Crux Mathematicorum

Volume/tome 50, issue/numéro 2
February/février 2024

Canadian Mathematical Society
Société mathématique du Canada

Crux Mathematicorum is a problem-solving journal at the secondary and university undergraduate levels, published online by the Canadian Mathematical Society. Its aim is primarily educational; it is not a research journal. Online submission:

`https://publications.cms.math.ca/cruxbox/`

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire publiée par la Société mathématique du Canada. Principalement de nature éducative, le Crux n'est pas une revue scientifique. Soumission en ligne:

`https://publications.cms.math.ca/cruxbox/`

ISSN 1496-4309 (Online)

ISSN 1496-4309 (électronique)

Supported by / Soutenu par :

- Intact Financial Corporation
- University of the Fraser Valley

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MathemAttic

No. 52

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2024**.

MA256. There exist positive integers whose value is quadrupled by moving the rightmost decimal digit into the leftmost position. Find the smallest such number.

MA257. A square of area 1 is divided into three rectangles which are geometrically similar (i.e., they have the same ratio of long to short sides) but no two of which are congruent. Let A, B and C be the areas of the rectangles, ordered from largest to smallest. Prove that $(AC)^2 = B^5$.

MA258. The three following circles are tangent to each other: the first has centre $(0, 0)$ and radius 4, the second has centre $(3, 0)$ and radius 1, and the third has centre $(-1, 0)$ and radius 3. Find the radius of a fourth circle tangent to each of these 3 circles.

MA259. Consider the equation $7a + 12b = c$ where a, b and c are nonnegative integers. For many values of c, it is possible to find one or more pairs (a, b) satisfying the equation. Given $c = 26$, for example, $(a, b) = (2, 1)$ is the only solution.

a) If $c = 365$, find all possible solutions (a, b), where a and b are nonnegative integers.

b) There are some values of c for which no solutions exist. For example, there is no pair (a, b) such that $7a + 12b = 20$, so $c = 20$ is one such case. Find the largest integer value of c for which there are no nonnegative integer solutions.

MA260. The expression $n!$ denotes the product $1 \cdot 2 \cdot 3 \cdots n$ and is read as "n factorial". For example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

a) The product $(2!)(3!)(4!)(5!)(6!)(7!)(8!)(9!)(10!)(11!)(12!)$ can be written in the form $M^2 N!$, where M, N are positive integers. Find a suitable value of N and justify your answer.

b) Prove that, for every $n \geq 1$, $(2!)(3!)(4!) \cdots ((4n)!)$ can be written as the product of a square and a factorial.

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2024**.

MA256. Il existe des entiers positifs dont la valeur est quadruplée lorsque l'on déplace le chiffre le plus à droite vers la position la plus à gauche. Trouver le plus petit nombre ayant cette propriété.

MA257. Un carré d'aire 1 est divisé en trois rectangles qui sont géométriquement similaires (i.e., les rapports entre les longueurs des côtés longs sur les longueurs des côtés courts sont les mêmes) mais aucun de ces rectangles est congru à un autre. Nommer A, B et C les aires des rectangles ordonnés du plus grand au plus petit. Prouver que $(AC)^2 = B^5$.

MA258. Trois cercles sont tangents les uns aux autres; le premier a son centre à $(0,0)$ et son rayon est 4; le second a son centre à $(3,0)$ et son rayon est 1; le troisime a son centre à $(-1,0)$ et son rayon est 3. Déterminer le rayon d'un quatrième cercle, tangent à chacun de ces 3 cercles.

MA259. Considérer l'équation $7a + 12b = c$, où a, b et c sont des entiers nonnégatifs. Étant donné c, il y a parfois une solution ou plus, (a,b), à l'équation. Par exemple, si $c = 26$, alors $(2,1)$ est la seule solution (a,b).

a) Si $c = 365$, déterminer toutes les solutions possibles (a,b), où a et b sont des entiers non négatifs.

b) Il existe certaines valeurs de c pour lesquelles il n'y a aucune solution (a,b) à l'équation précédente. Par exemple, si $c = 20$, l'équation $7a+12b = 20$ n'a aucune solution (a,b). Déterminer la plus grande valeur de c pour laquelle il n'existe aucune solution (a,b) où a et b sont des entiers non négatifs.

MA260. L'expression $n!$ représente le produit $1\cdot 2\cdot 3\cdots n$ et se lit "factorielle n". Par exemple, $5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$.

a) Le produit $(2!)(3!)(4!)(5!)(6!)(7!)(8!)(9!)(10!)(11!)(12!)$ peut s'écrire sous la forme $M^2N!$, où M et N sont des entiers strictement positifs. Déterminer une valeur appropriée de N et justifier votre réponse.

b) Montrer que pour tout $n \geq 1$, $(2!)(3!)(4!)\cdots((4n)!)$ peut s'écrire comme le produit d'un carré et d'une factorielle.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2023: 49(7), p. 340–341.

MA231. A grocery store clerk wants to make a large triangular pyramid of oranges. The bottom level is an equilateral triangle made up 1275 oranges. Each orange above the first level rests in a pocket formed by three oranges below. The stack is completed at the final level with a single orange. How many oranges are in the stack?

Originally question 9 from the 35th University of Alabama High School Mathematics Tournament: Team Competition, 2016.

We received 7 submissions, all correct and complete. We present the solution by Teodor Constantin.

What is the smallest orange pyramid that one can stack with *more* than one orange? If one takes 3 oranges to form the equilateral triangular base, and adds one more orange on top, a 4 orange pyramid is formed! In this case, $n = 2$ oranges were used for the side of the bottom triangle.

How about when $n = 3$? Since $n = 3$ is the number of oranges that can fit on the side of the equilateral triangle of the pyramid bottom, one can only fit 3 rows of oranges in this triangle, where the first row has 3 oranges, the second one has 2 oranges, and the third one has only 1 orange. Note that the total number or oranges that can be arranged in the bottom of the pyramid is 3+2+1 or $3{\cdot}4/2$, i.e., 6 oranges in all. As for the total number of oranges in the pyramid, it is important to note that 3 layers of oranges can be arranged in the following distribution:

3^{rd} **layer (bottom)**	2^{nd} **layer**	1^{st} **layer (top)**
3+2+1	2+1	1

Similarly, when $n = 4$, the bottom of the pyramid will contain $4 + 3 + 2 + 1$ or $4 \cdot 5/2$, i.e., 10 oranges in all. The pyramid will have 4 layers with the oranges being stacked as shown in the following table:

4^{th} **layer (bottom)**	3^{rd} **layer**	2^{nd} **layer**	1^{st} **layer (top)**
4+3+2+1	3+2+1	2+1	1

It is important to notice that regardless if n is odd or even, the pyramid will always contain n layers of oranges, with a decreasing number of oranges from bottom to top with only one orange on the top layer.

To generalize, for an arbitrary number n of oranges that can be arranged on the side of the triangle, the total number of oranges needed to cover the bottom layer will be $n + (n - 1) + \cdots + 2 + 1$. This sum is equal to $\frac{n(n+1)}{2}$. Given that the number of oranges on the bottom level is 1275, the following equality is obtained: $\frac{n(n+1)}{2} = 1275$. This quadratic equation can be written as $(n+51)(n-50) = 0$, which has only one positive solution, i.e. $n = 50$. The second solution, $n = -51$, is not a viable solution since it's a negative integer.

To find out the total number of oranges in the pyramid, the number of oranges in all layers has to be computed. The distribution of oranges is shown below:

50^{th} **layer (bottom)**	49^{th} **layer**	$\cdots$	1^{st} **layer (top)**
$50 + 49 + \cdots +1$	$49 + 48 + \cdots +1$	$\cdots$	1

Let S be the total number of oranges in the pyramid. The sum can be written as:

$$S = 1 \cdot 50 + 2 \cdot 49 + 3 \cdot 48 + \cdots + 49 \cdot 2 + 50 \cdot 1,$$

which is equivalent to

$$S = 1 \cdot (51-1) + 2 \cdot (51-2) + 3 \cdot (51-3) + \cdots + 49 \cdot (51-2) + 50 \cdot (51-50),$$

which forces number 51 to appear in all terms of the sum, as well as a distribution of perfect squares: $1, 2^2, \cdots, 50^2$. Therefore the sum becomes:

$$S = 51 \cdot (1 + 2 + \cdots + 50) - (1^2 + 2^2 + \cdots + 50^2).$$

Let $S_1 = 1 + 2 + \cdots + 50$ and $S_2 = 1^2 + 2^2 + \cdots + 50^2$. Then $S = 51 \cdot S_1 - S_2$.

The values for S_1 and S_2 can be found by using standard summation formulas, so that S becomes:

$$S = 51\frac{50 \cdot 51}{2} - \frac{50 \cdot 51 \cdot 101}{6} = \frac{50 \cdot 51 \cdot 52}{6}.$$

Therefore, the total number of oranges in the triangular pyramid stack is $S = 22,100$.

MA232. Determine the number of integers of the form $\overline{abc} + \overline{cba}$, where $\overline{abc}$ and $\overline{cba}$ are three-digit numbers with $ac \neq 0$.

Originally from Mathematics Competitions *Vol. 25, #2 (2012),* Heaven and Earth*, heavenly problem 14.*

There were 4 submissions, all of them complete and correct. We present the solution by Catherine Jian.

Since $\overline{abc} = 100a + 10b + c$, $\overline{cba} = 100c + 10b + a$, we have

$$\overline{abc} + \overline{cba} = 101(a+c) + 20b = 101m + 20b$$

where $m = a + c$.

Since b can be 0, 1, …, 9, there are 10 ways to choose b. Since a and c are digits and cannot be 0, $m = a + c$ can be an integer from 2 to 18, i.e. there are 17 ways to choose m. Therefore in total there are 170 possible integers that can be written as $101m + 20b$, or equivalently can be written as $\overline{abc} + \overline{cba}$ where $\overline{abc}$ and $\overline{cba}$ are three-digit numbers.

MA233. Determine a polynomial function $p(x)$ with the property that if a line is drawn and intersects the graph of $y = p(x)$ in two distinct points $(a, p(a))$ and $(b, p(b))$, then the y-intercept of the line is ab.

Inspired by John Cook blogpost linked here.

We received 3 submissions, all correct. We present the solution provided by Dimitrios Giotas Orfeas.

