1. Show that for all integers $a \geq 1$, $\lfloor \sqrt{a}+\sqrt{a+1}+\sqrt{a+2} \rfloor = \lfloor \sqrt{9a+8} \rfloor$.

SOLUTION: We will prove the following inequality

$$\sqrt{9a+8} < \sqrt{a} + \sqrt{a+1} + \sqrt{a+2} < \sqrt{9a+9}.$$

Consider the following:

$$\begin{aligned}(\sqrt{a}+\sqrt{a+2})^2 &= a+a+2+2\sqrt{a^2+2a} \\ &< 2a+2+2\sqrt{a^2+2a+1} \\ &= 4a+4 \\ &= (2\sqrt{a+1})^2\end{aligned}$$

Thus

$$\sqrt{a}+\sqrt{a+2} < 2\sqrt{a+1}$$

and so

$$\sqrt{a}+\sqrt{a+1}+\sqrt{a+2} < 3\sqrt{a+1} = \sqrt{9a+9}.$$

By the AM-GM inequality,

$$\begin{aligned} & \sqrt{a}+\sqrt{a+1}+\sqrt{a+2} \\ \geq\ & 3\sqrt[6]{a(a+1)(a+2)} \\ =\ & \sqrt{\sqrt[3]{729(a^3+3a^2+2a)}} \\ =\ & \sqrt{\sqrt[3]{729a^3+2187a^2+1458a)}} \\ =\ & \sqrt{\sqrt[3]{729a^3+1944a^2+1728a+512+(243a^2-270a-512)}} \\ =\ & \sqrt{\sqrt[3]{(9a+8)^3+(243a^2-270a-512)}} \\ >\ & \sqrt{\sqrt[3]{(9a+8)^3}} \text{ when } a \geq 3 \\ =\ & \sqrt{9a+8}\end{aligned}$$

And when $a = 1, 2$ we can verify numerically that

$$\sqrt{9a+8} < \sqrt{a}+\sqrt{a+1}+\sqrt{a+2}$$

This shows that for all positive integers

$$\sqrt{9a+8} < \sqrt{a} + \sqrt{a+1} + \sqrt{a+2} < \sqrt{9a+9}.$$

Taking the floor of the above inequality yields:

$\lfloor\sqrt{9a+8}\rfloor \leq \lfloor\sqrt{a}+\sqrt{a+1}+\sqrt{a+2}\rfloor \leq \lfloor\sqrt{9a+9}\rfloor$.

Notice that $\sqrt{9a+8}$ and $\sqrt{9a+9}$ are the square roots of consecutive integers. Thus, the floors of these will differ only when $9a+9$ is a perfect square.

When $9a+9$ is not a perfect square, the outer sides of the inequality are equal, so the middle is also the same. When $9a+9$ is a perfect square, we had $\sqrt{a}+\sqrt{a+1}+\sqrt{a+2} < \sqrt{9a+9}$ and so the left hand inequality holds with equality. Thus $\lfloor\sqrt{a}+\sqrt{a+1}+\sqrt{a+2}\rfloor = \lfloor\sqrt{9a+8}\rfloor$.

2. Given a set $S$, of integers, an *optimal partition of $S$* into sets $T, U$ is a partition which minimizes the value $|t-u|$, where $t$ and $u$ are the sum of the elements of $T$ and $U$ respectively.

   Let $P$ be a set of distinct positive integers such that the sum of the elements of $P$ is $2k$ for a positive integer $k$, and no subset of $P$ sums to $k$.

   Either show that there exists such a $P$ with at least 2020 different optimal partitions, or show that such a $P$ does not exist.

   SOLUTION: Consider the set

   $$P = \{1,3\} \cup \{10,20,30\} \cup \{100,200,300\} \cup \cdots \cup \{10^{11}, 2\cdot 10^{11}, 3\cdot 10^{11}\}.$$

   We claim $P$ has the desired properties. The sum of elements of $P$ is $666666666664 = 2k$ for $k = 333333333332$. Note that $k$ is 2 more than a multiple of 10. Since the only elements of $P$ which are not multiples of 10 are 1 and 3, it is not possible for a subset of $P$ to sum to $k$.

