The Canadian Junior Mathematical Olympiad

Official Solutions for CJMO 2024

J1. Centuries ago, the pirate Captain Blackboard buried a vast amount of treasure in a single cell of a 2×4 grid-structured island. You and your crew have reached the island and have brought special treasure detectors to find the cell with the treasure. For each detector, you can set it up to scan a specific subgrid $[a, b] \times [c, d]$ with $1 \leq a \leq b \leq 2$ and $1 \leq c \leq d \leq 4$. Running the detector will tell you whether the treasure is in the region or not, though it cannot say where in the region the treasure was detected. You plan on setting up Q detectors, which may only be run simultaneously after all Q detectors are ready. What is the minimum Q required to guarantee your crew can determine the location of Blackboard's legendary treasure?

Solution. We shall prove that $Q = 3$.

Let us first observe that $Q \leq 3$, that is, it is possible to complete the task with three detectors on the grid having 2 rows and 4 columns, as follows. The first detector scans the four cells in the first row; a second detector scans all four cells in the first two columns; and the third detector scans all four cells in the second and third columns. The following diagram shows which detectors cover each cell:

12	123	1 3	1
2	23	3	

Notice that no two cells would give the same response from all three detectors, so these three detectors suffice to distinguish the eight possible locations. This proves that $Q \leq 3$.

To see that $Q \geq 3$, observe that there are only 2×2 possible responses from any arrangement of two detectors. These four possible responses are not enough to distinguish eight possible locations. Therefore three detectors are needed. $\square$

J2. Let n be a positive integer. Let

$$I_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \min\left(\frac{1}{i}, \frac{1}{j}, \frac{1}{k}\right)$$

and $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Determine $I_n - H_n$ in terms of n.

Solution. Fix a positive integer ℓ with $1 \le \ell \le n$. Then $\min\left(\frac{1}{i}, \frac{1}{j}, \frac{1}{k}\right) = \frac{1}{\ell}$ precisely when one of $i, j, k = \ell$ and the others are at most ℓ. By inclusion-exclusion, the number of (i, j, k) that achieve this is $3\ell^2 - 3\ell + 1$. Consequently,

$$I_n = \sum_{\ell=1}^{n} (3\ell^2 - 3\ell + 1) \cdot \frac{1}{\ell} = \sum_{\ell=1}^{n} 3\ell - \sum_{\ell=1}^{n} 3 + \sum_{\ell=1}^{n} \frac{1}{\ell} = 3 \cdot \frac{n(n+1)}{2} - 3n + H_n$$

so

$$I_n - H_n = \frac{3n(n+1)}{2} - 3n = \frac{3n(n-1)}{2}.$$

□

J3. Let ABC be a triangle with incenter I. Suppose the reflection of AB across CI and the reflection of AC across BI intersect at a point X. Prove that XI is perpendicular to BC.

(The incenter is the point where the three angle bisectors meet.)

Solution. Suppose the reflection of AC across BI intersects BC at E. Define F similarly for the reflection of AB across CI. Also suppose CI intersects AB at M and BI intersects AC at N. Since CA and $CF = BC$ are reflections across CI, and so are MA and $MF = XM$, we have that A and F are reflections across CI. Similarly A and E are reflections across BI. Thus $\angle XFC = \angle BAC = \angle XEB$ if $\angle BAC$ is acute (and $\angle XFC = \angle XEB = \pi - \angle BAC$, when $\angle BAC$ is obtuse), so $XF = XE$. Moreover we also find that $IF = IA = IE$ by the aforementioned reflection properties, so thus XI is the perpendicular bisector of EF and is hence perpendicular to BC. $\square$

J4. Jane writes down 2024 natural numbers around the perimeter of a circle. She wants the 2024 products of adjacent pairs of numbers to be exactly the set $\{1!, 2!, \ldots, 2024!\}$. Can she accomplish this?

