The Canadian Mathematical Society La Société mathématique du Canada

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The Canadian Mathematical Society

in collaboration with

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

presents the

Sun Life Financial
Canadian Open Mathematics Challenge

Wednesday, November 21, 2007

Sun
Life Financial

Solutions

Part A

1. *Solution 1*

   If $a = 15$ and $b = -9$, then

   $$a^2 + 2ab + b^2 = (a+b)^2 = (15 + (-9))^2 = 6^2 = 36$$

   Solution 2

   If $a = 15$ and $b = -9$, then

   $$a^2 + 2ab + b^2 = 15^2 + 2(15)(-9) + (-9)^2 = 225 - 270 + 81 = 36$$

   Answer: 36

2. Since there are 60 seconds in a minute, the wind power generator turns $\dfrac{30}{60} = \dfrac{1}{2}$ of a revolution each second.

   Since a full revolution is 360°, then the generator turns $\frac{1}{2}(360^\circ) = 180^\circ$ each second.

   (Alternatively, the generator turns through $30 \times 360^\circ$ in one minute, so through $30 \times 360^\circ \div 60 = 180^\circ$ in one second.)

   Answer: 180

3. Since $AD = 4$ and AD is perpendicular to the x-axis, then A has y-coordinate 4.

   Suppose that the coordinates of A are $(a, 4)$. (This tells us also that D has coordinates $(a, 0)$.)

   Since A lies on the line $y = x + 10$, then $4 = a + 10$, or $a = -6$.

   Therefore, A has coordinates $(-6, 4)$ and D has coordinates $(-6, 0)$.

   Since $ABCD$ is a rectangle, then AB is parallel to the x-axis, so B has y-coordinate 4.

   Suppose that the coordinates of B are $(b, 4)$. (This tells us also that C has coordinates $(b, 0)$ since BC is perpendicular to the x-axis.)

   Since B lies on the line $y = -2x + 10$, then $4 = -2b + 10$ so $2b = 6$ or $b = 3$.

   Therefore, B has coordinates $(3, 4)$ and C has coordinates $(3, 0)$.

   Now the height of rectangle $ABCD$ equals the length of AD, so is 4.

   The width of rectangle $ABCD$ equals the length of CD, which is $3 - (-6) = 9$.

   Therefore, the area of rectangle is $9 \times 4 = 36$.

   Answer: 36

4. *Solution 1*

   Suppose that there were $3k$ boys and $2k$ girls in the school in June, for some positive integer k.

In September, there were thus $3k-80$ boys and $2k-20$ girls in the school. Since the new ratio is $7:5$, then

$$\begin{aligned}\frac{3k-80}{2k-20} &= \frac{7}{5}\\ 5(3k-80) &= 7(2k-20)\\ 15k-400 &= 14k-140\\ k &= 260\end{aligned}$$

Therefore, the total number of the students in the school in June was $3k+2k=5k=5(260)$, or 1300 students.

Solution 2

Suppose that there were b boys and g girls in the school in June.
In September, there were thus $b-80$ boys and $g-20$ girls in the school.
From the given information, we know that $\frac{b}{g}=\frac{3}{2}$ and $\frac{b-80}{g-20}=\frac{7}{5}$.
Eliminating fractions gives the equations $2b=3g$ and $5(b-80)=7(g-20)$ or $5b-400=7g-140$ or $5b-7g=260$.
Multiplying the second equation by 2 gives $10b-14g=520$, and substituting $10b=15g$ gives $g=520$.
Therefore, $b=\frac{3}{2}(520)=780$, so there were $b+g=780+520=1300$ students in the school in June.

ANSWER: 1300

5. *Solution 1*

When the nine numbers are placed in the array in any arrangement, the sum of the row sums is always $1+2+3+4+5+6+7+8+9=45$, because each of the nine numbers appears in exactly one row.
Similarly, the sum of the column sums is also always 45, as again each of the nine numbers appears in exactly one column.
Therefore, the grand sum S equals 90 plus the sum of the diagonal sums, and so depends only on the diagonal entries, labelled in the array below:

a		c
	e	
g		k

So $S=90+(a+e+k)+(c+e+g)=90+2e+a+c+g+k$. To make S as large as possible, we must make $2e+a+c+g+k$ as large as possible.

