The Canadian Mathematical Society | La Société mathématique du Canada

CMS
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The Canadian Mathematical Society

in collaboration with

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

presents the

Canadian Open Mathematics Challenge

Wednesday, November 23, 2005

Solutions

Part A

1. Determine the value of $10^2 - 9^2 + 8^2 - 7^2 + 6^2 - 5^2 + 4^2 - 3^2 + 2^2 - 1^2$.

Solution 1

Using differences of squares,

$$\begin{aligned} & 10^2 - 9^2 + 8^2 - 7^2 + 6^2 - 5^2 + 4^2 - 3^2 + 2^2 - 1^2 \\ = & (10-9)(10+9) + (8-7)(8+7) + (6-5)(6+5) + (4-3)(4+3) + (2-1)(2+1) \\ = & 1(10+9) + 1(8+7) + 1(6+5) + 1(4+3) + 1(2+1) \\ = & 10+9+8+7+6+5+4+3+2+1 \\ = & 55 \end{aligned}$$

(We can get the answer 55 either by computing the sum directly, or by using the fact that $1+2+3+4+5+6+7+8+9+10 = \frac{1}{2}(10)(11) = 55$.)

Solution 2

Computing directly,

$$\begin{aligned} & 10^2 - 9^2 + 8^2 - 7^2 + 6^2 - 5^2 + 4^2 - 3^2 + 2^2 - 1^2 \\ = & 100 - 81 + 64 - 49 + 36 - 25 + 16 - 9 + 4 - 1 \\ = & 19 + 15 + 11 + 7 + 3 \qquad \text{(computing difference of each pair)} \\ = & 55 \end{aligned}$$

Answer: 55

2. A bug in the xy-plane starts at the point $(1, 9)$. It moves first to the point $(2, 10)$ and then to the point $(3, 11)$, and so on. It continues to move in this way until it reaches a point whose y-coordinate is twice its x-coordinate. What are the coordinates of this point?

Solution 1

The bug starts at $(1, 9)$ and each time moves 1 unit to the right and 1 unit up.
Thus, after k moves, the bug will be at the point $(1 + k, 9 + k)$.
When its y-coordinate is twice its x-coordinate, we have $9 + k = 2(1 + k)$ or $9 + k = 2 + 2k$ or $k = 7$.
When $k = 7$, the bug is at point $(1 + 7, 9 + 7) = (8, 16)$, and the bug stops here.

Solution 2

The bug starts at $(1, 9)$ and each time moves 1 unit to the right and 1 unit up.
Thus, at any point to which the bug moves, the y-coordinate will be 8 more than

the x-coordinate, so every such point is of the form $(n, n+8)$.
For the y-coordinate to be twice the x-coordinate, $n+8=2n$ or $n=8$.
When $n=8$, the bug is at the point $(8,16)$, and the bug stops here.

Solution 3
We write out the sequence of points to which the bug moves and stop when we get to a point where the y-coordinate is twice the x-coordinate:

$$(1,9),(2,10),(3,11),(4,12),(5,13),(6,14),(7,15),(8,16)$$

Thus, the bug stops at $(8,16)$.

ANSWER: $(8,16)$

3. If $ax^3+bx^2+cx+d=(x^2+x-2)(x-4)-(x+2)(x^2-5x+4)$ for all values of x, what is the value of $a+b+c+d$?

Solution 1
We use the fact that $a+b+c+d=a(1^3)+b(1^2)+c(1)+d$, so $a+b+c+d$ must be equal to the right side of the given equation with x set equal to 1.
Thus,

$$a+b+c+d=(1^2+1-2)(1-4)-(1+2)(1^2-5+4)=0(-3)-3(0)=0$$

Solution 2
We simplify the right side of the given equation by factoring the two quadratic polynomials:

$$\begin{aligned}(x^2+x-2)(x-4)-(x+2)(x^2-5x+4) &= (x-1)(x+2)(x-4)-(x+2)(x-1)(x-4)\\ &= 0\end{aligned}$$

Therefore, $ax^3+bx^2+cx+d=0$ for all values of x. (In other words, $ax^3+bx^2+cx+d=0$ is the zero polynomial, so all of its coefficients are equal to 0.)
Therefore, $a=b=c=d=0$, so $a+b+c+d=0$.

Solution 3
We expand and simplify the right side:

$$\begin{aligned}ax^3+bx^2+cx+d &= (x^2+x-2)(x-4)-(x+2)(x^2-5x+4)\\ &= x^3+x^2-2x-4x^2-4x+8-(x^3+2x^2-5x^2-10x+4x+8)\\ &= x^3-3x^2-6x+8-(x^3-3x^2-6x+8)\\ &= 0\end{aligned}$$

Therefore, each of the coefficients of ax^3+bx^2+cx+d are 0, so $a=b=c=d=0$ and thus $a+b+c+d=0$.

Answer: 0

4. A fraction $\frac{p}{q}$ is in lowest terms if p and q have no common factor larger than 1.
How many of the 71 fractions $\frac{1}{72}, \frac{2}{72}, \ldots, \frac{70}{72}, \frac{71}{72}$ are in lowest terms?

Solution 1
First, we note that $72 = 2^3 \times 3^2$.
For one of the fractions $\frac{a}{72}$ to be in lowest terms, then a and 72 have no common factors.
In other words, a cannot be divisible by 2 or 3 (since 2 and 3 are the only prime numbers which are divisors of 72).
How many of the positive integers from 1 to 71 are not divisible by 2 or 3?
Of these integers, 35 of the them *are* divisible by 2 (namely, 2, 4, 6, ..., 70).
Also, 23 of them (namely, 3, 6, ..., 69) are divisible by 3.
But some numbers are counted twice in these lists: all of the multiples of both 2 and 3 (ie. the multiples of 6). These are 6, 12, ..., 66 – that is, 11 numbers in total.
So the number of positive integers from 1 to 71 which are divisible by 2 or 3 is $35+23-11=47$ (11 is subtracted to remove the double-counted numbers).
So the number of positive integers from 1 to 71 which are not divisible by 2 or 3 is $71-47=24$.
Therefore, 24 of the 71 fractions $\frac{1}{72}, \frac{2}{72}, \cdots, \frac{70}{72}, \frac{71}{72}$ are irreducible.

Solution 2
First, we note that $72 = 2^3 \times 3^2$.
For one of the fractions $\frac{a}{72}$ to be in lowest terms, then a and 72 have no common factors.
Since the only primes which are divisors of 72 are 2 and 3, then a and 72 have no common factors when a is not divisible by 2 or 3.
Look at the first few fractions in the list:

$$\frac{1}{72}, \frac{2}{72}, \frac{3}{72}, \frac{4}{72}, \frac{5}{72}, \frac{6}{72}, \frac{7}{72}, \frac{8}{72}, \frac{9}{72}, \frac{10}{72}, \frac{11}{72}, \frac{12}{72}$$

From this list, the ones which are in lowest terms are

$$\frac{1}{72}, \frac{5}{72}, \frac{7}{72}, \frac{11}{72}$$

So the 1st and 5th of each of the two sets of 6 fractions above are in lowest terms.
This pattern will continue, so if we include the fraction $\frac{72}{72}$ (which we know is not in lowest terms) at the end of the list, we obtain 12 sets of 6 fractions, and 2 fractions out of each set will be in lowest terms, giving $12 \times 2 = 24$ fractions in lowest terms.

(Why does this pattern continue? Each of the fractions in the list can be written in one of the following forms:

$$\frac{6k+1}{72}, \frac{6k+2}{72}, \frac{6k+3}{72}, \frac{6k+4}{72}, \frac{6k+5}{72}, \frac{6k+6}{72}$$

Since the numerators $6k+2$, $6k+4$ and $6k+6$ are divisible by 2 and the numerator $6k+3$ is divisible by 3, then none of the fractions with these as numerators is in lowest terms.
Also, $6k+1$ and $6k+5$ are never divisible by 2 or 3, so these corresponding fractions are always in lowest terms.
Thus, 2 out of each set of 6 fractions is in lowest terms.)

