1. Week 1

We give two entry level problems this week.

Problem A

Four dice are thrown. What is the chance that the product of the numbers equals 36?

Solution:

Problem 1 of the 1992 Dutch Mathematical Olympiad, Second Round which appeared in Crux at [1995; 192]. We present the solution by Edward T. H. Wang that appeared at [1997:13].

There are four ways of obtaining a product of 36 from four numbers between 1 and 6:

$$\{1,1,6,6\};\{1,2,3,6\};\{2,2,3,3\} \text{ and } \{1,3,3,4\}\,.$$

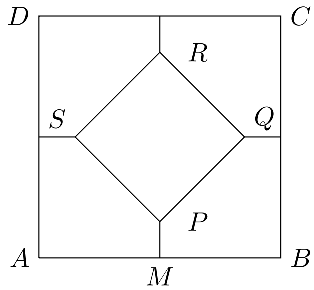
$\{1,1,6,6\}$ and $\{2,2,3,3\}$ can occur in $\frac{4!}{2!\cdot 2!}=6$ ways.
$\{1,2,3,6\}$ can occur in $4!=24$ ways.
$\{1,3,3,4\}$ can occur in $\frac{4!}{2!}=12$ ways.
Therefore, the probability is

$$\frac{6+6+24+12}{6^4}=\frac{1+1+4+2}{6^3}=\frac{8}{2^3\cdot 3^3}=\frac{1}{3^3}=\frac{1}{27}\,.$$

Editor's note: To see that these are the only possibilities, first not that each dice must be a divisor of 36, and hence can only be $1,2,3,4,6$. Now, you can split the counting into three cases: there are 2, 1 or no sixes. The first two cases are immediate, while in the third case it is easy to see that there must be two 3's, from where the two possibilities are obvious.

Problem B

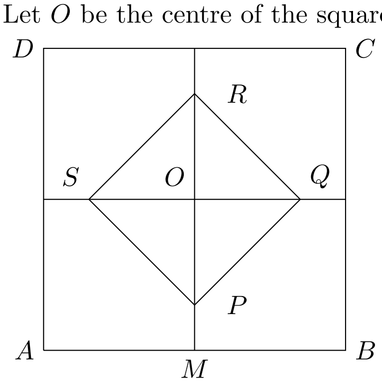
A linoleum company currently produces a product in which the pattern is a repetition of the figure. $ABCD$ and $PQRS$ are concentric squares. The diagonals of $PQRS$ are parallel to the sides of $ABCD$. If the length of AB is one unit and if the length of PQ is 1/2 unit, compute the length of PM where M is the midpoint of AB.



Solution:

Problem 2 of the 1992 Saskatchewan Senior Mathematics Contest which appeared in Crux at [1995:261] . We present the official solution that appeared at [1995:296].

Let O be the centre of the square.



Then,

$$OM = \frac{1}{2}$$

$$OP = \frac{1}{2}RP = \frac{1}{2}\sqrt{RQ^2 + PQ^2} = \frac{1}{2}\sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{4}\,.$$

Therefore,

$$PM = OM - OP = \frac{1}{2} - \frac{\sqrt{2}}{4} = \frac{2 - \sqrt{2}}{4}\,.$$

Editor's note: With the same picture, if the side of $ABCD$ is a units and the side of $PQRS$ is b units, find the length of PM as a formula in a, b.

2. Week 2

Problem Suppose that $a_1, a_2, \ldots, a_n$ are the numbers $1, 2, \ldots, n$ written in some order. Prove that

$$(a_1 - 1)^2 + (a_2 - 2)^2 + \ldots + (a_n - n)^2$$

is always even.

Solution:

Problem 4 of the Section A of the South African Mathematics Olympiad, Third Round, September 1995 which appeared in Crux at [1999:391]. We present the solution by Pierre Bornsztein that appeared at [2001:433].