Since a, c, e, g, k can be any of the numbers from 1 to 9, then S is largest when $e = 9$ and a, c, g, k are $5, 6, 7, 8$ in some order, for example in the configuration below:

5	1	6
2	9	3
7	4	8

Therefore, the maximum possible value of S is $90 + 2(9) + 8 + 7 + 6 + 5 = 90 + 44 = 134$.

Solution 2

Suppose that $a, b, c, d, e, f, g, h, k$ represent the numbers 1 to 9 in some order, and are entered in the array as shown:

a	b	c
d	e	f
g	h	k

The grand sum is thus

$$\begin{aligned} S &= (a+b+c)+(d+e+f)+(g+h+k)+(a+d+g)+(b+e+h)+(c+f+k)+ \\ &\qquad (a+e+k)+(c+e+g) \\ &= 4e+3a+3c+3g+3k+2b+2d+2f+2h \end{aligned}$$

To make S as large as possible, we should assign the values of $a, b, c, d, e, f, g, h, k$ so that the largest values go to the variables with the largest coefficients in the expression for S.

In other words, S will be maximized when $e = 9$, a, c, g, k equal $8, 7, 6, 5$ in some order, and b, d, f, h equal $4, 3, 2, 1$ in some order.

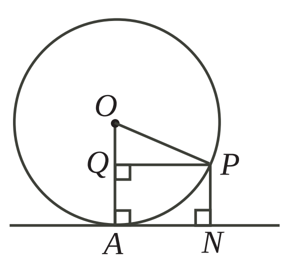
Therefore, the maximum possible value of S is

$$S = 4(9) + 3(8+7+6+5) + 2(4+3+2+1) = 36 + 3(36) + 2(10) = 134$$

ANSWER: 134

6. Suppose that r is the radius of the circle.

Join O to P and drop a perpendicular from P to Q on OA.



Since OA and PN are perpendicular to AN and PQ is perpendicular to OA, then $QPNA$ is a rectangle. Therefore, $QP = AN = 15$ and $QA = PN = 9$.
Since O is the centre of the circle and A and P are on the circumference, then $OA = OP = r$.
Since $OA = r$ and $QA = 9$, then $OQ = r - 9$.
Since $\triangle OQP$ is right-angled at Q, then, by the Pythagorean Theorem,

$$\begin{aligned} OP^2 &= OQ^2 + QP^2 \\ r^2 &= (r-9)^2 + 15^2 \\ r^2 &= r^2 - 18r + 81 + 225 \\ 18r &= 306 \\ r &= 17 \end{aligned}$$

Therefore, the radius is 17.

Answer: 17

7. From the second equation, $x + y = 7 - z$, so after squaring both sides, we obtain

$$x^2 + 2xy + y^2 = 49 - 14z + z^2 \qquad (*)$$

From the third equation, $x^2 + y^2 = 133 - z^2$, so using this and the first equation to substitute into $(*)$, we get

$$\begin{aligned} (133 - z^2) + 2(z^2) &= 49 - 14z + z^2 \\ 14z &= -84 \\ z &= -6 \end{aligned}$$

Substituting this value for z back into the first two equations, we get $xy = (-6)^2 = 36$ and $x + y = 7 - (-6) = 13$.
Therefore, $y = 13 - x$ and so $x(13 - x) = 36$ or $0 = x^2 - 13x + 36$.
This tells us that $0 = (x - 4)(x - 9)$ so $x = 4$ or $x = 9$.
If $x = 4$, then $y = 13 - x$ tells us that $y = 9$.
If $x = 9$, then $y = 13 - x$ tells us that $y = 4$.
Therefore, the solutions are $(x, y, z) = (4, 9, -6), (9, 4, -6)$.

Answer: $(4, 9, -6), (9, 4, -6)$

8. In order to travel from A to B along the segments without travelling along any segment more than once, we must always move up, down or to the right. (In other words, we can never travel to the left without retracing our steps.) To see this, we note that if we do travel along

a segment to the left, then we must have travelled along the other horizontal segment in this square to the right at an earlier stage. We would then need to travel back along one of these segments to get to B, thus retracing our steps.

Any route from A to B involves exactly 9 moves to the right and some number of moves up and down.

Any route from A to B involves exactly one more move down than moves up, as we start at the top of the grid and end up at the bottom. Therefore, the total number of up and down moves must be odd, as it equals $(x+1)+x=2x+1$, where x is the total number of up moves.

