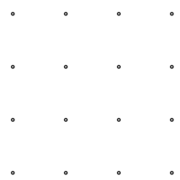
equations and dividing both sides by $2n^2$ yield $ab = 18$. But since $\{1, 2, a, b, n, 2n\}$ are the positive divisors of $2n$, $ab = 2n$. Therefore, $2n = 18$, from which we can conclude that $n = 9$ is a candidate which yields $\varphi(2n) = 6$ and $\varphi(6n) = 8$.

This can be easily verified, since the positive divisors of 18 are $\{1, 2, 3, 6, 9, 18\}$. Since the positive divisors of 54 are $\{1, 2, 3, 6, 9, 18, 27, 54\}$, $\varphi(54) = 8$. $\square$

B3 Given the following 4 by 4 square grid of points, determine the number of ways we can label ten different points $A, B, C, D, E, F, G, H, I, J$ such that the lengths of the nine segments

$$AB, BC, CD, DE, EF, FG, GH, HI, IJ$$

are in strictly increasing order.



Solution: The answer is 24.

First, we count the number of possible lengths of the segments. By the Pythagorean Theorem, the different lengths are $\sqrt{0^2+1^2} = 1, \sqrt{0^2+2^2} = 2, \sqrt{0^2+3^2} = 3, \sqrt{1^2+1^2} = \sqrt{2}, \sqrt{1^2+2^2} = \sqrt{5}, \sqrt{1^2+3^2} = \sqrt{10}, \sqrt{2^2+2^2} = \sqrt{8}, \sqrt{2^2+3^2} = \sqrt{13}, \sqrt{3^2+3^2} = \sqrt{18}$. These nine lengths are all different. Therefore, all nine lengths are represented among $AB, BC, CD, DE, EF, FG, GH, HI, IJ$. Furthermore, these nine lengths in increasing order are:

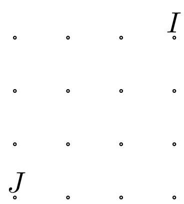
$$\sqrt{0^2+1^2} < \sqrt{1^2+1^2} < \sqrt{0^2+2^2} < \sqrt{1^2+2^2} < \sqrt{2^2+2^2}$$
$$< \sqrt{0^2+3^2} < \sqrt{1^2+3^2} < \sqrt{2^2+3^2} < \sqrt{3^2+3^2}.$$

Hence, the longest length must be a segment that goes from one corner to the diagonally-opposite corner.

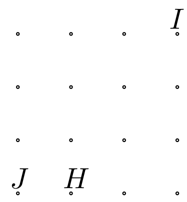
We will construct the ten points in the order $J, I, H, G, F, E, D, C, B, A$.

For simplicity, we place the points on the coordinate plane, with the bottom left corner at $(0,0)$ and the top right corner at $(3,3)$.

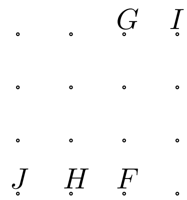
Note that J must be a corner of the grid, and there are four such corners. Furthermore, I must be the diagonally opposite corner from J. Without loss of generality, suppose $J = (0,0)$. Then $I = (3,3)$.



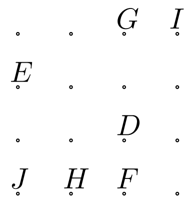
The point H has the property that $HI = \sqrt{3^2 + 2^2}$, i.e. H is a point which is distance three horizontally from I and distance two vertically from I, or vice versa. By symmetry along the diagonal JI, there two choices for H, namely $(0, 1)$ or $(1, 0)$. Without loss of generality, suppose $H = (1, 0)$.



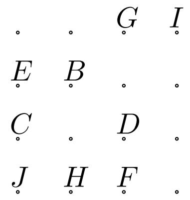
The segment GH has length $\sqrt{3^2 + 1^2}$. Hence, G is either $(0, 3)$ or $(2, 3)$. But if $G = (0, 3)$ then F is a point such that $FG = 3 = \sqrt{0^2 + 3^2}$. Then $F = (0, 0)$ or $(3, 3)$, which are already occupied by J, I, respectively. Therefore, G cannot be $(0, 3)$, and thus must be $(2, 3)$. Consequently, $F = (2, 0)$.



EF has length $\sqrt{8} = \sqrt{2^2 + 2^2}$. Hence, $E = (0, 2)$. DE has length $\sqrt{5} = \sqrt{2^2 + 1^2}$. Hence, $D = (2, 1)$.



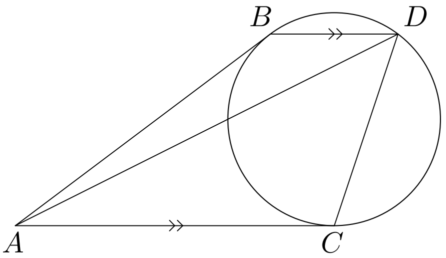
Then $C = (0, 1)$ and $B = (1, 2)$.



From B, there are three remaining points A such that $AB = 1$, namely $(1,1),(1,3),(2,2)$.

