The Canadian Mathematical Society La Société mathématique du Canada

CMS
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The Canadian Mathematical Society

in collaboration with

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

presents the

Canadian Open Mathematics Challenge

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Solutions

Part A

1. What is the value of $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)$?

Solution 1

$$\begin{aligned}\left(1+\tfrac{1}{2}\right)\left(1+\tfrac{1}{3}\right)\left(1+\tfrac{1}{4}\right)\left(1+\tfrac{1}{5}\right) &= \left(\tfrac{3}{2}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{6}{5}\right)\\ &= \left(\frac{\not 3}{2}\right)\left(\frac{\not 4}{\not 3}\right)\left(\frac{\not 5}{\not 4}\right)\left(\frac{6}{\not 5}\right)\\ &\quad \text{(simplifying numerators and denominators)}\\ &= \tfrac{6}{2}\\ &= 3\end{aligned}$$

Solution 2

$$\begin{aligned}\left(1+\tfrac{1}{2}\right)\left(1+\tfrac{1}{3}\right)\left(1+\tfrac{1}{4}\right)\left(1+\tfrac{1}{5}\right) &= \left(\tfrac{3}{2}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{6}{5}\right)\\ &= \tfrac{360}{120}\\ &= 3\end{aligned}$$

2. If $f(2x+1)=(x-12)(x+13)$, what is the value of $f(31)$?

Solution 1
Since $f(2x+1)=(x-12)(x+13)$, then

$$f(31)=f(2(15)+1)=(15-12)(15+13)=3(28)=84$$

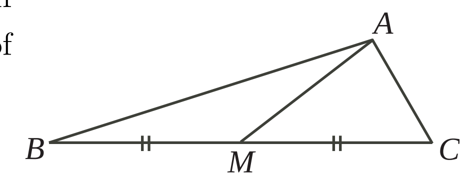
Solution 2
If $w=2x+1$, then $x=\dfrac{w-1}{2}$.
Since $f(2x+1)=(x-12)(x+13)$, then

$$f(w)=\left(\frac{w-1}{2}-12\right)\left(\frac{w-1}{2}+13\right)=\left(\frac{w-25}{2}\right)\left(\frac{w+25}{2}\right)$$

Therefore,

$$f(31)=\left(\frac{31-25}{2}\right)\left(\frac{31+25}{2}\right)=3(28)=84$$

3. In $\triangle ABC$, M is the midpoint of BC, as shown. If $\angle ABM = 15^\circ$ and $\angle AMC = 30^\circ$, what is the size of $\angle BCA$?



Solution

Since $\angle AMC = 30^\circ$, then $\angle AMB = 180^\circ - \angle AMC = 150^\circ$.
Since $\angle ABM = 15^\circ$ and $\angle AMB = 150^\circ$, then $\angle BAM = 180^\circ - \angle ABM - \angle AMB = 15^\circ$.
Since $\angle ABM = \angle BAM$, then $BM = MA$.
Since $BM = MA$ and $BM = MC$, then $MA = MC$, so $\angle MAC = \angle MCA$.
Thus, $\angle MCA = \frac{1}{2}(180^\circ - \angle AMC) = 75^\circ$.
Therefore, $\angle BCA = \angle MCA = 75^\circ$.

4. Determine all solutions (x, y) to the system of equations

$$\begin{aligned} \frac{4}{x} + \frac{5}{y^2} &= 12 \\ \frac{3}{x} + \frac{7}{y^2} &= 22 \end{aligned}$$

Solution 1

Subtracting 5 times the second equation from 7 times the first equation, we obtain

$$\begin{aligned} 7\left(\frac{4}{x} + \frac{5}{y^2}\right) - 5\left(\frac{3}{x} + \frac{7}{y^2}\right) &= 7(12) - 5(22) \\ \frac{13}{x} &= -26 \\ x &= -\tfrac{1}{2} \end{aligned}$$

Substituting $x = -\frac{1}{2}$ into the first equation, we obtain $\dfrac{4}{-\frac{1}{2}} + \dfrac{5}{y^2} = 12$ or $-8 + \dfrac{5}{y^2} = 12$ or $\dfrac{5}{y^2} = 20$ or $y^2 = \frac{1}{4}$.
Therefore, $y = \pm\frac{1}{2}$.
Thus, the solutions are $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.
(We can check by substitution that both of these solutions work.)

Solution 2

Subtracting 4 times the second equation from 3 times the first equation, we obtain

$$\begin{aligned} 3\left(\frac{4}{x}+\frac{5}{y^2}\right)-4\left(\frac{3}{x}+\frac{7}{y^2}\right) &= 3(12)-4(22) \\ -\frac{13}{y^2} &= -52 \\ y^2 &= \tfrac{1}{4} \\ y &= \pm\tfrac{1}{2} \end{aligned}$$

Substituting $y=\pm\frac{1}{2}$ into the first equation, we obtain $\dfrac{4}{x}+\dfrac{5}{\left(\pm\frac{1}{2}\right)^2}=12$ or $\dfrac{4}{x}+20=12$ or $\dfrac{4}{x}=-8$ or $x=-\frac{1}{2}$.

Thus, the solutions are $\left(-\frac{1}{2},\frac{1}{2}\right)$ and $\left(-\frac{1}{2},-\frac{1}{2}\right)$.

(We can check by substitution that both of these solutions work.)

5. In $\triangle ABC$, $BC=4$, $AB=x$, $AC=x+2$, and $\cos(\angle BAC)=\dfrac{x+8}{2x+4}$.
Determine all possible values of x.

Solution

Using the cosine law in $\triangle ABC$,

$$\begin{aligned} BC^2 &= AB^2+AC^2-2(AB)(AC)\cos(\angle BAC) \\ 4^2 &= x^2+(x+2)^2-2x(x+2)\frac{x+8}{2x+4} \\ 16 &= x^2+x^2+4x+4-x(x+8) \\ 0 &= x^2-4x-12 \\ 0 &= (x-6)(x+2) \end{aligned}$$

Therefore, $x=6$ or $x=-2$.

Since $AB=x$, then x must be positive, so $x=6$.

6. Determine the number of integers n that satisfy *all three* of the conditions below:

- each digit of n is either 1 or 0,
- n is divisible by 6, and
- $0<n<10^7$.

Solution 1

Since $0<n<10^7$, then n is a positive integer with fewer than 8 digits.