Let the polynomial be of the form

$$p(x) = mx^2 + nx. \tag{1}$$

Since the line e intersects the y-axis at $(0, ab)$, it is of the form

$$y = kx + ab. \tag{2}$$

Point $(a, p(a))$ belongs to e, thus

$$p(a) = ka + ab \tag{3}$$

Similarly, for point $(b, p(b))$, we have

$$p(b) = kb + ab \tag{4}$$

From equation (1), we get

$$p(a) = ma^2 + na \tag{5}$$

and

$$p(b) = mb^2 + nb \tag{6}$$

Combining equations (3) and (5), we get

$$\begin{aligned} ka + ab &= ma^2 + na \\ k + b &= ma + n \\ k - n &= ma - b \end{aligned} \tag{7}$$

Similarly, combining equations (4) and (6), we obtain

$$\begin{aligned} kb + ab &= mb^2 + nb \\ k + a &= mb + n \\ k - n &= mb - a \end{aligned} \tag{8}$$

From equations (7) and (8):

$$ma - b = mb - a \quad \Longleftrightarrow \quad m(a - b) = b - a \quad \Longleftrightarrow \quad m = -1.$$

Therefore, equation (7) can be rewritten as

$$k - n = -a - b \Rightarrow n = k + a + b.$$

Thus, the polynomial is in the form

$$p(x) = -x^2 + (k + a + b)x.$$

As an example, for $a = 1$, $b = 2$, $k = 3$, we have

$$p(x) = -x^2 + (3 + 1 + 2)x = -x^2 + 6x.$$

MA234. *Proposed by Ed Barbeau.*

Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are two distinct fractions of positive integers that are both less than $\frac{1}{2}$. Prove that the numerator of one of the fractions can be increased by 1 so that the sum of the two resulting fractions is less than 1.

There were only 2 submissions, both of them complete and correct. We present the solution by Richard Hess.

Without loss of generality, we can assume $\frac{a}{b} < \frac{c}{d}$ and let $b = 2a+x$ and $d = 2c+y$, where x and y are integers. Then the two fractions are:

$$\frac{a}{2a + x} = \frac{1}{2} - \frac{x}{4a + 2x} \quad \text{and} \quad \frac{c}{2c + y} = \frac{1}{2} - \frac{y}{4c + 2y}.$$

If we add 1 to a, the new sum becomes

$$S_1 = 1 + \frac{2 - x}{4a + 2x} - \frac{y}{4c + 2y}.$$

If we add 1 to c, the new sum becomes

$$S_2 = 1 + \frac{2 - y}{4c + 2y} - \frac{x}{4a + 2x}.$$

The largest we can make either sum occurs when $x = y = 1$ so that

$$S_1 = 1 + \frac{1}{4a + 2} - \frac{1}{4c + 2} = 1 - \frac{2(a - c)}{(2a + 1)(2c + 1)}$$

and

$$S_2 = 1 + \frac{1}{4c + 2} - \frac{1}{4a + 2} = 1 - \frac{2(c - a)}{(2a + 1)(2c + 1)}.$$

We cannot have $a = c$ for either case where $x = y = 1$, so that S_1 or S_2 will always be < 1 depending upon whether $a > c$ or $c > a$, respectively.

MA235. *Proposed by Aravind Mahadevan.*

In ΔABC, $\cos A \cos B + \sin A \sin B \sin C = 1$. Find $a : b : c$, where a, b, and c are the lengths of sides BC, CA and AB respectively.

There were 6 correct solutions. We present the solution by Miguel Amengual Covas, Bing Jian and Digby Smith, independently.

Since

$$1 = \cos A \cos B + \sin A \sin B \sin C \leq \cos A \cos B + \sin A \sin B = \cos(A - B) \leq 1,$$

the inequality must be an equality. Hence $A - B = 0$ and $\sin C = 1$. Therefore, $C = 90^\circ$ and $A = B = 45^\circ$, so that $a : b : c = 1 : 1 : \sqrt{2}$.

Comments from the editor. Richard Hess pointed out that the trigonometric expression can be written as

$$\cos(A - B) - \sin A \sin B(1 - \sin C).$$

The minuend is in $[-1, 1]$ and the subtrahend in $[0, 1]$, so the former must be 1 and the latter 0.

TEACHING PROBLEMS

No. 25

Ed Barbeau

The Trisection Problem

Often it is the multiple approaches that can be taken or a curious aspect of the forms of solution that underlie the inclusion of a particular example. In this case, the contribution addresses one of the more famous unsolved problems in mathematics. It is shared here also in the hope that some readers will be motivated to submit ideas to Teaching Problems in future. Interesting problems from your own teaching experiences are welcomed. Send them along to `mathemattic@cms.math.ca`*. Some Teaching Problems issues stress pedagogical value of problems and others the mathematical richness while most blend aspects of both. Here it is the mathematical side that figures prominently. The experience of working with an unsolved problem may well be a first for many students, thus enhancing the pedagogical value of the Trisection Problem.*

. . . .

After students learn about bisecting an angle using straightedge and compasses, the question of a similar construction for trisecting an angle may come up, and some students may attempt to find a method. Once I was contacted by a middle school teacher, one of whose students thought he had succeeded.

The proposed construction was pleasantly simple. Let POQ be the (acute) angle to be trisected. From any point A on OP, drop a perpendicular to meet OQ at B. Construct an equilateral triangle ABC with side AB with O and the vertex C on opposite sides of AB. Then it is claimed that $\angle COB$ is equal to one third of $\angle POQ$. If you check it out with a protractor the method is not bad at all, with numerical evidence suggesting that the error is within one or two degrees. In fact, it works for one acute angle; it is not hard to identify and check this angle.

However, there is a pedagogical difficulty here. One could don the mantle of authority and simply tell the student that it was rigorously proved long ago that no such method exists. It is more satisfactory to find an explanation that involves mathematics accessible to the student. I pose two problems for the reader and suggest solutions for them that I think can be improved upon.

1. Find an argument that the proposed trisection construction is faulty that involves facts of Euclidean geometry that the student might be expected to know. The more straightforward the argument the better.

2. Using standard high school mathematics, provide an analysis that identifies the situations for which the method delivers a trisection.

We are asked to refute a construction that purports to produce a trisection for *every* acute angle. We employ a proof by contradiction: assume that the construction works for every angle and derive from this a false statement. All we have to do is to find *at least one* angle for which it does not work. The following argument will begin with the assumption that it works for both angles 30° and 60° and derive inconsistent conclusions. (In Section 3, you will see how the assumption that it works for $POQ = 60°$ leads to a contradiction.)

In the diagram below, $\angle POQ = 60°$ and $\angle DOB = 30°$. We will suppose that $AB = 3$, from which we find that $BD = 1$, $AD = 2$ and $OB = \sqrt{3}$. Triangles ABC and DBE are equilateral, and $\angle CBQ = \angle EBQ = 30°$. Since $AC \| DE$, $CE = AD = 2$.

Assuming the method is valid, $\angle EOB = 10°$, so

$$\angle OEB = \angle EBQ - \angle EOB = 20°.$$

Also, by hypothesis, $\angle COQ = 20°$, so $\angle COE = 10°$. Since

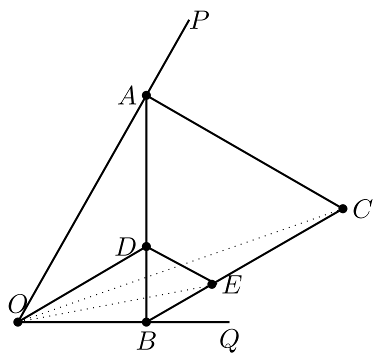
$$\angle OCE = \angle OEB - \angle COE = 10°,$$

triangle COE is isosceles with $OE = CE = 2$.

Consider triangle OBE, On the one hand, $\angle OBE$ is obtuse. On the other,

$$OB^2 + BE^2 = 3 + 1 = 4 = OE^2,$$

which can occur only if $\angle OBE = 90°$. Since these two statements are incompatible, the method fails for at least one of 30° and 60°.

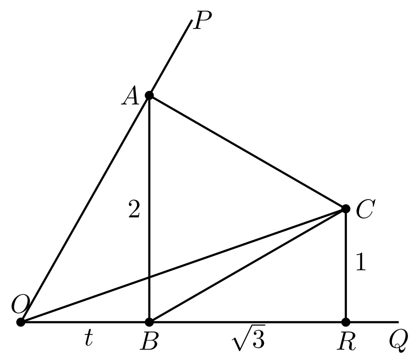


Now look at the general situation of a proper acute angle. In the diagram below, assume that $AB = 2$ and $OB = t$ with $t > 0$. From the diagram, we see that

$$\tan \angle COQ = \frac{1}{t + \sqrt{3}}$$

and

$$\tan \angle POQ = \frac{2}{t}.$$



It can be verified that

$$\tan 3\angle COQ = \frac{3t^2 + 6t\sqrt{3} + 8}{t^3 + 3t^2\sqrt{3} + 6t}.$$

Therefore

$$\begin{aligned}(t^3 + 3t^2\sqrt{3} + 6t) \cdot [\tan 3\angle COQ - \tan \angle POQ] &= (3t^2 + 6t\sqrt{3} + 8) - 2(t^2 + 3t\sqrt{3} + 6) \\ &= t^2 - 4 = (t-2)(t+2).\end{aligned}$$

This vanishes if and only if $t = 2$ and $\angle POQ = 45°$. (There is a degenerate situation when $\angle POQ = 90°$. Here $Q = B$ and $\angle COQ = 30°$.)

A different assignment of lengths suggested by J. Chris Fisher gives a more transparent relationship between the tangents of the angles POQ and COQ. Let $AB = \sqrt{3}$ and $OB = t$; then $CR = \sqrt{3}/2$, $BR = 3/2$ and $\tan \angle COQ = \sqrt{3}/(2t+3)$. Then

$$\tan 3\angle COQ = \frac{3\sqrt{3}(t^2 + 3t + 2)}{t(2t^2 + 9t + 9)} = \tan \angle POQ \left(\frac{3(t^2 + 3t + 2)}{2t^2 + 9t + 9} \right).$$

When $\angle POQ = 60°$, then $t = 1$ and

$$\tan 3\angle COQ = (9/10)\sqrt{3} = (9/10) \tan 60°.$$

In this case, the trisection method produces an angle of about 19.1°.