There are 10 vertical segments. Any choice of an odd number of these vertical segments uniquely determines a route from A to B, as we must start at A, travel to the top of the leftmost of these segments, travel down the segment, travel to the right to the bottom of the next segment, travel up it, and so on.

Therefore, the routes from A to B are in exact correspondence with choices of an odd number of the 10 vertical segments.

We compute the number of routes using n of these segments, for $n=1,3,5,7,9$. In each case, the length of the route will be $9+n$.

For $n=1$ and $n=9$, the number of routes is $\binom{10}{1}=\binom{10}{9}=10$.

For $n=3$ and $n=7$, the number of routes is $\binom{10}{3}=\binom{10}{7}=\frac{10(9)(8)}{3(2)(1)}=120$.

For $n=5$, the number of routes is $\binom{10}{5}=\frac{10(9)(8)(7)(6)}{5(4)(3)(2)(1)}=\frac{10(9)(8)(7)}{5(4)}=2(9)(2)(7)=252$.

Therefore, the route length with the maximum number of routes is when $n=5$. In this case, the route length is 14 and the number of routes is 252.

(Instead of going through all of the above calculations, we could have remarked that among the numbers $\binom{10}{n}$, the largest occurs when n is exactly half of 10.)

Answer: Length= 14, Number of Routes= 252

Part B

1. (a) Since $x-1$, $2x+2$, and $7x+1$ form an arithmetic sequence, then

$$\begin{aligned}(2x+2)-(x-1) &= (7x+1)-(2x+2)\\ x+3 &= 5x-1\\ 4 &= 4x\\ x &= 1\end{aligned}$$

so $x=1$.

(b) *Solution 1*

Since $x=1$, the first term of the sequence is 0.

Since the last term is 72, the sequence is arithmetic, and we are told that there is a middle term, then this middle term is equal to $\frac{0+72}{2}=36$.

(Note that if there was an even number of terms, there would not necessarily be a middle term. Since we are asked to find the middle term, we can safely assume that there is one!)

Solution 2

Since $x=1$, the first three terms of the sequence are 0, 4, 8.

Since the common difference is 4 and the first term is 0, the number of times that the difference needs to be added to get to the final term of 72 is $\frac{72-0}{4}=18$.

Therefore, 72 is the 19th term.

The middle term is thus the 10th term, or $0+4(10-1)=36$.

(c) Since $y-1$, $2y+2$, and $7y+1$ form a geometric sequence, then

$$\begin{aligned}\frac{2y+2}{y-1} &= \frac{7y+1}{2y+2}\\ (2y+2)^2 &= (y-1)(7y+1)\\ 4y^2+8y+4 &= 7y^2-6y-1\\ 0 &= 3y^2-14y-5\\ 0 &= (3y+1)(y-5)\end{aligned}$$

Therefore, $y=-\frac{1}{3}$ or $y=5$.

(d) If $y=-\frac{1}{3}$, the first three terms of the sequence are $-\frac{4}{3}$, $\frac{4}{3}$, $-\frac{4}{3}$.

In this case, the common ratio between successive terms is $\frac{\frac{4}{3}}{-\frac{4}{3}}=-1$.

Therefore, the 6th term in this sequence is $-\frac{4}{3}(-1)^5=\frac{4}{3}$.

If $y=5$, the first three terms of the sequence are 4, 12, 36.

In this case, the common ratio between successive terms is $\dfrac{12}{4} = 3$.

Therefore, the 6th term in this sequence is $4(3^5) = 4(243) = 972$.

2. (a) *Solution 1*

Since $\angle ABC = \angle BCD = 90^\circ$, then BA and CD are parallel, so $ABCD$ is a trapezoid.

Thus, the area of $ABCD$ is $\frac{1}{2}(24)(9 + 18) = 12(27) = 324$.

Solution 2

Since $\angle ABC = 90^\circ$, then the area of $\triangle ABC$ is $\frac{1}{2}(9)(24) = 9(12) = 108$.

Also, since $\angle BCD = 90^\circ$, then $\triangle ACD$ has height 24.

Therefore, the area of $\triangle ACD$ is $\frac{1}{2}(18)(24) = 9(24) = 216$.

Thus, the area of quadrilateral $ABCD$ is $108 + 216 = 324$.

(b) *Solution 1*

Since BA is parallel to CD, then $\angle ABD = \angle BDC$.

Since $\angle BEA = \angle DEC$ as well, then $\triangle ABE$ is similar to $\triangle CDE$.