Since n is divisible by 6, then n is even. Since each digit of n is either 1 or 0, then n must end with a 0.
Since n is divisible by 6, then n is divisible by 3, so n has the sum of its digits divisible by 3. Since each digit of n is 0 or 1 and n has at most 6 non-zero digits, then the sum of the digits of n must be 3 or 6 (that is, n contains either 3 or 6 digits equal to 1).

Since n has at most 7 digits, we can write n in terms of its digits as $abcdef0$, where each of a, b, c, d, e, f can be 0 or 1. (We allow n to begin with a 0 in this representation.)
If n contains 6 digits equal to 1, then there is no choice in where the 1's are placed so $n = 1111110$.
If n contains 3 digits equal to 1, then 3 of the 6 digits a through f are 1 (and the other 3 are 0). The number of such possibilities is $\binom{6}{3} = 20$.
Therefore, there are $20 + 1 = 21$ such integers n.

Solution 2
Since $0 < n < 10^7$, then n is a positive integer with fewer than 8 digits.
Since n is divisible by 6, then n is even. Since each digit of n is either 1 or 0, then n ends with a 0.
Since n is divisible by 6, then n is divisible by 3, so has the sum of its digits divisible by 3. Since each digit of n is 0 or 1 and n has at most 6 non-zero digits, then the sum of the digits of n must be 3 or 6 (that is, n contains either 3 or 6 digits equal to 1).

If n contains 6 digits equal to 1, then $n = 1111110$, since n has at most 7 digits.
If n contains 3 digits equal to 1, then n has between 4 and 7 digits, and must begin with 1.
If n has 4 digits, then n must be 1110.
If n has 5 digits, then n has the form $1abc0$ with 2 of a, b, c equal to 1. There are $\binom{3}{2} = 3$ such possibilities.
If n has 6 digits, then n has the form $1abcd0$ with 2 of a, b, c, d equal to 1. There are $\binom{4}{2} = 6$ such possibilities.
If n has 7 digits, then n has the form $1abcde0$ with 2 of a, b, c, d, e equal to 1. There are $\binom{5}{2} = 10$ such possibilities.
Therefore, there are $1 + 1 + 3 + 6 + 10 = 21$ possibilities for n.

7. Suppose n and D are integers with n positive and $0 \le D \le 9$.
Determine n if $\dfrac{n}{810} = 0.\overline{9D5} = 0.9D59D59D5\ldots$.

Solution

First, we note that $0.\overline{9D5} = \dfrac{9D5}{999}$ since

$$\begin{aligned} 1000(0.\overline{9D5}) &= 9D5.\overline{9D5} \\ 1000(0.\overline{9D5}) - 0.\overline{9D5} &= 9D5.\overline{9D5} - 0.\overline{9D5} \\ 999(0.\overline{9D5}) &= 9D5 \\ 0.\overline{9D5} &= \frac{9D5}{999} \end{aligned}$$

Alternatively, we could derive this result by noticing that

$$\begin{aligned} 0.\overline{9D5} &= 0.9D59D59D5\ldots \\ &= \frac{9D5}{10^3} + \frac{9D5}{10^6} + \frac{9D5}{10^9} + \cdots \\ &= \frac{\frac{9D5}{10^3}}{1 - \frac{1}{10^3}} \qquad \text{(summing the infinite geometric series)} \\ &= \frac{9D5}{1000 - 1} \\ &= \frac{9D5}{999} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{n}{810} &= \frac{9D5}{999} \\ 999n &= 810(9D5) \\ 111n &= 90(9D5) \\ 37n &= 30(9D5) \end{aligned}$$

Thus, $30(9D5)$ is divisible by 37. Since 30 is not divisible by 37 and 37 is prime, then $9D5$ must be divisible by 37.

The multiples of 37 between 900 and 1000 are 925, 962 and 999.

Thus, $9D5$ must be 925, so $D = 2$.

So $37n = 30(925)$ or $n = 30(25) = 750$.

8. What is the probability that 2 or more successive heads will occur at least once in 10 tosses of a fair coin?

Solution 1

For a given toss, we use T to represent a result of tails and H for heads.

There are $2^{10} = 1024$ possible sequences of outcomes when a fair coin is tossed 10 times.

Let t_n be the number of sequences of n tosses of a fair coin which *do not* contain 2 or more successive heads.

(So the number of sequences of length 10 that contain 2 or more successive heads is $1024 - t_{10}$ which means that the desired probability is $\frac{1024 - t_{10}}{1024}$.)

Note that $t_1 = 2$, as both T and H do not contain 2 successive H's.

Also, $t_2 = 3$. (These are the sequences TT, TH, HT.)

Consider a sequence of n tosses of a fair coin which do not contain 2 or more successive heads, where $n \geq 3$.

Such a sequence must begin with H or T.

If the sequence begins with H, the second toss must be T, and the last $n - 2$ can be any sequence of $n - 2$ tosses that does not contain 2 or more successive heads. There are t_{n-2} such sequences of $n - 2$ tosses, so there are t_{n-2} sequences of n tosses beginning with H and not containing 2 or more successive heads.

If the sequence begins with T, the last $n - 1$ tosses can be any sequence of $n - 1$ tosses that does not contain 2 or more successive heads. There are t_{n-1} such sequences of $n - 1$ tosses, so there are t_{n-1} sequences of n tosses beginning with T and not containing 2 or more successive heads.

Therefore, $t_n = t_{n-1} + t_{n-2}$, since each sequence begins with H or T.

Starting with $t_1 = 2$ and $t_2 = 3$, we can then generate the sequence t_n for $n = 1$ to $n = 10$ by adding the two previous terms to obtain the next term:

$$2, 3, 5, 8, 13, 21, 34, 55, 89, 144$$

So $t_{10} = 144$, so the desired probability is $\frac{1024 - 144}{1024} = \frac{880}{1024} = \frac{55}{64}$.

Solution 2

For a given toss, we use T to represent a result of tails and H for heads.

There are $2^{10} = 1024$ possible sequences of outcomes when a fair coin is tossed 10 times.

Let us count the number of such sequences which *do not* contain 2 or more successive H's, by grouping them by the number of H's that they contain. (Note that not containing 2 or more successive H's is equivalent to not containing the pair HH.)

If a sequence of length 10 consists of 0 H and 10 T's, there is only 1 possibility.
If a sequence of length 10 consists of 1 H and 9 T's, there are $\binom{10}{1} = 10$ possibilities.