Solution 1. Given any prime p and positive integer x, let $v_p(x)$ denote the highest power of p dividing x. We claim that Jane cannot write 2024 such numbers as that would imply that $1! \cdot 2! \cdots 2024!$ is the square of the product of the 2024 numbers. Let p be a prime and k be a natural number such that $k < p$, $kp \leq 2024$, and $(k+1)p > 2024$. Then note that

$$v_p(1! \cdot 2! \cdots 2024!) = (2024 - p + 1) + (2024 - 2p + 1) + \ldots + (2024 - kp + 1).$$

In particular, let p be in $(\frac{2024}{4}, \frac{2024}{2})$. By Bertrand's Postulate, such a prime p exists (and p must also be odd). Further, the corresponding k is either 2 or 3. Either way, $v_p(1! \cdot 2! \cdots 2024!)$ is odd from the above formula, and so $1! \cdot 2! \cdots 2024!$ cannot be a perfect square. $\square$

Solution 2. As in the first solution, we prove $1! \cdot 2! \cdots 2024!$ is not a perfect square. To do this, note that we can rewrite the product as $(1!)^2 \cdot 2 \cdot (3!)^2 \cdot 4 \cdots (2023!)^2 \cdot 2024$ which is

$$2 \cdot 4 \cdots 2024 \cdot (1! \cdot 3! \cdots 2023!)^2 = 1012! \cdot (2^{1012} \cdot 1! \cdot 3! \cdots 2023!)^2$$

so it is sufficient to verify $1012!$ is not a perfect square. This can be verified by either noticing the prime 1009 only appears as a factor of $1012!$ once, or by evaluating $v_2(1012!) = 1005$. $\square$

J5. Let N be the number of positive integers with 10 digits $\overline{d_9 d_8 \cdots d_1 d_0}$ in base 10 (where $0 \leq d_i \leq 9$ for all i and $d_9 > 0$) such that the polynomial

$$d_9 x^9 + d_8 x^8 + \cdots + d_1 x + d_0$$

is irreducible in $\mathbb{Q}$. Prove that N is even.

(A polynomial is irreducible in $\mathbb{Q}$ if it cannot be factored into two non-constant polynomials with rational coefficients.)

Solution. Let $f(x) = d_9 x^9 + d_8 x^8 + \cdots + d_1 x + d_0$. If $d_0 = 0$, then $f(x)$ is divisible by x and thus reducible, so we may ignore all such polynomials. The remaining polynomials all have nonzero leading and constant coefficients.

For any polynomial $p(x)$ of degree n with nonzero leading and constant coefficients, say $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, define $\overline{p}(x)$ to be the reversed polynomial $a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$. Observe that $\overline{p}(x)$ also has degree n and furthermore, $\overline{p}(x) = x^n \left(a_0 + a_1 \left(\frac{1}{x}\right) + \cdots + a_{n-1} \left(\frac{1}{x}\right)^{n-1} + a_n \left(\frac{1}{x}\right)^n \right) = x^n p\left(\frac{1}{x}\right)$.

Consider pairing each $f(x)$ with $\overline{f}(x)$ whenever $f(x) \neq \overline{f}(x)$. If $f(x)$ is reducible, it can be factored as $f(x) = g(x)h(x)$ where $\deg g, \deg h \geq 1$. Because the leading and constant coefficients of $f(x)$ are nonzero, so are the leading and constant coefficients of $g(x)$ and $h(x)$. Hence $\overline{g}(x)$ and $\overline{h}(x)$ are well defined with $\deg \overline{g} = \deg g \geq 1$ and $\deg \overline{h} = \deg h \geq 1$. Furthermore,

$$\overline{f}(x) = x^9 f\left(\frac{1}{x}\right) = x^9 g\left(\frac{1}{x}\right) h\left(\frac{1}{x}\right) = \left(x^{\deg g} g\left(\frac{1}{x}\right)\right)\left(x^{\deg h} h\left(\frac{1}{x}\right)\right) = \overline{g}(x)\overline{h}(x).$$

Thus $\overline{f}(x) = \overline{g}(x)\overline{h}(x)$ is a factorization of $\overline{f}(x)$ into two non-constant polynomials, so $\overline{f}(x)$ is also reducible. Therefore $f(x)$ is irreducible if and only if $\overline{f}(x)$ is irreducible, so considering each pair, there are an even number of irreducible polynomials with $f(x) \neq \overline{f}(x)$.

Finally, note that if $f(x) = \overline{f}(x)$, then $d_i = d_{9-i}$ for each i. In such a case, we have $f(-1) = (d_0 - d_9) + (d_2 - d_7) + (d_4 - d_5) + (d_6 - d_3) + (d_8 - d_1) = 0$, so by the Factor Theorem, $(x+1)$ is a factor of $f(x)$. Therefore these remaining polynomials are all reducible. $\square$