, and let E be the intersection point of diagonal lines AC and BD.

The area of the quadrilateral is the sum of the areas of $\triangle ABD$ and $\triangle CBD$. Both triangles have length BD and their heights sum to AC.

Thus, $S = \frac{1}{2} \times BD \times AE + \frac{1}{2} \times BD \times EC = \frac{1}{2} \times BD \times (AE + EC) = \frac{1}{2} \times BD \times AC$.

We also know that the area of the quadrilateral is the sum of the areas of $\triangle ABC$ and $\triangle ADC$. Both triangles are right-angled triangles with side lengths 30 and 40.

Thus, $S = \frac{1}{2} \times 30 \times 40 + \frac{1}{2} \times 30 \times 40 = 30 \times 40 = 1200$.

We have shown that $\frac{1}{2} \times BD \times AC = 1200$, which implies that $BD = \frac{2400}{AC}$.

Since AC is the hypotenuse of right-angled $\triangle ABC$, we know by the Pythagorean Theorem that $AC = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$. Since $AC = 50$, we conclude that $BD = \frac{2400}{50} = 48$.

We can also solve the problem using Similar Triangles, noting that $\triangle ABC$ is similar to $\triangle AEB$. We see that $\frac{EB}{AB} = \frac{BC}{AC}$, and so $EB = \frac{BC \times AB}{AC} = \frac{40 \times 30}{50} = \frac{1200}{50} = 24$.

Since $EB = 24$, we see that $BD = 2EB = 48$, and so once again we see that $BD = 48$.

Answer: **(C)**

Part C: 7 marks each

11. For each function $f(x)$, define $f^{n+1}(x) = f(f^n(x))$ for all $n \geq 1$.

For example, if $f(x) = x + 2$, then $f^3(3) = f(f^2(3)) = f(f(f(3))) = f(f(5)) = f(7) = 9$.

If $f(x) = \dfrac{-4}{x+2}$, what is the integer that is closest to $f^{2024}(2024)$?

(A)	-2024	**(C)**	0	**(E)**	4
(B)	-2	**(D)**	2	**(F)**	2024

Solution: We will prove that $f^3(x) = x$ for all values of x except for $x = -2$, at which the function is not defined.

Since $f(x) = \frac{-4}{x+2}$, we have

$$\begin{aligned} f^2(x) &= f(f(x)) \\ &= f\left(\frac{-4}{x+2}\right) \\ &= \frac{-4}{\frac{-4}{x+2} + 2} \\ &= \frac{-4}{\frac{-4}{x+2} + \frac{2x+4}{x+2}} \\ &= \frac{-4}{\frac{2x}{x+2}} \\ &= \frac{-4(x+2)}{2x} \\ &= \frac{-2x-4}{x}. \end{aligned}$$

Therefore, we have

$$f^3(x) = f(f^2(x)) = f\left(\frac{-2x-4}{x}\right) = \frac{-4}{\frac{-2x-4}{x} + 2} = \frac{-4}{\frac{-2x-4}{x} + \frac{2x}{x}} = \frac{-4}{\frac{-4}{x}} = \frac{-4x}{-4} = x$$

Since $f^3(x) = x$, this implies $f^4(x) = f(f^3(x)) = f(x)$. This implies $f^5(x) = f(f^4(x)) = f(f(x)) = f^2(x)$.

A similar argument shows that $f^6(x) = f^3(x)$, $f^7(x) = f^4(x)$, and so on.

Hence, the sequence $f(x), f^2(x), f^3(x), f^4(x), f^5(x), f^6(x), \ldots$ repeats every third term.

Since 2024 gives a remainder of 2 when divided by 3, we know that $f^{2024}(x) = f^2(x) = \frac{-2x-4}{x}$. And so

$$f^{2024}(2024) = f^2(2024) = \frac{-2 \times 2024 - 4}{2024} = -2 - \frac{4}{2024}$$

Since $\frac{4}{2024}$ is a very small number (approximately equal to 0.002), the integer closest to $f^{2024}(2024)$ is -2.

