The Canadian Mathematical Society
in collaboration with
The Center for Education in Mathematics and Computing

The Fourth Canadian Open Mathematics Challenge

Wednesday, November 24, 1999

Solutions

Part A

Note: All questions in part A were graded out of 5 points.

1. *Answer*
   495
   The average on this question was 3.7.

   Comments
   This question is most easily solved by adding the series from 'front to back' or by using the formula. Students could also have just added the terms mechanically to get the same answer.

2. *Answer*
   $-2 \leq x \leq 0$ or $x \geq 3, x \in \mathbf{R}$
   The average on this question was 3.9.

   Comments
   The intercepts of the function are -2, 0 or 3 which gives the answer $-2 \leq x \leq 0$ or $x \geq 3$. Generally speaking, students should have solved for the intercepts first and then used the diagram to read off the correct intervals. Students should take care in dealing with inequality signs. Many students had the signs going in the wrong direction.

3. *Solution* If we convert to base $\frac{2}{3}$, this gives,

$$\left(\frac{4}{9}\right)^{x}\left(\frac{8}{27}\right)^{1-x}=\frac{2}{3}$$

$$\left(\frac{2}{3}\right)^{2x}\left[\left(\frac{2}{3}\right)^{3}\right]^{1-x}=\frac{2}{3}$$

$$\left(\frac{2}{3}\right)^{2x}\left(\frac{2}{3}\right)^{3-3x}=\left(\frac{2}{3}\right)^{1}$$

Therefore $\left(\frac{2}{3}\right)^{-x+3}=\left(\frac{2}{3}\right)^{1}$
Since bases are equal, $-x+3=1$, $x=2$.
The average on this question was 3.7.

Comments
This question was generally well done. It should be noted that converting to base $\frac{2}{3}$ is by far the easiest way to approach the problem. The biggest mistake here was in the inappropriate use of the power rules for exponents.

4. *Solution*
Factoring equation 3, $(x+2y-z)(x+2y+z)=15$.
Substituting $x+2y-z=5$, $5(x+2y+z)=15$ or, $x+2y+z=3$.
Subtracting this from (2): $2x=8$
Therefore, $x=4$.
The average on this question was 3.4.

Comments
A variety of good solutions were given by students who used the first two equations to arrive at $y=4-x$ and $z=3-x$. This allowed substitutions into the third equation. The best way to proceed, howeve, was to factor the third equation as a difference of squares and then make the direct substitution on $x+2y-z=5$ as shown above.

5. *Solution*
Factoring gives, $2\sin^2 x(\sin x+3)-(\sin x+3)=0 \Leftrightarrow (2\sin^2 x-1)(\sin x+3)=0$.
Either $\sin x=-3$ which is inadmissible since $|\sin x|\leq 1$, or

$$2\sin^2 x=1$$

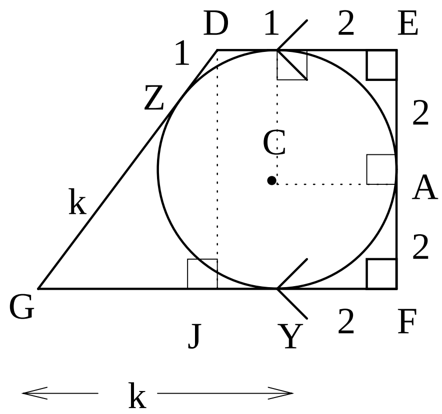
$$\sin x=\pm\frac{1}{\sqrt{2}}$$

Therefore, $x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$.
The average on this question was 1.8.

Comments
This problem could have been solved by either factoring directly or using the factor theorem. Students should make the comment in their solutions that $\sin x=-3$ is inadmissible. From there, the recognition that $\sin^2 x=\frac{1}{2}$ has four solutions should be an easy matter.

6. *Solution*

From C draw a line perpendicular to both DE and EF and label the diagram as shown. From D draw a line perpendicular to GF to meet the line at J.



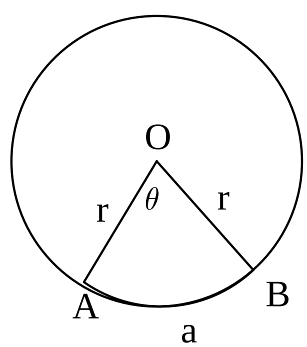
Since $DE = JF = 3$, $JY = 3 - 2 = 1$. Thus, $GJ = k - 1$. Since $\triangle DJG$ is right angled, $(k + 1)^2 = (k - 1)^2 + 4^2$ or, $k^2 + 2k + 1 = 16 + k^2 - 2k + 1$, $4k = 16$, $k = 4$. This makes $GF = 6$ and the area of trapezoid $DEFG$ is $\dfrac{(3 + 6)}{2}(4) = 18$.
The average on this question was 2.1.