Note that for each j between 1 and n we have

$$(a_j - j)^2 = a_j^2 - 2 \cdot a_j \cdot j + j^2 \,.$$

Therefore

$$(a_1 - 1)^2 + (a_2 - 2)^2 + \ldots + (a_n - n)^2 = \left(a_1^2 + a_2^2 + \ldots + a_n^2\right) + 2\,(1 \cdot a_1 + 2 \cdot a_2 + \ldots + n \cdot a_n) + \left(1^2 + 2^2 + \ldots + n^2\right) \,.$$

Since $a_1, a_2, \ldots, a_n$ are the numbers $1, 2, \ldots, n$ written in some order, we have

$$a_1^2 + a_2^2 + \ldots + a_n^2 = 1^2 + 2^2 + \ldots + n^2 \,.$$

Therefore,

$$(a_1 - 1)^2 + (a_2 - 2)^2 + \ldots + (a_n - n)^2 = 2\,(1 \cdot a_1 + 2 \cdot a_2 + \ldots + n \cdot a_n) + 2\left(1^2 + 2^2 + \ldots + n^2\right)$$

is even.

Editor's note: You can prove more generally that if $b_1, b_2, \ldots, b_n$ are some integers, and $a_1, a_2, \ldots, a_n$ are a rearrangement of $b_1, b_2, \ldots, b_n$, then, for all possible choices of signs below, the numbers

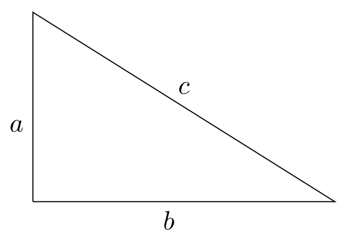
$$(a_1 \pm b_1)^2 + (a_2 \pm b_2)^2 + \ldots + (a_n \pm b_n)^2$$

are even.

3. Week 3

Problem Let $a < b < c$ be the sides of a right triangle, let $2p = a + b + c$ be its perimeter and S be its area. Prove that

$$p(p-c) = (p-a)(p-b) = S$$



Solution:

Problem 20 of the Baltic Way–92 contest, which appeared in Crux at [1996:159]. We present the solution by many solvers that appeared at [1997:462], slightly modified.

Since the triangle is a right triangle we have

$$2S = ab$$
$$c^2 = a^2 + b^2$$

Then

$$p(p-c) = \frac{1}{4}(a+b+c)(a+b-c) = \frac{1}{4}\left((a+b)^2 - c^2\right)$$
$$= \frac{1}{4}(a^2 + b^2 + 2ab - c^2) = \frac{ab}{2} = S\,.$$

Also

$$(p-a)(p-b) = p^2 - (a+b)p + ab = p^2 - (2p-c)p + 2S$$
$$= p^2 - 2p^2 + pc + 2S = 2S - p(p-c) = S\,.$$

This proves the claim.

Editor's note 1: In any triangle we have the Herron's formula:

$$S = \sqrt{p(p-a)(p-b)(p-c)}\,.$$

See if you can prove that.

Editor's note 2: See if you can prove the following more general problem:

Problem : Let ABC be a triangle with side lengths $BC = a, AC = b, AB = c$ and let $2p = a + b + c$. Then, the following are equivalent:

(i) $p(p-c) = (p-a)(p-b)$.
(ii) $p(p-c) = S$.

(iii) $(p-a)(p-b) = S$.
(iv) $p(p-c) = \frac{ab}{2}$.
(v) $(p-a)(p-b) = \frac{ab}{2}$.
(vi) $a^2 + b^2 = c^2$.
(vi) $\angle C = 90^\circ$.

4. Week 4

Problem Let d, d' be two divisors of n such that $d' > d$. Prove that

$$d' > d + \frac{d^2}{n}\,.$$

Solution:

Problem 1 from the Russia National Olympiad 2011: Grade 11, which appeared in Crux Mathematicorum [2011:496]. We present the solution by Kim Uyen Truong which appeared at [2013:18].

