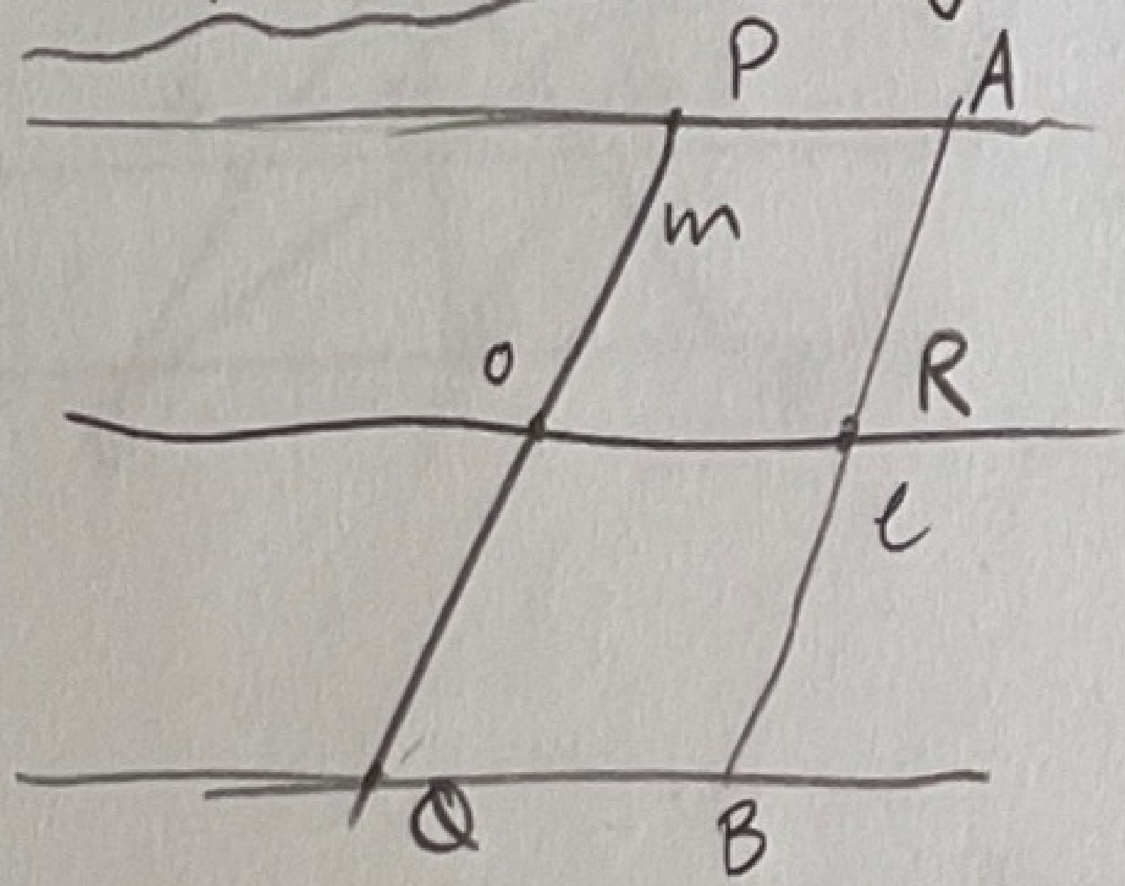
Problem P5 (maximum: 7 points)

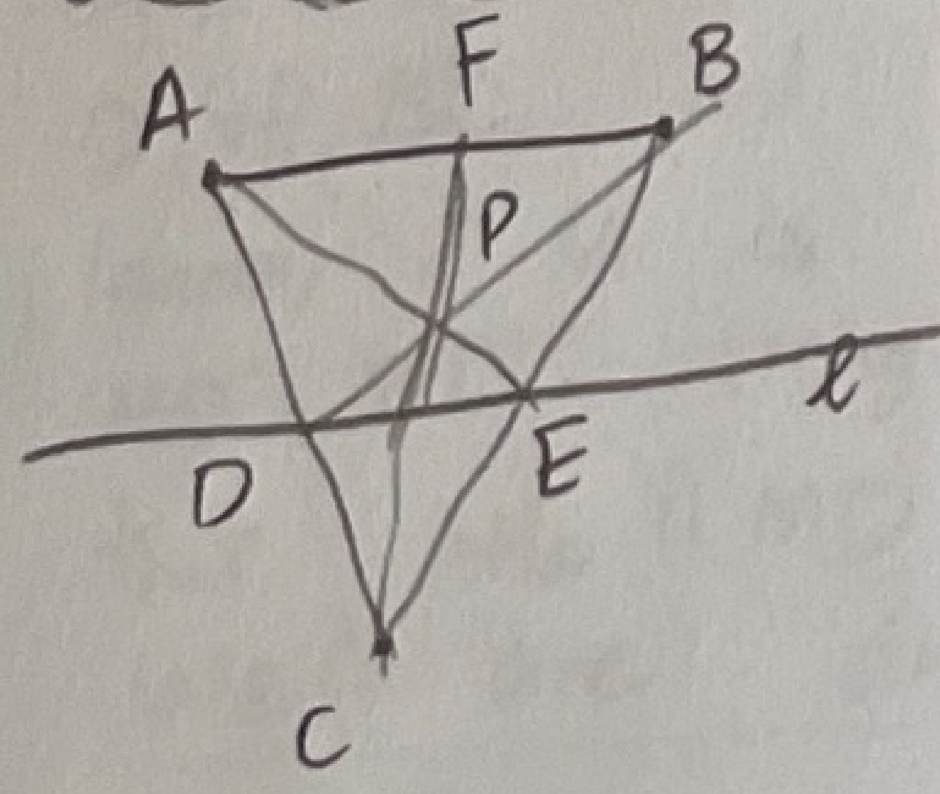
We will solve the problem by making more and more complicated operations that we can do.