Therefore, $\dfrac{DE}{BE} = \dfrac{CD}{AB} = \dfrac{18}{9} = 2$, so $DE : EB = 2 : 1$, as required.

Solution 2

As suggested by the diagram, we coordinatize the diagram.

Put C at the origin, D on the positive x-axis (with coordinates $(18, 0)$) and B on the positive y-axis (with coordinates $(0, 24)$).

Since $\angle ABC = 90^\circ$, then A has coordinates $(9, 24)$.

Therefore, the line through C and A has slope $\frac{24}{9} = \frac{8}{3}$ so has equation $y = \frac{8}{3}x$.

Also, the line through B and D has slope $\frac{-24}{18} = -\frac{4}{3}$, so has equation $y = -\frac{4}{3}x + 24$.

Point E lies at the point of intersection of these lines, so we combine the equations to find the coordinates of E, getting $\frac{8}{3}x = -\frac{4}{3}x + 24$ or $4x = 24$ or $x = 6$.

Therefore, E has y-coordinate $\frac{8}{3}(6) = 16$, so E has coordinates $(6, 16)$.

To show that $DE : EB = 2 : 1$, we can note that E lies one-third of the way along from B to D since the x-coordinate of E is one-third that of D (and the x-coordinate of B is 0), or since the y-coordinate of E is two-thirds that of B (and the y-coordinate of D is 0).

Alternatively, we could calculate the length BE (which is 10) and the length of ED (which is 20).

Using any of these methods, $DE : EB = 2 : 1$.

(c) *Solution 1*

From (b), $\triangle ABE$ is similar to $\triangle CDE$ and their sides are in the ratio $1 : 2$.

This also tells us that the height of $\triangle CDE$ is twice that of $\triangle ABE$.

Since the sum of the heights of the two triangles is 24, then the height of $\triangle CDE$ is $\frac{2}{3}(24) = 16$.

Therefore, the area of $\triangle DEC$ is $\frac{1}{2}(18)(16) = 144$.

Solution 2

From (b), the coordinates of E are $(6, 16)$.

Therefore, the height of $\triangle DEC$ is 16.

Therefore, the area of $\triangle DEC$ is $\frac{1}{2}(18)(16) = 144$.

(d) *Solution 1*

From (c), the area of $\triangle DEC$ is 144.

From Solution 2 of (a), the area of $\triangle ACD$ is 216.

The area of $\triangle DAE$ is the difference in these areas, or $216 - 144 = 72$.

Solution 2

Using the coordinatization from (b), the coordinates of A are $(9, 24)$, the coordinates of E are $(6, 16)$, and the coordinates of D are $(18, 0)$.

Using the "up products and down products" method, the area of the triangle is

$$\frac{1}{2}\left|\begin{array}{rr} 9 & 24 \\ 6 & 16 \\ 18 & 0 \\ 9 & 24 \end{array}\right| = \frac{1}{2}|9(16) + 6(0) + 18(24) - 24(6) - 16(18) - 0(9)|$$

$$= \frac{1}{2}|144 + 0 + 432 - 144 - 288 - 0|$$

$$= 72$$

3. (a) Alphonse should win.

If Alphonse starts by taking 1 stone, then by Rule #3, Beryl must remove at least 1 stone and at most $2(1) - 1 = 1$ stone. In other words, Beryl must remove 1 stone.

This in turn forces Alphonse to remove 1 stone, and so on.

Continuing in this way, Alphonse removes 1 stone from an odd-sized pile at each turn and Beryl removes 1 stone from an even-sized pile at each turn. Thus, Alphonse removes the last stone.

Therefore, Alphonse wins by removing 1 stone initially since 7 is odd.

(In fact, this argument shows that Alphonse should win whenever N is odd.)

(b) Beryl should win.

We make a chart in which the rows enumerate a possible combination of moves. Each move indicates the number of stones removed.

A1	B1	A2	B2	A3	B3	A4	B4	Winner
1	1	1	1	1	1	1	1	B
2	2	1	1	1	1			B
2	2	2	2					B
2	2	3	1					B
3	5							B
4	4							B
5	3							B
6	2							B
7	1							B

Thus, no matter what number of stones Alphonse removes initially, there is a move that Beryl can make which allows her to win. (There are possible combinations of moves where Alphonse wins that are not listed in this chart.) Therefore, Beryl should win when $N = 8$. Her strategy is:

- If Alphonse removes 3 or more stones, then he can remove the remaining stones in the pile and win.
- If Alphonse removes 1 or 2 stones, then he can win by using the table above, choosing a row for which she wins. In effect, Beryl repeats Alphonse's move on her first turn. This ensures that Alphonse receives a pile with an even number of stones and that he can remove no more than 3 stones on his next turn. Thus, she can win, as the table shows.