Let $f(t) = [3(t^2 + 3t + 2)]/[2t^2 + 9t + 9]$. Suppose that $\angle POQ = \theta$. If t is the value of OB that corresponds to θ, then $3/t$ is the value of OB that corresponds to the complement, $90° - \theta$. Then $f(3/t) \cdot f(t) = 1$. The function $f(t)$ increases from $2/3$ when $t = 0$ (and $\theta = 90°$) to $3/2$ when $t = \infty$ (and $\theta = 0$). However, the effect of the values of $f(t)$ in the accuracy of the trisection when θ is further from 45° and $f(t)$ is further from 1 is offset by the fact that when θ is close to 90° a large change in its tangent corresponds to a small change in the angle, and when θ is small, the error from a true trisection will also be small. For what angle is the deviation from a proper trisection maximum?

OLYMPIAD CORNER

No. 420

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **April 15, 2024**.

OC666. The squares of a 1×10 board are numbered 1 to 10 in order. Clarissa and Marissa start from square 1, jump 9 times to the other squares so that they visit each square once, and end up at square 10. Jumps forward and backward are allowed. Each jump of Clarissa was for the same distance as the corresponding jump for Marissa. Does this mean that they both visited squares in the same order?

OC667. In the convex quadrilateral $ABCD$, AB and CD are parallel. Moreover, $\angle DAC = \angle ABD$ and $\angle CAB = \angle DBC$. Is $ABCD$ necessarily a square?

OC668. Consider all 100-digit positive integers such that each digit is 2, 3, 4, 5, 6, 7. How many of these integers are divisible by 2^{100}?

OC669. Let $M_2(\mathbb{Z})$ be the set of 2×2 matrices with integer entries. Let $A \in M_2(\mathbb{Z})$ such that

$$A^2 + 5I = 0,$$

where $I \in M_2(\mathbb{Z})$ and $0 \in M_2(\mathbb{Z})$ denote the identity and null matrices, respectively. Prove that there exists an invertible matrix $C \in M_2(\mathbb{Z})$ with $C^{-1} \in M_2(\mathbb{Z})$ such that

$$CAC^{-1} = \begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix} \quad \text{or} \quad CAC^{-1} = \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}.$$

OC670. Prove that the arithmetic sequence $5, 11, 17, 23, 29, \ldots$ contains infinitely many primes.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2024**.

OC666. Les cases d'un tableau 1×10 sont numérotées de 1 à 10 dans l'ordre. Clarissa et Marissa partent de la case 1, sautent 9 fois vers les autres cases de façon à visiter chaque case une fois, et finissent à la case 10. Les sauts vers l'avant et vers l'arrière sont autorisés. Chaque saut de Clarissa a été effectué sur la même distance que le saut correspondant de Marissa. Cela signifie-t-il qu'elles ont toutes deux visité les cases dans le même ordre ?

OC667. Dans le quadrilatère convexe $ABCD$, AB et CD sont parallèles. De plus, $\angle DAC = \angle ABD$ et $\angle CAB = \angle DBC$. Est-ce que $ABCD$ est nécessairement un carré ?

OC668. Considérons tous les entiers positifs à 100 chiffres dont chacun des chiffres est 2, 3, 4, 5, 6, ou 7. Combien de ces entiers sont divisibles par 2^{100} ?

OC669. Soit $M_2(\mathbb{Z})$ l'ensemble des matrices 2×2 dont les composantes sont des entiers. Soit $A \in M_2(\mathbb{Z})$ tel que

$$A^2 + 5I = 0,$$

où $I \in M_2(\mathbb{Z})$ et $0 \in M_2(\mathbb{Z})$ désignent respectivement les matrices identité et nulle. Montrez qu'il existe une matrice inversible $C \in M_2(\mathbb{Z})$ avec $C^{-1} \in M_2(\mathbb{Z})$ telle que

$$CAC^{-1} = \begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix} \quad \text{ou} \quad CAC^{-1} = \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}.$$

OC670. Montrez que la suite arithmétique $5, 11, 17, 23, 29, \ldots$ contient une infinité de nombres premiers.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2023: 49(7), p. 359–360.

OC641. Let a, b, c be integers. Prove that there exists a positive integer n such that the number $n^3 + an^2 + bn + c$ is not a perfect square.

Originally Poland Mathematics Olympiad, 8th Problem, First Round 2017.

We received 12 submissions. We present 3 solutions.

Solution 1, by Oliver Geupel.

Let $f(x) = x^3 + ax^2 + bx + c$. We will show the stronger result that, for some integer n in $\{1, 2, 3, 4\}$, the number $f(n)$ is congruent to 2 or 3 modulo 4. Since every perfect square is congruent to either 0 or 1 modulo 4, the required property follows immediately. Observing that

$$f(4) \equiv c \pmod 4,$$

we need to consider only $c \in \{0, 1\}$. Next,

$$f(2) \equiv 2b + c \pmod 4$$

allows us to restrict attention to $b \in \{0, 2\}$. The following table surveys all cases, where all congruences are implicitly modulo 4.

$(b, c) \pmod 4$	$a \equiv 0$	$a \equiv 1$	$a \equiv 2$	$a \equiv 3$
$(0, 0)$	$f(3) \equiv 3$	$f(1) \equiv 2$	$f(1) \equiv 3$	$f(3) \equiv 2$
$(0, 1)$	$f(1) \equiv 2$	$f(1) \equiv 3$	$f(3) \equiv 2$	$f(3) \equiv 3$
$(2, 0)$	$f(1) \equiv 3$	$f(3) \equiv 2$	$f(3) \equiv 3$	$f(1) \equiv 2$
$(2, 1)$	$f(3) \equiv 2$	$f(3) \equiv 3$	$f(1) \equiv 2$	$f(1) \equiv 3$

This completes the proof.

Solution 2, by Vivek Mehra.

We prove the stronger result that $P(x) = ax^3 + bx^2 + cx + d$ cannot be a perfect square for all positive integers x (where a, b, c, d are given integers with $a > 0$). Suppose $P(x) = ax^3 + bx^2 + cx + d$ is a square for all positive integers x.

Let

$$P(x) = (F(x))^2 G(x), \qquad F(x), G(x) \in \mathbb{Z}[x],$$

where $G(x)$ is square-free in $\mathbb{Z}[x]$. Then $\deg G \in \{1, 3\}$. Suppose $\deg G = 1$. Then it is easy to see that $G(x)$ is not a square for infinitely many positive integers x. Also $(F(x))^2 = 0$ for only finitely many x. So $G(y)$ is not a square for some positive integer y such that $F(y) \neq 0$ and then $P(y) = (F(y))^2 G(y)$ is not a perfect square leading to a contradiction. So $\deg G = 3$. But by the argument given in solution to ***Crux*** problem OC124 (Vol 40 no. 5, p. 198-199), we have that $\deg G$ cannot be ≥ 2 and so $\deg G \neq 3$. So we get that $P(x)$ cannot be a square for all positive integers x.

Solution 3, by UCLan Cyprus Problem Solving Group.

This is known in an even more general setting: If $P(x) \in \mathbb{Z}[x]$ satisfies that $P(n)$ is a perfect k-th power whenever $n \in \mathbb{N}$, then $P(x) = Q(x)^k$ for some $Q(x) \in \mathbb{Z}[x]$. (See e.g. Part Eight, Problem 114 of G. Pólya and G. Szegő, *Problems and theorems in analysis Vol. II*, 1976.)

We provide a proof in our particular instance. It doesn't generalise but we believe it is simpler than the more general proof.

Let $P(x) = x^3 + ax^2 + bx + c$. We claim that there is an integer n and a prime $p \geq 5$ such that $p \mid Q(n)$.

If $c = 0$, then this is possible by taking any $p \geq 5$ and $n = p$. If $c \neq 0$, then

$$P(6cm) = c[6Q(m) + 1]$$

for some cubic polynomial $Q(x)$. So the claim holds by taking $n = 6cm$ for some m such that $Q(m) \neq 0$ and any prime p with $p \mid 6Q(m) + 1$.

Pick a prime p as in the above paragraph. If $p^2 \nmid P(n)$, then we are done so assume that $p^2 \mid P(n)$. Then

$$P(n+p) - P(n) \equiv p(3n^2 + 2an + b) \equiv pP'(n) \pmod{p^2} .$$

If $p \nmid P'(n)$, then $p \mid P(n+p)$ but $p^2 \nmid P(n+p)$. This implies that $P(n+p)$ is not a perfect square and we are done.

So we may assume that $p \mid P(n)$ and $p \mid P'(n)$. This implies that over $\mathbb{F}_p$ we can write

$$P(x) = (x-n)^2(x-m)$$

for some integer m. Let $k \not\equiv (n-m) \pmod{p}$ be a number which is not a perfect square modulo p. Such a number exists because every $p \geq 5$ has at least two non-quadratic residues. Then

$$P(k+m) \equiv k(k+m-n)^2 \pmod{p}$$

so it is not a perfect square modulo p. But then $P(k+m)$ is not a perfect square.

OC642. Determine if there exist positive integers n and k such that

$$\frac{n}{11^k - n}$$

is the square of an integer.

Originally Czech-Slovakia Math Olympiad, 4th Problem, Category A, Regional Round 2018.

We received 15 correct submissions. We present 2 solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

There is no such number. Indeed, assume by contradiction that $\frac{n}{11^k - n} = m^2$ for some positive integer m. Then $n = \frac{11^k m^2}{m^2+1}$.

Pick any prime p such that $p \mid (m^2+1)$. Then $p \nmid m^2$, so $p \mid 11^k$ and therefore $p = 11$. Then $m^2 \equiv -1 \pmod{11}$ which is a contradiction as -1 is not a perfect square modulo 11 (as $11 \equiv 3 \pmod 4$).

Solution 2, by Oliver Geupel.

We show that there are no such numbers. Suppose that k, m, and n are positive integers with the property that $n = \left(11^k - n\right) m^2$. Then $\left(m^2+1\right) n = 11^k m^2$. Since m^2 and m^2+1 are relatively prime, we deduce that m^2+1 is a divisor of 11^k. Hence, there is a positive integer q such that $m^2+1 = 11^q$. If q were even, say, $q = 2s$, we would obtain $(11^s - m)(11^s + m) = 1$, which is impossible. Thus q is odd. With the notation $q = 2s+1$, we find that

$$m^2 = 11^q - 1 = 10 \sum_{j=0}^{2s} 11^j,$$

so that m is divisible by 10. Let $m = 10t$. We conclude

$$0 \equiv 10t^2 \equiv \sum_{j=0}^{2s} 11^j \equiv 2s+1 \pmod{10},$$

a contradiction.