If a sequence of length 10 consists of 2 H's and 8 T's, then we start with

$$\_T\_T\_T\_T\_T\_T\_T\_T\_$$

Each of the two H's must be placed in separate spaces. We can then eliminate any unused spaces to obtain a sequence of 8 T's and 2 H's containing no consecutive H's (and we get all such sequences this way). There are $\binom{9}{2} = 36$ ways of positioning the H's, and so 36 such sequences.

In a similar way, with 3 H's and 7 T's, there are $\binom{8}{3} = 56$ such sequences.
With 4 H's and 6 T's, there are $\binom{7}{4} = 35$ such sequences.
With 5 H's and 5 T's, there are $\binom{6}{5} = 6$ such sequences.

Therefore, there is a total of $1 + 10 + 36 + 56 + 35 + 6 = 144$ sequences of 10 tosses which do not contain 2 or more successive H's.
Thus, there are $1024 - 144 = 880$ sequences of 10 tosses which do contain 2 or more successive H's. Therefore, the probability that a given sequence contains 2 or more successive H's is $\frac{880}{1024} = \frac{55}{64}$.

Part B

1. Piotr places numbers on a 3 by 3 grid using the following rule, called "Piotr's Principle":

   For any three adjacent numbers in a horizontal, vertical or diagonal line, the middle number is always the average (mean) of its two neighbours.

   (a) Using Piotr's principle, determine the missing numbers in the grid to the right. (You should fill in the missing numbers in the grid in your answer booklet.)

3		19
8		

Solution

Since the average of 3 and 19 is $\frac{1}{2}(3 + 19) = 11$, then 11 goes between the 3 and 19.
The number which goes below 8 is the number whose average with 3 is 8, so 13 goes below 8.
The average of 13 and 19, or 16, goes in the middle square.
The number which goes to the right of the 16 is the number whose average with 8 is 16, or 24.
The number which goes below 24 is the number whose average with 19 is 24, or 29.
The number which goes between 13 and 29 is their average, which is 21.

Therefore, the completed grid is

3	11	19
8	16	24
13	21	29

.

(We can check that each line obeys Piotr's Principle.)

Note

There are other orders in which the squares can be filled.

(b) Determine, with justification, the total of the nine numbers when the grid to the right is completed using Piotr's Principle.

x		
5		23

Note

When we have the three numbers a, X, b on a line, then X is the average of a and b, so $X = \frac{1}{2}(a + b)$.
When we have the three numbers a, b, X on a line, then b is the average of a and X, so $b = \frac{1}{2}(a + X)$ or $2b = a + X$ or $X = 2b - a$.
These facts will be useful as we solve (b) and (c).

Solution 1

The average of 5 and 23 is $\frac{1}{2}(5+23) = 14$, which goes in the square between the 5 and 23.
Since the average of the numbers above and below the 5 equals 5, then their sum is $2(5) = 10$. (Note that we do not need to know the actual numbers, only their sum.)
Similarly, the sum of the numbers above and below the 14 is $2(14) = 28$ and the sum of the numbers above and below the 23 is $2(23) = 46$.
Therefore, the sum of the numbers in the whole grid is $5 + 10 + 14 + 28 + 23 + 46 = 126$.

Solution 2

The average of 5 and 23 is $\frac{1}{2}(5+23) = 14$, which goes in the square between the 5 and 23.
Since the average of the x and the number below the 5 is 5, then the number below the 5 is $10 - x$.
Since the average of the x and the bottom right number is 14, then the bottom right number is $28 - x$.
The average of $10 - x$ and $28 - x$ is $\frac{1}{2}(10 - x + 28 - x) = 19 - x$, which goes in the middle square on the bottom row.
Since the average of $19 - x$ and the number above the 14 is 14, then the number above the 14 is $2(14) - (19 - x) = 9 + x$.
Since the average of $28 - x$ and the number above the 23 is 23, then the number above the 23 is $2(23) - (28 - x) = 18 + x$.

Thus, the completed grid is

x	$9 + x$	$18 + x$
5	14	23
$10 - x$	$19 - x$	$28 - x$

and so the sum of the entries is
$x + 9 + x + 18 + x + 5 + 14 + 23 + 10 - x + 19 - x + 28 - x = 126$.

(c) Determine, with justification, the values of x and y when the grid to the right is completed using Piotr's Principle.

x	7	
9		y
		20

Solution

The centre square is the average of 9 and y and is also the average of x and 20.
Comparing these facts, $\frac{1}{2}(9 + y) = \frac{1}{2}(x + 20)$ or $9 + y = x + 20$ or $x - y = -11$.
The number in the top right corner gives an average of 7 when combined with x (so equals $2(7) - x = 14 - x$) and gives an average of y when combined with 20 (so equals $2y - 20$).
Therefore, $14 - x = 2y - 20$ or $x + 2y = 34$.
Subtracting the first equation from the second, we obtain $3y = 45$ or $y = 15$.
Substituting back into the first equation, we obtain $x = 4$.

We check by completing the grid. Starting with

4	7	
9		15
		20

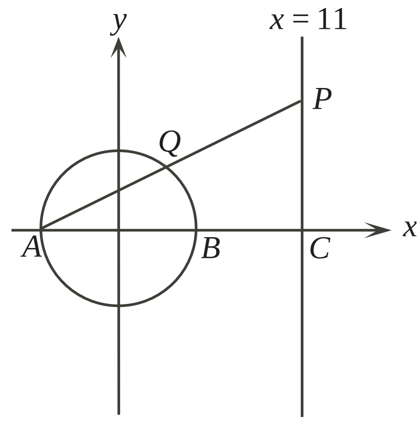
gives, after some work,

4	7	10
9	12	15
14	17	20

, which does obey Piotr's Principle.

Therefore, $x = 4$ and $y = 15$.

2. In the diagram, the circle $x^2+y^2 = 25$ intersects the x-axis at points A and B. The line $x = 11$ intersects the x-axis at point C. Point P moves along the line $x = 11$ above the x-axis and AP intersects the circle at Q.



(a) Determine the coordinates of P when $\triangle AQB$ has maximum area. Justify your answer.

Solution

Since the circle has equation $x^2 + y^2 = 25$, then to find the coordinates of A and B, the x-intercepts of the circle, we set $y = 0$ to obtain $x^2 = 25$ or $x = \pm 5$. Therefore, A and B have coordinates $(-5, 0)$ and $(5, 0)$, respectively.