(c) *Solution 1*

We show that Beryl has a winning strategy if and only if $N = 2^m$, with m a positive integer.

First, if N is odd, we know that Alphonse has a winning strategy as in (a) (Alphonse removes 1 stone, forcing Beryl to remove 1 stone, and so on).
Second, if $N = 2$, then Beryl wins as Alphonse must remove 1 stone to begin, so Beryl removes the remaining stone.

Next, we show that if $N = 2k$, then the player who has the winning strategy for $N = k$ also has a winning strategy for $N = 2k$. This will tell us that Beryl has a winning strategy for $N = 2, 4, 8, 16, \ldots$ (in general, for $N = 2^m$) and that Alphonse has a winning strategy if $N = 2^m q$ where q is an odd integer (since Alphonse wins for $N = q, 2q, 4q, \ldots$). Since every even integer can be written in one of these two forms, this will complete our proof.

So consider $N = 2k$.

- If either player removes an odd number of stones from an even-sized pile (leaving an odd-sized pile), then they can be forced to lose, as the other player can then remove 1 stone from an odd-sized pile and force a win as in (a). So if Alphonse removes an even number of stones to start, then Beryl should next remove an even number of stones (so that Alphonse can't immediately force her to lose), so the pile size will always remain even and each player's move will always be to remove an even number of stones.

- Suppose that Alphonse has a winning strategy for $N = k$ of the form $a_1, b_1, a_2, b_2, \ldots, a_j$. Here, we mean that Alphonse removes a_1 stones on his first turn and responds to Beryl's first move b_1 by removing a_2 and so on. (Of course, a_2 will depend on b_1 which could take a number of values, and so on.) Since these are valid moves, then $1 \leq a_1 < k$, and $b_1 < 2a_1$, and $a_2 < 2b_1$, and so on.
  Then $2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2a_j$ will be a winning strategy for Alphonse for $N = 2k$ since $1 < 2a_1 < 2k$, and $2b_1 < 2(2a_1)$, and $2a_2 < 2(2b_1)$, and so on, so this is a valid sequence of moves and they exhaust the pile with Alphonse taking the last stone.
  In other words, to win when $N = 2k$, Alphonse consults his winning strategy for $N = k$. He removes twice his initial winning move for $N = k$. If Beryl removes $2b$ stones next, Alphonse then removes $2a$ stones, where a is his winning response to Beryl removing b stones in the $N = k$ game. This guarantees that he will win.
- Suppose that Beryl has a winning strategy for $N = k$.
  By an analogous argument, Beryl has a winning strategy for $N = 2k$, for if Alphonse removes $2a$ stones, then she removes $2b$ stones, where b is her winning responding move to Alphonse removing a stones in the $N = k$ game.

Therefore, Beryl wins if and only if $N = 2^m$, with m a positive integer.

Solution 2
We show that Beryl has a winning strategy if and only if $N = 2^m$, with m a positive integer.

Suppose first that N is not a power of 2.
We can write N as a sum of distinct powers of 2, in the form $N = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_j}$, where $k_1 > k_2 > \cdots > k_j \geq 0$. (In essence, we are writing N in binary.) Since N is not itself a power of 2, then this representation uses more than one power of 2 (that is, $j \geq 2$).
We will show that Alphonse has a strategy where he can always reduce the number of powers of 2 being used, while Beryl can never reduce the number of powers of 2 being used. This will show that Alphonse can always remove the final stone, as only he can reduce the number of powers of 2 being used to 0.
Alphonse's strategy is to initially remove the smallest power of 2 from the representation of N (that is, he removes 2^{k_j} stones).

On her first turn, Beryl thus receives a pile with $2^{k_1}+2^{k_2}+\cdots+2^{k_{j-1}}$ stones. By rule #3, she must remove fewer than $2(2^{k_j}) = 2^{k_j+1}$ stones. Since $k_{j-1} > k_j$, then $k_{j-1} \geq k_j + 1$, so Beryl must remove fewer than $2^{k_{j-1}}$ stones.