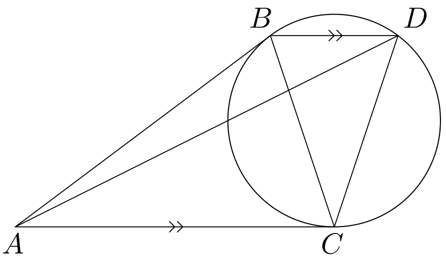
By our construction, the points J, H and A were the only points where there was more than one choice. Every other point was determined from our construction. There were 4 choices for J, 2 choices for H and 3 choices for A. Hence, the number of ways to select 10 points that satisfy the condition given in the problem is $4 \times 3 \times 2 = 24$. The answer is 24. $\square$

B4 In the following diagram, two lines that meet at a point A are tangent to a circle at points B and C. The line parallel to AC passing through B meets the circle again at D. Join the segments CD and AD. Suppose $AB = 49$ and $CD = 28$. Determine the length of AD.



Solution 1: The answer is $AD = 63$.

Join the segment BC. Since the two lines are both tangent to the circle, $AB = AC$. Therefore, $\angle ABC = \angle ACB$.



Furthermore, since BD is parallel to AC, $\angle ACB = \angle DBC$. Since AC is tangent to the circle at C, by the tangent-chord theorem, $\angle BDC = \angle ACB$. Hence, we have the following sequence of equal angles:

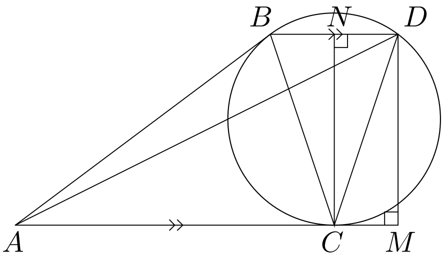
$$\angle ABC = \angle ACB = \angle CBD = \angle CDB.$$

Furthermore, $AB = AC$ and $CB = CD$. Therefore, ΔABC is similar to ΔCBD. Hence.

$$\frac{AB}{BC} = \frac{CB}{BD}.$$

Since $AB = 49$ and $BC = CD = 28$, $BD = BC^2/AB = 28^2/49 = 4^2 = 16$.

Let M be the foot of the perpendicular from D on AC and N the foot of the perpendicular on BD from C.



Since $CB = CD$, N is the midpoint of BD. Since BD is parallel to CM, $NDMC$ is a rectangle. Therefore, $CM = ND = \frac{1}{2} \cdot BD = 8$. We now determine the length of DM.

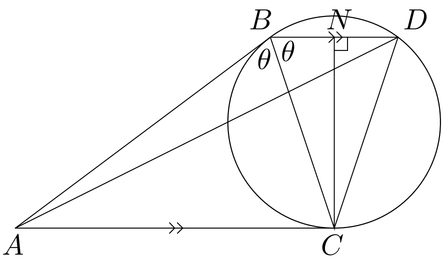
$$DM = NC = \sqrt{DC^2 - DN^2} = \sqrt{28^2 - 8^2} = \sqrt{784 - 64} = \sqrt{720}.$$

Therefore,

$$AD = \sqrt{AM^2 + MD^2} = \sqrt{(AC + CM)^2 + MD^2} = \sqrt{(49 + 8)^2 + 720} = \sqrt{3969} = 63.$$

Hence, the answer is 63. □

Solution 2: As in Solution 1, join segment BC and conclude that $BC = 28$ and $BD = 16$. Also as in Solution 1, $\angle ABC = \angle ACB = \angle CBD = \angle CDB$. Let θ be this angle.



Let N be the foot of the perpendicular on BD from C. As in Solution 1, N is the midpoint of BD. Therefore, $BN = 8$. We can now determine $\cos\theta$ from ΔCBN, which is

$$\cos\theta = \frac{BN}{BC} = \frac{8}{28} = \frac{2}{7}.$$

(Alternatively, we can use ΔABC to determine $\cos\theta$.) Note that $\angle ABD = 2\theta$. We then apply the cosine law on ΔABD to determine AD. By the cosine law, we have

$$AD^2 = BA^2 + BD^2 - BA \cdot BD \cdot \cos(2\theta).$$

We first determine $\cos(2\theta)$, which by the double-angle formula is

$$\cos(2\theta) = 2\cos^2\theta - 1 = 2 \cdot (2/7)^2 - 1 = -41/49.$$

Therefore,

$$AD^2 = 49^2 + 16^2 - 2(49)(16)(-41/49) = 2401 + 256 + 2 \cdot 16 \cdot 41 = 3969.$$

Hence, $AD = \sqrt{3969} = 63$. $\square$

Solution 3: Let θ be defined as in Solution 2. Then as shown in Solution 2, $\cos\theta = 2/7$. Then note that

$$\angle BCD = 180 - \angle CBD - \angle CDB = 180 - 2\theta.$$

Therefore, $\angle ACD = \angle ACB + \angle BCD = \theta + (180 - 2\theta) = 180 - \theta$. We now apply the cosine law on ΔACD.

$$AD^2 = CA^2 + CD^2 - 2 \cdot CA \cdot CD \cdot \cos\angle ACD = 49^2 + 28^2 - 2 \cdot 49 \cdot 28 \cdot \cos(180 - \theta)$$

$$= 2401 + 784 + 2 \cdot 49 \cdot 28 \cdot \cos\theta = 3185 + 2 \cdot 49 \cdot 28 \cdot \frac{2}{7} = 3185 + 4 \cdot 7 \cdot 28 = 3969.$$

Therefore, $AD = \sqrt{3969} = 63$. $\square$

Part C

C1 Let $f(x) = x^2$ and $g(x) = 3x - 8$.