Since $\triangle AQB$ has a base AB of constant length and a variable height, then the area of $\triangle AQB$ is maximized when the height of $\triangle AQB$ is maximized (that is, when Q is furthest from AB).

To maximize the height of $\triangle AQB$, we would like Q to have as large a y-coordinate as possible. Thus, we would like Q to be at the "top" of the circle – that is, at the place where the circle intersects the y-axis.

Since the circle has equation $x^2 + y^2 = 25$, then setting $x = 0$, we obtain $y^2 = 25$ or $y = \pm 5$, so Q has coordinates $(0, 5)$ as Q lies above the x-axis.

Therefore, P lies on the line through $A(-5, 0)$ and $Q(0, 5)$. This line has slope 1 and y-intercept 5, so has equation $y = x + 5$.

Since P has x-coordinate 11 and lies on the line with equation $y = x + 5$, then P has coordinates $(11, 16)$.

(b) Determine the coordinates of P when Q is the midpoint of AP. Justify your answer.

Solution

Suppose the coordinates of P are $(11, p)$.

We will determine p so that the midpoint of PA lies on the circle. (This is equivalent to finding P so that the point on the circle is the midpoint of P and A.)

Since A has coordinates $(-5, 0)$, then for Q to be the midpoint of AP, Q must have coordinates $\left(\frac{1}{2}(-5+11), \frac{1}{2}(0+p)\right) = (3, \frac{1}{2}p)$.

For $(3, \frac{1}{2}p)$ to lie on the circle,

$$\begin{aligned} 3^2 + \left(\tfrac{1}{2}p\right)^2 &= 25 \\ \tfrac{1}{4}p^2 &= 16 \\ p^2 &= 64 \\ p &= \pm 8 \end{aligned}$$

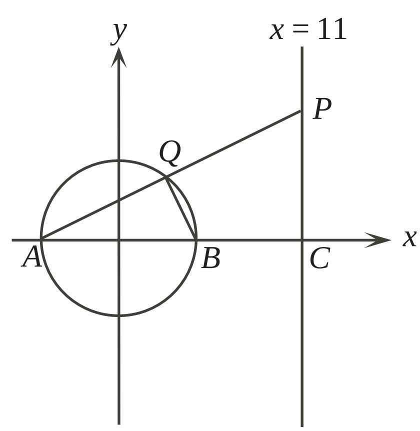
Since P must lie above the x-axis, then $p = 8$.

Therefore, P has coordinates $(11, 8)$.

(c) Determine the coordinates of P when the area of $\triangle AQB$ is $\frac{1}{4}$ of the area of $\triangle APC$. Justify your answer.

Solution 1

Join Q to B.



Since AB is a diameter of the circle, then $\angle AQB = 90°$.

Thus, since $\triangle AQB$ and $\triangle ACP$ are both right-angled and share a common angle at A, then $\triangle AQB$ and $\triangle ACP$ are similar.

Since the area of $\triangle ACP$ is 4 times the area of $\triangle AQB$, then the sides of $\triangle ACP$ are $\sqrt{4} = 2$ times as long as the corresponding sides of $\triangle AQB$.

Since $AB = 10$, then $AP = 2AB = 20$.

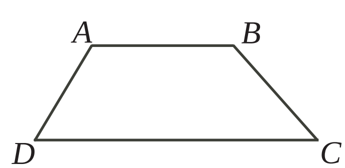
We also know that $AC = 16$ (since C has coordinates $(11, 0)$ and A has coordinates $(-5, 0)$).
Therefore, by the Pythagorean Theorem, $PC = \sqrt{AP^2 - AC^2} = \sqrt{20^2 - 16^2} = 12$.
Thus, P has coordinates $(11, 12)$.

Solution 2
Let the coordinates of P be $(11, p)$, and the coordinates of Q be (a, b). Thus, the height of $\triangle AQB$ is b.
The area of $\triangle AQB$ is $\frac{1}{2}(AB)(b) = 5b$ since $AB = 10$.
The area of $\triangle APC$ is $\frac{1}{2}(AC)(p) = 8p$ since $AC = 16$.
Since the area of $\triangle AQB$ is $\frac{1}{4}$ that of $\triangle APC$, then $5b = 2p$ or $b = \frac{2}{5}p$.
This tells us that Q must be $\frac{2}{5}$ of the way along from A to P.
Since A has x-coordinate -5 and P has x-coordinate 11, then Q has x-coordinate $-5 + \frac{2}{5}(11 - (-5)) = \frac{7}{5}$. Therefore, Q has coordinates $\left(\frac{7}{5}, \frac{2}{5}p\right)$.
Since the circle has equation $x^2 + y^2 = 25$, then

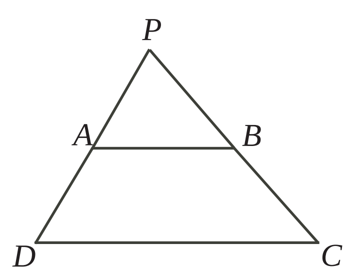
$$\begin{aligned} \left(\tfrac{7}{5}\right)^2 + \left(\tfrac{2}{5}p\right)^2 &= 25 \\ \tfrac{49}{25} + \tfrac{4}{25}p^2 &= \tfrac{625}{25} \\ 4p^2 &= 576 \\ p^2 &= 144 \end{aligned}$$

and so $p = 12$ since $p > 0$. Therefore, P has coordinates $(11, 12)$.

3. (a) In the diagram, trapezoid $ABCD$ has parallel sides AB and DC of lengths 10 and 20 and sides AD and BC of lengths 6 and 8. Determine the area of trapezoid $ABCD$.



Solution 1
Extend DA and CB to meet at P.



Since AB is parallel to DC, then $\angle PAB = \angle PDC$ and $\angle PBA = \angle PCD$.
Therefore, $\triangle PAB$ is similar to $\triangle PDC$.

Since $AB = \frac{1}{2}DC$, then the sides of $\triangle PAB$ are $\frac{1}{2}$ the length of the corresponding sides of $\triangle PDC$.

Therefore, $PA = AD = 6$ and $PB = BC = 8$.

Thus, the sides of $\triangle PDC$ have lengths 12, 16 and 20. Since $12^2 + 16^2 = 20^2$, then $\triangle PDC$ is right-angled at P by the Pythagorean Theorem.

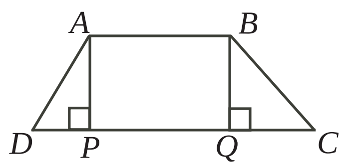
Thus, the area of $\triangle PDC$ is $\frac{1}{2}(12)(16) = 96$.