   The set $T = \{3, 30, 300, \ldots, 3\cdot 10^11\}$ sums to $k+1$ which means $T$ and $P-T$ are an optimal partition. For each $3\cdot 10^k, k \geq 1$ in $T$, we could instead put $10^k$ and $2\cdot 10^k$ and get another optial partition. Since there are 11 values of $k$ for which we could make this change, there are $2^11 > 2020$ different optimal partitions of $P$.

3. Let $N$ be a positive integer and $A = a_1, a_2, \ldots, a_N$ be a sequence of real numbers. Define the sequence $f(A)$ to be

   $$f(A) = \left(\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}, \cdots, \frac{a_{N-1}+a_N}{2}, \frac{a_N+a_1}{2}\right)$$

and for $k$ a positive integer define $f^k(A)$ to be $f$ applied to $A$ consecutively $k$ times (i.e. $f(f(\cdots f(A))))$

Find all sequences $A = (a_1, a_2, \ldots, a_N)$ of integers such that $f^k(A)$ contains only integers for all $k$.

SOLUTION: Let $M(A) = (a_1+a_2+\cdots a_N)/N$ and let $S(A) = |M(A) - a_1| + |M(A) - a_2| + \cdots + |M(A) - a_N|$.

Then

$$\begin{aligned} S(A) &= (\tfrac{1}{2}|M(A) - a_1| + \tfrac{1}{2}|M(A) - a_2|) + (\tfrac{1}{2}|M(A) - a_2| + \tfrac{1}{2}|M(A) - a_3|) + \cdots \\ &\geq \tfrac{1}{2}|M(A) - \tfrac{a_1+a_2}{2}| + \tfrac{1}{2}|M(A) - \tfrac{a_2+a_3}{2}| + \cdots \\ &= S(f(A)) \end{aligned}$$

And equality holds only when $A$ is a constant sequence.

If $f^k(A)$ has only integer values for all $k$, then $N \cdot S(A)$ must always be an integer. Since this is non-increasing positive integer value, it must eventually be constant. Thus, the sequence $A$ must eventually be constant. If $A$ is a constant sequence, then $f^{-1}(A)$ must either equal $A$, or be a sequence of the form $x, y, x, y, \ldots$, where $N$ is even and $x, y$ have the same parity. When $x \neq y$, there is no sequence $f^{-1}(x, y, x, y, \cdots)$.

Thus $A$ must be a constant integer seequence, or a sequence of the form $x, y, x, y, \ldots, y$.

4. Determine all graphs $G$ with the following two properties:

- $G$ contains at least one Hamilton path.
- For any pair of vertices, $u, v \in G$, if there is a Hamilton path from $u$ to $v$ then the edge $uv$ is in the graph $G$.

Solution: Consider a graph $G$ with the desired properties and a Hamilton path $(v_1, v_2, \ldots, v_n)$. Then the edge $v_1 v_n$ must also be in $G$. If $n = 2$ then $G$ is a graph with a single edge. If $n \geq 3$ then $(v_1, v_2, \ldots v_n)$ is a Hamilton cycle. If $G$ contains no other edges, then it satisfies the given properties, so all cycle graphs satisfy the desired properties.

Suppose $G$ has more edges than just a cycle. We call an edge from $v_i$ to $v_j$ in $G$ a chord of length $j - i$, where $j - i$ is calculated module $n$. We prove the following two lemmas:

- If $G$ has a chord of length $k$ then $G$ has all chords of length $k$.
- If $G$ has a chord of length $2 \leq k \leq n-2$ then $G$ has all chords of length $k+2m$ for $m$ an integer.

