2024 CMO Qualifying Repêchage

Problems

1. **[10 points]** Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the functional equation

$$f(x + f(xy)) = f(x)(1 + y).$$

2. **[10 points]** Call a natural number N *good* if its base 3 expansion has no consecutive digits that are the same. For example, 289 is good since its base 3 representation is 101201_3. Find the 2024^{th} smallest good number (0 is *not* considered to be a natural number). Your answer should be in base 10.

3. **[10 points]** Let $\triangle ABC$ be an acute triangle with $AB < AC$. Let H be its orthocentre and M be the midpoint of arc $\widehat{BAC}$ on the circumcircle. It is given that B, H, M are collinear, the length of the altitude from M to AB is 1, and the length of the altitude from M to BC is 6. Determine all possible areas for $\triangle ABC$.

4. **[10 points]** A sequence $\{a_i\}$ is given such that $a_1 = \frac{1}{3}$ and for all positive integers n

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}.$$

   Prove that

$$\frac{1}{2} - \frac{1}{3^{2^{n-1}}} < a_1 + a_2 + \cdots + a_n < \frac{1}{2} - \frac{1}{3^{2^n}},$$

   for all positive integers n.

5. **[10 points]** Let S be the set of 25 points (x, y) with $0 \leq x, y \leq 4$. A triangle whose three vertices are in S is chosen at random. What is the expected value of the square of its area?

6. **[10 points]** For certain real constants p, q, r, we are given a system of equations

$$\begin{cases} a^2 + b + c = p \\ a + b^2 + c = q \\ a + b + c^2 = r \end{cases}$$

   What is the maximum number of solutions of real triplets (a, b, c) across all possible p, q, r? Give an example of the p, q, r that achieves this maximum.

7. **[20 points]**

   (a) In triangle ABC, let I be the incentre. Let H be the orthocentre of triangle BIC. Show that AH is parallel to BC if and only if H lies on the circle with diameter AI.

   (b) In triangle ABC, let I be the incentre, O be the circumcentre, and H be the orthocentre. It is given that $IO = IH$. Show that one of the angles of triangle ABC must be equal to 60 degrees.

8. **[20 points]** A sequence of Xs and Os is given, such that no three consecutive characters in the sequence are all the same, and let N be the number of characters in this sequence. Maia may swap two consecutive characters in the sequence. After each swap, any consecutive block of three or more of the same character will be erased (if there are multiple consecutive blocks of three or more characters after a swap, then they will be erased at the same time), until there are no more consecutive blocks of three or more of the same character. For example, if the original sequence were $XXOOXOXO$ and Maia swaps the fifth and sixth character, the end result will be $XXOOOXXO \to XXXXO \to O$. Find the maximum value N for which Maia can't necessarily erase all the characters after a series of swaps. Partial credit will be awarded for correct proofs of lower and upper bounds on N.