Answer: **(B)**

12. Let $t_1, t_2, t_3, \ldots, t_{12}$ be a sequence of positive integers with $t_1 < t_2 < t_3 < \ldots < t_{12}$.

    This sequence has the property that each term t_n (with $3 \leq n \leq 12$) is the sum of the two previous terms.

    If $t_7 = 184$, what is the value of t_{12}?

(A)	2024	**(C)**	2032	**(E)**	2042
(B)	2027	**(D)**	2037	**(F)**	2047

Solution: Let $t_1 = x$ and $t_2 = y$ for some positive integers x and y. Since our sequence is increasing, we know that $x < y$.

Since each term is the sum of the two previous terms, we have $t_3 = t_1 + t_2 = x + y$ and $t_4 = t_2 + t_3 = y + (x + y) = x + 2y$.

Continuing, we can show that $t_5 = 2x + 3y$, $t_6 = 3x + 5y$, $t_7 = 5x + 8y$, $t_8 = 8x + 13y$, $t_9 = 13x + 21y$, $t_{10} = 21x + 34y$, $t_{11} = 34x + 55y$, and $t_{12} = 55x + 89y$.

We are given that $t_7 = 5x + 8y = 184$ and our goal is to determine the value of $t_{12} = 55x + 89y$.

Observe that $t_{12} = 55x+89y = 55x+88y+y = 11(5x+8y)+y = 11\times 184+y = 2024 + y$. Thus, to calculate the value of t_{12}, we just need to calculate the value of y.

We know that x and y are positive integers with $x < y$ for which $5x + 8y = 184$. Since $5x$ is a positive multiple of 5, we know that $184 - 8y = 8(23 - y)$ must also be a positive multiple of 5. It suffices to check the values of $y > 0$ for which $23 - y$ is a positive multiple of 5, namely $y = 3$, $y = 8$, $y = 13$, and $y = 18$.

These correspond to the solutions $(x, y) = (32, 3)$, $(x, y) = (24, 8)$, $(x, y) = (16, 13)$, and $(x, y) = (8, 18)$. Only one of these solutions, namely $(x, y) = (8, 18)$ satisfies the property that $0 < x < y$.

Therefore, we conclude that $y = 18$, and so t_{12} must equal $2024 + y = 2042$.

We can check that the sequence $8, 18, 26, 44, 70, 114, 184, 298, 482, 780, 1262, 2042$ satisfies the conditions of the problem.

Answer: $\boxed{\textbf{(E)}}$

13. We play the following game where you start with 2 points.

You are given a fair coin that lands Heads with probability $\frac{1}{2}$ and Tails with probability $\frac{1}{2}$.

You flip this coin six consecutive times. Each time you flip Heads, the number of points *multiplies* by 10. Each time you flip Tails, the number of points *increases* by 2. Your final score is the number of points you have at the end of six coin flips.

For example, if you flip Heads, Tails, Tails, Tails, Tails, Heads, then your final score is $(2 \times 10 + 2 + 2 + 2 + 2) \times 10 = 280$.

And if you flip Heads, Heads, Tails, Heads, Tails, Tails, then your final score is $(2 \times 10 \times 10 + 2) \times 10 + 2 + 2 = 2024$.

What is the probability that your final score is more than 2024?

(A) $\frac{11}{32}$ **(C)** $\frac{1}{2}$ **(E)** $\frac{5}{8}$
(B) $\frac{3}{8}$ **(D)** $\frac{19}{32}$ **(F)** $\frac{21}{32}$

Solution: Suppose you have a score of N points. If you flip Heads then Tails then you receive $10N + 2$ points, and if you flip Tails then Heads then you receive $(N + 2) \times 10 = 10N + 20$ points.