Suppose $G$ has a chord of length $k$ and let the edge $v_1 v_{k+1}$ be in $G$. Then the path $v_2, v_3, \ldots, v_k, v_{k+1}, v_1, v_n, v_{n-1}, \ldots, v_{k+2}$ is a Hamilton path, and so the edge $v_2 v_{k+2}$ is in the graph. Repeating this process proves the first lemma.

Next consider the path $(v_1, v_{k+1}, v_{k+2}, v_2, v_3, \ldots, v_k v_n, v_{n-1}, \ldots, v_{k+3})$. By the first lemma, all of these edges are in $G$ and so this is a Hamilton path. This shows that the $v_1 v_{k+3}$ is in the graph. A similar construction show the edge $v_1 v_{k-1}$ is also in the graph. Repeating these processes, combined with the first lemma, proves the second lemma.

If $n$ is odd, then an edge $v_i v_j$ gives a chord of length $j-i$ and $i-j$, one of which is odd and one of which is even (modulo $n$), and so $G$ would be a complete graph.

If $n$ is even and $k$ is odd, this give a complete bipartite graph. If $G$ had another edge then this would be an even chord and $G$ would have all even chords and be a complete graph.

If $n$ is even and $k$ is even, then the edge $v_1 v_3$ is in $G$. If $n = 4$ then this is a complete bipartite graph. If $n > 4$ We see that $v_2, v_1, v_3, v_4, \ldots, v_n$ is also a Hamilton path in $G$. On this path $v_2 v_4$ is a chord of length 3 and $v_3 v_5$ is a chord of length 2, both of which are in $G$. Thus the graph is a complete graph.

Thus the graphs with the desired properties are all graphs which are cycles, complete bipartite graphs, or complete graphs.

5. We define the following sequences:

   - Sequence $A$ has $a_n = n$.
   - Sequence $B$ has $b_n = a_n$ when $a_n \not\equiv 0 \pmod 3$ and $b_n = 0$ otherwise.
   - Sequence $C$ has $c_n = \sum_{i=1}^{n} b_i$.
   - Sequence $D$ has $d_n = c_n$ when $c_n \not\equiv 0 \pmod 3$ and $d_n = 0$ otherwise.

- Sequence $E$ has $e_n = \sum_{i=1}^{n} d_i$.

Prove that the terms of sequence $E$ are exactly the perfect cubes.

**SOLUTION:**

Observe that the sequence $\{b_n\}$ is defined as:

$$b_n = \begin{cases} 0 & \text{if } n \equiv (0 \mod 3) \\ n & \text{otherwise.} \end{cases}$$

Considering $n$ modulo 3, we can compute $c_n$ as:

$$c_n = \begin{cases} 3k^2 + 3k + 1 = 3k(k+1) + 1 & \text{if } n = 3k + 1 \\ 3(k+1)^2 & \text{if } n = 3k + 2 \\ 3k^2 & \text{if } n = 3k. \end{cases}$$

To determine $\{d_n\}$, we replace all multiples of 3 with zeroes. This is occurs when $n = 3k$ or $n = 3k + 1$, so $\{d_n\}$ is of the form

$$1, 0, 0, 7, 0, 0, 19, 0, 0, 37, 0, ...,$$

and $e_n$ is of the form

$$1, 1, 1, 8, 8, 8, 27, 27, 27, 64, 64, ...$$

Noting that $e_n$ increases on every $n = 3k + 1$ index, we redefine $n$ as cycling between $3k + 1$, $3k + 2$, $3k + 3$ for values of $k$, so that For

$n = 3k + i$, $k \in \{0\} \cup \mathbb{N}$, and $i = \{1, 2, 3\}$

$$\begin{aligned} e_n &= \sum_{r=0}^{k} (3r^2 + 3r + 1) \\ &= 3\sum_{r=1}^{k} r^2 + 3\sum_{r=1}^{k} r + \sum_{r=0}^{k} 1 \\ &= 3\frac{k(k+1)(2k+1)}{6} + 3\frac{k(k+1)}{2} + k + 1 \\ &= \frac{2k^3 + 3k^2 + k}{2} + \frac{3k^2 + 3k}{2} + k + 1 \\ &= \frac{2k^3 + 6k^2 + 4k}{2} + k + 1 \\ &= k^3 + 3k^2 + 2k + k + 1 \\ &= (k+1)^3 \end{aligned}$$

6. In convex pentagon $ABCDE$, $AC$ is parallel to $DE$, $AB$ is perpendicular to $AE$, and $BC$ is perpendicular to $CD$. If $H$ is the orthocentre of triangle $ABC$ and $M$ is the midpoint of segment $DE$, prove that $AD$, $CE$ and $HM$ are concurrent.

