2017 Canadian Mathematical Olympiad

Sun Life Financial

Official Problem Set

1. Let a, b, and c be non-negative real numbers, no two of which are equal. Prove that

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}>2.$$

2. Let f be a function from the set of positive integers to itself such that, for every n, the number of positive integer divisors of n is equal to $f(f(n))$. For example, $f(f(6))=4$ and $f(f(25))=3$. Prove that if p is prime then $f(p)$ is also prime.

3. Let n be a positive integer, and define $S_n=\{1,2,\ldots,n\}$. Consider a non-empty subset T of S_n. We say that T is balanced if the median of T is equal to the average of T. For example, for $n=9$, each of the subsets $\{7\}$, $\{2,5\}$, $\{2,3,4\}$, $\{5,6,8,9\}$, and $\{1,4,5,7,8\}$ is balanced; however, the subsets $\{2,4,5\}$ and $\{1,2,3,5\}$ are not balanced. For each $n\geq 1$, prove that the number of balanced subsets of S_n is odd.
(To define the median of a set of k numbers, first put the numbers in increasing order; then the median is the middle number if k is odd, and the average of the two middle numbers if k is even. For example, the median of $\{1,3,4,8,9\}$ is 4, and the median of $\{1,3,4,7,8,9\}$ is $(4+7)/2=5.5$.)

4. Points P and Q lie inside parallelogram $ABCD$ and are such that triangles ABP and BCQ are equilateral. Prove that the line through P perpendicular to DP and the line through Q perpendicular to DQ meet on the altitude from B in triangle ABC.

5. One hundred circles of radius one are positioned in the plane so that the area of any triangle formed by the centres of three of these circles is at most 2017. Prove that there is a line intersecting at least three of these circles.