(a) (2 marks) Determine the values of $f(2)$ and $g(f(2))$.

(b) (4 marks) Determine all values of x such that $f(g(x)) = g(f(x))$.

(c) (4 marks) Let $h(x) = 3x - r$. Determine all values of r such that $f(h(2)) = h(f(2))$.

Solution:

(a) The answers are $f(2) = 4$ and $g(f(2)) = 4$.

Substituting $x = 2$ into $f(x)$ yields $f(2) = 2^2 = 4$.

Substituting $x = 2$ into $g(f(x))$ and noting that $f(2) = 4$ yields $g(f(2)) = g(4) = 3 \cdot 4 - 8 = 4$. □

(b) The answers are $x = 2$ and $x = 6$.

Note that

$$f(g(x)) = f(3x - 8) = (3x - 8)^2 = 9x^2 - 48x + 64$$

and

$$g(f(x)) = g(x^2) = 3x^2 - 8.$$

Therefore, we are solving

$$9x^2 - 48x + 64 = 3x^2 - 8.$$

Rearranging this into a quadratic equation yields

$$6x^2 - 48x + 72 = 0 \Rightarrow 6(x^2 - 8x + 12) = 0.$$

This factors into $6(x - 6)(x - 2) = 0$. Hence, $x = 2$ or $x = 6$. We now verify these are indeed solutions.

If $x = 2$, then $f(g(2)) = f(3(2) - 8) = f(-2) = (-2)^2 = 4$ and $g(f(2)) = 4$ by part(a). Hence, $f(g(2)) = g(f(2))$. Therefore, $x = 2$ is a solution.

If $x = 6$, then $f(g(6)) = f(3 \cdot 6 - 8) = f(10) = 10^2 = 100$ and $g(f(6)) = g(6^2) = g(36) = 3 \cdot 36 - 8 = 108 - 8 = 100$. Hence, $f(g(6)) = g(f(6))$. Therefore, $x = 6$ is also a solution. □

(c) The answers are $r = 3$ and $r = 8$.

We first calculate $f(h(2))$ and $h(f(2))$ in terms of r.

$$f(h(2)) = f(3 \cdot 2 - r) = f(6 - r) = (6 - r)^2$$

and

$$h(f(2)) = h(2^2) = h(4) = 3 \cdot 4 - r = 12 - r.$$

Therefore, $(6 - r)^2 = 12 - r \Rightarrow r^2 - 12r + 36 = 12 - r$. Re-arranging this yields

$$r^2 - 11r + 24 = 0,$$

which factors as

$$(r - 8)(r - 3) = 0.$$

Hence, $r = 3$ or $r = 8$. We will now verify that both of these are indeed solutions.

If $r = 3$, then $h(x) = 3x - 3$. Then $f(h(2)) = f(3 \cdot 2 - 3) = f(3) = 9$ and $h(f(2)) = h(2^2) = h(4) = 3 \cdot 4 - 3 = 9$. Therefore, $f(h(2)) = h(f(2))$. Consequently, $r = 3$ is a solution. From the result of part (b), we also verified that $r = 8$ is a solution. □

C2 We fill a 3×3 grid with 0s and 1s. We score one point for each row, column, and diagonal whose sum is *odd*.

1	1	0
1	0	1
0	1	1

1	1	1
1	0	1
0	1	1

For example, the grid on the left has 0 points and the grid on the right has 3 points.

(a) (2 marks) Fill in the following grid so that the grid has exactly 1 point. No additional work is required. Many answers are possible. You only need to provide one.

Solution: Any of the following is a solution:

0	0	0
1	1	0
1	1	0

1	1	0
1	1	0
0	0	0

0	1	1
0	1	1
0	0	0

0	0	0
0	1	1
0	1	1

0	1	1
1	1	0
1	0	1

1	0	1
1	1	0
0	1	1

1	0	1
0	1	1
1	1	0

1	1	0
0	1	1
1	0	1

(b) (4 marks) Determine all grids with exactly 8 points.

Solution: Note that there are three rows, three columns and two diagonals. Hence, every row, column and diagonal has an odd sum.

We will consider two cases; the first case is when the middle number is 0 and second case is when the middle number is 1.

Case 1: If the middle number is 0, then let A, B, C, D be the values provided in the following squares.