Since d, d' are divisors of n, there exists some integers k, m such that

$$n = kd = md'\,.$$

Since $d < d'$ we have $k > m$ and hence $k \geq m + 1$. Therefore,

$$\begin{aligned} & d' > d + \frac{d^2}{n} \Leftrightarrow \\ & \frac{n}{m} > \frac{n}{k} + \frac{n}{k^2} \Leftrightarrow \\ & \frac{1}{m} > \frac{1}{k} + \frac{1}{k^2} \Leftrightarrow \\ & k^2 > km + m\,. \end{aligned}$$

But the last inequality is true since

$$km + m \leq k(k-1) + k - 1 = k^2 - 1 < k^2\,.$$

Therefor

$$d' > d + \frac{d^2}{n}\,,$$

as claimed.

5. Week 5

Problem a, b, c are positive integers such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1 .$$

Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{41}{42} .$$

Solution:

Problem 3 from the Final Round of the mathematics competition in Finland, 2000-01, which appeared in Crux Mathematicorum [2004:245]. We present the solution by Edward T.H. Wang which appeared at [2006:162].

Let $S = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Because of complete symmetry, we can assume without loss of generality that $a \leq b \leq c$. We clearly have $a \geq 2$. We split the problem into three cases.

Case 1: $a \geq 4$. Then, $b, c \geq 4$ and hence

$$S \leq \frac{3}{4} < \frac{41}{42} .$$

This shows that claim in this case.

Case 2: $a = 3$. If $c = 3$ then $b = 3$ and hence $S = 1$ a contradiction. Therefore, we must have $b \geq 3$ and $c \geq 4$ which gives

$$S \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{4} = \frac{11}{12} < \frac{41}{42} .$$

This shows that claim in this case.

Case 3: $a = 2$. Then,

$$\frac{1}{b} + \frac{1}{c} < \frac{1}{2} .$$

It follows immediately that $b \geq 3$.

Subcase 3a: $b \geq 5$. Then, $c \geq 5$ and hence

$$S \leq \frac{1}{2} + \frac{1}{5} + \frac{1}{5} = \frac{9}{10} < \frac{41}{42} .$$

This shows that claim in this subcase.

Subcase 3b: $b = 4$. We cannot have $c = 4$ and hence $c \geq 5$. It follows that

$$S \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20} < \frac{41}{42} .$$

This shows that claim in this subcase.

Subcase 3c: $b = 3$. Then

$$\frac{1}{c} < \frac{1}{6}$$

and hence $c \geq 7$. Therefore,

$$S \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42} ,$$

proving the claim in the last subcase.

6. Week 6

Problem

Find all positive integers n for which $7^n - 1$ is a multiple of $6^n - 1$.

Solution:

Problem 1 of the Czechoslovakia Mathematical Olympiad 1993, which appeared in Crux at [1996:109]. We present the solution by Mansur Boase and Edward T.H. Wang, that appeared at [1997:395], slightly modified.

We show that no such n exists.

Assume by contradiction that there exists some n such that 6^n-1 divides 7^n-1. Then, since

$$6^n - 1 = (6-1)(6^{n-1} + 6^{n-2} + \ldots + 1)$$

we get that $7^n - 1$ is a multiple of 5.

Now, a simple computation shows that

$$7^4 - 1 = (7^2-1)(7^2+1) = 50 \cdot (7^2 - 1)\,.$$

Therefore, $7^4 - 1$ is a multiple of 5.

By the remainder theorem,

$$n = 4q + r$$

for non-negative integers q, r with $0 \leq r \leq 3$. Then

$$\begin{aligned}\underbrace{7^{4q+r} - 1}_{\text{multiple of } 5} &= 7^{4q+r} - 7^r + 7^r - 1 = 7^r\left(7^{4q} - 1\right) + 7^r - 1 \\ &= 7^r\left((7^4)^q - 1^q\right) + 7^r - 1 \\ &= 7^r \underbrace{(7^4-1)}_{\text{multiple of } 5} \left(7^{4(q-1)} + 7^{4(q-2)} + \ldots + 7^4 + 1\right) + 7^r - 1\end{aligned}$$

from which it follows that $7^r - 1$ is a multiple of 5. Checking the four possibilities $r \in \{0,1,2,3\}$ we see that only $r = 0$ works.