   **Solution.** Let $P$ denote the intersection of lines $AE$ and $CD$ and let $Q$ denote the midpoint of $AC$. Since $H$ is the orthocentre of triangle $ABC$, it follows that $CH \perp AB$ and $AH \perp BC$. Combining this with the fact that $AE \perp AB$ and $BC \perp CD$ yields that $AH \| CD$ and $CH \| AE$. This implies that $AHCP$ is a parallelogram and consequently that $PH$ passes through the midpoint $Q$ of $AC$. Since $DE \| AC$, it follows that triangle $PED$ is similar to triangle $PAC$. This implies that $\angle PEM = \angle PED = \angle PAC = \angle PAQ$ and that

$$\frac{AQ}{AP} = \frac{AC}{2 \cdot AP} = \frac{ED}{2 \cdot EP} = \frac{EM}{EP}.$$

   Hence triangles $PAQ$ and $PEM$ are similar and $\angle EPM = \angle APQ$. Therefore the point $M$ lies on the line through $P$, $Q$ and $H$ and it suffices to show that $PH$, $CE$ and $AD$ are concurrent. Since $DE \| AC$, $AH \| CD$ and $CH \| AE$, it follows that triangles $PED$ and $HCA$ are similar with corresponding sides parallel. Therefore

$$\frac{CH}{EP} = \frac{AH}{DP}.$$

Let $X$ and $Y$ be the intersections of $CE$ and $AD$ with $HP$, respectively. Because $CH \| EP$ and $AH \| DP$, it follows that triangles $EXP$ and $CXH$ are similar and that triangles $PYD$ and $HYA$ are similar. Considering the ratios of similarity yields that

$$\frac{HX}{XP} = \frac{CH}{EP} = \frac{AH}{DP} = \frac{HY}{YP}.$$

Since points $X$ and $Y$ both lie on segment $HP$, it follows that $X = Y$. Therefore $CE$, $AD$ and $HP$ are concurrent at the point $X$, as desired. This implies that $AD$, $CE$ and $HM$ are concurrent.

7. Let $a, b, c$ be positive real numbers with $ab + bc + ac = abc$. Prove that

$$\frac{bc}{a^{a+1}} + \frac{ac}{b^{b+1}} + \frac{ab}{c^{c+1}} \geq \frac{1}{3}.$$

**Solution 1.** Since the desired inequality is symmetric in $a, b, c$, it may be assumed without the loss of generality that $a \geq b \geq c$. Further $0 < ab + ac = (a-1)bc$ implies that $a > 1$, and by a similar argument it follows that $b > 1$ and $c > 1$. Combining these results yields that $ab \geq ac \geq bc$ and $a^{a+1} \geq b^{b+1} \geq c^{c+1}$. Applying Chebyshev's inequality and the above inequality yields that

$$\begin{aligned} \frac{bc}{a^{a+1}} + \frac{ac}{b^{b+1}} + \frac{ab}{c^{c+1}} &\geq \frac{1}{3}(bc + ac + ab)\left(\frac{1}{a^{a+1}} + \frac{1}{b^{b+1}} + \frac{1}{c^{c+1}}\right) \\ &= \frac{1}{3} \cdot abc\left(\frac{1}{a} \cdot a^{-a} + \frac{1}{b} \cdot b^{-b} + \frac{1}{c} \cdot c^{-c}\right). \end{aligned}$$

Rearranging the condition yields that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Applying Weighted AM-GM with weights $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ yields that

$$\frac{1}{a} \cdot a^{-a} + \frac{1}{b} \cdot b^{-b} + \frac{1}{c} \cdot c^{-c} \geq (a^{-a})^{\frac{1}{a}}(b^{-b})^{\frac{1}{b}}(c^{-c})^{\frac{1}{c}} = \frac{1}{abc}.$$

Applying this to the inequality derived above yields the desired result.