A	B	C
	0	D

Then since each row, column and diagonal has an odd sum, each term diametrically opposite from A, B, C, D has a different value from A, B, C, D, respectively. Denote $\overline{0} = 1$ and $\overline{1} = 0$. Then we have the following values in the grid:

A	B	C
$\overline{D}$	0	D
$\overline{C}$	$\overline{B}$	$\overline{A}$

Note that $X + \overline{X} = 1$ for any value X. Then note the sum of $A, B, C, \overline{A}, \overline{B}, \overline{C}$ is $1 + 1 + 1 = 3$. Hence, one of $A + B + C$ and $\overline{A} + \overline{B} + \overline{C}$ is even. Therefore, either the top row or bottom row sum to an even number. Hence, there are no grids with 8 points in this case.

<u>Case 2:</u> If the middle number is 1, then again, let A, B, C, D be the values provided in the following squares.

A	B	C
	1	D

Then since each row, column and diagonal has an odd sum, each term diagonally opposite from A, B, C, D has the same value as A, B, C, D, respectively. Then we have the following values in the grid:

A	B	C
D	1	D
C	B	A

Since $A + B + C$ and $A + D + C$ are both odd, $B = D$.

A	B	C
B	1	B
C	B	A

Hence, the only remaining restriction is that $A+B+C$ is odd. Since $A, B, C = 0$ or 1, $A+B+C = 1$ or 3. The only triples (A, B, C) that give this result are $(A, B, C) = (1,0,0), (0,1,0), (0,0,1)$ or $(1,1,1)$. The following are the grids corresponding to these results, which completes the problem. □

1	0	0
0	1	0
0	0	1

0	1	0
1	1	1
0	1	0

0	0	1
0	1	0
1	0	0

1	1	1
1	1	1
1	1	1

(c) (4 marks) Let E be the number of grids with an even number of points, and O be the number of grids with an odd number of points. Prove that $E = O$.

Solution 1: Consider the set of all grids. Pair the grids so that each grid G is paired with the grid G^* formed by switching the top-left number of G. (By switching, we mean if the top left number of G is 0, we switch it to a 1. If the top left number of G is 1, we switch it to a 0.) The following is an example of the action provided by G^*.

$G =$

0	0	1
1	1	1
1	0	1

$G^* =$

1	0	1
1	1	1
1	0	1

Note that the sum of the elements in the top row, the left most column and the diagonal going from the top-left to the bottom-right switches parity, i.e. switches either from odd to even, or even to odd and the sum of the elements of the other rows / columns / diagonals remain unchanged. Hence, the total number of rows/columns/diagonals which have odd sum in G and G^* differ by an odd number. Hence, exactly one of G, G^* has an even number of points and the other has an odd number of points. Since each grid lies in exactly one pair, there is the same number of grids with an even number of points as grids with an odd number of points, i.e. $E = O$. □

Comment: The solution also applies if we switch any one of the four corners of the grid.

Solution 2: Note that the grid consisting of all zeros has an even number of points, namely zero. Note that for any grid, switching the centre square keeps the parity of the number of points the same. Switching any of the four side squares keeps the parity of the number of points the same. As in Solution 1, switching the centre changes the parity of the number of points the same.

Therefore, if a grid has $0, 2$ or 4 of its corners as 1, then the number of points of the grid is even. If a grid has 1 or 3 of its corner as 1, then number of points in the grid is odd.

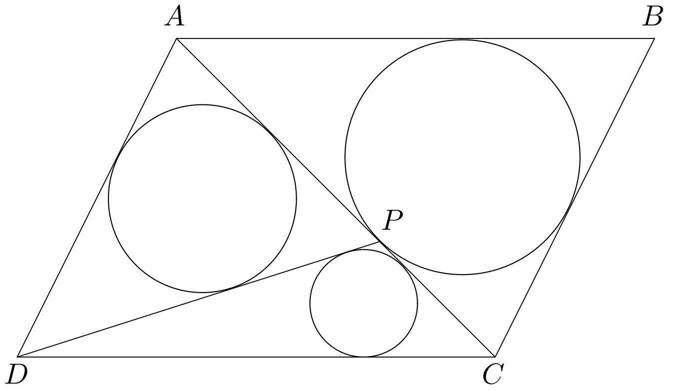
We will count the number of grids of based on the number of corner squares containing 1.

There are five non-corner squares. Therefore, there are 2^5 grids with zero corners containing 1.

There are $\binom{4}{1} = 4$ ways to choose one corner to be 1. Therefore, there are 4×2^5 grids with one corner containing 1. Similarly, there are $\binom{4}{2} \times 2^5 = 6 \times 2^5$ grids with two corners containing 1, $\binom{4}{3} \times 2^5 = 4 \times 2^5$ grids with three corners containing 1 and 2^5 grids with four corners containing 1.