ANSWER: 24

5. An office building has 50 storeys, 25 of which are painted black and the other 25 of which are painted gold. If the number of gold storeys in the top half of the building is added to the number of black storeys in the bottom half of the building, the sum is 28. How many gold storeys are there in the top half of the building?

Solution 1
Let G be the number of gold storeys in the top half of the building.
Then there are $25 - G$ black storeys in the top half of the building.
Since there are 25 black storeys in total, then the number of black storeys in the bottom half of the building is $25 - (25 - G) = G$.
Since the sum of the number of gold storeys in the top half of the building and the number of black storeys in the bottom half of the building is 28, then $G + G = 28$, or $G = 14$.
Thus, there are 14 gold storeys in the top half of the building.

Solution 2
Let G and g be the number of gold storeys in the top and bottom halfs of the building, and B and b the number of black storeys in the top and bottom halfs of the building.
Then $G + B = 25$ and $g + b = 25$, looking at the top and bottom halfs of the building.
Also, $G + g = 25$ and $B + b = 25$, since 25 of the storeys are painted in each colour.
Also, $G + b = 28$ from the given information, or $b = 28 - G$.
Since $B + b = 25$, then $B + 28 - G = 25$, so $B = G - 3$.
Since $G + B = 25$, then $G + G - 3 = 25$ or $2G = 28$ or $G = 14$.
Thus, there are 14 gold storeys in the top half of the building.

ANSWER: 14

6. In the grid shown, each row has a value assigned to it and each column has a value assigned to it. The number in each cell is the sum of its row and column values. For example, the "8" is the sum of the value assigned to the 3rd row and the value assigned to the 4th column. Determine the values of x and y.

3	0	5	6	-2
-2	-5	0	1	y
5	2	x	8	0
0	-3	2	3	-5
-4	-7	-2	-1	-9

Solution 1

First, we label the values assigned to the five columns A, B, C, D, E and the values assigned to the five rows a, b, c, d, e.

Look at the sub-grid

0	1
x	8

.

Since the 0 is in row 2 and column 3, then $0 = b + C$.

Similarly, $1 = b + D$, $8 = c + D$ and $x = c + C$.

But then $0 + 8 = (b + C) + (c + D) = (c + C) + (b + D) = x + 1$ or $x = 7$.

In a similar way, we can show by looking at the sub-grid

1	y
8	0

that we must have $1 + 0 = y + 8$ or $y = -7$.

Thus, $x = 7$ and $y = -7$.

(In fact, in any sub-grid of the form

p	q
r	s

, we must have $p + s = q + r$.)

Solution 2

First, we label the values assigned to the five columns A, B, C, D, E and the values assigned to the five rows a, b, c, d, e.

Suppose that we try $A = 0$.

Looking at the "3" in the first row and first column, $A + a = 3$, so $a = 3$.

Since $a = 3$ and the entry in the first row and second column is 0, then $a + B = 0$, or $B = -3$.

Similarly, $C = 2$, $D = 3$ and $E = -5$.

Since $A = 0$ and the entry in the second row and first column is -2, then $b + A = 0$, then $b = -2$.

Since $y = b + E$, then $y = -2 + (-5) = -7$.

Since $A = 0$ and the entry in the third row and first column is 5, then $c + A = 5$, so $c = 5$.

Since $x = c + C$, then $x = 5 + 2 = 7$.

Thus, $x = 7$ and $y = -7$.

Solution 3

First, we label the values assigned to the five columns A, B, C, D, E and the values assigned to the five rows a, b, c, d, e.

If we choose five entries from the table which include one from each row and one from each column, then the sum of these entries is constant no matter how we choose the entries, as it is always equal to $A+B+C+D+E+a+b+c+d+e$.

Here are three ways in which this can be done (looking at the bolded numbers):

3	0	5	6	-2
-2	$\mathbf{-5}$	0	1	y
5	2	x	**8**	0
0	-3	**2**	3	-5
-4	-7	-2	-1	$\mathbf{-9}$

3	0	5	6	$\mathbf{-2}$
-2	-5	0	**1**	y
5	2	**x**	8	0
0	$\mathbf{-3}$	2	3	-5
$\mathbf{-4}$	-7	-2	-1	-9

3	0	5	6	-2
-2	-5	0	1	**y**
5	**2**	x	8	0
0	-3	2	**3**	-5
-4	-7	$\mathbf{-2}$	-1	-9

Therefore, $3+(-5)+2+8+(-9)=(-4)+(-3)+x+1+(-2)=3+y+2+(-2)+3$ or $-1=x-8=y+6$.

Thus, $x=7$ and $y=-7$.

Solution 4

First, we label the values assigned to the five columns A, B, C, D, E and the values assigned to the five rows a, b, c, d, e.

Consider the first two entries in row 1.

We have $3=A+a$ and $0=B+a$.

Subtracting these, we obtain $3=3-0=(A+a)-(B+a)=A-B$.

Notice that whenever we take entries in columns 1 and 2 from the same row, their difference will always equal $A-B$, which is equal to 3.

Similarly, since the difference between the 0 and the 5 in the first row is 5, then every entry in column 3 will be 5 greater than the entry in column 2 from the same row.

Thus, $x=2+5=7$.

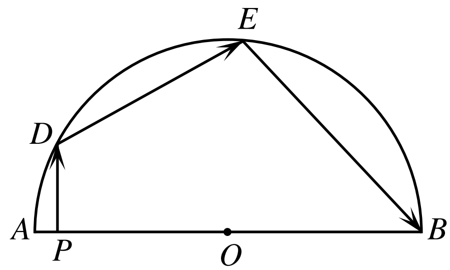
Also, since the difference between the 6 and the -2 in the first row is 8, then every entry in column 5 will be 8 less than the entry in column 4 from the same row.

Thus, $y=1-8=-7$.

Therefore, $x=7$ and $y=-7$.

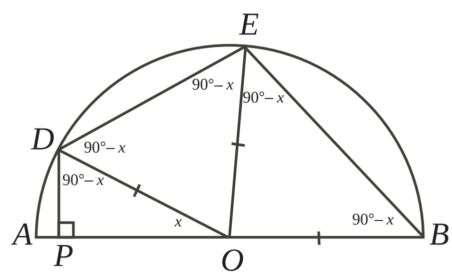
Answer: $x=7$ and $y=-7$

7. In the diagram, the semi-circle has centre O and diameter AB. A ray of light leaves point P in a direction perpendicular to AB. It bounces off the semi-circle at point D in such a way that $\angle PDO = \angle EDO$. (In other words, the angle of incidence equals the angle of reflection at D.) The ray DE then bounces off the circle in a similar way at E before finally hitting the semicircle again at B. Determine $\angle DOP$.



Solution 1

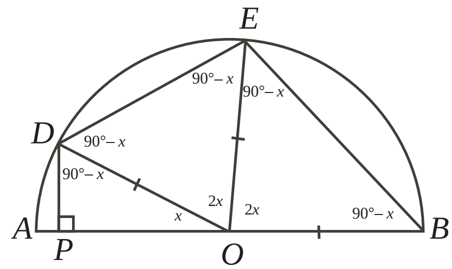
Join D and E to O, and let $\angle DOP = x$.
Since $DP \perp AB$, then $\angle PDO = 90^\circ - x$.
Since the angle of incidence equals the angle of reflection at D, then $\angle EDO = \angle PDO = 90^\circ - x$.