Comments

This question could be done in a variety of ways. The easiest way is to use properties of tangents to a circle and the Pythagorean theorem. When solving problems of this type, it is almost always a matter of dropping perpendiculars and using simple properties of circles and triangles to get the appropriate equations. There were a number of very unusual and insightful solutions to this problem

7. *Solution*

We are given that $a + 2r = 12$. Therefore $a = 12 - 2r$. The formula for the area (A) of a sector is, $A = \dfrac{1}{2}ar$ where a is arc lenght and r is radius. Using the formula for area,



$$A = \frac{1}{2}(12 - 2r)r$$

$$A = -r^2 + 6r$$

To maximize the area we complete the square or use calculus to find $r = 3$. Thus the radius that maximizes the area is $r = 3$.
The average for this question was 1.6.

Comments

This question was nicely done by a large number of competitors. The formula for the area of a sector can be easily derived to be $A = \dfrac{1}{2}ar$. From there if we use the relationship, $a + 2r = 12$ it is not difficult to get the required expression for the area of the sector of the circle. It was graifying to see the number of students who solved this problem correctly.

8. *Solution*
If we rewrite $\dfrac{14k+17}{k-9}$ in the following way,

$$\frac{14k+17}{k-9} = \frac{[14(k-9)+126]+17}{k-9} = \frac{14(k-9)}{k-9} + \frac{143}{k-9} = 14 + \frac{143}{k-9}, k \neq 9$$

Since $143 = 1.11.13$, it is not difficult to see that $k - 9 = qd$ where d is a number contained in $1.11.13$ and neither q nor d equals 1. The smallest possible value for d and q is 11 and 2 respectively. This makes $k = 31$.
The average on this question was 0.9.

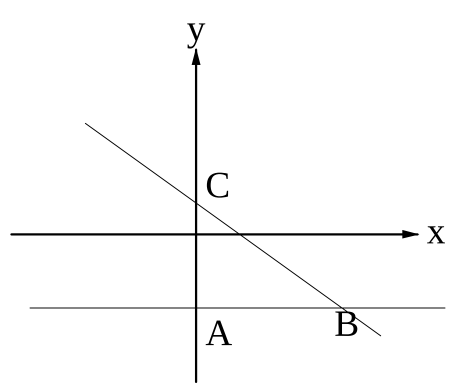
Comments
It was delighted to see how many competitors solved this problem and the variety of solutions. Some students made the observation that $d|k - 9$ and $d|14k + 17$ and so $d|14k + 17 - 14(k - 9)$ or $d|143$. This also leads quickly to the solution.

Part B

Note: All questions in part B were graded out of 10 points.

1.

(a) *Solution*
The line $y = \frac{-3}{4}x + b$ meets the y-axis at C and the line $y = -4$ at B. Since the slope of this line is $\frac{-3}{4}$, we let AC be $3a$ and AB be $4a$. Then the area of the triangle is given by



$$\frac{1}{2}(3a)(4a) = 24$$

$$6a^2 = 24$$

$$a^2 = 4$$

$$a = \pm 2$$

Then $3a = \pm 6$. Now the coordinates of C are $(0, b)$, so $AC = 6+4$. Then $b+4 = \pm 6$
$b = 2$ or -10

Comments
The best way of solving this problem was to recognize that a slope of $\frac{-3}{4}$ means that the ratio of the height to the base for the triangle is $3a : 4a$. If the triangle has an area of 24, then the actual lenghts of the sides are 6 and 8. This gives $b = 2$ or $b = -10$. In part (b), the triangle is right angled with a hypotenuse of 10. This gives the radius 5. Students should always draw diagrams for problems of this type. Many students lost marks here because they didn't explain how they arrived at their answers and did not visualize a solution.

(b) *Solution*
Since either of the triangles have side lenghts 6 and 8 then the hypotenuse has a lenght of 10. The semi-circle must have a radius of 5.
The average on this question was 3.4.

2. *Solution*
Assume that we can expand and compare coefficients. Expanding,

$$(x + r)(x^2 + px + q) = x^3 + (p + r)x^2 + (pr + q)x + qr$$

Comparing coefficeints,

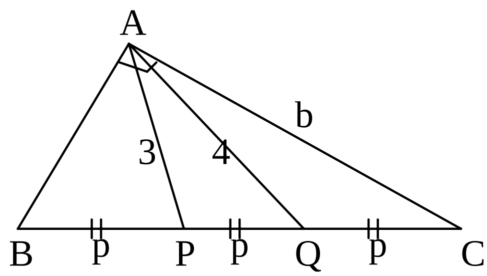
$$p + r = b \quad (1)$$
$$pr + q = c \quad (2)$$
$$qr = d \quad (3)$$

If $bd + cd$ is odd, so is $d(b + c)$. From this, d and $b + c$ are both odd. From (3), if d is odd then q and r are both odd.(4)
Adding (1) and (2), $b+c = p+r+pr+q = (q+r)+p(1+r)$. Since $b+c$ is odd then $(q+r)+p(1+r)$ is also odd. From (4), if q and r are both odd then $q+r$ is even. This implies that $p(1 + r)$ must be odd but this is not possible because r is odd and $r + 1$ is then even making $p(1 + r)$ both odd and even at the same time. This contradiction implies that our original assumption was incorrect and thus $x^3 + bx^2 + cx + d$ cannot be expressed in the form $(x + r)(x^2 + px + q)$.
The average on this question was 2.1.