When she removes these stones, the $2^{k_{j-1}}$ will be removed from the representation of the number of remaining stones, but will be replaced by at least one (if not more) smaller powers of 2.

Thus, Beryl cannot reduce the number of powers of 2 in the representation.

Suppose that Alphonse thus receives a pile with $2^{k_1}+2^{k_2}+\cdots+2^{k_{j-2}}+2^{d_1}+2^{d_2}+\cdots+2^{d_h}$ stones, with $k_1 > k_2 > \cdots > k_{j-2} > d_1 > d_2 > \cdots > d_h$ and $h \geq 1$.

This means that Beryl removed $B = 2^{k_{j-1}} - (2^{d_1}+2^{d_2}+\cdots+2^{d_h})$ stones.

But $B > 0$ and B is divisible by 2^{d_h} (since $B = 2^{d_h}\left(2^{k_{j-1}-d_h} - (2^{d_1-d_h}+2^{d_2-d_h}+\cdots+2^0\right)$ and each of the exponents initially were larger than d_h), so $B \geq 2^{d_h}$.

Therefore, Alphonse can remove 2^{d_h} stones on his turn (that is, the smallest power of 2 in the representation of the number of remaining stones) since 2^{d_h} stones satisfies Rule #3, and so his strategy can continue.

Therefore, Alphonse has a winning strategy if N is not a power of 2.

If N is a power of 2, then Alphonse on his first turn cannot decrease the number of powers of 2 in the representation of N. (This is a similar argument to the one above for Beryl's first turn.) On Beryl's first turn, though, she can reduce the number of powers of 2 (as in Alphonse's second turn above).

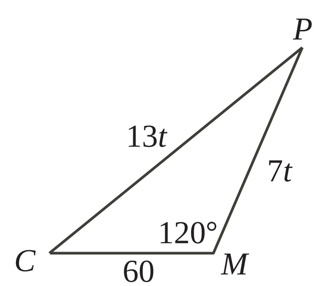
Therefore, the roles are reversed, and Beryl can always reduce the number of powers of 2, while Alphonse cannot. Therefore, Beryl has a winning strategy when N is a power of 2.

Therefore, Beryl has a winning strategy if and only if N is a power of 2.

4. (a) *Solution 1*

In t seconds, the mouse runs $7t$ metres and the cat runs $13t$ metres.

Using this, we get a triangle with the cat and mouse meeting at point P.

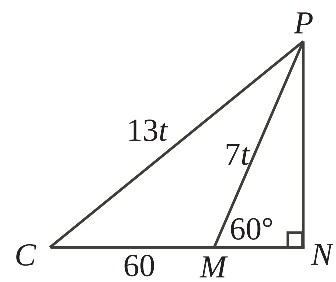


By the cosine law,

$$\begin{aligned} CP^2 &= CM^2 + MP^2 - 2(CM)(MP)\cos(\angle CMP) \\ (13t)^2 &= 60^2 + (7t)^2 - 2(60)(7t)\cos(120^\circ) \\ 169t^2 &= 3600 + 49t^2 - 120(7t)(-\tfrac{1}{2}) \\ 169t^2 &= 3600 + 49t^2 + 60(7t) \\ 120t^2 - 420t - 3600 &= 0 \\ 2t^2 - 7t - 60 &= 0 \\ (2t-15)(t+4) &= 0 \end{aligned}$$

Therefore, $t = \frac{15}{2}$ or $t = -4$.
Since t represents a time, then $t > 0$, so $t = \frac{15}{2}$.

Solution 2
In t seconds, the mouse runs $7t$ metres and the cat runs $13t$ metres.
Using this, we get a triangle with the cat and mouse meeting at point P. Drop a perpendicular from P to N on CM extended.