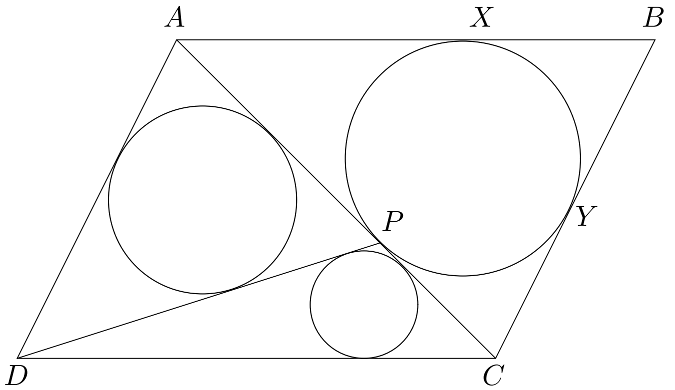
Hence, there are $2^5(1 + 6 + 1) = 8 \times 2^5$ grids with an even number of points and $2^5(4+4) = 8 \times 2^5$ grids with an odd number of points. Therefore, $E = O$, as desired. $\square$

C3 Let $ABCD$ be a parallelogram. We draw in the diagonal AC. A circle is drawn inside ΔABC tangent to all three sides and touches side AC at a point P.



(a) (2 marks) Prove that $DA + AP = DC + CP$.

Solution: Let the circle inside ΔABC touch AB, BC at X, Y, respectively.



Then by equal tangents, we have

$$DA + AP = DA + AX = DA + AB - BX$$

and

$$DC + CP = DC + CY = DC + CB - BY.$$

By equal tangents, we have $BX = BY$. Since opposite sides of a parallelogram have equal lengths, $AB = DC$ and $DA = CB$. Therefore, $DA + AB - BX = DC + CB - BY$. Consequently, $DA + AP = DC + CP$, as desired. □

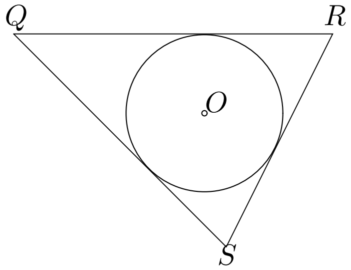
(b) (4 marks) Draw in the line DP. A circle of radius r_1 is drawn inside ΔDAP tangent to all three sides. A circle of radius r_2 is drawn inside ΔDCP tangent to all three sides. Prove that

$$\frac{r_1}{r_2} = \frac{AP}{PC}.$$

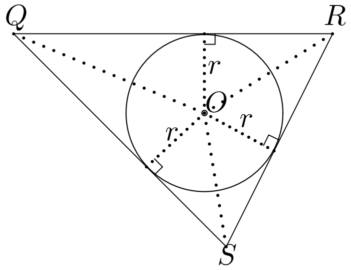
Solution 1: Consider the triangles ΔAPD and ΔCPD and note that the heights of these triangles to side AP, PC are the same. Therefore,

$$\frac{AP}{PC} = \frac{[APD]}{[CPD]},$$

where $[\cdots]$ denotes area.



Given any triangle QRS with a circle on the inside touching all three sides, let O be the centre of the circle and r the radius of the circle. Then the distance from O to each of the sides QR, RS, SQ is the same, and is the radius of the circle. Join OQ, OR, OS.



Then

$$[QRS] = [OQR] + [ORS] + [OSQ] = \frac{r \cdot QR}{2} + \frac{r \cdot RS}{2} + \frac{r \cdot SQ}{2}$$
$$= \frac{r}{2} \cdot (QR + RS + SQ) = \frac{r}{2} \cdot (\text{Perimeter of } \Delta QRS)\,.$$

Then

$$[APD] = \frac{r_1}{2} \cdot (\text{Perimeter of } \Delta APD)$$

and

$$[CPD] = \frac{r_2}{2} \cdot (\text{Perimeter of } \Delta CPD)$$

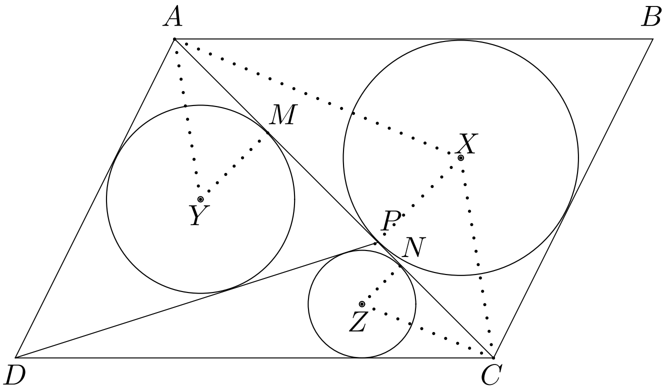
Then

$$\frac{AP}{PC} = \frac{[APD]}{[CPD]} = \frac{r_1}{r_2} \cdot \frac{\text{Perimeter of } \Delta APD}{\text{Perimeter of } \Delta CPD}.$$

Hence, to prove that $AP/PC = r_1/r_2$, it suffices to show that $\Delta APD, \Delta CPD$ have the same perimeter.