Since $\triangle PAB$ is right-angled at P, its area is $\frac{1}{2}(6)(8) = 24$.

Therefore, the area of trapezoid $ABCD$ is $96 - 24 = 72$.

Solution 2

Drop perpendiculars from A and B to P and Q on DC.



Let $AP = BQ = h$ and let $DP = x$.

Since $AB = 10$ and $ABQP$ is a rectangle, then $PQ = 10$.

Since DC has length 20, then $QC = 20 - x - 10 = 10 - x$.

Using the Pythagorean Theorem in $\triangle DPA$, we obtain $x^2 + h^2 = 6^2$.

Using the Pythagorean Theorem in $\triangle CQB$, we obtain $(10 - x)^2 + h^2 = 8^2$.

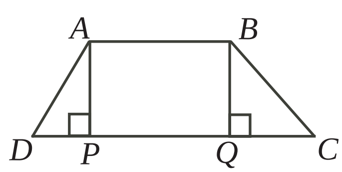
Subtracting the first of these equations from the second, we obtain $100 - 20x = 28$ or $20x = 72$ or $x = \frac{18}{5}$.

Substituting back into the first equation, $h^2 = 36 - \left(\frac{18}{5}\right)^2 = \frac{576}{25}$ so $h = \frac{24}{5}$.

Therefore, the area of trapezoid $ABCD$ is $\frac{1}{2}\left(\frac{24}{5}\right)(10 + 20) = 72$.

Solution 3

Drop perpendiculars from A and B to P and Q on DC.



Cut out rectangle $ABQP$ and join the two remaining pieces along the cut line.

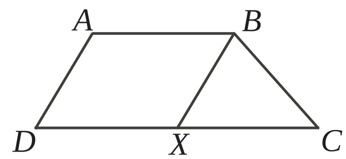
The remaining shape is a triangle DCX with side lengths $DX = 6$, $XC = 8$ and $DC = 20 - 10 = 10$. Since $6^2 + 8^2 = 10^2$, then $\triangle DCX$ is right-angled by the Pythagorean Theorem.

Since $\sin(\angle XDC) = \dfrac{XC}{DC} = \dfrac{8}{10} = \dfrac{4}{5}$, then the length of the altitude from X to DC is

$$XD \sin(\angle XDC) = 6\left(\tfrac{4}{5}\right) = \tfrac{24}{5}$$

which is also the height of the original trapezoid.
(We could also determine the length of this altitude by calculating the area of $\triangle XDC$ in two ways: once as $\frac{1}{2}(DX)(XC) = 24$ and once as $\frac{1}{2}h(DC) = 5h$.)
Therefore, the area of the original trapezoid is $\frac{1}{2}\left(\frac{24}{5}\right)(10+20) = 72$.
(Alternatively, the area of the original trapezoid is the sum of the areas of rectangle $ABQP$ ($\frac{24}{5} \times 10 = 48$) and $\triangle XDC$ (24), for a total of 72.)

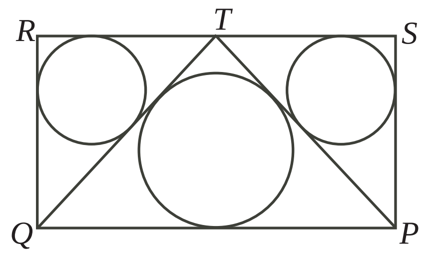
Solution 4
Draw BX from B to X on DC so that BX is parallel to AD.



Then $ABXD$ is a parallelogram so $BX = AD = 6$ and $DX = AB = 10$.
Therefore, $XC = DC - DX = 10$.
Thus, $\triangle BXC$ has sides of length 6, 8 and 10. Since $6^2 + 8^2 = 10^2$, then $\triangle BXC$ is right-angled at B by the Pythagorean Theorem. The area of $\triangle BXC$ is thus $\frac{1}{2}(6)(8) = 24$.
Join B to D. BD divides the area of $ABXD$ in half.
Also, BX divides the area of $\triangle BDC$ in half, since it is a median.
Therefore, the areas of $\triangle ABD$, $\triangle BDX$ and $\triangle XBC$ are all equal.
So the area of trapezoid $ABCD$ is $3(24) = 72$.

Solution 5
Let X be the midpoint of DC. Join X to A and B.
Then $AB = DX = XC = 10$.
Since $AB = DX$ and AB is parallel to DX, then AD and BX are parallel and equal, so $BX = 6$.
Since $AB = XC$ and AB is parallel to XC, then AX and BC are parallel and equal, so $AX = 6$.
Therefore, the trapezoid is divided into three triangles, each of which has side lengths 6, 8 and 10.
A triangle with side lengths 6, 8 and 10 is right-angled (since $6^2 + 8^2 = 10^2$), so has area $\frac{1}{2}(6)(8) = 24$.
Therefore, the area of the trapezoid is $3 \times 24 = 72$.

(b) In the diagram, $PQRS$ is a rectangle and T is the midpoint of RS. The inscribed circles of $\triangle PTS$ and $\triangle RTQ$ each have radius 3. The inscribed circle of $\triangle QPT$ has radius 4. Determine the dimensions of rectangle $PQRS$.



Solution 1

Let $RT = a$ (so $RS = 2a$) and $RQ = b$.

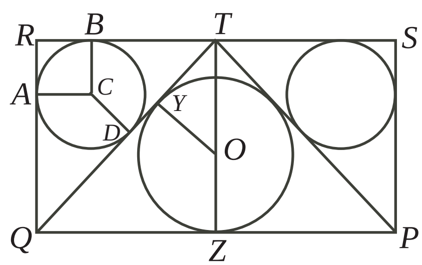
Drop a perpendicular from T to Z on QP. By symmetry, Z is also the point of tangency of the middle circle to QP.

Let O be the centre of the circle inscribed in $\triangle QTP$ and C be the centre of the circle inscribed in $\triangle RTQ$.

Let A, B and D be the points of tangency of the circle with centre C to QR, RT and QT, respectively.

Let Y be the point of tangency of the circle with centre O to QT.

Join C to A, B and D, and O to Y. These radii are perpendicular to QR, RT, QT, and QT, respectively.



We know that $OY = OZ = 4$.

Since $TZ = b$, then $TO = b - 4$.

Since $QZ = a$, then $QY = a$ (equal tangents).