Editor's Comment. Roy Barbara and Konstantine Zelator proved the more general case: if $n, k \in \mathbb{N}$ and p is a prime, $p \equiv 3 \pmod 4$, then $\frac{n}{p^k - n}$ is not the square of an integer. Giuseppe Fera also made the same remark. The proof is done by contradiction and by using some considerations about the prime factorization of an integer. We leave the details to the reader. Also, Roy Barbara proved a stronger result: under the same hypothesis, the number $\frac{n}{p^k - n}$ is not the square of a rational number.

OC643. Find the smallest natural number n such that for every 3-colouring of the numbers $1, 2, \dots, n$ there are two (different) numbers of the same colour such that their positive difference is a perfect square.

Originally Czech-Slovakia Math Olympiad, 6th Problem, Category A, Final Round 2018.

We received 4 solutions, of which 3 were correct and complete. We present the solution by UCLan Cyprus Problem Solving Group.

We will show that the smallest such n is $n = 29$.

Note first that $n \geq 29$ since in the following 3-colouring of the first 28 numbers, no two numbers of the same colour differ by a perfect square: colour the numbers in $\{1, 4, 6, 9, 12, 14, 19, 24, 27\}$ red, the ones in $\{2, 5, 7, 10, 15, 17, 20, 22, 25, 28\}$ green, and the ones in $\{3, 8, 11, 13, 16, 18, 21, 23, 26\}$ blue.

To see that we cannot colour the first 29 numbers in such a way, observe that 10 and 17 must be similarly coloured. Indeed if they are differently coloured, then 1 must take a different colour to both, and so must 26. But then 1 and 26 would have the same colour, a contradiction. Similarly, the following pairs of numbers should be similarly coloured $(11, 18), (12, 19), (13, 20)$. (For the pair $(13, 20)$ we look at the numbers 4 and 29.)

Suppose without loss of generality that $10, 17$ are coloured red. Then $11, 18$ must have a different colour, say green. Then $12, 19$ cannot be red (since $19 - 10 = 9$), but they also cannot be green (since $12 - 11 = 1$). Say that they are both coloured blue. But then there is no available colour for 13 and 20. They cannot be red as $17 - 13 = 4$, they cannot be green as $20 - 11 = 9$ and they cannot be blue as $20 - 19 = 1$.

OC644. In a 2018×2018 chessboard, some of the cells are painted white, the rest are black. It is known that from this chessboard one can cut out a 10×10 square, all cells of which are white, and a 10×10 square, all cells of which are black. What is the smallest d for which it can be guaranteed that a 10×10 square can be cut out of it, in which the number of black and white cells differs by no more than d?

Originally Moscow Math Olympiad, 2nd Problem, Grade 10, 2018.

We received 2 correct submissions. We present both approaches.

Solution 1, by UCLan Cyprus Problem Solving Group.

For each 10×10 square S, let W_S be the number of white squares in S and B_S the number of black squares in S. Let $d_S = W_S - B_S$. We know that there is a square S_1 with $d_{S_1} = 100$ and a square S_2 with $d_{S_2} = -100$.

Create a graph with vertices the 10×10 squares, where two such squares are neighbours if and only if they intersect in a 9×10 or 10×9 rectangle. In this graph there is a path from S_1 to S_2.

So we can find a sequence of 10×10 squares, $T_1, T_2, \ldots, T_k$ such that the sequence $d_{T_1}, d_{T_2}, \ldots, d_{T_k}$ starts with 100, ends with -100 and consecutive elements differ by at most 20. So one of these elements must belong in the set $\{-10, -9, \ldots, 9, 10\}$.

So we can find a 10×10 square in which the number of white and black squares differ by at most 10.

We can not improve on this. Consider for example the following 'diagonal colouring' of the square: Colour the cell (i, j) (where $1 \leq i, j \leq 100$) white if $i + j \leq 100$ and black otherwise. Then every 10×10 square is also diagonally coloured. The diagonals have sizes $1, 2, \ldots, 9, 10, 9, \ldots, 2, 1$ with all cells in a diagonal having the same colour. The first few (maybe none) of these diagonals are all coloured white and the rest black. The colour which appears in the diagonal of size 10 occurs at least 10 more times than the other colour in this square.

Solution 2, by Oliver Geupel.

We prove that the smallest d is equal to 10.

Assume that we have a frame that encloses a 10×10 square. A move is a translation of the frame on the board by one column to the left or right or by one row up or down. Starting with the frame on a 10×10 white square, an appropriate finite sequence of moves, say, of m moves, will bring the frame to enclosing a 10×10 black square. Consider the corresponding sequence $b_0, b_1, b_2, \ldots, b_m$ where $b_0 = 0$ and, for $1 \leq k \leq m$, b_k is the number of black cells inside the frame after the k^{th} move. In each move, any 10 cells are replaced by other 10 cells, so that $|b_k - b_{k-1}| \leq 10$ for $1 \leq k \leq m$. Since $b_0 = 0$ and $b_m = 100$, there exists an index k with the property that $45 \leq b_k \leq 55$. It follows that in the framed 10×10 square after the k^{th} move the numbers of black and white cells differ by at most 10.

We finish the proof by paiting the cells of a board so that the smallest number d with the desired property is equal to 10. Let us paint the 10 left columns all white and call them the white region. Further, paint the 2007 right columns all black and call them the black region. Finally, paint the cells of the column between the black and the white region alternating black and white. Then, for every 10×10 square which is fully contained in the white or black region, the numbers of black and white cells differ by 100. A square that is not completely contained in one of the regions, consists of some n white columns, some $9 - n$ black columns and an alternatingly coloured border column, resulting in $10n + 5$ white and $95 - 10n$ black cells. For $n \in \{4, 5\}$ we reach the minimum $d = 10$.

OC645. Prove that if $a, b, c \geq 0$ and $a + b + c = 3$, then

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}.$$

Originally Romania Math Olympiad, 2nd Problem, Grade 9, Final Round 2018.

We received 15 submissions. We present the solution by Theo Koupelis.

The desired inequality is equivalent to

$$\frac{a-1}{1+b}+\frac{b-1}{1+c}+\frac{c-1}{1+a}\geq 0,$$

or after clearing denominators,

$$\begin{aligned}&(a^2-1)(c+1)+(b^2-1)(a+1)+(c^2-1)(b+1)\geq 0 \Longleftrightarrow\\&a^2c+b^2a+c^2b+[a^2+b^2+c^2-(a+b+c)-3]\geq 0.\end{aligned}$$

But

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=3+2(a+b+c)-2(ab+bc+ca).$$

Substituting, we get the equivalent inequality

$$(a^2c+c)+(b^2a+a)+(c^2b+b)\geq 2ac+2ab+2bc,$$

which is obvious by AM-GM. Equality occurs when $a=b=c=1$.

Reading a Math Book

Solutions to No. 2

Yagub Aliyev

The statements of the problems in this section originally appear in 2023: 49(5), p. 266-273. The problems were selected from [1].

S6. Prove that if the three vertices A, A', A'' of a changing triangle are moving along three fixed and concurrent lines u, u', u'', respectively, and its two sides $A'A''$ and $A''A$ are rotating about two fixed points O and O', respectively, then the third side AA' is also rotating about a fixed point O'', which is on the line OO'. [1, Problem 51]

Solution.

The triangles obtained by moving the vertices of $\triangle AA'A''$ along the lines u, u', u'' are in central perspective from the point of intersection of u, u', u''. Desargues' theorem then guarantees that such triangles are in axial perspective from the line OO'.

S7. Prove that the lines joining a point on a circle with the endpoints of a chord of the circle divide the diameter perpendicular to the chord harmonically. ([1], Problem 32).

Solution 1.

Let AB be the given chord, CD the diameter and E the given point. Let lines AE and BE intersect line CD at points G and F. The result follows from the easily provable fact that EC and ED are internal and external bisectors of $\angle GEF$:

$$\frac{EG}{EF} = \frac{CG}{CF} \quad \text{and} \quad \frac{EG}{EF} = \frac{DG}{DF} \quad \text{so} \quad \frac{GC}{GD} = \frac{FC}{FD}.$$

Solution 2, by Oliver Geupel.

Denote the given circle, the point on it, the chord, and the perpendicular diameter by Γ, P, AB, and CD, respectively. Suppose that the lines AP and BP meet the line CD at points Q and R, respectively. Without loss of generality assume that the points A and P lie on a common semicircle CD of Γ. Then Q is an exterior point and R an interior point of the diameter CD.

Consider the problem in the complex plane where Γ is the unit circle and where a lowercase letter denotes the affix of a point designated by the corresponding uppercase letter (so, the affix of the point $P = (a, b)$ is the complex number $p = a + bi$). We may assume that $b = \overline{a} = 1/a$, $c = 1$, and $d = -1$. It is known that the intersection of two arbitrary chords XY and ZW has affix

$$\frac{xy(z+w)-zw(x+y)}{xy-zw}. \tag{1}$$

Therefore, since $AP\cap CD=Q$ and $BP\cap CD=R$,

$$q=\frac{ap(c+d)-cd(a+p)}{ap-cd}=\frac{a+p}{ap+1}$$

and

$$r=\frac{bp(c+d)-cd(b+p)}{bp-cd}=\frac{b+p}{bp+1}=\frac{1+ap}{p+a}=\frac{1}{q}.$$

Hence,

$$\frac{CQ}{DQ}=\frac{q-1}{q+1}=\frac{1-\frac{1}{q}}{1+\frac{1}{q}}=\frac{1-r}{r+1}=\frac{CR}{DR},$$

which completes the proof.

S8. Given are a fixed circle and two fixed points A and B on it. On the same circle two arbitrary points C and D are chosen. Let $M=AC\cap BD$ and $N=AD\cap BC$. Prove that as the points C and D change, the line MN passes through a fixed point of the plane. [1, Problem 162]

Solution by Oliver Geupel.