Since DO and EO are both radii, then $DO = EO$, so $\triangle EDO$ is isosceles, and so $\angle DEO = \angle EDO = 90^\circ - x$.
Since the angle of incidence equals the angle of reflection at E, then $\angle DEO = \angle BEO = 90^\circ - x$.
Since EO and BO are both radii, then $EO = BO$, so $\triangle BEO$ is isosceles, and so $\angle EBO = \angle BEO = 90^\circ - x$.
Consider quadrilateral $PDEB$.
We have $\angle DPB = 90^\circ$, $\angle PDE = (90^\circ - x) + (90^\circ - x) = 180^\circ - 2x$, $\angle DEB = (90^\circ - x) + (90^\circ - x) = 180^\circ - 2x$, and $\angle EBP = 90^\circ - x$.
Since the sum of the angles in the quadrilateral is 360°, then
$90^\circ + 180^\circ - 2x + 180^\circ - 2x + 90^\circ - x = 360^\circ$ or $540^\circ - 5x = 360^\circ$ or $5x = 180^\circ$ or $x = 36^\circ$.
Therefore, $\angle DOP = x = 36^\circ$.

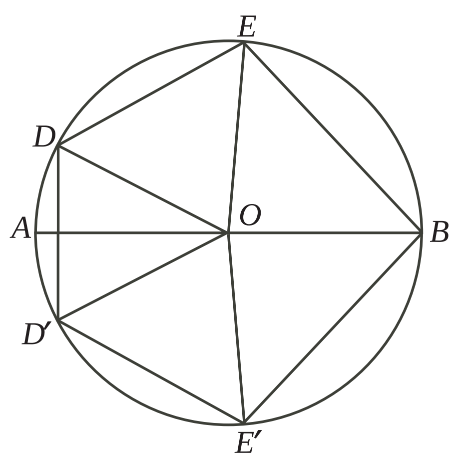
Solution 2

Join D and E to O, and let $\angle DOP = x$.
Since $DP \perp AB$, then $\angle PDO = 90^\circ - x$.
Since the angle of incidence equals the angle of reflection at D, then $\angle EDO = \angle PDO = 90^\circ - x$.



Since DO and EO are both radii, then $DO = EO$, so $\triangle EDO$ is isosceles, and so $\angle DEO = \angle EDO = 90^\circ - x$. Also, $\angle DOE = 180^\circ - 2(90^\circ - x) = 2x$.
Since the angle of incidence equals the angle of reflection at E, then $\angle DEO = \angle BEO = 90^\circ - x$.
Since EO and BO are both radii, then $EO = BO$, so $\triangle BEO$ is isosceles, and so $\angle EBO = \angle BEO = 90^\circ - x$. Also, $\angle EOB = 180^\circ - 2(90^\circ - x) = 2x$.
Since POB is a straight line, then $\angle POD + \angle DOE + \angle EOB = 180^\circ$ or $x + 2x + 2x = 180^\circ$ or $5x = 180^\circ$ or $x = 36^\circ$.
Therefore, $\angle DOP = x = 36^\circ$.

Solution 3
Reflect the diagram across AB to complete the circle and form the pentagon $DEBE'D'$. (Note that DPD' is a straight line since $\angle DPO = \angle D'PO = 90^\circ$.)



Since DO, EO, BO, $E'O$ and $D'O$ are all radii, then $DO = EO = BO = E'O = D'O$.
Let $\angle DOP = x$. Since $DP \perp AB$, then $\angle PDO = 90^\circ - x$.
By reflection, $\angle D'OP = \angle DOP = x$, so $\angle DOD' = 2x$.
Since the angle of incidence equals the angle of reflection at D, then $\angle EDO = \angle PDO = 90^\circ - x$.
Since DO and EO are both radii, then $DO = EO$, so $\triangle EDO$ is isosceles, and so $\angle DEO = \angle EDO = 90^\circ - x$. Also, $\angle DOE = 180^\circ - 2(90^\circ - x) = 2x$.
Since the angle of incidence equals the angle of reflection at E, then $\angle DEO = \angle BEO = 90^\circ - x$.
Since EO and BO are both radii, then $EO = BO$, so $\triangle BEO$ is isosceles, and so $\angle EBO = \angle BEO = 90^\circ - x$. Also, $\angle EOB = 180^\circ - 2(90^\circ - x) = 2x$.

Therefore, the triangles DOE, EOB, BOE', $E'OD'$ and $D'OD$ are all congruent by side-angle-side. Therefore, pentagon $DEBE'D'$ is a regular pentagon.
Thus, $\angle DOD' = \frac{1}{5}(360°) = 72°$ since the central angles of each of the five sides of the pentagon are equal.
Since $\triangle DOD'$ is isosceles and OP is perpendicular to DD', then $\angle POD = \frac{1}{2}\angle DOD' = 36°$.
Thus, $\angle POD = 36°$.

Answer: $\angle DOP = 36°$

8. The number 18 *is not* the sum of any 2 consecutive positive integers, but *is* the sum of consecutive positive integers in at least 2 different ways, since $5+6+7=18$ and $3+4+5+6=18$. Determine a positive integer less than 400 that *is not* the sum of any 11 consecutive positive integers, but *is* the sum of consecutive positive integers in at least 11 different ways.

Solution
Suppose that the positive integer N is the sum of an odd number of consecutive integers, say $2k+1$ consecutive integers. Then for some integer a,

$$N = (a-k) + (a-(k-1)) + \cdots + (a-1) + a + (a+1) + \cdots + (a+k) = (2k+1)a$$

Thus, $2k+1$ is a divisor of N (ie. the number of integers in the representation is a divisor of N).
Next, suppose that N is the sum of an even number of consecutive integers, say $2k$ consecutive integers. Then for some integer b,

$$N = (b-k)+(b-(k-1))+\cdots+(b-1)+b+(b+1)+\cdots(b+(k-1)) = 2kb-k = k(2b-1) = \frac{1}{2}(2k)(2b-1)$$

Thus, k is a divisor of N and $2k$ is not a divisor of N (since $2b-1$ is odd and so has no factor of 2).

We would like to find a positive integer N which is not the sum of 11 consecutive positive integers (and so is not a multiple of 11) but is the sum of consecutive positive integers in 11 different ways.

Let's consider the number of integers in each of the ways in which we write N as the sum of consecutive integers. Note that if N is the sum of m consecutive positive integers, then N is at least $1+2+\cdots+m$. We make a table of what properties N must have for N to be the sum of m consecutive integers for $m=2$ to $m=10$:

m	N at least	Property of N
2	3	Divisible by 1, not by 2
3	6	Divisible by 3
4	10	Divisible by 2, not by 4
5	15	Divisible by 5
6	21	Divisible by 3, not by 6 (ie. divisible by 3, not by 2)
7	28	Divisible by 7
8	36	Divisible by 4, not by 8
9	45	Divisible by 9
10	55	Divisible by 5, not by 10 (ie. divisible by 5, not by 2)

How can we combine as many of these as possible? If we make N at least 55 and divisible by 5, 7 and 9 and not divisible by 2, then N will be the sum of 2, 3, 5, 6, 7, 9 and 10 consecutive positive integers (7 representations in total). In this case, N must be divisible by $5 \times 7 \times 9 = 315$. So following this line of thought, if N is less than 400, then we must have $N = 315$. Now, 315 is also

- the sum of 15 consecutive positive integers since 315 is divisible by 15 and is at least 120 (ie. $1 + 2 + 3 + \cdots + 15$),
- the sum of 14 consecutive positive integers since 315 is divisible by 7, not by 14, and is at least 105 (ie. $1 + 2 + 3 + \cdots + 14$)
- the sum of 18 consecutive positive integers since 315 is divisible by 9, not by 18, and is at least 171 (ie. $1 + 2 + 3 + \cdots + 18$)
- the sum of 21 consecutive positive integers since 315 is divisible by 21 and is at least 231 (ie. $1 + 2 + 3 + \cdots + 21$)

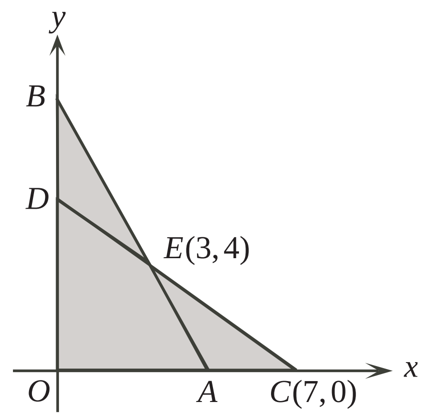
So 315 is the sum of consecutive positive integers in at least 11 ways, and is not the sum of 11 consecutive positive integers. (In fact, 315 is the unique answer, but we are not asked to justify this.)

