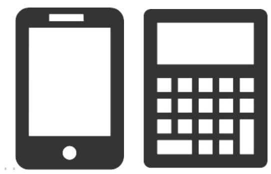
The Sun Life Financial
Canadian Open Mathematics Challenge
November 6/7, 2013

STUDENT INSTRUCTION SHEET

General Instructions



Mobile phones and calculators are NOT permitted.

1) Do not open the exam booklet until instructed to do so by your supervising teacher.
2) Take the first five minutes to fill in the exam cover sheet. Be sure to fill in all information fields and write legibly.
3) Once you have completed the exam and given it to your supervising teacher you may leave the exam room.
4) The contents of the COMC 2013 exam and your answers and solutions must not be publically discussed (including web chats) for at least 24 hours.

Exam Format

There are three parts to the COMC to be completed in 2 hours and 30 minutes:

PART A: Consists of 4 basic questions worth 4 marks each.

PART B: Consists of 4 intermediate questions worth 6 marks each.

PART C: Consists of 4 advanced questions worth 10 marks each.

Diagrams are not drawn to scale; they are intended as aids only.

Work and Answers

All solution work and answers are to be presented in this booklet in the boxes provided. Marks are awarded for completeness and clarity. For sections A and B, it is not necessary to show your work in order to receive full marks. However, if your answer or solution is incorrect, any work that you do and present in this booklet will be considered for part marks. For section C, you must show your work and provide the correct answer or solution to receive full marks.

It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2 + \sqrt{7}$, etc., rather than as 12.566, 4.646, etc. The names of all award winners will be published on the Canadian Mathematical Society web site.

CMS SMC

Sun Life Financial

The 2013 Sun Life Financial Canadian Open Mathematics Challenge

Please print clearly and **complete all information below***. Failure to print legibly or provide complete information may result in your exam being disqualified. This exam is not considered valid unless it is accompanied by your test supervisor's signed form.*

First Name:

Last Name:

Are you currently registered in full-time attendance at an elementary, secondary or Cégep school, or home schooled and have been since September 15th of this year?

☐ Yes ☐ No

Are you a Canadian Citizen or a Permanent Resident of Canada (regardless of current address)?

☐ Yes ☐ No

Grade:

☐ 8 ☐ 9 ☐ 10
☐ 11 ☐ 12 ☐ Cégep
Other: ________

T-Shirt Size:

☐ XS ☐ S ☐ M
☐ L ☐ XL ☐ XXL

Date of Birth:

y y y y m m d d

Gender: (Optional)

☐ Male ☐ Female

E-mail Address:

Signature: ___________________________________

For official use only:

A1	A2	A3	A4

B1	B2	B3	B4

C1	C2	C3	C4

TOTAL

_______ Marker initials _______ Data entry initials

Part A: Question 1 (4 marks)

Determine the positive integer n that satisfies the following equation:

$$\frac{1}{2^{10}} + \frac{1}{2^9} + \frac{1}{2^8} = \frac{n}{2^{10}}.$$

Your Solution:

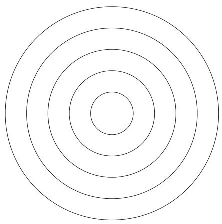
Part A: Question 2 (4 marks)

Determine the *positive* integer k for which the parabola $y = x^2 - 6$ passes through the point (k, k).

Your Solution:

Part A: Question 3 (4 marks)

In the figure below, the circles have radii 1, 2, 3, 4, and 5. The total area that is contained inside an *odd* number of these circles is $m\pi$ for a positive number m. What is the value of m?



Your Solution:

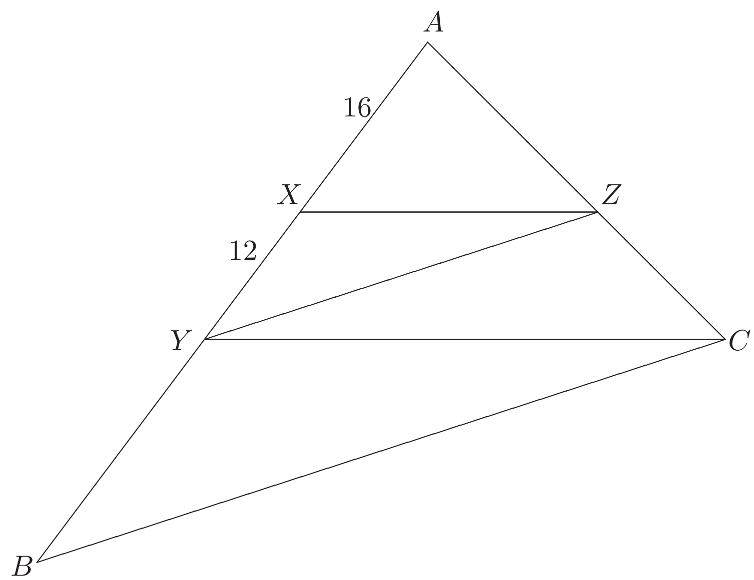
Part A: Question 4 (4 marks)

A positive integer is said to be bi-digital if it uses two different digits, with each digit used exactly twice. For example, 1331 is bi-digital, whereas 1113, 1111, 1333, and 303 are not. Determine the exact value of the integer b, the number of bi-digital positive integers.