We prove that MN passes through the intersection T of the tangents at A and B. Consider the problem in the complex plane where A, B, C, and D lie on the unit circle. Let a, b, c, d, m, n, and t be the affixes of A, B, C, D, M, N, and T, respectively. Using equation (1) from the previous problem, we get

$$t=\frac{2ab}{a+b},\qquad m=\frac{f(c,d)}{g(c,d)},\qquad n=\frac{f(d,c)}{g(d,c)},$$

where

$$f(x,y)=ax(b+y)-by(a+x)$$

and

$$g(x,y)=ax-by.$$

The points M, N, and T are collinear if

$$(t-m)(\overline{t}-\overline{n})=(t-n)(\overline{t}-\overline{m}). \tag{2}$$

For a, b, c and d lie on the unit circle, we have

$$\overline{a}=\frac{1}{a},\ \overline{b}=\frac{1}{b},\ \overline{c}=\frac{1}{c},\ \overline{d}=\frac{1}{d},$$

which gives

$$\overline{t}=\frac{2}{a+b},\qquad \overline{m}=\frac{h(c,d)}{g(c,d)},\qquad \overline{n}=\frac{h(d,c)}{g(d,c)},$$

where

$$h(x, y) = a - b + x - y.$$

The collinearity condition thus simplifies to

$$F(c, d)G(c, d) = F(d, c)G(d, c), \tag{3}$$

where

$$F(x, y) = 2ab \cdot g(x, y) - (a + b) \cdot f(x, y)$$

and

$$G(x, y) = 2g(y, x) - (a + b) \cdot h(y, x).$$

With a little algebra, we easily obtain

$$F(x, y) = (a - b)\left(ab(x + y) - (a + b)xy\right)$$

and

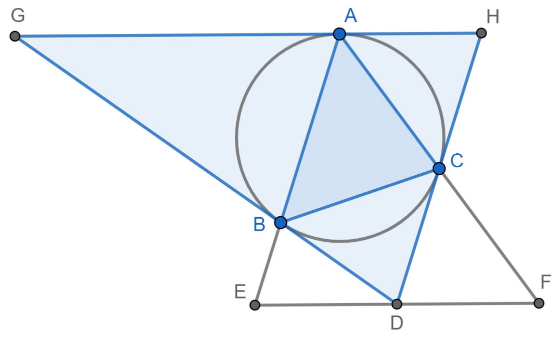
$$G(x, y) = (a - b)(x + y - a - b).$$

This shows that the polynomials F and G are symmetric in the variables x and y. Hence (1) holds, which completes the proof.

Note that the problem can also be solved using Pascal's theorem by applying it to the hexagon $AACBBD$.

S9. Let the tangent lines of the circumcircle of $\triangle ABC$ at the points B and C intersect at point D. Through D draw the line that is parallel to the tangent line of the circle at A. This line intersects lines AB and AC at points E and F, respectively. Show that D is the midpoint of the segment EF. [1, Problem 159]

Solution.



Denote the intersection points of the tangent of the circumcircle at point A with the tangents at B and C by G and H, respectively. It is obvious that $GA = GB$, $HA = HC$, and $DB = DC$. By similarity of triangles BDE and BGA, and of triangles CDF and CHA, we obtain

$$\frac{BD}{BG} = \frac{DE}{GA}$$

and

$$\frac{CD}{CH} = \frac{DF}{HA}.$$

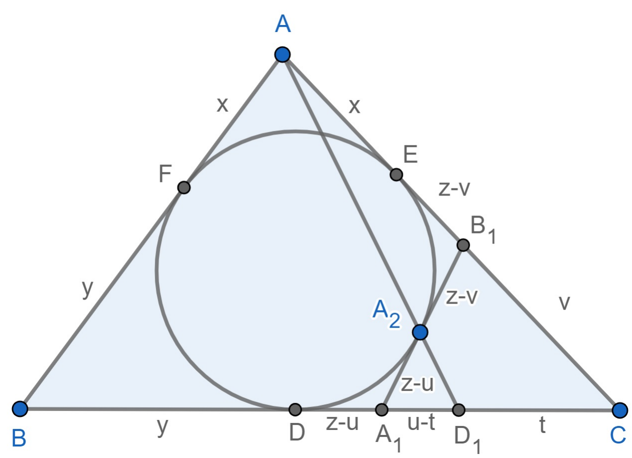
This means that $BD = DE$ and $CD = DF$, whence $DE = DF$.

S10. Given are a triangle ABC and its incircle touching the side BC at the point D. From a point A_1 on the side BC, a second tangent A_1A_2 of the incircle is drawn and the tangency point A_2 is connected with A by a line. The line AA_2 intersects the side BC at D_1. Prove that

$$\frac{DB}{DC} \cdot \frac{D_1B}{D_1C} = \left(\frac{A_1B}{A_1C}\right)^2.$$

[1, Problem 35]

Solution.



Denote $AB = c = x + y$, $AC = b = x + z$, and $BC = a = y + z$ as in the picture. Denote also $CA_1 = u$, $CD_1 = t$, and $CB_1 = v$. Then $DA_1 = z - u$, $A_1D_1 = u - t$, and $EB_1 = z - v$. By Menelaus' theorem,

$$\frac{t}{u-t} \cdot \frac{z-u}{z-v} \cdot \frac{z-v+x}{z+x} = 1.$$

By solving this for t, we find:

$$t = -\frac{u\left(vx + zv - zx - z^2\right)}{uv - ux - uz - vx - 2zv + 2zx + 2z^2}. \tag{4}$$

On the other hand, it is known that

$$(a+b+c)uv - 2ab(u+v) + ab(a+b-c) = 0,$$

(see [3, p. 101, Exercise 47]) which can be written as

$$(x+y+z)uv-(y+z)(x+z)(u+v)+(y+z)(x+z)z=0.$$

By solving this for x, we find

$$x=-\frac{uvy+uvz-uyz-u\,z^2-vyz-v\,z^2+z^2y+z^3}{uv-yu-uz-yv-zv+zy+z^2}. \qquad (5)$$

By substituting (5) in (4), and using computer assistance we can find

$$t=\frac{y\,u^2}{(u-z)^2+yz},$$

which we can substitute into the claimed identity

$$\frac{y}{z}\cdot\frac{(y+z-t)}{t}=\frac{(y+z-u)^2}{u^2},$$

to check that it is correct.

References

[1] M.P. Chernyaev, *Problems in synthetic geometry*, Rostov University, Rostov-on-Don, 1961 (in Russian).

[2] Y. Aliyev, Reading a Math Book: No. 2 (M.P. Chernyaev: Problems in synthetic geometry), *Crux Mathematicorum*, 49(5), May 2023, 266–272.

[3] M.B. Balk, V.G. Boltyanskiy, Geometriya mass, Library Kvant 61, 1987 (in Russian).

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2024**.

4911. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Given a triangle ABC with $\angle BAC = 60^\circ$, let P denote one of the points where its circumcircle intersects the perpendicular bisector of AC, and T denote the foot of the perpendicular from P to the bisector of $\angle BAC$. Prove that PT is tangent to the nine-point circle of ΔABC at T.

4912. *Proposed by Michel Bataille.*

Let P be a point inside an equilateral triangle ABC with side a. Prove that PA, PB and PC are the sides of a triangle $\mathcal{T}$ and that $\mathcal{T}$ has an angle of 60° if and only one of its medians has length $\frac{a}{2}$.

4913. *Proposed by Albert Natian.*

Suppose the continuous function f satisfies the integral equation

$$\int_0^{xf(7)} f\left(\frac{tx^2}{f(7)}\right) dt = 3f(7)\,x^4.$$

Find $f(7)$.

4914. *Proposed by Ivan Hadinata.*

Let $\mathbb{R}_{\geq 0}$ be the set of all non-negative real numbers. Find all possible monotonically increasing $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

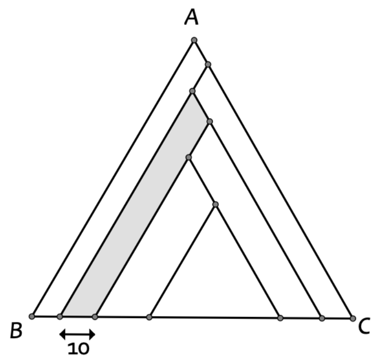
$$f(x^2 + y + 1) = xf(x) + f(y) + 1, \quad \forall x, y \in \mathbb{R}_{\geq 0}.$$

4915. *Proposed by Michel Bataille.*

Let $S_n = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+n+1)}$, where n is a nonnegative integer. Find real numbers a, b, c such that $\displaystyle\lim_{n\to\infty} \left(n^3 S_n - (an^2 + bn + c)\right) = 0$.

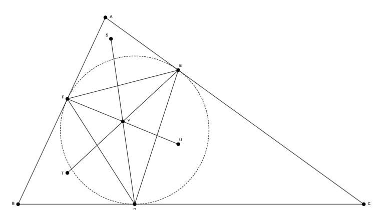
4916. *Proposed by Arsalan Wares.*

Equilateral triangle ABC is split into 5 isosceles trapezoids and a smaller equilateral triangle, all of the same area. One of the side lengths of the shaded trapezoid is 10. Determine the exact length of AB.



4917. *Proposed by Pericles Papadopoulos.*

Let D, E and F be the points of contact of the incircle of a triangle ABC with the sides BC, AC and AB, respectively. Let S, T and U be the orthocenters of the triangles EAF, FBD and DCE, respectively. Prove that SD, TE and UF concur at a point.



4918. *Proposed by Yagub Aliyev.*

Let $L = \lim_{\lambda\to+\infty} \dfrac{\lambda^{x^2}}{\int_a^b \lambda^{t^2}dt}$.

a) Show that if $0 \le a \le x < b$, then $L = 0$.

b) Show that if $0 \le a < x = b$, then $L = +\infty$.

4919. *Proposed by Daniel Sitaru.*

If $A, B \in M_6(\mathbb{R})$ are matrices such that

$$A^2 + B^2 = AB + A + B - I_6,$$

then

$$\det(BA - AB) \ge 0$$

4920. *Proposed by Ángel Plaza.*

If $k > 1$ and $n \in \mathbb{N}$, evaluate $\displaystyle\int_0^1 \frac{\log(1 + x^k + x^{2k} + \cdots + x^{nk})}{x} dx.$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2024**.

4911. *Proposée par Mihaela Berindeanu, modifié par le comité de rédaction.*

Soit ABC un triangle tel que $\angle BAC = 60°$ et soient P un des points d'intersection du cercle circonscrit et de la bissectrice orthogonale de AC, puis T le point d'arrivée de la perpendiculaire de P vers la bissectrice de $\angle BAC$. Démontrer que PT est tangent en T au cercle des neuf points de ΔABC.