Since $10N + 20 > 10N + 2$, we observe that given any sequence of flips (e.g. HHTHTT) where Heads = H and Tails = T, we can *increase* the final score by switching any consecutive HT to TH, and *decrease* the final score by switching any consecutive TH to HT. Each switch affects the final score but does not change the number of Heads and Tails in our sequence.

As an example, if we switch H**HT**HTT to H**TH**HTT, the final score increases from 2024 to 2204, and if we switch HH**TH**TT to HH**HT**TT, the number of points decreases from 2024 to 2006.

If a sequence has 4 Heads and 2 Tails, we see that the *minimum* final score occurs when the sequence is HHHHTT, and the final score is $2 \times 10 \times 10 \times 10 \times 10 + 2 + 2 = 20004 > 2024$.

Similarly, if the sequence has 5 Heads or 6 Heads, we can easily show that the final score is guaranteed to exceed 2024.

If a sequence has 2 Heads and 4 Tails, we see that the *maximum* final score occurs when the sequence is TTTTHH, and the final score is $(2+2+2+2+2) \times 10 \times 10 = 1000 < 2024$.

Similarly, if the sequence has 1 Head or 0 Heads, we can easily show that the final score is guaranteed to be less than 2024.

So the only interesting scenario to consider is when the sequence has 3 Heads and 3 Tails. There are $\binom{6}{3} = \frac{6!}{3!3!} = \frac{720}{6\times 6} = 20$ different ways that a sequence of 6 flips has 3 Heads and 3 Tails.

If the sequence starts with T, the minimum final score occurs when the sequence is THHHTT, with a final score of $(2+2)\times 10\times 10\times 10+2+2 = 4004$, and if the sequence starts with HT, the minimum final score occurs when the sequence is HTHHTT, with a final score of $(2\times 10+2)\times 10\times 10+2+2 = 2204$.

So all sequences with 3 Heads and 3 Tails that begin with T or HT are guaranteed to have a final score greater than 2024.

It suffices to consider the four scenarios where the sequence starts with HH.

If the sequence is HHHTTT, the final score is $2\times 10\times 10\times 10+2+2+2 = 2006$.
If the sequence is HHTHTT, the final score is $(2\times 10\times 10+2)\times 10+2+2 = 2024$.
If the sequence is HHTTHT, the final score is $(2\times 10\times 10+2+2)\times 10+2 = 2042$.
If the sequence is HHTTTH, the final score is $(2\times 10\times 10+2+2+2)\times 10 = 2060$.

Of the $\binom{6}{3} = 20$ different ways that a sequence of 6 flips has 3 Heads and 3 Tails, we see that for all but two sequences (HHHTTT, HHTHTT), the final score is more than 2024.

There are $\binom{6}{K} = \frac{6!}{K!(6-K)!}$ ways that a sequence of 6 flips has K Heads and $6-K$ Tails. By our analysis above, there are $\binom{6}{4} + \binom{6}{5} + \binom{6}{6} + (\binom{6}{3} - 2) =$

$15 + 6 + 1 + 18 = 40$ ways in which the final score is more than 2024.

Since all $2^6 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 64$ sequences are equally likely to occur, the desired probability is $\frac{40}{64} = \frac{5}{8}$.

Answer: **(E)**

14. For each positive integer n, define $S(n)$ to be the sum of the positive divisors of n, and define $P(n)$ to be the number of prime divisors of n.

    For example, $S(20) = 1 + 2 + 4 + 5 + 10 + 20 = 42$, and $P(20) = 2$ because the only prime divisors of 20 are 2 and 5.

    If n is a positive integer for which $S(n) > 4n$, what is the minimum possible value of $P(n)$?

    (A) 1 **(C)** 3 **(E)** 5
    (B) 2 **(D)** 4 **(F)** 6

Solution: Let $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of n, where each p_i is prime, and each a_i is a positive integer.

Every divisor of n must only contain the primes in the set $\{p_1, p_2, \ldots, p_k\}$, and be of the form $p_1^{b_1} \cdot p_2^{b_2} \cdots p_k^{b_k}$, where $0 \leq b_i \leq a_i$ for each $1 \leq i \leq k$.