(Note: A good way to write a solution to this problem would be to first figure out in rough that 315 was the answer, and then begin the solution by claiming that 315 is the answer. We could then demonstrate that 315 works by showing that it can be represented in the correct number of ways. While this approach is perfectly correct, it would not give much of a clue as to how the answer was obtained.)

Answer: 315

Part B

1. A line with slope -3 intersects the positive x-axis at A and the positive y-axis at B. A second line intersects the x-axis at $C(7, 0)$ and the y-axis at D. The lines intersect at $E(3, 4)$.



(a) Find the slope of the line through C and E.

Solution

Since C has coordinates $(7, 0)$ and E has coordinates $(3, 4)$, then the slope of the line through C and E is

$$\frac{0-4}{7-3} = \frac{-4}{4} = -1$$

(b) Find the equation of the line through C and E, and the coordinates of the point D.

Solution 1

Since the line through C and E has slope -1 and passes through the point $(7, 0)$, then the line has equation $y - 0 = (-1)(x - 7)$ or $y = -x + 7$.
From the equation of the line, $y = 7$ is the y-intercept of the line.
Since D is the point where this line crosses the y-axis, then D has coordinates $(0, 7)$.

Solution 2

Since the line through C and E has slope -1 and passes through the point $(3, 4)$, then the line has equation $y - 4 = (-1)(x - 3)$ or $y = -x + 7$.
From the equation of the line, $y = 7$ is the y-intercept of the line.
Since D is the point where this line crosses the y-axis, then D has coordinates $(0, 7)$.

(c) Find the equation of the line through A and B, and the coordinates of the point B.

Solution

Since the line through A and B has slope -3 and passes through the point $E(3, 4)$, then

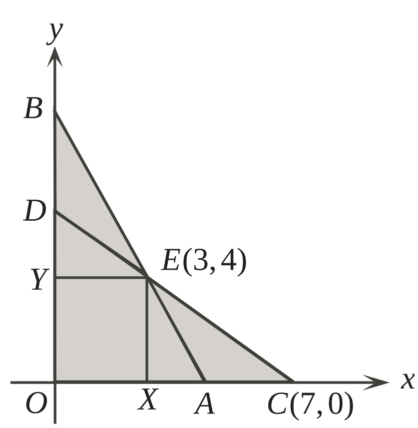
the line has equation $y - 4 = (-3)(x - 3)$ or $y = -3x + 13$.
From the equation of the line, $y = 13$ is the y-intercept of the line.
Since B is the point where this line crosses the y-axis, then B has coordinates $(0, 13)$.

(d) Determine the area of the shaded region.

Solution 1
The area of the shaded region is the sum of the areas of $\triangle DOC$ and $\triangle BDE$.
$\triangle DOC$ is right-angled at O, so the area of $\triangle DOC$ is $\frac{1}{2}(DO)(OC) = \frac{1}{2}(7)(7) = \frac{49}{2}$.
We can consider $\triangle BDE$ as having base BD of length $13 - 7 = 6$ and height equal to the distance of E from the y-axis (a distance of 3).
Therefore, the area of $\triangle BDE$ is $\frac{1}{2}(6)(3) = 9$.
Thus, the area of the shaded region is $\frac{49}{2} + 9 = \frac{67}{2}$.

Solution 2
The area of the shaded region is the sum of the areas of $\triangle BOA$ and $\triangle AEC$.
$\triangle BOA$ is right-angled at O, so the area of $\triangle BOA$ is $\frac{1}{2}(BO)(OA)$.
Point A is the point where the line $y = -3x + 13$ crosses the x-axis, so it has x-coordinate which satisfies $-3x + 13 = 0$, ie. $x = \frac{13}{3}$.
Therefore, the area of $\triangle BOA$ is $\frac{1}{2}(13)\left(\frac{13}{3}\right) = \frac{169}{6}$.
We can consider $\triangle AEC$ as having base AC of length $7 - \frac{13}{3} = \frac{8}{3}$ and height equal to the distance of E from the x-axis (a distance of 4).
Therefore, the area of $\triangle AEC$ is $\frac{1}{2}(4)\left(\frac{8}{3}\right) = \frac{16}{3}$.
Thus, the area of the shaded region is $\frac{169}{6} + \frac{16}{3} = \frac{201}{6} = \frac{67}{2}$.

Solution 3
Drop perpendiculars from E to point X on the x-axis and to point Y on the y-axis.



Then Y has coordinates $(0, 4)$, X has coordinates $(3, 0)$, and $OXEY$ is a rectangle.
The area of the shaded region is thus the sum of the areas of $\triangle BYE$, rectangle $OXEY$ and $\triangle EXC$.

Since $\triangle BYE$ is right-angled at Y, its area is $\frac{1}{2}(BY)(YE) = \frac{1}{2}(13-4)(3-0) = \frac{27}{2}$.
The area of rectangle $OYEX$ is $3 \times 4 = 12$.
Since $\triangle EXC$ is right-angled at X, its area is $\frac{1}{2}(EX)(XC) = \frac{1}{2}(4-0)(7-3) = 8$.
Therefore, the area of the shaded region is $\frac{27}{2} + 12 + 8 = \frac{67}{2}$.

2. (a) Determine all possible ordered pairs (a, b) such that

$$\begin{aligned} a - b &= 1 \\ 2a^2 + ab - 3b^2 &= 22 \end{aligned}$$

Solution 1
Factoring the left side of the second equation, we get $2a^2 + ab - 3b^2 = (a-b)(2a+3b)$.
Since $a - b = 1$, we get $(1)(2a + 3b) = 22$ or $2a + 3b = 22$.
So we now have $a - b = 1$ and $2a + 3b = 22$.
Adding 3 times the first equation to the second equation, we get $5a = 25$ or $a = 5$.
Substituting back into the first equation, we get $b = 4$.
Thus, the only solution is $(a, b) = (5, 4)$.

Solution 2
From the first equation, $a = b + 1$.
Substituting into the second equation, we obtain

$$\begin{aligned} 2(b+1)^2 + (b+1)(b) - 3b^2 &= 22 \\ (2b^2 + 4b + 2) + (b^2 + b) - 3b^2 &= 22 \\ 5b &= 20 \\ b &= 4 \end{aligned}$$

Substituting back into the first equation, we get $a = 5$, so the only solution is $(a, b) = (5, 4)$.

Solution 3
From the first equation, $b = a - 1$.
Substituting into the second equation, we obtain

$$\begin{aligned} 2a^2 + a(a-1) - 3(a-1)^2 &= 22 \\ 2a^2 + (a^2 - a) - (3a^2 - 6a + 3) &= 22 \\ 5a &= 25 \\ a &= 5 \end{aligned}$$

Substituting back into the first equation, we get $b = 4$, so the only solution is $(a, b) = (5, 4)$.