Your Solution:

Part B: Question 1 (6 marks)

Given a triangle ABC, X, Y are points on side AB, with X closer to A than Y, and Z is a point on side AC such that XZ is parallel to YC and YZ is parallel to BC. Suppose $AX = 16$ and $XY = 12$. Determine the length of YB.



Your Solution:

Part B: Question 2 (6 marks)

There is a unique triplet of positive integers (a, b, c) such that $a \leq b \leq c$ and

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}.$$

Determine $a + b + c$.

Your Solution:

Part B: Question 3 (6 marks)

Teams A and B are playing soccer until someone scores 29 goals. Throughout the game the score is shown on a board displaying two numbers – the number of goals scored by A and the number of goals scored by B. A mathematical soccer fan noticed that several times throughout the game, the sum of all the digits displayed on the board was 10. (For example, a score of $12 : 7$ is one such possible occasion). What is the maximum number of times throughout the game that this could happen?

Your Solution:

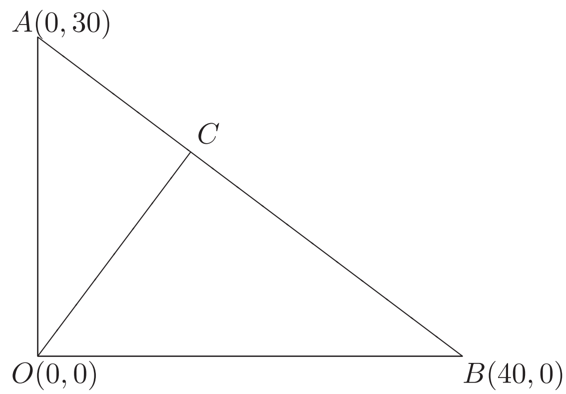
Part B: Question 4 (6 marks)

Let a be the largest real value of x for which $x^3 - 8x^2 - 2x + 3 = 0$. Determine the integer closest to a^2.

Your Solution:

Part C: Question 1 (10 marks)

In the diagram, $\triangle AOB$ is a triangle with coordinates $O = (0, 0)$, $A = (0, 30)$, and $B = (40, 0)$. Let C be the point on AB for which OC is perpendicular to AB.



(a) Determine the length of OC.

(b) Determine the coordinates of point C.

(c) Let M be the centre of the circle passing through O, A, and B. Determine the length of CM.

Your Solution:

Part C: Question 2 (10 marks)

(a) Determine all real solutions to $a^2 + 10 = a + 10^2$.

(b) Determine two positive real numbers $a, b > 0$ such that $a \neq b$ and $a^2 + b = b^2 + a$.

(c) Find all triples of real numbers (a, b, c) such that $a^2 + b^2 + c = b^2 + c^2 + a = c^2 + a^2 + b$.

Your Solution:

Part C: Question 3 (10 marks)

Alphonse and Beryl play the following game. Two positive integers m and n are written on the board. On each turn, a player selects one of the numbers on the board, erases it, and writes in its place any positive divisor of this number as long as it is different from any of the numbers previously written on the board. For example, if 10 and 17 are written on the board, a player can erase 10 and write 2 in its place. The player who cannot make a move loses. Alphonse goes first.

(a) Suppose $m = 2^{40}$ and $n = 3^{51}$. Determine which player is always able to win the game and explain the winning strategy.

(b) Suppose $m = 2^{40}$ and $n = 2^{51}$. Determine which player is always able to win the game and explain the winning strategy.

Your Solution:

Part C: Question 4 (10 marks)

For each real number x, let $[x]$ be the largest integer less than or equal to x. For example, $[5] = 5$, $[7.9] = 7$ and $[-2.4] = -3$. An *arithmetic progression* of length k is a sequence $a_1, a_2, \ldots, a_k$ with the property that there exists a real number b such that $a_{i+1} - a_i = b$ for each $1 \leq i \leq k-1$.

Let $\alpha > 2$ be a given irrational number. Then $S = \{[n \cdot \alpha] : n \in \mathbb{Z}\}$, is the set of all integers that are equal to $[n \cdot \alpha]$ for some integer n.

(a) Prove that for any integer $m \geq 3$, there exist m distinct numbers contained in S which form an arithmetic progression of length m.

(b) Prove that there exist no infinite arithmetic progressions contained in S.

Your Solution:

Canadian Mathematical Society
Société mathématique du Canada

Sun Life Financial

Canadian Open Mathematics Challenge