Since $CA = CB = 3$ and $RACB$ is a rectangle (as it has three right angles), then $RACB$ is a square and $RA = RB = 3$.

Therefore, $AQ = b - 3$ and $BT = a - 3$.

By equal tangents, $TD = BT = a - 3$ and $QD = QA = b - 3$.

Now, $TY = QT - QY = QD + TD - QY = (b - 3) + (a - 3) - a = b - 6$.

Therefore, $\triangle TOY$ is right-angled at Y with sides of length $TO = b - 4$, $TY = b - 6$ and $OY = 4$.

By the Pythagorean Theorem, $4^2 + (b - 6)^2 = (b - 4)^2$ or $4b = 36$ or $b = 9$.

Therefore, $TY = 9 - 6 = 3$ and $\tan(\angle OTY) = \dfrac{OY}{TY} = \dfrac{4}{3}$.

Also, $\frac{4}{3} = \tan(\angle QTZ) = \dfrac{a}{b}$.

Since $b = 9$, $a = 12$.

Therefore, the rectangle is 24 by 9.

Solution 2

Let $RT = a$ (so $RS = 2a$) and $RQ = b$.

Drop a perpendicular from T to Z on QP. By symmetry, Z is also the point of tangency of the middle circle to QP, and $QZ = a$.

Since the incircle of $\triangle QRT$ has radius 3, then so does the incircle of $\triangle QZT$.

Let O be the centre of the circle inscribed in $\triangle QTP$ and C' the centre of the circle inscribed in $\triangle QZT$.

Since C' and O both lie on the angle bisector of $\angle TQP$, then $\tan(\angle C'QP) = \tan(\angle OQP)$.

Since C' is 3 units from the line TZ, then $\tan(\angle C'QP) = \dfrac{3}{a-3}$ and $\tan(\angle OQP) = \dfrac{4}{a}$ so $\dfrac{3}{a-3} = \dfrac{4}{a}$ or $3a = 4a - 12$ or $a = 12$.

We can calculate $b = 9$ as in Solution 1, to obtain that the rectangle is 24 by 9.

Solution 3

Let $RT = a$ (so $RS = 2a$) and $RQ = b$.

We calculate the areas of each of $\triangle QRT$ and $\triangle QTP$ in two ways: once using the standard $\frac{1}{2}bh$ formula and once using the less well-known Area $= sr$ formula, where s is the semi-perimeter of the triangle (that is, half of the perimeter) and r is the inradius (that is, the radius of the inscribed circle).

In $\triangle QRT$, $RT = a$, $RQ = b$, $QT = \sqrt{a^2+b^2}$ and the inradius is 3, so

$$\tfrac{1}{2}ab = \tfrac{1}{2}(a + b + \sqrt{a^2+b^2})(3)$$

In $\triangle QTP$, $QT = TP = \sqrt{a^2+b^2}$, $QP = 2a$, the height is b, and the inradius is 4, so

$$\tfrac{1}{2}(2a)b = \tfrac{1}{2}(2\sqrt{a^2+b^2} + 2a)(4)$$

Simplifying these equations, we get the system of equations

$$\begin{aligned} ab &= 3(a+b) + 3\sqrt{a^2+b^2} \\ ab &= 4a + 4\sqrt{a^2+b^2} \end{aligned}$$

Eliminating the ab terms, we obtain $3b - a = \sqrt{a^2+b^2}$.

Squaring both sides, we obtain $9b^2 - 6ab + a^2 = a^2 + b^2$ or $8b^2 - 6ab = 0$.

Since $b \neq 0$, $\dfrac{a}{b} = \dfrac{4}{3}$.

Substituting $a = \frac{4}{3}b$ into the first equation yields

$$\begin{aligned}
\tfrac{4}{3}b^2 &= 4b + 3b + 3\sqrt{\tfrac{16}{9}b^2 + b^2} \\
\tfrac{4}{3}b^2 &= 7b + \sqrt{25b^2} \\
\tfrac{4}{3}b^2 &= 7b + 5b \qquad \text{(since } b > 0\text{)} \\
4b^2 &= 36b \\
b &= 9
\end{aligned}$$

since $b \neq 0$. Therefore, $a = 12$ and the rectangle is 24 by 9.

Solution 3

We use the notation and diagram from Solution 1.

Since RS and QP are parallel, then $\angle BTD = \angle YQZ$.

Since C and O are the centres of inscribed circles, then C lies on the angle bisector of $\angle BTD$ and O lies on the angle bisector of $\angle YQZ$.

Therefore, $\angle BTC = \frac{1}{2}\angle BTD = \frac{1}{2}\angle YQZ = \angle OQZ$.

Therefore, $\triangle BTC$ and $\triangle ZQO$ are similar, as each is right-angled.

Thus, $BT : QZ = BC : OZ = 3 : 4$.

Suppose that $BT = 3x$ and $QZ = 4x$.

Then $RT = RB + BT = AC + 3x = 3 + 3x$, since $RBCA$ is a square.

But $RT = QZ$ so $4x = 3 + 3x$ or $x = 3$.

Let $QA = y$.

Then $QT = QD + DT = QA + BT$ by equal tangents, so $QT = y + 3x = y + 9$.

Since $\triangle QRT$ is right-angled, then

$$\begin{aligned}
QR^2 + RT^2 &= QT^2 \\
(y+3)^2 + 12^2 &= (y+9)^2 \\
y^2 + 6y + 9 + 144 &= y^2 + 18y + 81 \\
12y &= 72 \\
y &= 6
\end{aligned}$$

Therefore, since $RT = 3 + 3x$, then $RS = 24$ and $RQ = 3 + y = 9$, so the rectangle is 24 by 9.

4. (a) Determine, with justification, the fraction $\dfrac{p}{q}$, where p and q are positive integers and $q < 100$, that is closest to, but not equal to, $\frac{3}{7}$.

Solution

We would like to find positive integers p and q with $q < 100$ which minimizes

$$\left|\frac{p}{q} - \frac{3}{7}\right| = \left|\frac{7p-3q}{7q}\right| = \frac{|7p-3q|}{7q}$$

To minimize such a fraction, we would like to make the numerator small while making the denominator large.

Since the two fractions $\frac{p}{q}$ and $\frac{3}{7}$ are not equal, the numerator of their difference cannot be 0. Since the numerator is a positive integer, its minimum possible value is 1.

We consider the largest possible values of q (starting with 99) and determine if $7p - 3q$ can possibly be equal to 1 or -1.