Since each b_i can range from 0 to a_i, there are $a_i + 1$ options for each term $p_i^{b_i}$. And this is true for each index i from 1 to k. Therefore, we observe that $S(n)$, the sum of all of the positive divisors of n, must equal

$$S(n) = (p_1^0 + p_1^1 + \ldots + p_1^{a_1}) \cdot (p_2^0 + p_2^1 + \ldots + p_2^{a_2}) \cdots (p_k^0 + p_k^1 + \ldots + p_k^{a_k}).$$

Consider the first term $(p_1^0 + p_1^1 + \ldots + p_1^{a_1})$, which is the sum of the first $a_1 + 1$ terms of a geometric sequence with first term p_1^0 and common ratio p_1. Letting T be this total, we have the following two equations:

$$\begin{aligned} T &= p_1^0 + p_1^1 + p_1^2 + p_1^3 + \ldots + p_1^{a_1} \\ p_1 T &= p_1^1 + p_1^2 + p_1^3 + p_1^4 + \ldots + p_1^{a_1+1} \end{aligned}$$

The second equation is formed by multiplying the first equation by p_1. If we subtract the first equation from the second, we get

$$p_1T - T = (p_1^1 + p_1^2 + p_1^3 + p_1^4 + \ldots + p_1^{a_1+1}) - (p_1^0 + p_1^1 + p_1^2 + p_1^3 + \ldots + p_1^{a_1}) = p_1^{a_1+1} - p_1^0.$$

Therefore, $p_1T - T = p_1^{a_1+1} - p_1^0$, which is equivalent to $T(p_1 - 1) = p_1^{a_1+1} - 1$, or $T = \frac{p_1^{a_1+1}-1}{p_1-1}$.

We have shown that the first term of our product $S(n)$ is equal to $\frac{p_1^{a_1+1}-1}{p_1-1}$, and we get a similar formula for all of the other terms. Therefore,

$$S(n) = \frac{p_1^{a_1+1}-1}{p_1-1} \cdot \frac{p_2^{a_2+1}-1}{p_2-1} \cdots \frac{p_k^{a_k+1}-1}{p_k-1}.$$

Since $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, we have

$$\frac{S(n)}{n} = \frac{S(n)}{p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}} = \frac{p_1^{a_1+1}-1}{p_1^{a_1}(p_1-1)} \cdot \frac{p_2^{a_2+1}-1}{p_2^{a_2}(p_2-1)} \cdots \frac{p_k^{a_k+1}-1}{p_k^{a_k}(p_k-1)}.$$

And this simplifies to

$$\frac{S(n)}{n} = \frac{p_1 - \frac{1}{p_1^{a_1}}}{p_1-1} \cdot \frac{p_2 - \frac{1}{p_2^{a_2}}}{p_2-1} \cdots \frac{p_k - \frac{1}{p_k^{a_k}}}{p_k-1} < \frac{p_1-0}{p_1-1} \cdot \frac{p_2-0}{p_2-1} \cdots \frac{p_k-0}{p_k-1}.$$

In the question, we are given that $S(n) > 4n$, i.e., $\frac{S(n)}{n} > 4$. Our goal is to find the minimum possible value of $P(n)$, which equals k, the number of prime divisors of n.

We will show that if $k = 3$, then $\frac{S(n)}{n}$ must be less than 4. Observe that each fraction $\frac{p_i-0}{p_i-1}$ is *maximized* when we make the prime p_i as *small* as possible.

If n has only 2, 3, and 5 as its prime divisors, then the above analysis shows that

$$\frac{S(n)}{n} < \frac{p_1-0}{p_1-1} \cdot \frac{p_2-0}{p_2-1} \cdot \frac{p_3-0}{p_3-1} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{4} = 3.75 < 4.$$

Thus, if $k = 3$, then $\frac{S(n)}{n} < 3.75$, which implies that we cannot have $S(n) > 4n$. A similar argument shows that $S(n) < 4n$ if $k = 2$ or $k = 1$.