Operation I: given two lines draw both angle bisectors.



Proof. Let lines ℓ and m intersect at O. Draw 2 lines parallel to ℓ, distance 1 away from ℓ, intersecting m at P, Q. Draw one line parallel to m distance 1 away from m intersecting ℓ at R. Let this line intersect the two previous lines we drew at A and B as shown to left. $OPAR$, $OQBR$ are rhombi so OA, OB are bisectors of $\angle POR$, $\angle QOR$ respectively.

Operation II: given two points draw their midpoint.



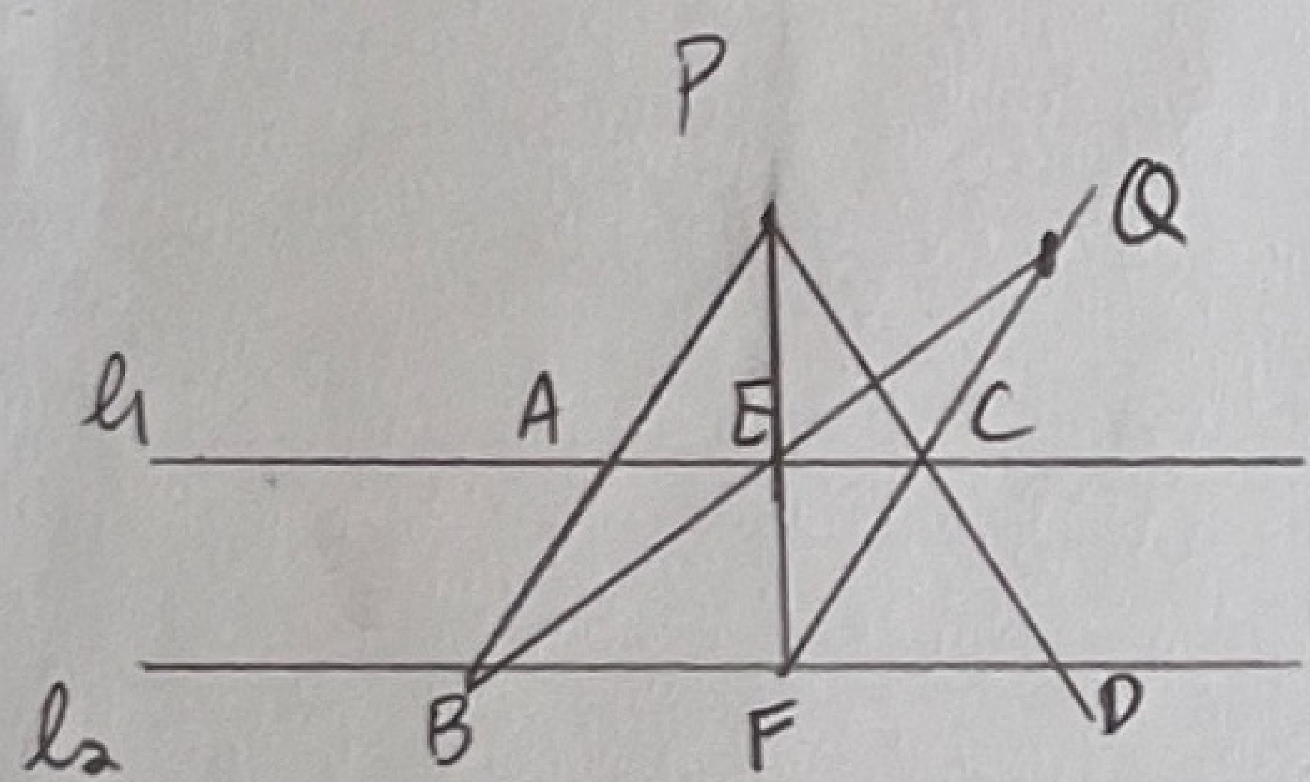
Let A and B be points, and ℓ a parallel line to AB of distance 1. Let C be arbitrary point on opposite side of ℓ as AB. Let $D = \ell \cap AC$, $E = \ell \cap BC$, $P = AE \cap BD$, $F = CP \cap AB$. By Ceva's,

$$\frac{AF}{FB} \cdot \frac{BE}{EC} \cdot \frac{DC}{DA} = 1 \text{ but } \frac{BE}{EC} = \frac{AD}{DC} \text{ so}$$

$AF = FB$.

Problem P5 (continued from previous page)

Operation III: given a point and a line ~~n~~draw a line through the point parallel to the line.



Let P be the point, ℓ_1 be the line. Draw another line ℓ_2 on the opposite side of ℓ_1 as P distance 1 away.

Draw arbitrary lines PAB, PCD such that $A, C \in \ell_1$, $B, D \in \ell_2$ as shown. Let E be AC's midpoint, then let $F = PE \cap \ell_2$ and $Q = BE \cap FC$.

Note that $\triangle QEC \sim \triangle QBF$, $\triangle PAE \sim \triangle PBF$ so $\Rightarrow$ $\frac{QE}{QB} = \frac{EC}{BF} = \frac{AE}{BF} = \frac{PA}{PB}$ so $\triangle BAE \sim \triangle BPQ$. Thus, $PQ \parallel AE$ as desired.

Now, in $\triangle ABC$, using Operation I, draw incenter A and A excenter I_A. Draw using Operation II midpoint D of BC and M of II_A. By incenter-excenter lemma, M is on the perpendicular bisector of BC so $MD \perp BC$. Using operation III, draw H_A on BC such that $AH_A \parallel MD \Rightarrow AH_A \perp BC$. Do similarly for B, C. We are done, by intersecting AH_A and BH_B.