If $q = 99$, $7p - 3q = 7p - 297$, which cannot equal ± 1 since the nearest multiple of 7 to 297 is 294.

If $q = 98$, $7p - 3q = 7p - 294$, which cannot equal ± 1 since $7p - 294$ is always divisible by 7.

If $q = 97$, $7p - 3q = 7p - 291$, which cannot equal ± 1 since the nearest multiple of 7 to 291 is 294.

If $q = 96$, $7p - 3q = 7p - 288$, which equals -1 if $p = 41$.

If $p = 41$ and $q = 96$, the difference between the fractions $\dfrac{3}{7}$ and $\dfrac{p}{q}$ is $\dfrac{1}{7(96)}$.

If $q > 96$, the numerator of $\dfrac{|7p-3q|}{7q}$ is always at least 2, so the difference is at least $\dfrac{2}{7(99)} > \dfrac{1}{7(96)}$.

If $q < 96$, the difference between $\dfrac{p}{q}$ and $\dfrac{3}{7}$ is at least $\dfrac{1}{7(95)} > \dfrac{1}{7(96)}$.

So $\frac{41}{96}$ minimizes the difference, so it is the closest fraction to $\frac{3}{7}$ under the given conditions.

(b) The *baseball sum* of two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is defined to be $\frac{a+c}{b+d}$. (A *rational number* is a fraction whose numerator and denominator are both integers and whose denominator is not equal to 0.) Starting with the rational numbers $\frac{0}{1}$ and $\frac{1}{1}$ as Stage 0, the baseball sum of each consecutive pair of rational numbers in a stage is inserted between the pair to arrive at the next stage. The first few stages of this process are shown below:

STAGE 0:	$\frac{0}{1}$								$\frac{1}{1}$
STAGE 1:	$\frac{0}{1}$				$\frac{1}{2}$				$\frac{1}{1}$
STAGE 2:	$\frac{0}{1}$		$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$		$\frac{1}{1}$
STAGE 3:	$\frac{0}{1}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{1}$

Prove that

(i) no rational number will be inserted more than once,

Solution

Consider two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ which occur next to each other at a given stage with $\frac{a}{b} < \frac{c}{d}$.

Note that this means that $ad < bc$ or $bc - ad > 0$.

The rational number that will be inserted between them at the next stage is $\frac{a+c}{b+d}$.

Now

$$\frac{a}{b} < \frac{a+c}{b+d} \Leftrightarrow a(b+d) < b(a+c) \Leftrightarrow 0 < bc - ad$$

which we know to be true, and

$$\frac{a+c}{b+d} < \frac{c}{d} \Leftrightarrow d(a+c) < c(b+d) \Leftrightarrow 0 < bc - ad$$

which is again true.

Therefore, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

This tells us that every rational number which is inserted at any given stage is *strictly* between the two rational numbers on either side.

Therefore, once a given rational number is inserted, every other rational number which is inserted must be either strictly larger or strictly smaller, as the list at each stage must be strictly increasing.

Therefore, no rational number will be inserted more than once.

(ii) no inserted fraction is reducible,

Solution

First, we prove a lemma.

Lemma

If $\frac{a}{b} < \frac{c}{d}$ are consecutive rational numbers in a given stage, then $bc - ad = 1$.

Proof

At Stage 0, the two fractions obey this property.

Assume that the property holds for all fractions in Stage k.

Consider two consecutive fractions $\frac{a}{b} < \frac{c}{d}$ at Stage k.

The fraction that will be inserted between $\frac{a}{b}$ and $\frac{c}{d}$ at Stage $k+1$ is $\frac{a+c}{b+d}$, giving $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Note that $b(a+c) - a(b+d) = bc - ad = 1$ and $c(b+d) - d(a+c) = bc - ad = 1$.

This tells us that each pair of consecutive fraction at Stage $k+1$ obeys this property.

Therefore, by induction, the required property holds.

Suppose then that a fraction $\frac{kp}{kq}$ $(k, p, q \in \mathbb{Z}^+)$ is inserted between $\frac{a}{b}$ and $\frac{c}{d}$.

By the Lemma, we must have $b(kp) - a(kq) = 1$ or $k(bp - aq) = 1$.

Therefore, k divides 1, so $k = 1$.

Thus, any inserted fraction can only have a common factor of 1 between its numerator and denominator, so is irreducible. Thus, no inserted fraction is reducible.

(iii) every rational number between 0 and 1 will be inserted in the pattern at some stage.

Solution 1

We note first that every rational number of the forms $\frac{1}{n}$ and $\frac{n-1}{n}$ for $n \geq 2$ do enter the pattern as the first and last new entry in Stage $n-1$. (These rational numbers enter between $\frac{0}{1}$ and $\frac{1}{n-1}$, and $\frac{n-2}{n-1}$ and $\frac{1}{1}$, respectively.)

Assume that there are rational numbers between 0 and 1 which are not inserted in the pattern at the some stage.

Suppose that $\frac{p}{q}$ with $p, q \in \mathbb{Z}^+$ and $\gcd(p,q) = 1$ is such a rational number with minimal denominator. (Note that all irreducible fractions with denominators 1, 2, 3 are inserted already.)

Since $\gcd(p, q) = 1$, the Diophantine equation $py - qx = 1$ has solutions.
In fact, this Diophantine equation has a unique solution with $0 \le y < q$. We can say further that $0 < y < q$ since $q > 1$. (If $y = 0$, we have $-qx = 1$ so q would have to be 1.)
Since $0 < y < q$ and $qx = py - 1$, then $0 < qx < pq$ so $0 < x < p$ as well.

Suppose that $(x, y) = (a, b)$ is this unique solution.
Note that $pb - qa = 1$ so $pb > qa$ so $\frac{a}{b} < \frac{p}{q}$.

Consider the fraction $\frac{p-a}{q-b}$.
Its numerator and denominator are each a positive integer since $0 < a < p$ and $0 < b < q$. Also, note that $b(p - a) - a(q - b) = bp - aq = 1$.
Among other things, this tells us that $\frac{a}{b} < \frac{p-a}{q-b}$.

If we can prove that $\frac{a}{b}$ and $\frac{p-a}{q-b}$ are consecutive at some stage, then $\frac{p}{q}$ will be inserted between them at the next stage.

Since each of $\frac{a}{b}$ and $\frac{p-a}{q-b}$ has a denominator less than q, it appears in the pattern as $\frac{p}{q}$ has the smallest denominator among those fractions which do not appear.