We now show that if $n = 2^{100} \times 3^{100} \times 5^{100} \times 7^{100}$, then $S(n) > 4n$. This will prove that $k = 4$ is the desired answer.

When a_i is large, the fraction $\frac{p_1 - \frac{1}{p_1^{a_1}}}{p_1-1}$ is essentially equal to $\frac{p_1-0}{p_1-1}$.

And so, for this value of n, we have

$$\frac{S(n)}{n} \sim \frac{p_1-0}{p_1-1} \cdot \frac{p_2-0}{p_2-1} \cdot \frac{p_3-0}{p_3-1} \cdot \frac{p_4-0}{p_4-1} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} = \frac{105}{24} = 4.375 > 4.$$

We have found a positive integer n with $k = 4$ prime divisors for which $S(n) > 4n$, and proven that if n has at most three prime divisors, then we must have $S(n) < 4n$. Thus, $k = 4$ is the minimum possible value of $P(n)$.

Interesting note: if $n = 2^5 \times 3^3 \times 5^1 \times 7^1$, we can show that $S(n) = 4n$. And so any multiple of this number, say $n = 2^6 \times 3^3 \times 5^1 \times 7^1$, is guaranteed to satisfy $S(n) > 4n$.

Answer: $\boxed{\textbf{(D)}}$

15. In $\triangle ABC$, let O be the circumcentre of the triangle and let I be the incentre of the triangle.

    Let R be the radius of the circumcircle, and let r be the radius of the incircle.

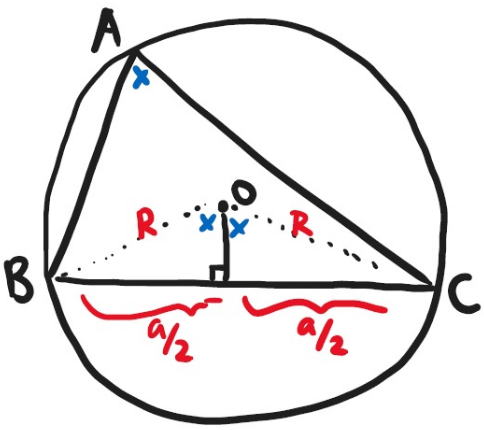
    If $\sin A = \frac{3}{5}$ and $\angle AIO = 90°$, what is the value of the ratio $\frac{R}{r}$?

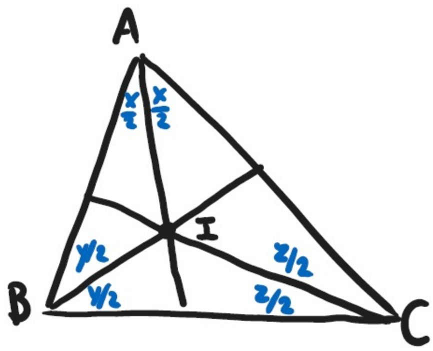
(A) $\frac{5}{3}$	**(C)** $\frac{5}{2}$	**(E)** 4
(B) 2	**(D)** 3	**(F)** 5

Solution: Let $\angle A = x$, $\angle B = y$, and $\angle C = z$. Thus, $x + y + z = 180°$. We will prove that $\frac{R}{r} = \frac{1}{1-\cos x}$.

Let a, b, and c be the lengths of the sides BC, AC, and AB, respectively.

We use trigonometry to determine formulas for AI and AO, which will enable us to determine an expression for the fraction $\frac{R}{r}$.





Draw the circumcircle of $\triangle ABC$, where the circumcircle has radius R. Note that $\angle BOC = 2\angle BAC = 2x$, by the Central Angle Theorem. By splitting $\triangle BOC$ into two right-angled triangles, we observe that $\sin x = \frac{a/2}{R} = \frac{a}{2R}$, implying that $R = \frac{a}{2\sin x}$.