**Solution 2.** Rearranging the condition yields that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$.

Applying Weighted AM-GM with weights $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ yields that

$$\begin{aligned} \frac{bc}{a^{a+1}} + \frac{ac}{b^{b+1}} + \frac{ab}{c^{c+1}} &= abc\left(\frac{1}{a}\cdot\frac{1}{a^{a+1}} + \frac{1}{b}\cdot\frac{1}{b^{b+1}} + \frac{1}{c}\cdot\frac{1}{c^{c+1}}\right) \\ &\geq abc\left(\frac{1}{a^{a+1}}\right)^{\frac{1}{a}}\left(\frac{1}{b^{b+1}}\right)^{\frac{1}{b}}\left(\frac{1}{c^{c+1}}\right)^{\frac{1}{c}} \\ &= \frac{1}{a^{\frac{1}{a}}b^{\frac{1}{b}}c^{\frac{1}{c}}}. \end{aligned}$$

It suffices to show that $3 \geq a^{\frac{1}{a}}b^{\frac{1}{b}}c^{\frac{1}{c}}$. Since $\log x$ is a concave function, applying Jensen's inequality with weights $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ yields that

$$\frac{1}{a}\cdot\log a + \frac{1}{b}\cdot\log b + \frac{1}{c}\cdot\log c \leq \log\left(\frac{1}{a}\cdot a + \frac{1}{b}\cdot b + \frac{1}{c}\cdot c\right) = \log 3.$$

Since $\log x$ is increasing, this implies the desired inequality.

8. Find all pairs $(a, b)$ of positive rational numbers such that $\sqrt[b]{a} = ab$.

**Answer.** $(a,b) = \left(\left(\frac{q}{q+1}\right)^{q}, \frac{q}{q+1}\right)$ where $q \in \mathbb{N}$; $(a,b) = \left(\left(\frac{q}{q+1}\right)^{q+1}, \frac{q+1}{q}\right)$ where $q \in \mathbb{N}$; and $(a,b) = (a,1)$ where $a \in \mathbb{Q}$.

**Solution.** Let $b = c/d$ where $c, d \in \mathbb{N}$ and $\gcd(c,d) = 1$. The equation now rearranges to $a^d = (ab)^c$ which implies that $a^d$ is the $c$th power of a rational number and, since $\gcd(c,d) = 1$, that there exists an $r \in \mathbb{Q}$ such that $a = r^c$. Substituting this into the equation yields that $r^{d-c} = c/d$. Letting $|c-d| = n$ yields that either $r^n = c/d$ or $r^{-n} = c/d$ which both imply, since $\gcd(c,d) = 1$, that $c$ and $d$ are each the $n$th power of a positive integer. Letting $c = p^n$ and $d = q^n$ for some $p, q \in \mathbb{N}$ yields that $|p^n - q^n| = n$ and, if $n = 0$ and $p = q$, then since $\gcd(c,d) = 1$, it must follows that $p = q = 1$ which yields the solution $(a,b) = (a,1)$ where $a \in \mathbb{Q}$. If $p \neq q$, then $n = |p^n - q^n| \geq 2^n - 1$. If $n \geq 2$, then $2^n - 1 > n$, which is a contradiction. Considering the case when $n = 1$ yields the solution sets $(a,b) = \left(\left(\frac{q}{q+1}\right)^{q}, \frac{q}{q+1}\right)$ and $(a,b) = \left(\left(\frac{q}{q+1}\right)^{q+1}, \frac{q+1}{q}\right)$ for each $q \in \mathbb{N}$.