(b) Determine all possible ordered triples (x, y, z) such that

$$\begin{aligned} x^2 - yz + xy + zx &= 82 \\ y^2 - zx + xy + yz &= -18 \\ z^2 - xy + zx + yz &= 18 \end{aligned}$$

Solution 1

If we add the second equation to the third equation, we obtain

$$\begin{aligned} y^2 - zx + xy + yz + z^2 - xy + zx + yz &= -18 + 18 \\ y^2 + 2yz + z^2 &= 0 \\ (y+z)^2 &= 0 \\ y + z &= 0 \\ z &= -y \end{aligned}$$

Substituting back into the three equations, we obtain

$$\begin{aligned} x^2 + y^2 &= 82 \\ 2xy &= -18 \\ -2xy &= 18 \end{aligned}$$

Thus, $x^2 + y^2 = 82$ and $xy = -9$.
Therefore, $(x+y)^2 = x^2 + 2xy + y^2 = 82 + (-18) = 64$, so $x + y = \pm 8$.

If $x + y = 8$, then $y = 8 - x$ and so since $xy = -9$, then $x(8 - x) = -9$ or $x^2 - 8x - 9 = 0$ or $(x - 9)(x + 1) = 0$ so $x = 9$ or $x = -1$.
Since $x + y = 8$, then if $x = 9$, we have $y = -1$ and $z = -y = 1$.
Since $x + y = 8$, then if $x = -1$, we have $y = 9$ and $z = -y = -9$.

If $x+y = -8$, then $y = -8-x$ and so since $xy = -9$, then $x(-8-x) = -9$ or $x^2+8x-9 = 0$ or $(x + 9)(x - 1) = 0$ so $x = -9$ or $x = 1$.
Since $x + y = -8$, then if $x = -9$, we have $y = 1$ and $z = -y = -1$.
Since $x + y = -8$, then if $x = 1$, we have $y = -9$ and $z = -y = 9$.

Therefore, the four solutions are $(x, y, z) = (9, -1, 1), (-1, 9, -9), (-9, 1, -1), (1, -9, 9)$.

Solution 2

If we add the first equation to the second equation, we obtain

$$\begin{aligned} x^2 - yz + xy + zx + y^2 - zx + xy + yz &= 82 - 18 \\ x^2 + 2xy + y^2 &= 64 \\ (x+y)^2 &= 64 \\ x + y &= \pm 8 \end{aligned}$$

Similarly, adding the first equation to the third equation, we obtain $x^2 + 2xz + z^2 = 100$ or $x + z = \pm 10$.

Also, adding the second equation to the third equation, we obtain $y^2 + 2yz + z^2 = 0$ or $y + z = 0$, and so $z = -y$.

Using $x + z = \pm 10$ and $z = -y$, we obtain $x - y = \pm 10$.

Thus, we have $x + y = \pm 8$ and $x - y = \pm 10$.

If $x + y = 8$ and $x - y = 10$, then adding these equations, we get $2x = 18$ or $x = 9$ and so $y = -1$ and $z = -y = 1$.

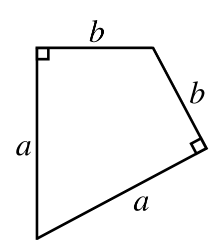
If $x + y = 8$ and $x - y = -10$, then adding these equations, we get $2x = -2$ or $x = -1$ and so $y = 9$ and $z = -y = -9$.

If $x + y = -8$ and $x - y = 10$, then adding these equations, we get $2x = 2$ or $x = 1$ and so $y = -9$ and $z = -y = 9$.

If $x + y = -8$ and $x - y = -10$, then adding these equations, we get $2x = -18$ or $x = -9$ and so $y = 1$ and $z = -y = -1$.

Therefore, the four solutions are $(x, y, z) = (9, -1, 1), (-1, 9, -9), (-9, 1, -1), (1, -9, 9)$.

3. Four tiles identical to the one shown, with $a > b > 0$, are arranged without overlap to form a square with a square hole in the middle.



(a) If the outer square has area $(a + b)^2$, show that the area of the inner square is $(a - b)^2$.

Solution 1

Each tile can be split into two right-angled triangles along a diagonal, each with legs of lengths a and b.

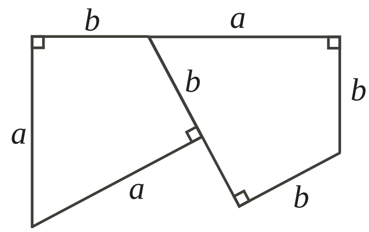
The area of each of these triangles is $\frac{1}{2}ab$, so the area of each tile is ab.

If the outer square has area $(a+b)^2$ and this is partially covered with four tiles each of area ab, then the area of the leftover portion (ie. the square hole) is

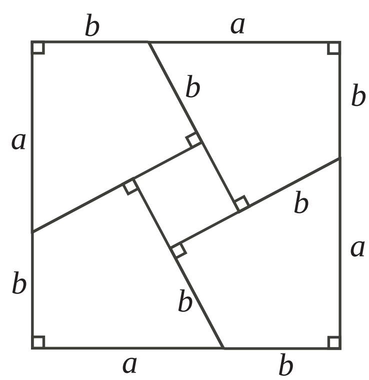
$$(a+b)^2 - 4ab = a^2 + 2ab + b^2 - 4ab = a^2 - 2ab + b^2 = (a-b)^2$$

Solution 2

If the outer square has area $(a+b)^2$, then the side length of the outer square is $a+b$.
In order to get a side length of $a+b$, we need to line up the "a" side of a tile with the "b" side of a second tile, as shown.



(Note that the tiles do fit together in this way, since each is a quadrilateral with two right angles, so the remaining two angles add to 180°, that is, a straight line.)
We can complete the square as follows:



Now the inner hole is clearly a rectangle (as it has four right angles) and is in fact a square as its four sides are all of length $a-b$ (as each of its sides are the remaining portion of a line segment of length a when a segment of length b is cut off from one end).
Since the inner square has side length $a-b$, then its area is $(a-b)^2$.

(b) Determine the smallest integer value of N for which there are prime numbers a and b such that the ratio of the area of the inner square to the area of the outer square is $1 : N$.

Solution

From (a), the ratio of the area of the inner square to the area of the outer square is $\dfrac{(a-b)^2}{(a+b)^2}$.
We would like to find integers N for which there are prime numbers a and b such that $\dfrac{(a-b)^2}{(a+b)^2} = \dfrac{1}{N}$ (and in fact find the minimum such N).
Taking the positive square root of both sides, we obtain $\dfrac{a-b}{a+b} = \dfrac{1}{\sqrt{N}}$.

Since the left side is a rational number (since a and b are integers), then $\sqrt{N}$ must be rational, so N must be a perfect square.
Suppose $N = k^2$, for some positive integer k.
Thus, we have $\frac{a-b}{a+b} = \frac{1}{k}$ or $a + b = k(a - b)$ or $(k - 1)a = (k + 1)b$.

Since we would like to find the smallest value of N which works, then we try to find the smallest value of k which works.
Does $k = 1$ work? Are there prime numbers a and b so that $0 = 2b$? No, since this means $b = 0$.
Does $k = 2$ work? Are there prime numbers a and b so that $a = 3b$? No, since here a is a multiple of 3, so the only possible prime value of a is 3, which would make $b = 1$, which is not a prime.
Does $k = 3$ work? Are there prime numbers a and b so that $2a = 4b$ (ie. $a = 2b$)? No, since here a is a multiple of 2, so the only possible prime value of a is 2, which would make $b = 1$, which is not a prime.
Does $k = 4$ work? Are there prime numbers a and b so that $3a = 5b$?
Yes: $a = 5$ and $b = 3$.
Therefore, the smallest value of k which works is $k = 4$, so the smallest value of N which works is $N = 16$.

(c) Determine, with justification, all positive integers N for which there are odd integers $a > b > 0$ such that the ratio of the area of the inner square to the area of the outer square is $1 : N$.