It follows that

$$r = 4q\,.$$

But then, we get that

$$6^n - 1 = 6^{4q} - 1 = (6^2)^{2q} - 1^{2q} = \underbrace{(6^2-1)}_{\text{multiple of } 7} (6^{2q(2q-1)} + 6^{2q(2q-2)} + \ldots + 6^2 + 1)$$

is a multiple of 7. Since $6^n - 1$ divides $7^n - 1$, it follows that $7^n - 1$ is a multiple of 7. But this is not possible for a positive integer n.

Since we got a contradiction, our assumption is wrong. Therefore, there is no such integer n.

Editor's note: The solution is much easier and faster to write if you are familiar with modular arithmetic. Indeed, the argument is the following:

$$6^n - 1 \equiv 0 \pmod{5}$$

implies that

$$7^n - 1 \equiv 0 \pmod{5}.$$

Checking the powers of 7 modulo 5, we immediately see that n is a multiple of 4 and hence even. But then

$$6^n - 1 \equiv (-1)^n - 1 = 0 \pmod{7}$$

implying that $7^n - 1$ is a multiple of 7, which is the contradiction.

7. Week 7

Problem

Which is larger

$$\left(\sqrt[3]{2}-1\right)^{\frac{1}{3}} \qquad \text{or} \qquad \sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}} \qquad ?$$

Solution:

Problem 1 of the 1996 Quickies list, which appeared in Crux at [2001:78] . We present the official solution that appeared at [2001:78], slightly modified .

It is an identity of Ramanujan that they are equal.

Let $a=\sqrt[3]{\frac{1}{3}}, b=\sqrt[3]{\frac{2}{3}}$. Then,

$$\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}=a^2-ab+b^2=\frac{a^3+b^3}{a+b}=\frac{\frac{1}{3}+\frac{2}{3}}{a+b}=\frac{1}{a+b}\,.$$

Next, note that

$$a+b=\sqrt[3]{\frac{1}{3}}(1+\sqrt[3]{2})\,.$$

Therefore,

$$\begin{aligned}\left(\sqrt[3]{2}-1\right)(a+b)^3&=\left(\sqrt[3]{2}-1\right)\left(1+\sqrt[3]{2}\right)^3\frac{1}{3}\\&=\frac{1}{3}\left(\sqrt[3]{2}-1\right)\left(1+3\sqrt[3]{2}+3\sqrt[3]{4}+2\right)\\&=\left(\sqrt[3]{2}-1\right)\left(1+\sqrt[3]{2}+\sqrt[3]{4}\right)=(\sqrt[3]{2})^3-1^3=1\,.\end{aligned}$$

Taking cubic roots we get

$$\left(\sqrt[3]{2}-1\right)^{\frac{1}{3}}(a+b)=1$$

and hence

$$\left(\sqrt[3]{2}-1\right)^{\frac{1}{3}}=\frac{1}{a+b}=\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}\,,$$

as claimed.

8. Week 7

This week we look at a diophantine equation.

Problem Find all integers x, y such that

$$x^3 + x^2 + x = y^2 + y$$

Solution:

Problem 4 from the 2011 Croatia Team Selection Test, Day 2, which appeared in Crux Mathematicorum [2012:54]. We present the solution by Oliver Geupel which appeared at [2013:169], modified by the editor.

We claim that $x = 0$.

Let us assume by contradiction that $x \neq 0$.

Note first that $y^2 + y \geq 0$ and $x^2 + x + 1 > 0$. Therefore, as

$$x(x^2 + x + 1) = y^2 + y \geq 0$$

we get $x > 0$.

The equation can be written as:

$$x^3 = y^2 - x^2 + y - x = (y - x)(y + x) + y - x = (y - x)(y + x + 1)\,. \tag{1}$$

We claim that $\gcd(y - x, y + x + 1) = 1$. Indeed, if $\gcd(y - x, y + x + 1) \neq 1$, there exists a prime p such that $p|y - x$ and $p|y + x + 1$.

Then

$$p|(y - x)(y + x + 1) = x^3$$

and hence, since p is prime, $p|x$. But then, as $p|y - x$ we also get $p|y$. Therefore, $p|x, p|y$ and $p|x + y + 1$ which gives $p|1$, a contradiction.