The same argument shows that $R = \frac{b}{2\sin y} = \frac{c}{2\sin z}$. Specifically, we have shown that $c = 2R\sin z$.

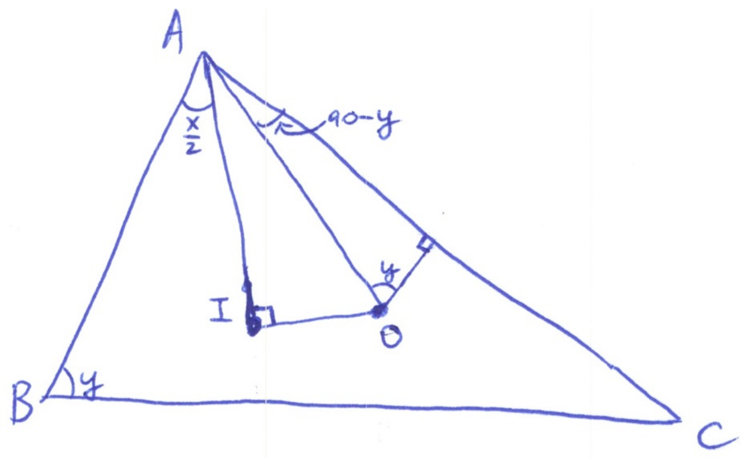
Draw the internal angle bisectors of the triangle, which meet at the incentre I, by definition. We see that $\angle BAI = \frac{x}{2}$ and $\angle ABI = \frac{y}{2}$, which implies that $\angle AIB = 180^\circ - \frac{x}{2} - \frac{y}{2} = 180^\circ - \frac{x+y}{2} = 180^\circ - \frac{180^\circ - z}{2} = 90^\circ + \frac{z}{2}$.

By applying the Sine Law on $\triangle ABI$, we see that $\frac{AI}{\sin\frac{y}{2}} = \frac{AB}{\sin(90^\circ + \frac{z}{2})} = \frac{AB}{\cos\frac{z}{2}}$, implying that $AI = \frac{c\sin\frac{y}{2}}{\cos\frac{z}{2}}$.

Using the double-angle formula $\sin z = 2\sin\frac{z}{2}\cos\frac{z}{2}$, we see that

$$\begin{aligned} \frac{AI}{AO} &= \frac{1}{AO}\cdot AI \\ &= \frac{1}{AO}\cdot\frac{c\sin\frac{y}{2}}{\cos\frac{z}{2}} \\ &= \frac{(2R\sin z)\sin\frac{y}{2}}{AO\cos\frac{z}{2}} \\ &= \frac{2R\cdot(2\sin\frac{z}{2}\cos\frac{z}{2})\sin\frac{y}{2}}{R\cos\frac{z}{2}} \\ &= 4\sin\frac{y}{2}\sin\frac{z}{2}. \end{aligned}$$

We now determine the measure of $\angle IAO$ in terms of y and z. Without loss of generality, assume that $y \geq z$.



By the Central Angle Theorem, $\angle AOC = 2\angle ABC = 2y$. By splitting $\triangle AOC$ into two right-angled triangles, we observe that $\angle OAC = 90^\circ - y$.

Therefore, $\angle IAO = \angle BAC - \angle BAI - \angle OAC = x - \frac{x}{2} - (90^\circ - y) = \frac{x}{2} - 90^\circ + y = \frac{180^\circ - y - z}{2} - 90^\circ + y = 90^\circ - \frac{y}{2} - \frac{z}{2} - 90^\circ + y = \frac{y-z}{2}$.