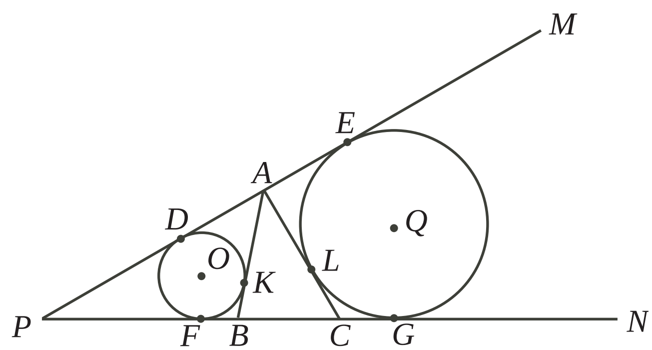
Solution

Suppose that N is a positive integer for which there are odd integers $a > b > 0$ such that $\frac{(a-b)^2}{(a+b)^2} = \frac{1}{N}$.
Then, as in (b), N must be a perfect square, say $N = k^2$, for some positive integer k.
Since a and b are odd, then set $a = 2A + 1$ and $b = 2B + 1$, for some integers A and B.
Thus we have $\frac{(2A-2B)^2}{(2A+2B+2)^2} = \frac{1}{k^2}$ or $\frac{A-B}{A+B+1} = \frac{1}{k}$ or $k(A - B) = A + B + 1$.
If A and B have the same parity (ie. both even or both odd), then $A - B$ is even so the left side is even and $A + B + 1$ is odd, so the right side is odd. Since we cannot have an odd number equal to an even number, then this cannot happen.
Thus, A and B must have opposite parity (ie. one even and the other odd). In this case, $A - B$ is odd and $A + B + 1$ is even. Since $k(A - B) = A + B + 1$, then k is even.
Therefore, N must be an even perfect square.

We must now check if every even perfect square is a possible value for N.

Suppose $N = (2m)^2$.
Using our substitutions from above, can we find integers A and B so that $2m(A - B) = A + B + 1$?
If $A = m$ and $B = m-1$, then $A-B = 1$ and $A+B+1 = 2m$, so $2m(A-B) = A+B+1$.
So if $a = 2A + 1 = 2m + 1$ and $b = 2B + 1 = 2m - 1$, then $\dfrac{(a-b)^2}{(a+b)^2} = \dfrac{1}{(2m)^2} = \dfrac{1}{N}$.
Therefore, the positive integers N which have the required property are all even perfect squares.

4. Triangle ABC has its base on line segment PN and vertex A on line PM. Circles with centres O and Q, having radii r_1 and r_2, respectively, are tangent to the triangle ABC externally and to each of PM and PN.



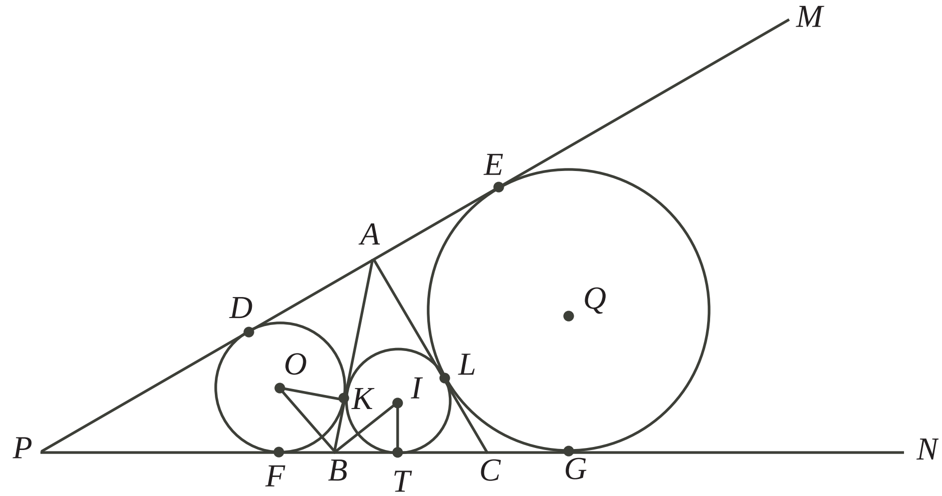
(a) Prove that the line through K and L bisects the perimeter of triangle ABC.

Solution
We must show that $KB + BC + CL = KA + AL$.
Since BK and BF are tangents to the left circle from the same point B, then $BK = BF$.
Since CL and CG are tangents to the right circle from the same point C, then $CL = CG$.
Since AK and AD are tangents to the left circle from the same point A, then $AK = AD$.
Since AL and AE are tangents to the right circle from the same point A, then $AL = AE$.
Therefore, $KB + BC + CL = FB + BC + CG = FG$ and $KA + AL = DA + AE = DE$.
Now $FG = PG - PF$ and $DE = PE - PD$.
Since PE and PG are tangents to the right circle from the same point P, then $PE = PG$.
Since PD and PF are tangents to the left circle from the same point P, then $PD = PF$.
Therefore, $FG = PG - PF = PE - PD = DE$, so $KB + BC + CL = KA + AL$, ie. the line through K and L bisects the perimeter of triangle ABC.

(b) Let T be the point of contact of BC with the circle inscribed in triangle ABC.
Prove that $(TC)(r_1) + (TB)(r_2)$ is equal to the area of triangle ABC.

Solution 1

Let I be the centre of the circle inscribed in $\triangle ABC$, T be the point of contact of this circle with BC, and r the radius of this circle.
Join O to K and B, and I to B and T.

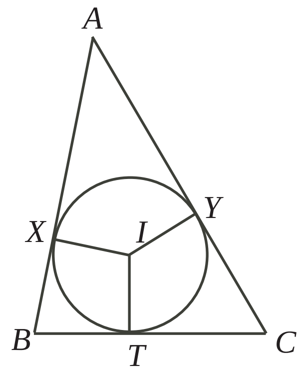


(Note that the circle with centre I is not necessarily tangent to AB at K or AC at L.)
Note that OK is perpendicular to KB and IT is perpendicular to BC.
Now OB bisects $\angle FBK$ and IB bisects $\angle KBC$, since the circles with centres O and I are tangent to FB and BK, and BA and BC, respectively.
Now $\angle KOB = 90^\circ - \angle KBO = 90^\circ - \frac{1}{2}\angle FBK = \frac{1}{2}(180^\circ - \angle FBK) = \frac{1}{2}\angle KBC = \angle IBT$, so $\triangle OKB$ is similar to $\triangle BTI$.
Therefore, $\dfrac{BK}{KO} = \dfrac{IT}{TB}$ or $\dfrac{BK}{r_1} = \dfrac{r}{TB}$ or $r_1 = \dfrac{(TB)(BK)}{r}$.
Similarly, $r_2 = \dfrac{(TC)(LC)}{r}$.
Therefore,

$$(TC)(r_1) + (TB)(r_2) = \frac{(TC)(TB)(BK)}{r} + \frac{(TB)(TC)(LC)}{r} = \frac{(TB)(TC)}{r}(BK + LC)$$

Let $BC = a$, $AB = c$, $AC = b$, and let s be the semi-perimeter of $\triangle ABC$ (that is, s is half of the perimeter of $\triangle ABC$).
Now, from (a), since $KB + BC + LC = s$, then $BK + LC = s - BC = s - a$.
Therefore, $(TC)(r_1) + (TB)(r_2) = \dfrac{(TB)(TC)}{r}(s - a)$.

We can now focus entirely on $\triangle ABC$.
Let X and Y be the points where the circle with centre I is tangent to sides AB and AC, respectively.