Therefore, $\gcd(y - x, y + x + 1) = 1$.

(1) then implies that there exists integers m, n such that

$$y - x = n^3$$
$$y + x + 1 = m^3\,.$$

Moreover, as $x > 0$ we have $m^3 > n^3$ and hence $m > n$. It follows that $m \geq n+1$. Plugging this into (1) we get

$$x = mn\,.$$

Therefore,

$$y = n^3 + mn$$
$$y = m^3 - mn - 1$$

and hence

$$(m - n)(m^2 + mn + n^2) = m^3 - n^3 = 2mn + 1\,.$$

Now, since $m > n$ we have $m - n \geq 1$. Also,

$$m^2 + n^2 \geq 2mn = 2x > 0\,.$$

Therefore

$$2mn + 1 = (m - n)(m^2 + mn + n^2) \geq 1 \cdot 3mn = 3mn\,,$$

which gives $mn = 1$. Since m, n are integers, this implies that $m = n = \pm 1$, which contradicts $m > n$.

Since we got a contradiction, our assumption that $x \neq 0$ is wrong. Therefore, $x = 0$.

Now, since $x = 0$, the original equation gives

$$y^2 + y = 0$$

and hence $y = 0$ or $y = -1$.

Therefore, there are two solution

$$(x, y) = (0, 0) \qquad \text{and} \qquad (x, y) = (0, -1)\,.$$

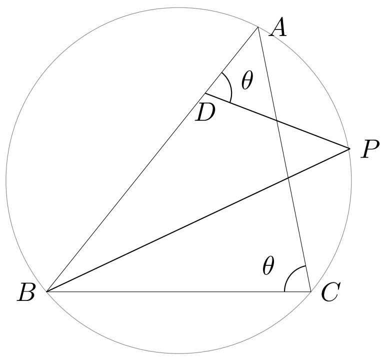
9. Week 9

Problem

Let ABC be a triangle and let D be a point on the side AB such that $AB = 4AD$. P is a point on circle passing through A, B, C such that P and C are on the same side of AB and

$$\angle ADP = \angle ACB \,.$$

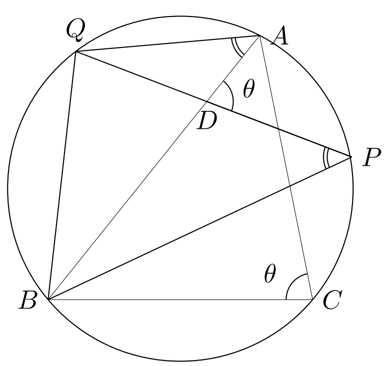
Show that $PB = 2PD$.



Solution:

Problem 2 from the Round 2 of the British Mathematical Olympiad 2002-03, slightly rephrased, which appeared in Crux Mathematicorum [2006:215]. We present the solution by Michel Bataille which appeared at [2007:221].

Extend PD to meet the circle at Q.



We have

$$\angle QAB = \frac{1}{2}\overset{\frown}{QB} = \angle QPB$$

Chasing angles, we have

$$\begin{aligned}\angle ADQ = \angle BDP = 180^\circ - \theta &= 180^\circ - \frac{1}{2}A\overset{\frown}{Q}B \\ &= \frac{1}{2}A\overset{\frown}{C}B = \angle AQB\,.\end{aligned}$$

Therefore, we have

$$\angle ADQ = \angle AQB$$
$$\angle QAD = \angle BAQ$$

which shows that the triangles ΔADQ and ΔAQB are similar. We therefore get

$$\frac{AD}{AQ} = \frac{AQ}{AB}\,.$$

Using $AB = 4AD$ we get

$$AB = 2AQ\,. \tag{2}$$

Next, by the above we have

$$\angle PDB = \angle AQB\,.$$

Since

$$\angle DPB = \angle QAB\,.$$

we get that the triangles ΔPDB and ΔAQB are also similar. Therefore

$$\frac{DP}{AQ} = \frac{PB}{AB}\,.$$

Combining this relation to (2) we get

$$PB = 2DP\,.$$

This proves the claim.