By part (a), we have $DA+AP = DC+CP$. The perimeter of ΔAPD is $DA+AP+PD = DC + CP + PD$, which is the perimeter of ΔCPD. This solves the problem. $\square$

Solution 2: Let X, Y, Z be the centres of the circles inside ΔABC, ΔAPD and ΔCPD, respectively, M the point where the circle inside ΔADP touch AC and N the point where the circle inside ΔCDP touch AC. Note that XP, YM and ZN are each perpendicular to AC.



Note also that AY bisects $\angle DAC$, CZ bisects $\angle DCA$, AX bisects $\angle BAC$ and CX bisects $\angle BCA$. Since AD is parallel to BC, $\angle DAC = \angle BCA$. Therefore, $\angle CAY = \angle ACX$, which implies that $\angle MAY = \angle PCX$. Since ΔAYM and ΔCXP are both right-angled triangles, $\Delta AYM \sim \Delta CXP$. Similarly, $\Delta CZN \sim \Delta AXP$. Therefore,

$$\frac{AM}{MY} = \frac{CP}{PX}, \quad \text{and} \quad \frac{CN}{NZ} = \frac{AP}{PX}.$$

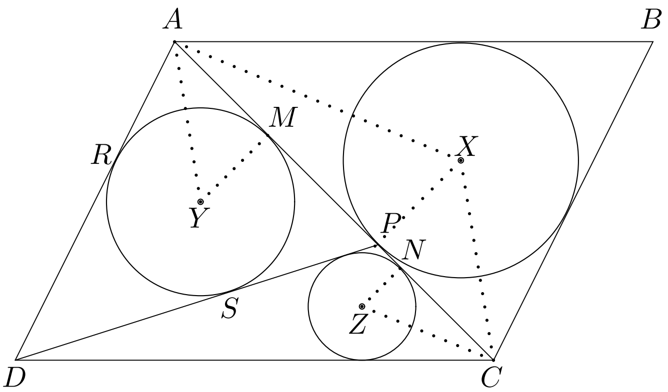
Note that $MY = r_1$ and $NZ = r_2$. This yields

$$\frac{AM}{r_1} = \frac{CP}{PX}, \quad \text{and} \quad \frac{CN}{r_2} = \frac{AP}{PX}.$$

Dividing the second equation by the first equation yields

$$\frac{AP}{PC} = \frac{AM}{CN} \cdot \frac{r_1}{r_2}.$$

Therefore, to solve the problem, it suffices to show that $AM = CN$.



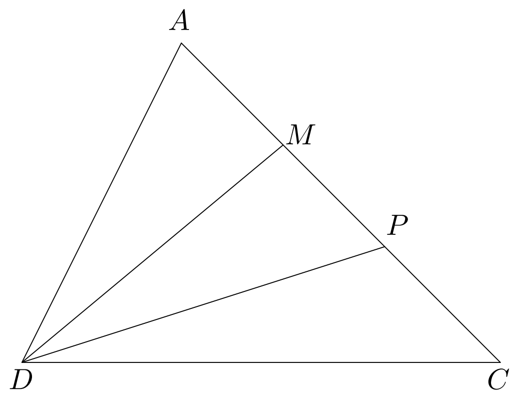
Let the circle inside ΔDAP touch AD, DP at R, S, respectively. Then note that $AR = AM, DR = DS$ and $PM = PS$. Therefore, $DA + AP = DR + RA + AM + MP = DS + AM + AM + SP = 2AM + DP$. Similarly, $DC + CP = 2CN + DP$. By part (a), $DA + AP = DC + CP$. Therefore, $2AM + DP = 2CN + DP$, from which we can conclude that $AM = CN$. This solves the problem. □

(c) (4 marks) Suppose $DA + DC = 3AC$ and $DA = DP$. Let r_1, r_2 be the two radii defined in (b). Determine the ratio r_1/r_2.

Solution: The answer is $r_1/r_2 = 4/3$.

Solution 1: By part (b). $r_1/r_2 = AP/PC$. Let $x = AP$ and $y = PC$. The answer is the ratio x/y.

By part (a), $DA + AP = DC + CP$. Let $s = DA + AP = DC + CP$. Then $DA = s - x$ and $DC = s - y$. Since $DA + DC = 3AC$, $(s - x) + (s - y) = 3(x + y)$. Hence, $2s = 4(x + y)$. Therefore, $s = 2(x + y)$. Therefore, $DA = x + 2y$ and $DC = 2x + y$. Since $DP = DA$, $DP = x + 2y$.