4912. *Proposée par Michel Bataille.*

Soit P un point à l'intérieur d'un triangle ABC équilatéral de côté a. Démontrer que PA, PB and PC sont les côtés d'un triangle $\mathcal{T}$, puis que $\mathcal{T}$ a un angle de $60°$ si et seulement si une de ses médianes est de longueur $\frac{a}{2}$.

4913. *Proposée par Albert Natian.*

Supposer que la fonction continue f est solution de l'équation intégrale

$$\int_0^{xf(7)} f\left(\frac{tx^2}{f(7)}\right) dt = 3f(7)\, x^4.$$

Déterminer $f(7)$.

4914. *Proposée par Ivan Hadinata.*

Soit $\mathbb{R}_{\geq 0}$ l'ensemble des nombres réels non négatifs. Déterminer toutes les fonctions non décroissantes $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ telles que

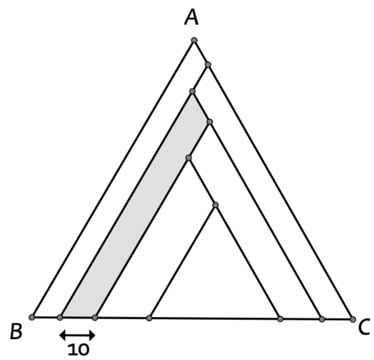
$$f(x^2 + y + 1) = xf(x) + f(y) + 1, \quad \forall x, y \in \mathbb{R}_{\geq 0}.$$

4915. *Proposée par Michel Bataille.*

Soit $S_n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+n+1)}$, où n est un entier non négatif. Déterminer des nombres réels a, b, c tels que $\lim_{n\to\infty} \left(n^3 S_n - (an^2 + bn + c)\right) = 0$.

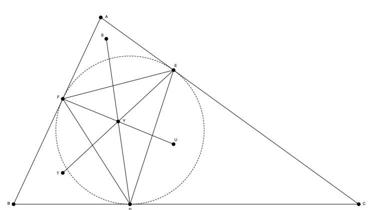
4916. *Proposée par Arsalan Wares.*

Le triangle équilatéral ABC est divisé en 5 trapézoïdes isocèles et un triangle équilatéral, tous de même surface. Un côté d'un des trapézoïdes est de longueur 10, tel qu'indiqué. Déterminer la valeur exacte de la longueur de AB.



4917. *Proposée par Pericles Papadopoulos.*

Soient D, E et F les points de contact du cercle inscrit du triangle ABC avec ses côtés BC, AC et AB, respectivement, puis soient S, T et U les orthocentres des triangles EAF, FBD et DCE, respectivement. Démontrer que SD, TE et UF sont concourantes.



4918. *Proposée par Yagub Aliyev.*

Soit $L = \lim_{\lambda\to+\infty} \frac{\lambda^{x^2}}{\int_a^b \lambda^{t^2} dt}$.

a) Démontrer que si $0 \le a \le x < b$, alors $L = 0$.

b) Démontrer que si $0 \le a < x = b$, alors $L = +\infty$.

4920. *Proposée par Daniel Sitaru.*

Si $A, B \in M_6(\mathbb{R})$ sont des matrices telles que

$$A^2 + B^2 = AB + A + B - I_6,$$

alors

$$\det(BA - AB) \geq 0$$

4920. *Proposée par Ángel Plaza.*

Si $k > 1$ et $n \in \mathbb{N}$, évaluer $\displaystyle\int_0^1 \frac{\log(1 + x^k + x^{2k} + \cdots + x^{nk})}{x} dx.$

SOLUTIONS

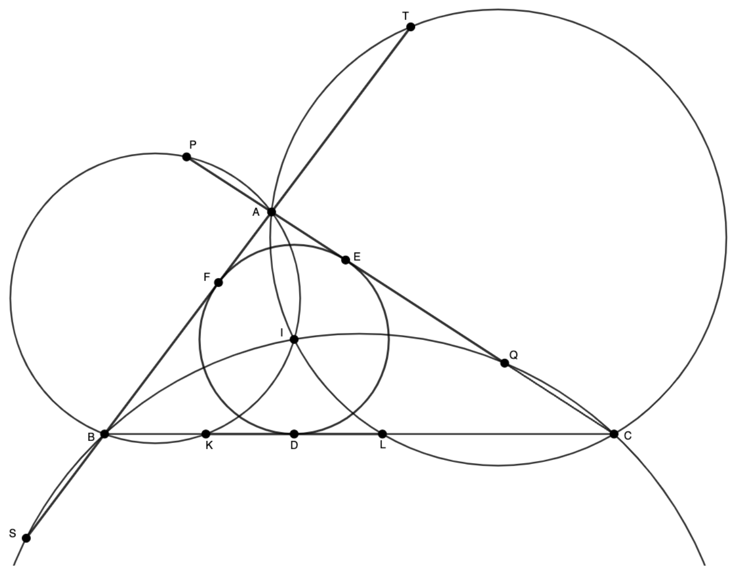
No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2023: 49(7), p. 375–380.

4861. *Proposed by Pericles Papadopoulos.*

Let I be the incenter of a triangle ABC and let D, E and F be the points of contact of the incircle of the triangle with the side BC, AC and AB respectively. The circle AIB meets the sides BC and AC at points K and P respectively; the circle AIC meets the sides BC and AB at points L and T respectively; the circle BIC meets the sides AC and AB at points Q and S, respectively. Prove the following:

- $KL + PQ + ST = AB + BC + AC$
- Points D, E and F are the midpoints of KL, PQ and ST respectively.



All 16 submissions that we received are correct; we provide a composite of the similar solutions from Michal Adamaszek, Oliver Geupel, Antoine Mhanna and the UCLan Cyprus Problem Solving Group.

As usual, let r be the radius of the incircle of ΔABC, $a = BC$, $b = CA$, $c = AB$, and $2s = a+b+c$. Let's begin with an outline of the argument. Since A, B, K, I are concyclic and D is on the extension of the ray BK, then $\angle DKI = \angle BAI = \frac{A}{2}$. Since also $ID = IF = r$, then the right-angled triangles IKD and IAF are congruent. We therefore have $KD = AF$, where $AF = s - a$ is the length of the tangent from the vertex A to the incircle. We similarly have

$$\begin{aligned}KD = LD = AF = AE = s - a\\ QE = PE = BD = BF = s - b\\ TF = SF = CE = CD = s - c.\end{aligned}$$

Note that $KL = KD + DL, PQ = PE + EQ$, and $ST = SF + FT$. Both parts of the problem now follow immediately.

For a detailed proof, one must observe that since the incenter I is always inside the triangle, any circle through two vertices and I must intersect the sides through the third vertex at points interior to one side and exterior to the other (except, of course, when the circle is tangent to those two sides). Any informal argument would therefore depend on how the five relevant points are arranged on each circle. Geupel avoided the difficulty through the use of directed angles. Here is his argument:

We use the notation $\angle(p, q)$ to denote the directed angle from the line p to the line q, taken modulo $180°$. Because A, B, I, and K are concyclic and also A, C, I, and L are concyclic, we can calculate

$$\angle(LK, IK) = \angle(BK, IK) = \angle(BA, IA) = \angle(IA, CA) = \angle(IL, CL) = \angle(IL, KL).$$

Hence, triangle IKL is isosceles with $\angle IKL = \angle ILK = A/2$, and the foot D of its altitude DI is the midpoint of KL. Analogously, E and F are the midpoints of PQ and ST, respectively. Triangles AEI and KDI are congruent by comparison of two angles and $EI = DI$. Therefore, $KD = AE = s - a$ and $KL = 2KD = 2(s - a)$. With the similar identities $PQ = 2(s - b)$ and $ST = 2(s - c)$, we finally conclude that

$$KL + PQ + ST = 2(s - a + s - b + s - c) = 2s = AB + BC + AC.$$

4862. *Proposed by Michel Bataille.*

Let m be a nonnegative integer. Find

$$\lim_{n\to\infty} \frac{1}{2^n n^m} \sum_{k=0}^{n} \binom{m+k}{k}\binom{m+n+1}{n-k}.$$

We received 10 submissions, 8 of which were correct. We present the joint solution of Ulrich Abel and Vitaliy Kushnirevych.

By taking the Cauchy product, we see that the sum

$$S_m(n) := \sum_{k=0}^{n} \binom{m+k}{k}\binom{m+n+1}{n-k}$$

is the coefficient of x^n in the power series expansion of

$$\sum_{k=0}^{\infty}\binom{m+k}{k}x^k\cdot\sum_{k=0}^{\infty}\binom{m+n+1}{k}x^k=(1-x)^{-m-1}\cdot(1+x)^{m+n+1}$$

where both series are absolutely convergent for $|x|<1$. Using $1+x=2+(x-1)$ and the binomial formula, this coefficient is

$$\begin{aligned}S_m(n)&=\frac{1}{n!}\,\frac{d^n}{dx^n}\bigg|_{x=0}(-1)^{m+1}\sum_{k=0}^{m+n+1}\binom{m+n+1}{k}2^k\,(x-1)^{n-k}\\&=\frac{1}{n!}\,(-1)^{m+1}\sum_{k=0}^{m+n+1}\binom{m+n+1}{k}2^k\,(n-k)^{\underline{n}}\,(-1)^{-k}\end{aligned}$$

with the falling factorial

$$(n-k)^{\underline{n}}:=(n-k)\,(n-1-k)\ldots(1-k)=\begin{cases}n! & (k=0),\\ 0 & (1\le k\le n),\\ (-1)^n\,\frac{(k-1)!}{(k-1-n)!} & (k>n).\end{cases}$$

After some manipulations (namely, pulling out the $k=0$ term and writing $k=\ell+n+1$ in the tail) we obtain

$$S_m(n)=(-1)^{m+1}+2^{n+1}\sum_{\ell=0}^{m}(-1)^{m-\ell}\binom{m+n+1}{m-\ell}\binom{\ell+n}{\ell}2^\ell.$$

Observing that $\binom{m+n+1}{m-\ell}\binom{\ell+n}{\ell}$ is a polynomial of degree m in the variable n with leading coefficient $1/\left((m-\ell)!\ell!\right)$ we infer that

$$\lim_{n\to\infty}\frac{1}{2^n n^m}S_m(n)=\frac{2}{m!}\sum_{\ell=0}^{m}(-1)^{m-\ell}\binom{m}{\ell}2^\ell=\frac{2}{m!}\,(2-1)^m=\frac{2}{m!}.$$

Editor's Comments. We chose to feature this approach because it tells us that

$$S_m(n)=(-1)^{m+1}+\frac{2^{n+1}}{m!}P_m(n)$$

where

$$P_m(n)=\sum_{\ell=0}^{m}(-1)^{m-\ell}\binom{m}{\ell}\frac{(n+m+1)!}{(n+\ell+1)n!}2^\ell$$

is a monic polynomial of degree m with integer coefficients. For example,

$$\begin{aligned}P_0(n)&=1\\P_1(n)&=n\\P_2(n)&=n^2+n+2\\P_3(n)&=n^3+3n^2+8n\\P_4(n)&=n^4+6n^3+23n^2+18n+24\\P_5(n)&=n^5+10n^4+55n^3+110n^2+184n\end{aligned}$$

and so on. The majority of solvers rewrote $S_m(n)$ in terms of the quantity

$$I_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{m+k+1} = \int_0^1 x^m (1+x)^n \, dx$$

and continued their analyses from there. A few others found the generating-function identity

$$\sum_{n=0}^{\infty} S_m(n) x^n = \frac{1}{(1-x)(1-2x)^{m+1}} = \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} \binom{m+i}{i} 2^i \Big) x^n$$

which lends itself very well to elementary reasoning.