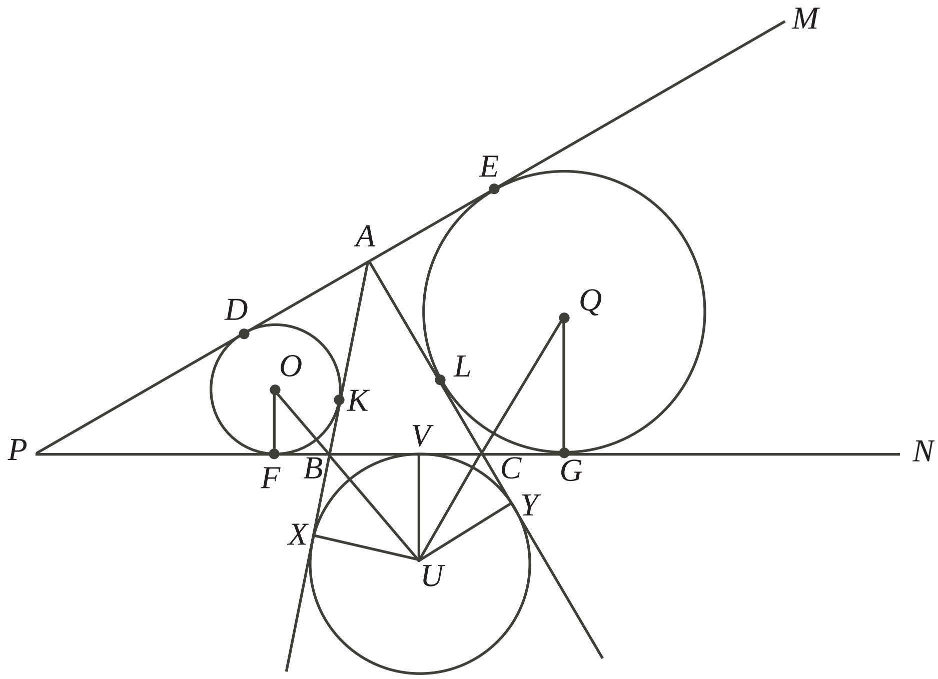


Using tangent arguments as in (a), we see that $AX = AY$, $BX = BT$ and $CY = CT$.
Since $AX + AY + BX + BT + CY + CT = 2s$, then $BT + AY + YC = s$,
so $TB = s - (AY + YC) = s - AC = s - b$.
Similarly, $TC = s - c$.
Therefore, $(TC)(r_1) + (TB)(r_2) = \dfrac{(s-b)(s-c)(s-a)}{r} = \dfrac{s(s-a)(s-b)(s-c)}{sr}$.
Let $|\triangle ABC|$ denote the area of $\triangle ABC$.
Then $s(s-a)(s-b)(s-c) = |\triangle ABC|^2$ by Heron's formula.
Also,

$$sr = \frac{1}{2}r(AB + BC + AC) = \frac{1}{2}(IX)(AB) + \frac{1}{2}(IT)(BC) + \frac{1}{2}(IY)(AC) \quad (*)$$

since IX, IT and IY are all radii of the circle with centre I.
Since IX, IT and IY are perpendicular to AB, BC and AC, respectively, then the three terms on the right side of $(*)$ are the areas of $\triangle IAB$, $\triangle IBC$ and $\triangle ICA$, respectively, and so their sum is $|\triangle ABC|$, ie. $sr = |\triangle ABC|$.
Thus, $(TC)(r_1) + (TB)(r_2) = \dfrac{|\triangle ABC|^2}{|\triangle ABC|} = |\triangle ABC|$, as required.

Solution 2
Join O to F and B and Q to C and G.
Since the circle with centre O is tangent to PB and AB at F and K, then OF is perpendicular to PB and OB bisects $\angle FBK$.
Similarly, QG is perpendicular to CN and QC bisects $\angle GCL$.
Extend AB and AC through B and C, respectively, and construct the circle which is tangent to AB extended, BC, and AC extended, and lies outside $\triangle ABC$. This circle is called an excircle of $\triangle ABC$.
The centre of this excircle, which we label U, is on the angle bisector of the angle formed by AB extended and BC, as the circle is tangent to these two lines, so U lies on OB extended. Similarly, U lies on QC extended.



Let $AB = c$, $AC = b$, $BC = a$, let s be the semi-perimeter (that is, half of the perimeter) of $\triangle ABC$, and let $|\triangle ABC|$ denote the area of $\triangle ABC$.

Then the radius of the excircle, which we will denote r_A is equal to $\dfrac{|\triangle ABC|}{s-a}$. (See the end of this solution for a proof of this fact.)

Let V be the point where the excircle is tangent to BC.

Then UV is perpendicular to BC.

Thus, $\triangle OFB$ is similar to $\triangle UVB$ and $\triangle QGC$ is similar to $\triangle UVC$ (since they have opposite angles which are equal and right angles).

This tells us that $\dfrac{OF}{FB} = \dfrac{UV}{VB}$ or $\dfrac{r_1}{FB} = \dfrac{r_A}{VB}$ and $\dfrac{QG}{GC} = \dfrac{UV}{VC}$ or $\dfrac{r_2}{GC} = \dfrac{r_A}{VC}$.

Therefore, since $FB = BK$ and $CG = CL$ and $KB + BC + CL = s$ by (a), then

$$(VB)(r_1)+(VC)(r_2) = r_A(FB+CG) = r_A(KB+LC) = r_A(s-BC) = r_A(s-a) = |\triangle ABC|$$

Now suppose that the excircle is tangent to AB extended and AC extended at X and Y, respectively.

Then $AX = AY$, and $AX = AB + BX = AB + BV$ and $AY = AC + CY = AC + CV$ (by equal tangents from B and C), so $AX + AY = AB + AC + BV + VC = AB + AC + BC = 2s$, ie. $AX = AY = s$.

Thus, $VB = BX = AX - AB = s - c$ and similarly $VC = s - b$.

But $TB = s - b = VC$ and $TC = s - c = VB$ (see Solution 1), so

$$|\triangle ABC| = (TC)(r_1) + (TB)(r_2)$$

as required.

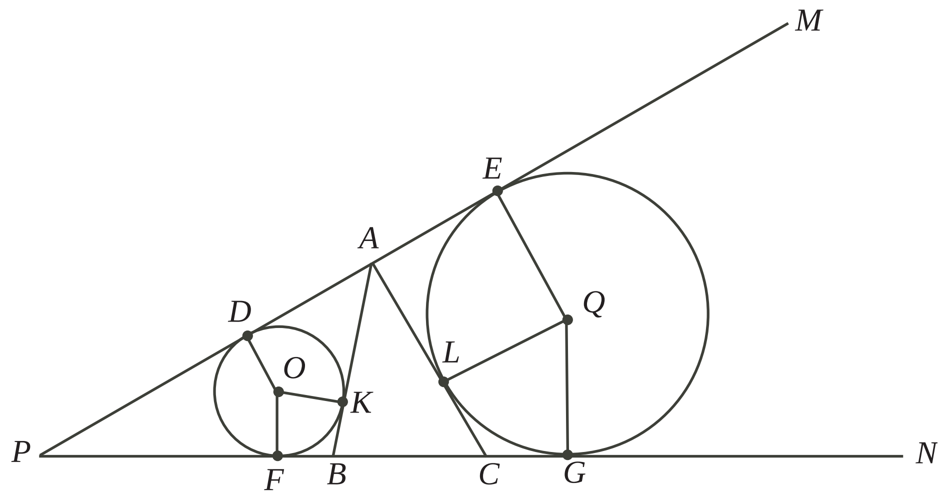
(Why is $r_A = \dfrac{|\triangle ABC|}{s-a}$?

Join X, Y and V to U. Note that $\angle AXU = \angle AYU = \angle BVU = \angle CVU = 90^\circ$.
Then $AXUY$ is a shape of the same type as in Problem 3, so its area is equal to $AX \cdot UX = sr_A$.
Similarly, the areas of $BVUX$ and $CVUY$ are $r_A(s-c)$ and $r_A(s-b)$.
Thus,

$$|\triangle ABC| = \text{Area of } AXUY - \text{Area of } BVUX - \text{Area of } CVUY = r_A(s-(s-c)-(s-b))$$

But, $s-(s-c)-(s-b) = b+c-s = a+b+c-a-s = 2s-a-s = s-a$, so $|\triangle ABC| = r_A(s-a)$, which is what we wanted to show.)