Since we are given that $\angle AIO = 90^\circ$, we have $\cos(\frac{y-z}{2}) = \frac{AI}{AO}$. Earlier, we showed that $\frac{AI}{AO} = 4\sin\frac{y}{2}\sin\frac{z}{2}$. Therefore,

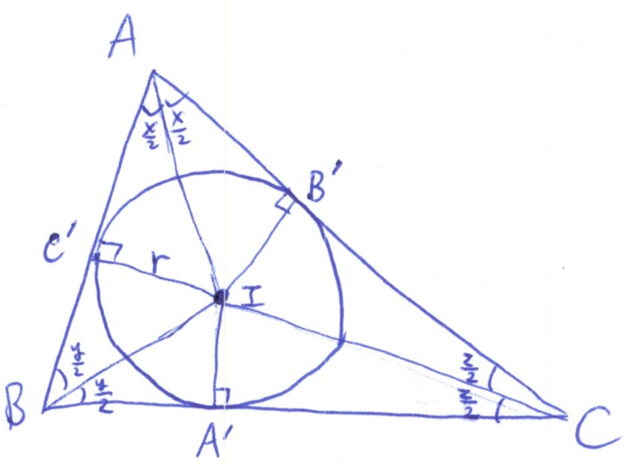
$$\begin{aligned} \frac{AI}{AO} &= 4\sin\frac{y}{2}\sin\frac{z}{2} \\ &= \cos\left(\frac{y-z}{2}\right) \\ &= \cos\left(\frac{y}{2} - \frac{z}{2}\right) \\ &= \cos\frac{y}{2}\cos\frac{z}{2} + \sin\frac{y}{2}\sin\frac{z}{2}. \end{aligned}$$

This implies $3\sin\dfrac{y}{2}\sin\dfrac{z}{2} = \cos\dfrac{y}{2}\cos\dfrac{z}{2}$, which implies that

$$\begin{aligned} 2\sin\frac{y}{2}\sin\frac{z}{2} &= 3\sin\frac{y}{2}\sin\frac{z}{2} - \sin\frac{y}{2}\sin\frac{z}{2} \\ &= \cos\frac{y}{2}\cos\frac{z}{2} - \sin\frac{y}{2}\sin\frac{z}{2} \\ &= \cos\left(\frac{y}{2} + \frac{z}{2}\right) = \cos\left(\frac{y+z}{2}\right) \\ &= \cos\left(\frac{180^\circ - x}{2}\right) = \cos\left(90^\circ - \frac{x}{2}\right) \\ &= \sin\frac{x}{2}. \end{aligned}$$

We now determine our formula for $\frac{R}{r}$ to answer our question.

We just need one final diagram.



From the picture above, we have $\frac{r}{AI} = \sin\frac{x}{2}$, which implies $r = AI\sin\frac{x}{2}$. Therefore,

$$\frac{r}{R} = \frac{AI\sin\frac{x}{2}}{AO} = \sin\frac{x}{2}\cdot\frac{AI}{AO} = \sin\frac{x}{2}\cdot\left(4\sin\frac{y}{2}\sin\frac{z}{2}\right) = 4\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}.$$

Earlier, we showed that the condition $\angle AIO = 90°$ implies $2\sin\frac{y}{2}\sin\frac{z}{2} = \sin\frac{x}{2}$. Therefore,

$$\begin{aligned}\frac{r}{R} &= 4\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\\ &= 2\sin\frac{x}{2}\cdot\left(2\sin\frac{y}{2}\sin\frac{z}{2}\right)\\ &= 2\sin\frac{x}{2}\cdot\sin\frac{x}{2}\\ &= 2\sin^2\left(\frac{x}{2}\right)\\ &= 1-\cos(x).\end{aligned}$$

The final formula is true due to the double-angle formula $\cos(2t) = 1 - 2\sin^2(t)$ with $t = \frac{x}{2}$.

We have now proven that $\frac{r}{R} = 1 - \cos(x)$. In the question, we are given that $\sin A = \sin x = \frac{3}{5}$.

Since $\cos^2 x + \sin^2 x = 1$, we have $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

Therefore, $\frac{r}{R} = 1 - \frac{4}{5} = \frac{1}{5}$, implying $\frac{R}{r} = 5$.

Answer: **(F)**