4863. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

In a parallelogram $ABCD$, let E be the point where the diagonal BD is tangent to the incircle of ΔABD. If r_1 and r_2 are the inradii of the triangles DEC and BEC, prove that $\frac{r_1}{r_2} = \frac{DE}{EB}$.

We received solutions from 17 solvers, all correct. The following is by Miguel Amengual Covas.

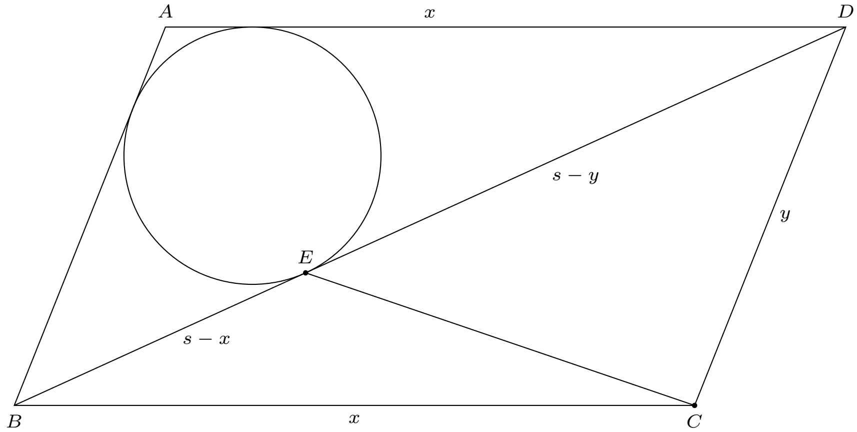
Solution. Let s be the semiperimeter of $\triangle ABD$ and let the two pairs of equal sides BC, DA and AB, CD of $ABCD$ be labeled x and y, respectively.

In $\triangle ABD$, then

$$BE = s - x \qquad ED = s - y,$$

implying that triangles BEC and DEC have equal perimeters (being the common perimeter $= s + CE$). Since the area of a triangle is equal to the product of the inradius and the semiperimeter, the areas of triangles with equal perimeters are proportional to the inradii of the triangles, hence

$$\frac{[DEC]}{[BEC]} = \frac{r_1}{r_2}.$$



Now, the areas of triangles with equal altitudes are proportional to the bases of the triangles, hence
$$\frac{[DEC]}{[BEC]} = \frac{DE}{EB},$$
and therefore
$$\frac{DE}{EB} = \frac{r_1}{r_2},$$
as desired.

4864. *Proposed by Goran Conar.*

Let a, b, c be side-lengths of an arbitrary three-dimensional box, and D the length of its main diagonal. Prove

$$\sqrt{1+a} + \sqrt{1+b} + \sqrt{1+c} \geq \frac{(a+b+c)^2}{D^2} \cdot \sqrt{1 + \frac{D^2}{a+b+c}}.$$

When does the equality occur?

We received 7 submissions, all correct. We presents the solution by Theo Koupelis.

Using Minkowski's inequality we get

$$\sqrt{1+a} + \sqrt{1+b} + \sqrt{1+c} \geq \sqrt{9 + (\sqrt{a} + \sqrt{b} + \sqrt{c})^2}.$$

With $D^2 = a^2 + b^2 + c^2$, it is sufficient then to show that

$$(a^2+b^2+c^2)^2 \left[9 + (\sqrt{a} + \sqrt{b} + \sqrt{c})^2\right] \geq (a+b+c)^4 + (a^2+b^2+c^2)(a+b+c)^3.$$

This is obvious because

$$3(a^2+b^2+c^2) \geq (a+b+c)^2$$

by AM-GM, and

$$(a^2+b^2+c^2)(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 \geq (a+b+c)^3$$

by Hölder's inequality. Equality occurs when $a = b = c$.

4865. *Proposed by George Apostolopoulos.*

Let ABC be an acute triangle with inradius r and circumradius R. Prove that

$$\frac{(\sec A)^{\cos A} + (\sec B)^{\cos B} + (\sec C)^{\cos C}}{\sec A + \sec B + \sec C} < \frac{5R - r}{12r}.$$

We receive 10 submissions, of which 2 were incomplete or incorrect. We present 2 solutions.

Solution 1, by Theo Koupelis, slightly modified by the Editorial Board.

Let $f(x) = 1/x^x$, where $x > 0$.

We have $f'(x) = -(1 + \ln x)/x^x$ and $f''(x) = -[1 - x(\ln x + 1)^2]/x^{x+1}$. Thus, the function $f(x)$ is concave in $(0, 1)$, with a maximum equal to $e^{1/e}$ at $x = e^{-1}$, and continuously decreasing, and convex for $x > 1$, with $f(1) = 1$.

Using Jensen's inequality we get

$$(\sec A)^{\cos A} + (\sec B)^{\cos B} + (\sec C)^{\cos C} \leq 3 \cdot \left(\frac{3}{\cos A + \cos B + \cos C} \right)^{(\cos A + \cos B + \cos C)/3}.$$

As $e^{1/e} \approx 1.444667 < \frac{3}{2}$,

$$(\sec A)^{\cos A} + (\sec B)^{\cos B} + (\sec C)^{\cos C} \leq 3 \cdot e^{1/e} < \frac{9}{2}.$$

But

$$\cos A + \cos B + \cos C = 1 + 2 \cos \frac{A+B}{2} \cdot \left(\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right),$$

and thus $1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$. The maximum occurs when the triangle is equilateral and the minimum is approached when two of the angles tend to $\frac{\pi}{2}^-$.

Also, from Cauchy-Schwarz inequality we get

$$\sec A + \sec B + \sec C \geq \frac{9}{\cos A + \cos B + \cos C} \geq 6.$$

Therefore,

$$\frac{(\sec A)^{\cos A} + (\sec B)^{\cos B} + (\sec C)^{\cos C}}{\sec A + \sec B + \sec C} < \frac{9}{12} \leq \frac{5R - r}{12r},$$

where in the last step we used Euler's inequality $R \geq 2r$.

Solution 2, by Marie-Nicole Gras.

We put

$$F = \frac{1}{\cos A^{\cos A}} + \frac{1}{\cos B^{\cos B}} + \frac{1}{\cos C^{\cos C}}.$$

We consider the function $f(x) = x^x = \exp\big(x \log(x)\big)$, where $x = \cos A$; since $\triangle ABC$ is acute, we have $0 < x < 1$.

We have $f'(x) = \big(\log(x) + 1\big) f(x)$; for $0 < x < \mathrm{e}^{-1}$, then $f'(x) < 0$, $f'(\mathrm{e}^{-1}) = 0$ and for $x > \mathrm{e}^{-1}$, then $f'(x) > 0$; we deduce that $x^x \geq (\mathrm{e}^{-1})^{\mathrm{e}^{-1}}$, and

$$\frac{1}{x^x} \leq e^{\mathrm{e}^{-1}} \approx 1.444667.$$

We deduce that $F < 4.5$.

The Cauchy-Schwarz inequality implies

$$\left(\cos A + \cos B + \cos C\right)\left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}\right) \geq 9;$$

it follows

$$\frac{F}{\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}} < 4.5 \times \frac{\cos A + \cos B + \cos C}{9} = \frac{\cos A + \cos B + \cos C}{2} \cdot$$

However, it is well known that $\cos A + \cos B + \cos C = \frac{R+r}{R}$; then, to prove the required inequality, it is enough to prove that

$$\frac{R+r}{2R} \leq \frac{5R-r}{12r} \cdot \qquad (1)$$

Let $H = 5R^2 - 7Rr - 6r^2$; then $H = (5R+3r)(R-2r)$, and Euler's inequality implies $F \geq 0$, whence (1) and the result.

4866. *Proposed by Ivan Hadinata.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equation

$$f(xy + f(f(y))) = xf(y) + y$$

holds for all real numbers x and y.

We received 21 submissions and they were all complete and correct. We present the following two solutions by the majority of solvers.

We show that the only solution to the given functional equation is $f(x) = x$. By taking $x = 0$, we have $f(f(f(y))) = y$ for each $y \in \mathbb{R}$, and thus f is bijective. By taking $y = 0$, we have $0 = f(f(f(0)) = xf(0)$ for all $x \in \mathbb{R}$, and thus $f(0) = 0$. To show that $f(x) = x$, there are multiple different ways.

Solution 1.

For each $y \neq 0$, by taking $x = -f(f(y))/y$, we get

$$0 = f(0) = -\frac{f(f(y))f(y)}{y} + y$$

which leads to

$$f(f(y))f(y) = y^2. \qquad (1)$$

Since f is bijective, by replacing y with $f(y)$ in equation (1), we get

$$yf(f(y)) = f(f(f(y)))f(f(y)) = f(y)^2 \qquad (2)$$

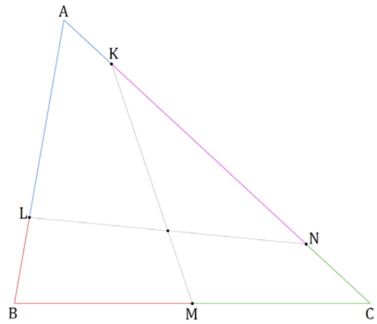
for each $y \neq 0$. Taking the ratio between equation (1) and equation (2), for each $y \neq 0$, we have $f(y)/y = y^2/f(y)^2$, which implies that $f(y) = y$.