Solution 3
Let $AB = c$, $AC = b$, $BC = a$, and let s denote the semi-perimeter of $\triangle ABC$ (that is, half of its perimeter).
Then by (a), $AK + AL = KB + BC + LC = s$.
Since PM and PN are tangent to both circles, then the line through O and Q passes through P.
Join O to D, F and K, and Q to L, E and G.



In each case, the centre of a circle is being joined to a point where the circle is tangent to a line, so creates a right angle.
Therefore, $\triangle POF$ is similar to $\triangle PQG$, so $\dfrac{PF}{OF} = \dfrac{PG}{QG}$ or $r_1(PG) = r_2(PF)$.
Each of the shapes $PDOF$, $ADOK$, $BFOK$, $AEQL$, $CGQL$ and $PEQG$ has two right angles and two pairs of equal sides (ie. each is a shape as in Problem 3).
The area of each of these shapes is the product of the lengths of two of the sides which meet at a right angle.
We use $|PEQG|$ to denote the area of the shape $PEQG$, and so on.

Therefore,

$$\begin{aligned}
|PEQG| &= |\triangle ABC| + |PDOF| + |ADOK| + |BFOK| + |AEQL| + |CGQL| \\
(PG)(QG) &= |\triangle ABC| + (PF)(OF) + (AK)(OK) + (KB)(OK) + (AL)(QL) + (CL)(LQ) \\
|\triangle ABC| &= r_2(PG) - r_2(AL) - r_2(CL) - r_1(PF) - r_1(AK) - r_1(KB) \\
&= r_2(PG - AL - CL) + r_1(-PF - AK - KB) \\
&= r_2(PG - CG - AL) + r_1(-PF - AB) \qquad \text{(equal tangents)} \\
&= r_2(PC - AL) + r_1(-PF - AB) \\
&= r_2(PF + FB + BC - AL) + r_1(-PF - AB) \\
&= r_2(PF + BK + a - (s - AK)) + r_1(-PF - AB) \\
&\qquad \text{(since } AK + AL = s\text{, and } BK = FB \text{ by equal tangents)} \\
&= r_2(PF + AK + BK + a - s) + r_1(-PF - AB) \\
&= r_2(PF + AB + a - s) + r_1(-PF - AB) \\
&= r_2(PF) + r_2(c + a - s) + r_1(-PF - AB) \\
&= r_1(PG) + r_2(a + b + c - b - s) + r_1(-PF - AB) \qquad \text{(since } r_1(PG) = r_2(PF)\text{)} \\
&= r_2(2s - b - s) + r_1(PG - PF - AB) \\
&= r_2(s - b) + r_1(GF - AB) \\
&= r_2(s - b) + r_1(FB + BC + CG - c) \\
&= r_2(s - b) + r_1(KB + BC + CL - c) \qquad \text{(equal tangents)} \\
&= r_2(s - b) + r_1(s - c)
\end{aligned}$$

As in Solution 1, $TB = s - c$ and $TC = s - b$.
Therefore, $|\triangle ABC| = (TC)(r_1) + (TB)(r_2)$, as required.

Solution 4
Let $\angle ABC = 2\beta$, $\angle ACB = 2\gamma$ and $\angle MPN = 2\theta$.
Then $\angle PAB = 2\beta - 2\theta$ and $\angle MAC = 2\gamma + 2\theta$, using external angles in $\triangle PAB$ and $\triangle PAC$. Also, $\angle ABP = 180^\circ - 2\beta$.
Since the circle with centre O is tangent to AP and AK, then O lies on the bisector of $\angle PAK$, so $\angle KAO = \beta - \theta$. Similarly, $\angle LAQ = \gamma + \theta$ and $\angle KBO = 90^\circ - \beta$.
Since $\triangle OKB$ is right-angled at K (since AB is tangent to the circle with centre O at K), then $\angle KOB = \beta$.
Thus, $\tan(\angle KAO) = \tan(\beta - \theta) = \dfrac{KO}{AK}$ and $\tan(\angle KOB) = \tan(\beta) = \dfrac{KB}{KO}$.

Therefore,

$$\begin{aligned}
AB &= AK + KB \\
AB &= \frac{KO}{\tan(\beta-\theta)} + KO\tan(\beta) \\
AB &= r_1\left[\frac{1+\tan(\beta)\tan(\theta)}{\tan(\beta)-\tan(\theta)} + \tan(\beta)\right] \qquad \text{(since } KO = r_1\text{)} \\
AB &= r_1\left[\frac{1+\tan(\beta)\tan(\theta)}{\tan(\beta)-\tan(\theta)} + \frac{\tan^2(\beta)-\tan(\beta)\tan(\theta)}{\tan(\beta)-\tan(\theta)}\right] \\
AB &= r_1\left[\frac{1+\tan^2(\beta)}{\tan(\beta)-\tan(\theta)}\right] \\
r_1 &= \frac{AB(\tan(\beta)-\tan(\theta))}{1+\tan^2(\beta)} \\
r_1 &= \frac{AB(\tan(\beta)-\tan(\theta))}{\sec^2(\beta)} \\
r_1 &= AB\sin(\beta)\cos(\beta) - AB\cos^2(\beta)\tan(\theta) \\
r_1 &= \tfrac{1}{2}AB\sin(2\beta) - AB\cos^2(\beta)\tan(\theta)
\end{aligned}$$

But $AB\sin(2\beta)$ is the length of the height, h, of $\triangle ABC$ from A to BC.
Thus $r_1 = \frac{1}{2}h - AB\cos^2(\beta)\tan(\theta)$.
Similarly, $r_2 = \frac{1}{2}h + AC\cos^2(\gamma)\tan(\theta)$.

Since the circle with centre I is tangent to AB and BC, then I lies on the angle bisector of $\angle ABC$, so $\angle IBT = \beta$, so $\tan(\beta) = \dfrac{IT}{TB}$.
Thus, $TB = \dfrac{IT}{\tan(\beta)} = \dfrac{r}{\tan(\beta)}$.
Similarly, $TC = \dfrac{r}{\tan(\gamma)}$.
Therefore,

$$\begin{aligned}
r_1(TC) + r_2(TB) &= TC\left[\tfrac{1}{2}h - AB\cos^2(\beta)\tan(\theta)\right] + TB\left[\tfrac{1}{2}h + AC\cos^2(\gamma)\tan(\theta)\right] \\
&= \tfrac{1}{2}h(TC+TB) + \tan(\theta)\left[-TC\cdot AB\cos^2(\beta) + TB\cdot AC\cos^2(\gamma)\right] \qquad (*)
\end{aligned}$$

The first term on the right side of $(*)$ equals $\frac{1}{2}h(BC)$ which equals the area of $\triangle ABC$.
Considering the second factor of the second term, we obtain

$$\begin{aligned}
& TB\cdot AC\cos^2(\gamma) - TC\cdot AB\cos^2(\beta) \\
=\ & \frac{r}{\tan(\beta)}AC\cos^2(\gamma) - \frac{r}{\tan(\gamma)}AB\cos^2(\beta) \\
=\ & \frac{r}{2\tan(\beta)\tan(\gamma)}\left(2AC\tan(\gamma)\cos^2(\gamma) - 2AB\tan(\beta)\cos^2(\beta)\right) \\
=\ & \frac{r}{2\tan(\beta)\tan(\gamma)}\left(2AC\sin(\gamma)\cos(\gamma) - 2AB\sin(\beta)\cos(\beta)\right) \\
=\ & \frac{r}{2\tan(\beta)\tan(\gamma)}\left(AC\sin(2\gamma) - AB\sin(2\beta)\right)
\end{aligned}$$

But $AC\sin(2\gamma) = AB\sin(2\beta) = h$, so this second factor equals 0, so the second term of the right side of $(*)$ equals 0.

Therefore, $r_1(TC) + r_2(TB)$ equals the area of $\triangle ABC$, as required.