Comments
There were a variety of gorgeous solutions to this problem. We provide one solution in its entirety. The method of proof here is that of contradiction. In essence we assume that is possible to compare coefficients by expanding the left side. From this, we show that this leads to a contradiction. Since there is a contridictory conclusion, our original assumption must in fact have been false and thus it is not possible to compare coefficients, as is required. A large number of students developed a large varity of proofs, some of which were quite unique and very interesting and in fact correct.

3. *Solution*

Label $\triangle ABP$ as shown. From $\triangle ABP$, $3^2 = p^2 + c^2 - 2pc\cos B$. Because $\angle BAC$ is a right angle, $\cos B = \frac{c}{3p}$



so $9 = p^2 + c^2 - 2pc\left(\frac{c}{3p}\right)$

or, $9 = p^2 + \frac{1}{3}c^2$ (1)

Following the same procedure in $\triangle ACQ$ we have,

$16 = p^2 + \frac{1}{3}b^2$ (2)

Adding (1) and (2) gives, $25 = 2p^2 + \frac{1}{3}(b^2 + c^2)$.

Since $b^2 + c^2 = 9p^2$, $25 = 2p^2 + \frac{1}{3}(9p^2) = 5p^2$.

Therefore, $p = \sqrt{5}(p > 0)$ and $BC = 3\sqrt{5}$ or $\sqrt{45}$.

Since $p = \sqrt{5}$, substituting in equation (1) and (2) gives $AB = \sqrt{12}$ and $AC = \sqrt{33}$.

The average on this question was 1.0.

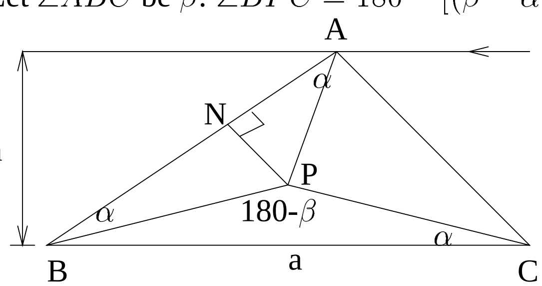
Comments

This problem could be approached in a wide variety of ways. We provide one of the many possible proofs above. Many students attempted this proof by drawing lines parallel to AB throught P and Q and then using the side splitting theorem. This leads to a simple application of Pythagoras and a nice system of equations to solved. The number of students who presented unique solutions was quite gratifying.

4. *Solution*

Let P be a point inside $\triangle ABC$ such that $\angle PAB = \angle PBA = \angle PCB = \alpha$.

Let $\angle ABC$ be β. $\angle BPC = 180 - [(\beta - \alpha) + \alpha] = 180 - \beta$.



From P draw a line perpendicular to AB meeting AB at N. Applying the Sine Law in $\triangle PBC$,

$$\frac{PB}{\sin\alpha} = \frac{a}{\sin(180 - \beta)} = \frac{a}{\sin\beta}$$

Therefore, $PB = \frac{a\sin\alpha}{\sin\beta}$ (1)

From $\triangle BPN$, $\cos\alpha = \dfrac{BN}{PB} = \dfrac{AB}{2PB}$ or $PB = \dfrac{AB}{2\cos\alpha}$ (2)
Equating our two expressions, (1) and (2), we have

$$\frac{a\sin\alpha}{\sin\beta} = \frac{AB}{2\cos\alpha}$$

or

$$2\sin\alpha\cos\alpha = \frac{AB\sin\beta}{a}$$

$$\sin 2\alpha = \frac{AB\sin\beta}{a}$$

Since $AB\sin\beta = h$ then $\sin 2\alpha = \frac{h}{a}$
Since $h < \frac{\sqrt{3}}{2}a$, $\sin 2\alpha < \frac{\sqrt{3}}{2}$ (by substitution)
Then $2\alpha < \frac{\pi}{3}$ and $\alpha < \frac{\pi}{6}$ or $2\alpha > \frac{2\pi}{3}$ and $\alpha > \frac{\pi}{3}$.
But $\alpha > \frac{\pi}{3}$ is impossible because the sum of the angles in the triangle is π. Hence there is one value of α for any given a and h.
The average on this question was 0.1.

Comments
This was a very hard problem. It can be done in three or four ways. A number of competitors got this question correct. We provide here just one solution. Some students solved the problem but only one or two students made an attempt at dealing with restrictions, $h \leq \dfrac{\sqrt{3}}{2}a$. This problem could have been attempted using coordinates but was awkward and hard mechanically.