Since $\angle PMN = 60^\circ$, then $\triangle PMN$ is a 30°-60°-90° triangle.
Therefore, $MN = \frac{1}{2}PM = \frac{7}{2}t$ and $PN = \sqrt{3}MN = \frac{7\sqrt{3}}{2}t$.
This gives us right-angled $\triangle CPN$ with $CP = 13t$, $PN = \frac{7\sqrt{3}}{2}t$, and $CN = 60 + \frac{7}{2}t$.
By the Pythagorean Theorem,

$$\begin{aligned} CP^2 &= CN^2 + NP^2 \\ (13t)^2 &= \left(60 + \tfrac{7}{2}t\right)^2 + \left(\tfrac{7\sqrt{3}}{2}t\right)^2 \\ 169t^2 &= 3600 + 420t + \tfrac{49}{4}t^2 + \tfrac{147}{4}t^2 \\ 169t^2 &= 3600 + 420 + 49t^2 \\ 120t^2 - 420t - 3600 &= 0 \\ 2t^2 - 7t - 60 &= 0 \\ (2t-15)(t+4) &= 0 \end{aligned}$$

Therefore, $t = \frac{15}{2}$ or $t = -4$.
Since t represents a time, then $t > 0$, so $t = \frac{15}{2}$.

(b) *Solution 1*

We coordinatize the situation, as suggested in the diagram with C having coordinates $(-60, 0)$ and M having coordinates $(0, 0)$.

Suppose that the cat intercepts the mouse at point $P(x, y)$.

Since the cat runs at 13 m/s and the mouse at 7 m/s, then $\frac{CP}{MP} = \frac{13}{7}$. Thus,

$$\begin{aligned}
\frac{\sqrt{(x+60)^2+y^2}}{\sqrt{x^2+y^2}} &= \frac{13}{7} \\
\frac{(x+60)^2+y^2}{x^2+y^2} &= \frac{169}{49} \\
49((x+60)^2+y^2) &= 169(x^2+y^2) \\
0 &= 120x^2 - 2(49)(60)x - 49(60^2) + 120y^2 \\
0 &= x^2 - 49x - 49(30) + y^2
\end{aligned}$$

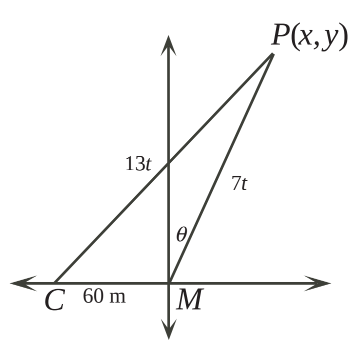
Since this equation is of the form $0 = x^2 + ax + y^2 + by + c$ and there is at least one point whose coordinates satisfy the equation (for example, setting $y = 0$ gives a quadratic equation with positive discriminant), then it is the equation of a circle, so all points of interception lie on a circle. (We could also complete the square to obtain the equation $(x - \frac{49}{2})^2 + y^2 = (\frac{91}{2})^2$, which is the equation of the circle with centre $(\frac{49}{2}, 0)$ and radius $\frac{91}{2}$.)

Solution 2

We coordinatize the situation, as suggested in the diagram, with C having coordinates $(-60, 0)$ and M having coordinates $(0, 0)$.

Suppose that the cat intercepts the mouse at point $P(x, y)$.

Suppose that the cat intercepts the mouse after t seconds and that the mouse runs in the direction θ East of North. (θ here could be negative. We can assume that $-90^\circ \leq \theta \leq 90^\circ$ to keep the situation in the upper half of the plane. If θ did not lie in this range, then P would be in the lower half plane and we could reflect it in the x-axis and use this argument.)



As in (a),

$$\begin{aligned} CP^2 &= CM^2 + MP^2 - 2(CM)(MP)\cos(\angle CMP) \\ (13t)^2 &= 60^2 + (7t)^2 - 2(60)(7t)\cos(90^\circ + \theta) \\ 120t^2 &= 3600 + 120(7t)\sin\theta \\ t^2 - 7t\sin\theta &= 30 \end{aligned}$$

But $MP^2 = 49t^2 = x^2 + y^2$ and $x = 7t\cos(90^\circ - \theta) = 7t\sin\theta$, so

$$\begin{aligned} \frac{x^2+y^2}{49} - x &= 30 \\ x^2 - 49x - 49(30) + y^2 &= 0 \end{aligned}$$

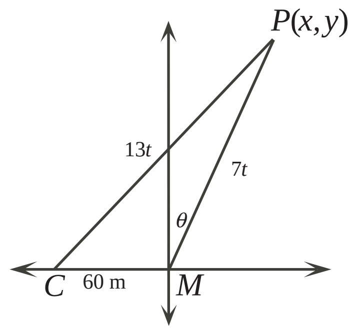
so all possible points P lie on a circle, as in Solution 1.

Solution 3

We coordinatize the situation, as suggested in the diagram with C having coordinates $(-60, 0)$ and M having coordinates $(0, 0)$.