Note next that $\frac{a}{b}$ cannot be $\frac{0}{1}$.
(If it was, $a = 0$ so $bp - aq = 1$ gives $bp = 1$ so $p = 1$.
We know that every fraction with $p = 1$ enters the pattern, so $p \neq 1$.)
Also, $\frac{p-a}{q-b}$ cannot be $\frac{1}{1}$.
(If it was, then $p - a = q - b$ so $b(p - a) - a(q - b) = 1$ gives $(b - a)(q - b) = 1$ so $b - a = 1$.
Since $p - a = q - b$ then $q - p = b - a = 1$.
But every fraction of the form $\frac{n-1}{n}$ enters the pattern, so $q - p \neq 1$.)

This tells us that each of $\frac{a}{b}$ and $\frac{p-a}{q-b}$ actually entered the pattern at some stage.

There are now three cases: $b < q - b$, $b > q - b$ and $b = q - b$.

- Assume that $b < q - b$.
  Consider the point when the fraction $\frac{p-a}{q-b}$ was inserted into the pattern.
  Suppose that $\frac{p-a}{q-b}$ was inserted immediately between $\frac{m}{n} < \frac{M}{N}$.

(That is, $m+M=p-a$ and $n+N=q-b$).
Note that $0<n<q-b$. (n and N cannot be 0 since denominators cannot be 0.)

Since $\frac{m}{n}<\frac{p-a}{q-b}$ are consecutive fractions at this stage, we must also have $(p-a)n-(q-b)m=1$.

But since $\gcd(p-a,q-b)=1$, the Diophantine equation $(p-a)Y-(q-b)X=1$ must have a unique solution with $0<Y<q-b$.
But $(X,Y)=(a,b)$ is such a solution since $0<b<q-b$ and $b(p-a)-a(q-b)=1$ (from above).
Therefore, $\frac{a}{b}$ must be the fraction immediately to the left of $\frac{p-a}{q-b}$ at the stage where $\frac{p-a}{q-b}$ enters, which means that $\frac{p}{q}$ will be inserted into the pattern, contradicting our assumption.

- Assume that $q-b<b$.
  Consider the point when the fraction $\frac{a}{b}$ was inserted into the pattern.
  Suppose that $\frac{a}{b}$ was inserted immediately between $\frac{m}{n}<\frac{M}{N}$ (that is, $m+M=a$ and $n+N=b$).
  Note that $0<n<b$. (n and N cannot be 0 since denominators cannot be 0.)

  Since $\frac{m}{n}<\frac{a}{b}$ are consecutive fractions at this stage, we must also have $an-bm=1$.

  But since $\gcd(a,b)=1$, the Diophantine equation $aY-bX=1$ must have a unique solution with $0<Y<b$.
  But $(X,Y)=(p-a,q-b)$ is such a solution since $0<q-b<b$ and $b(p-a)-a(q-b)=1$ (from above).
  Therefore, $\frac{p-a}{q-b}$ must be the fraction immediately to the left of $\frac{a}{b}$ at the stage where $\frac{a}{b}$ enters, which means that $\frac{p}{q}$ will be inserted into the pattern, contradicting our assumption.

- Assume that $q-b=b$.
  In this case, $q=2b$.
  But $bp-aq=1$ so $b(p-2a)=1$, and so $b=1$ giving $q=2$.
  But we know that every irreducible fraction with denominator 2 does enter. (Namely, the fraction $\frac{1}{2}$.)
  So this case cannot occur.

Thus, every rational number between 0 and 1 will be inserted in the pattern at some stage.

Solution 2

Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive (irreducible) fractions at some stage.

Define $S\left(\frac{a}{b}, \frac{c}{d}\right) = a + b + c + d$, the sum of the numerators and denominators of the consecutive fractions.

We consider the minimum value of S at a given stage.

When $\frac{a+c}{b+d}$ is inserted with $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$, the two new sums are

$$S\left(\frac{a}{b}, \frac{a+c}{b+d}\right) = a + b + a + c + b + d = 2a + 2b + c + d$$

and

$$S\left(\frac{a+c}{b+d}, \frac{c}{d}\right) = a + c + b + d + c + d = a + b + 2c + 2d$$

each of which is larger than $S\left(\frac{a}{b}, \frac{c}{d}\right)$.

So the minimum value of these sums must increase from one stage to the next.

Suppose that the fraction $\frac{a}{b}$ between 0 and 1 with $a, b \in \mathbb{Z}^+$ and $\gcd(a, b) = 1$ is never inserted into the pattern.

At any given Stage, since the fraction $\frac{a}{b}$ does not occur, it must be strictly between two consecutive fractions, say $\frac{m_1}{n_1} < \frac{a}{b} < \frac{m_2}{n_2}$. (Since $\frac{a}{b}$ never occurs, we must be able to find such a pair of fractions at every Stage.)

(We know that $m_2 n_1 - n_2 m_1 = 1$ from (b).)

Thus, $m_1 b < n_1 a$ and $n_2 a < m_2 b$. Since each of these quantities is a positive integer, $n_1 a - m_1 b \geq 1$ and $m_2 b - n_2 a \geq 1$.

Now

$$\begin{aligned}
& m_2 + n_2 + m_1 + n_1 \\
= \; & (m_2 + n_2)(1) + (m_1 + n_1)(1) \\
\leq \; & (m_2 + n_2)(n_1 a - m_1 b) + (m_1 + n_1)(m_2 b - n_2 a) \\
= \; & m_2 n_1 a + n_1 n_2 a - m_1 m_2 b - m_1 n_2 b + m_1 m_2 b + m_2 n_1 b - m_1 n_2 a - n_1 n_2 a \\
= \; & a(m_2 n_1 - m_1 n_2) + b(m_2 n_1 - m_1 n_2) \\
= \; & a + b
\end{aligned}$$

But for a fixed fraction $\frac{a}{b}$, $a + b$ is fixed and the minimum possible value of

$$m_2 + n_2 + m_1 + n_1$$

increases from one stage to the next, so eventually $a + b < m_2 + n_2 + m_1 + n_1$, a contradiction, since once this happens we cannot find two fractions in that Stage between which to put $\frac{a}{b}$.
That is, there will be a stage beyond which we cannot find two consecutive fractions with $\frac{a}{b}$ between them. This means that $\frac{a}{b}$ must actually occur in the pattern.
Therefore, every rational number between 0 and 1 will be inserted in the pattern at some stage.