Drop the perpendicular from D to AC and let the perpendicular intersect AC at M. Since $DA = DP$, M is the midpoint of AP. Therefore, $MP = x/2$. By the Pythagorean Theorem, we have $MD^2 + MC^2 = DC^2$ and $MD^2 + MP^2 = DP^2$. Therefore, $DC^2 - MC^2 = DP^2 - MP^2$. Therefore,

$$(2x + y)^2 - (x/2 + y)^2 = (x + 2y)^2 - (x/2)^2.$$

Simplifying this yields

$$4x^2 + 4xy + y^2 - \frac{x^2}{4} - xy - y^2 = x^2 + 4xy + 4y^2 - \frac{x^2}{4}.$$

Hence, $3x^2 - xy - 4y^2 = 0$. Factoring this yields $(3x - 4y)(x + y) = 0$. Since x, y are lengths, $x + y \neq 0$. Therefore, $3x - 4y = 0$. Therefore, $x/y = 4/3$. $\square$

Solution 2: We define x, y as in Solution 1. Then we have $DA = x+2y$ and $DC = 2x+y$ and $DP = x + 2y$. Consider triangles ΔADP and ΔCDP. Then

$$\cos\angle APD = \frac{PA^2 + PD^2 - AD^2}{2 \cdot PA \cdot PD} = \frac{x^2 + (x + 2y)^2 - (x + 2y)^2}{2 \cdot x \cdot (x + 2y)} = \frac{x^2}{2x(x + 2y)} = \frac{x}{2(x + 2y)}$$

and

$$\cos\angle CPD = \frac{PC^2 + PD^2 - CD^2}{2 \cdot PC \cdot PD} = \frac{y^2 + (x + 2y)^2 - (2x + y)^2}{2 \cdot y \cdot (x + 2y)} = \frac{-3x^2 + 4y^2}{2y(x + 2y)}.$$

Since $\angle APD$ and $\angle CPD$ sum to $180°$, their cosine values are negatives of each other. Hence,

$$\frac{-x}{2(x + 2y)} = \frac{-3x^2 + 4y^2}{2y(x + 2y)} \Rightarrow -x = \frac{-3x^2 + 4y^2}{y}.$$

This simplifies to $3x^2 - xy - 4y^2 = 0$. Factoring this yields $(3x - 4y)(x + y) = 0$. As in Solution 1, we get $x/y = 4/3$. $\square$

Solution 3: We define x, y as in Solution 1. Then we have $DA = x+2y$ and $DC = 2x+y$ and $DP = x + 2y$. We now determine $\cos\angle DAP$ using cosine law in both ΔDAP and ΔDAC.

$$\cos\angle DAP = \frac{AD^2 + AP^2 - DP^2}{2 \cdot AD \cdot AP}$$

$$= \frac{(x + 2y)^2 + x^2 - (x + 2y)^2}{2 \cdot (x + 2y) \cdot x} = \frac{x^2}{2x(x + 2y)} = \frac{x}{2(x + 2y)}$$

and

$$\cos\angle DAC = \frac{AD^2 + AC^2 - DC^2}{2 \cdot AD \cdot AC}$$

$$= \frac{(x + 2y)^2 + (x + y)^2 - (2x + y)^2}{2 \cdot (x + 2y)(x + y)} = \frac{-2x^2 + 2xy + 4y^2}{2(x + 2y)(x + y)} = \frac{-(x - 2y)(x + y)}{(x + 2y)(x + y)} = \frac{-x + 2y}{x + 2y}.$$

Therefore,

$$\frac{x}{2(x + 2y)} = \frac{-x + 2y}{x + 2y}.$$

Hence, $x = 2(-x + 2y)$. This simplifies to $x/y = 4/3$. $\square$

C4 For any positive integer n, an n-tuple of positive integers $(x_1, x_2, \cdots, x_n)$ is said to be *super-squared* if it satisfies both of the following properties:

(1) $x_1 > x_2 > x_3 > \cdots > x_n$.

(2) The sum $x_1^2 + x_2^2 + \cdots + x_k^2$ is a perfect square for each $1 \le k \le n$.

For example, $(12, 9, 8)$ is super-squared, since $12 > 9 > 8$, and each of 12^2, $12^2 + 9^2$, and $12^2 + 9^2 + 8^2$ are perfect squares.

(a) (2 marks) Determine all values of t such that $(32, t, 9)$ is super-squared.

Solution: The only answer is $t = 24$.

Note that $32^2 + t^2 = 1024 + t^2$ and $32^2 + t^2 + 9^2 = 1105 + t^2$ are perfect squares. Then there exist positive integers a, b such that

$$\begin{aligned} 1024 + t^2 &= a^2 \\ 1105 + t^2 &= b^2. \end{aligned}$$

Subtracting the first equation from the second equation gives

$$b^2 - a^2 = 81 \Rightarrow (b - a)(b + a) = 81.$$

The only ways 81 can be written as the product of two distinct positive integers is $81 = 1 \times 81$ and $81 = 3 \times 27$.

If $(b - a, b + a) = (1, 81)$, then $b - a = 1$ and $b + a = 81$. Summing these two equations yield $2b = 82$. Therefore, $b = 41$. Hence, $a = 40$. Therefore, $t^2 = a^2 - 32^2 = 40^2 - 32^2 = 8^2(5^2 - 4^2) = 8^2 \cdot 3^2$. Hence, $t = 24$.

We now verify that $(32, 24, 9)$ is indeed super-squared. Clearly, the tuple is strictly decreasing, i.e. satisfies condition (1). Finally, $32^2 + 24^2 = 8^2(4^2 + 3^2) = 8^2 \cdot 5^2 = 40^2$ and $32^2 + 24^2 + 9^2 = 40^2 + 9^2 = 1681 = 41^2$. Therefore, the tuple also satisfies condition (2).