Suppose that the cat intercepts the mouse at point $P(x, y)$.

Suppose that the cat intercepts the mouse after t seconds and that the mouse runs in the direction θ East of North. (θ here could be negative. We can assume that $-90^\circ \le \theta \le 90^\circ$ to keep the situation in the upper half of the plane. If θ did not lie in this range, then P would be in the lower half plane and we could reflect it in the x-axis and use this argument.)



If the mouse decides to run due East, then it will be caught when $-60 + 13t = 7t$ or $t = 10$, so will be caught at $B(70, 0)$.

If the mouse decides to run due West, then it will be caught when $-60 + 13t = -7t$ or $t = 3$, so will be caught at $A(-21, 0)$.

The positions above the x-axis where the mouse will be caught should be exactly symmetric with the positions below the x-axis where the mouse will be caught. Therefore, if these positions lie on a circle, then a diameter of this circle should lie on the x-axis.

Since the only positions on the x-axis where the mouse will be caught are $A(-21, 0)$ and

$B(70,0)$, then these must be endpoints of the diameter.
Therefore, the circle will have centre E with coordinates $\left(\frac{1}{2}(-21+70),0\right) = \left(\frac{49}{2},0\right)$ and radius $\frac{1}{2}(70-(-21)) = \frac{91}{2}$.
From the diagram above, the coordinates of the point P of intersection will be

$$(7t\cos(90^\circ - \theta), 7t\sin(90^\circ - \theta)) = (7t\sin\theta, 7t\cos\theta)$$

If we can show that $PE = \frac{91}{2}$ for every value of θ, then we will have shown that every point of intersection lies on the circle with centre E and radius $\frac{91}{2}$.
Now

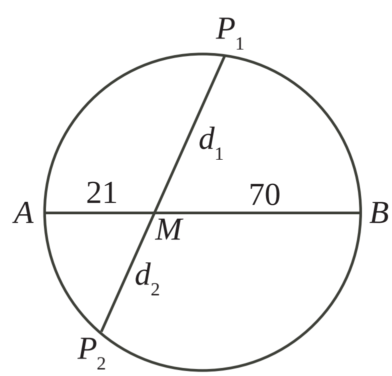
$$\begin{aligned} PE^2 &= \left(7t\sin\theta - \tfrac{49}{2}\right)^2 + (7t\cos\theta)^2 \\ &= 49t^2\sin^2\theta - 7(49)t\sin\theta + \tfrac{49^2}{4} + 49t^2\cos^2\theta \\ &= 49t^2(\sin^2\theta + \cos^2\theta) - 7(49)t\sin\theta + \tfrac{49^2}{4} \\ &= 49t^2 - 7(49)t\sin\theta + \tfrac{49^2}{4} \end{aligned}$$

From Solution 2, $t^2 - 7t\sin\theta = 30$, so

$$PE^2 = 49(30) + \tfrac{49^2}{4} = 49\left(30 + \tfrac{49}{4}\right) = 49\left(\tfrac{169}{4}\right) = \tfrac{7^2 13^2}{4} = \left(\tfrac{91}{2}\right)^2$$

so $PE = \frac{91}{2}$, as required.
Therefore, all points of intersection lie on a circle.

(c) From (b), we know that the points of intersection lie on the circle with diameter AB, where A has coordinates $(-21,0)$ and B has coordinates $(70,0)$.
Suppose that the mouse is intercepted at point P_1 after running d_1 metres and at point P_2 after running d_2 metres.



By the Intersecting Chords Theorem, $d_1d_2 = 21(70)$.
By the Arithmetic Mean-Geometric Mean Inequality,

$$\frac{d_1+d_2}{2} \geq \sqrt{d_1d_2} = \sqrt{21(70)} = 7\sqrt{30}$$

Therefore, $d_1 + d_2 \geq 2(7\sqrt{30}) = 14\sqrt{30}$.

(The Arithmetic Mean-Geometric Mean Inequality (known as the AM-GM Inequality) comes from the fact that if d_1 and d_2 are non-negative, then $(d_1 - d_2)^2 \geq 0$.
Thus, $d_1^2 + 2d_1d_2 + d^2 \geq 4d_1d_2$, so $\left(\frac{d_1 + d_2}{2}\right)^2 \geq d_1d_2$, so $\frac{d_1 + d_2}{2} \geq \sqrt{d_1d_2}$.)