If $(b-a, b+a) = (3, 27)$, then $b-a = 3$ and $b+a = 27$. Summing these two equations gives $2b = 30$. Therefore, $b = 15$. Hence, $a = 12$. Therefore, $t^2 = a^2 - 32^2 = 12^2 - 32^2 < 0$. Hence, there are no solutions for t in this case.

Therefore, $t = 24$ is the only solution.

(b) (2 marks) Determine a super-squared 4-tuple (x_1, x_2, x_3, x_4) with $x_1 < 200$.

Solution: Note that if $(x_1, \cdots, x_n)$ is super-squared, then $(ax_1, \cdots, ax_n)$ is also super-squared for any positive integer a. We will show that this tuple satisfies both (1) and (2) to show that it is indeed super-squared. Clearly, since $x_1 > x_2 > \cdots > x_n$, $ax_1 > ax_2 > \cdots ax_n$. Since $x_1^2+x_2^2+\cdots+x_k^2$ is a perfect square, $x_1^2+x_2^2+\cdots+x_k^2 = m^2$ for some positive integer m. Therefore, $(ax_1)^2+\cdots+(ax_k)^2 = (am)^2$. Hence, $(ax_1, \cdots, ax_n)$ is super-squared.

From the example in the problem statement, $(12, 9, 8)$ is super-squared. Therefore, $12(12, 9, 8) = (144, 108, 96)$ is also super-squared. Note that $12^2 + 9^2 + 8^2 = 17^2$. Hence, $144^2 + 108^2 + 96^2 = 204^2 = 12^2 \cdot 17^2$.

Note that $13^2 \cdot 17^2 = (12^2 + 5^2) \cdot 17^2 = 12^2 \cdot 17^2 + 5^2 \cdot 17^2 = 12^2 \cdot 17^2 + 85^2$. Therefore, $221^2 = 13^2 \times 17^2 = 144^2 + 108^2 + 96^2 + 85^2$. And so we conclude that $(144, 108, 96, 85)$ is super-squared.

Comment: The list of all super-squared 4-tuples (x_1, x_2, x_3, x_4) with $x_1 < 200$ is

$$(132, 99, 88, 84), (144, 108, 75, 28), (144, 108, 96, 85), (156, 117, 104, 60), (180, 96, 85, 60),$$

$$(180, 135, 120, 32), \quad \text{and} \quad (192, 144, 100, 69).$$

(c) (6 marks) Determine whether there exists a super-squared 2012-tuple.

Solution: There does indeed exist a super-squared 2012-tuple.

We will show that there exists a super-squared n-tuple for any positive integer $n \geq 3$. We will prove this by induction on n. In the problem statement and in part (b), we showed that this statement holds for $n = 3, 4$.

Suppose there exists a super-squared k-tuple $(x_1, x_2, \cdots, x_k)$ for some positive integer $k \geq 3$. We will show from this k-tuple that there exists a super-squared $(k + 1)$-tuple.

Let a, b, c be a tuple of positive integers such that $a^2 + b^2 = c^2$. We will provide the additional conditions on (a, b, c) shortly.

Let r be the positive integer such that $x_1^2 + x_2^2 + \cdots + x_k^2 = r^2$. As in part (b), we note that if $(x_1, \cdots, x_k)$ is super-squared, then $(ax_1, \cdots, ax_k)$ is also super-squared and $(ax_1)^2+\cdots+(ax_k)^2 = (ar)^2$. Then we claim that $(ax_1, \cdots, ax_k, br)$ satisfies property (2) of super-squared. Clearly, $(ax_1)^2 + \cdots + (ax_t)^2$ is a perfect square, since $(ax_1, \cdots, ax_k)$ is super-squared, for all $1 \leq t \leq k$. To prove the claim, it remains to show that $(ax_1)^2 + \cdots + (ax_k)^2 + (br)^2$ is a perfect square. This is clear since this quantity is equal to $(ar)^2 + (br)^2 = r^2(a^2 + b^2) = (cr)^2$. This proves the claim.

To make the tuple $(ax_1, \cdots, ax_k, br)$ super-squared, we require that $ax_k > br$, or equivalently, $a/b > r/x_k$. Note that r, x_k are determined from the tuple $(x_1, \cdots, x_k)$. Hence, it suffices to show that there exists a Pythagorean triple (a, b, c), with $a^2 + b^2 = c^2$ such that $a/b > r/x_k$. In general, we need to show that a/b can be arbitrarily large.

Note that $(a, b, c) = (m^2 - 1, 2m, m^2 + 1)$ is a Pythagorean triple for any positive integer m. This is clear since $(m^2-1)^2+(2m)^2 = m^4-2m^2+1+4m^2 = m^4+2m^2+1 = (m^2+1)^2$. In such a case,

$$\frac{a}{b} = \frac{m^2 - 1}{2m} = \frac{m}{2} - \frac{1}{2m} > \frac{m}{2} - 1,$$

which can be made arbitrarily large. This completes the induction proof. □