Solution 2.

Choosing $y = 1$ leads to $f(x + c) = xf(1) + 1$, where $c = f(f(1))$. Replacing x with $x - c$, we obtain
$$f(x) = (x - c) f(1) + 1.$$
It follows that $0 = f(0) = -cf(1) + 1$. Hence, $f(x) = ax$, where $a = f(1)$. Inserting this into the functional equation, we get
$$a\left(xy + a^2 y\right) = axy + y.$$
It is easy to derive that $a = 1$. We conclude that $f(x) = x$.

4867. *Proposed by Thanos Kalogerakis.*

Consider a triangle ABC with $|AC| > |BC| > |AB|$ and let M be the midpoint of BC. Let K, L and N be points on the sides of ABC (see the figure) such that the points K, L, M, N divide the perimeter of ABC into 4 equal parts. Prove that KM bisects LN.



We received 17 submissions, all of which are correct and complete. We present here two solutions with different approaches, slightly modified by the Editorial Board.

Solution 1, by Madhav R. Modak.

Let a, b, c be the lengths of the sides BC, CA, AB. Thus $b > a > c$. Let, with A as origin, $\mathbf{u}, \mathbf{v}$ denote the unit vectors along $\overline{AB}$ and $\overline{AC}$ respectively. Let $t = (a + b + c)/4$. By data, $MC + CN = t$ so that
$$CN = t - a/2 = (b + c - a)/4.$$
Similarly, $BL = (b + c - a)/4$. Also,
$$AN = b - CN = (a + 3b - c)/4$$
and
$$AL = c - BL = (a - b + 3c)/4.$$
Next,
$$AK = b - CK = b - (t + CN) = (b - c)/2.$$

So the position vectors of points B, C, L, N, K, and M are respectively

$$\begin{array}{lll} \mathbf{b} = c\,\mathbf{u}, & \mathbf{c} = b\,\mathbf{v}, & \mathbf{l} = \dfrac{a-b+3c}{4}\,\mathbf{u}, \\ \mathbf{n} = \dfrac{a+3b-c}{4}\,\mathbf{v}, & \mathbf{k} = \dfrac{b-c}{2}\,\mathbf{v}, & \mathbf{m} = \dfrac{1}{2}(\mathbf{b}+\mathbf{c}) = \dfrac{c}{2}\,\mathbf{u} + \dfrac{b}{2}\,\mathbf{v}. \end{array}$$

Let P be the midpoint of LN, so that its position vector is

$$\mathbf{p} = \frac{1}{2}(\mathbf{l}+\mathbf{n}) = \frac{a-b+3c}{8}\mathbf{u} + \frac{a+3b-c}{8}\mathbf{v}.$$

It follows that

$$\mathbf{p} - \mathbf{k} = \frac{a-b+3c}{8}\left(\mathbf{u}+\mathbf{v}\right) \quad \text{and} \quad \mathbf{m} - \mathbf{p} = \frac{-a+b+c}{8}\left(\mathbf{u}+\mathbf{v}\right).$$

Therefore, $\overline{KP}$ is parallel to $\overline{PM}$, i.e. the points K, P, M are collinear, which achieves the proof. This proves also that the line KM is parallel to the bisector of the angle $\angle BAC$ of direction vector $\mathbf{u}+\mathbf{v}$.

Solution 2, by Bing Jian.

Let P be the midpoint of LN. We need to prove that P, K and M are collinear and we will leverage barycentric coordinates. In the absolute barycentric coordinate system relative to the vertices A, B, C, we have

$$A = (1,0,0),\ B = (0,1,0),\ C = (0,0,1) \text{ and } M = \frac{1}{2}\cdot(B+C) = (0, \frac{1}{2}, \frac{1}{2}).$$

Now let's find the barycentric coordinates of L, N and K.

Let $|BC| = a$, $|AC| = b$, and $AB| = c$ with $b > a > c$. Since points K, L, M, N divide the perimeter of ABC into 4 equal parts, we have

$$|BL| = |CN| = \frac{b+c-a}{4},\ |AK| = \frac{b-c}{2},$$

which gives the following barycentric coordinates:

$$K = \left(\frac{b+c}{2b}, 0, \frac{b-c}{2b}\right),\ N = \left(\frac{b+c-a}{4b}, 0, \frac{3b+a-c}{4b}\right),\ L = \left(\frac{b+c-a}{4c}, \frac{3c+a-b}{4c}, 0\right),$$

and therefore

$$P = \frac{1}{2}\cdot(N+L) = \left(\frac{(b+c-a)(b+c)}{8bc}, \frac{3c+a-b}{8c}, \frac{3b+a-c}{8b}\right).$$

Then we notice that

$$K - M = \left(\frac{b+c}{2b}, -\frac{1}{2}, -\frac{c}{2b}\right),$$

$$K - P = \frac{3c+a-b}{4c}\cdot\left(\frac{b+c}{2b}, -\frac{1}{2}, -\frac{c}{2b}\right) = \frac{3c+a-b}{4c}\cdot(K-M),$$

which implies P, K and M are collinear. In fact, we have

$$P = \frac{b+c-a}{4c} \cdot K + \frac{3c+a-b}{4c} \cdot M.$$

This completes our proof.

4868. *Proposed by Michel Bataille.*

Let $k \in [-1, 1]$ and let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 1$. Find the minimal and maximal values of $a^3 + b^3 + c^3 + kabc$.

We received 17 submissions, 9 correct and complete. We present the solution by Marie-Nicole Gras, slightly altered by the editor.

For all k, if $a = 1$, $b = 0$, $c = 0$ then $a^3 + b^3 + c^3 + kabc = 1$, and if $a = -1$, $b = 0$, $c = 0$ then $a^3 + b^3 + c^3 + kabc = -1$. We will prove that if $k \in [-2, 2]$ and $a^2 + b^2 + c^2 = 1$, then

$$-1 \leq a^3 + b^3 + c^3 + kabc \leq 1, \tag{1}$$

which will prove that -1 (respectively 1) is the lower (respectively upper) bound.

It is obvious that if a, b, c, k are nonnegative, then for all $u, v, w, t = \pm 1$,

$$-a^3 - b^3 - c^3 - kabc \leq ua^3 + vb^3 + wc^3 + tkabc \leq a^3 + b^3 + c^3 + kabc.$$

To prove (1), it sufficies to prove that, if a, b, c, k are nonnegative, then

$$a^3 + b^3 + c^3 + kabc \leq 1$$

and since $0 \leq k \leq 2$, it sufficies to prove that $G = a^3 + b^3 + c^3 + 2abc \leq 1$.

By the Cauchy-Schwarz inequality

$$(a^2 + b^2)(a^2 + c^2) \geq (a^2 + bc)^2 \quad \text{and} \quad b^2 + a^2)(b^2 + c^2) \geq (b^2 + ac)^2,$$

so that

$$\begin{aligned} G &= a(a^2 + bc) + b(b^2 + ac) + c^3 \\ &\leq a\sqrt{(a^2 + b^2)(a^2 + c^2)} + b\sqrt{b^2 + a^2}\sqrt{b^2 + c^2} + c\sqrt{c^4} \\ &\leq \sqrt{a^2 + b^2 + c^2}\sqrt{(a^2 + b^2)(a^2 + c^2) + (b^2 + a^2)(b^2 + c^2) + c^4} \\ &= 1 \times \sqrt{a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2} \\ &= \sqrt{(a^2 + b^2 + c^2)^2} \\ &= a^2 + b^2 + c^2 = 1. \end{aligned}$$

Editor's Comments. Vivek Mehra noted that a generalization of this problem appeared as ***Crux*** Problem 3654 with a solution in Vol. 38, No. 6.

4869. *Proposed by Leonard Giugiuc and Mohamed Amine Ben Ajiba.*

Let ABC be a non-obtuse triangle with area 1 and side-lengths a, b, c. Let n be a fixed non-negative real number. Find the minimum value of

$$\frac{2n}{a^2+b^2+c^2}+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

There were 4 correct solutions and 3 incomplete solutions. We will present several approaches.

Solution by UCLan Cyprus Problem Solving Group.

When $n \geq 3$, the minimum value is $\sqrt{n+1}$, and when $n \leq 3$, the minimum is $\frac{1}{4}(n+5)$.

Since the triangle is acute, the square of each side does not exceed the sum of the squares of the other two sides. Thus $x = \frac{1}{2}(b^2+c^2-a^2)$, $y = \frac{1}{2}(c^2+a^2-b^2)$ and $z = \frac{1}{2}(a^2+b^2-c^2)$ are all nonnegative and $(a^2, b^2, c^2) = (y+z, z+x, x+y)$.

The area of the triangle is given by one quarter of the squareroot of

$$2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4) = xy+yz+zx.$$

The problem is to find the minimum value of

$$P = \left(\frac{n}{x+y+z}+\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\right),$$

subject to the conditions that $x, y, z > 0$ and $xy+yz+zx = 4$.

Let $s = x+y+z$ and $p = xyz$. Then

$$\begin{aligned} P &= \frac{n}{s}+\frac{(x^2+y^2+z^2)+3(xy+yz+zx)}{(x+y)(y+z)(z+x)} \\ &= \frac{n}{s}+\frac{(x+y+z)^2+(xy+yz+zx)}{(x+y+z)(xy+yz+zx)-xyz} \\ &= \frac{n}{s}+\frac{s^2+4}{4s-p} \geq \frac{n}{s}+\frac{s^2+4}{4s} \\ &= \frac{n+1}{s}+\frac{s}{4} \geq \sqrt{n+1}. \end{aligned}$$

The last inequality is due to the arithmetic-geometric means inequality. Equality holds if and only if $xyz = 0$ and $s^2 = 4(n+1)$.

Suppose, wolog, $z = 0$. Then $c^2 = x+y = s = 2\sqrt{n+1}$, $a^2 = y$ and $b^2 = x$. Since also $a^2b^2 = xy+yz+zx = 4$, a^2 and b^2 are roots of the quadratic polynomial, $t^2-(2\sqrt{n+1})t+4$, whose discriminant is $4(n-3)$. Therefore, when $n \geq 3$, the inequality is satisfied when

$$(a, b, c) = \left(\sqrt{\sqrt{n+1}+\sqrt{n-3}}, \sqrt{\sqrt{n+1}-\sqrt{n-3}}, \sqrt{2\sqrt{n+1}}\right).$$

When $n \leq 3$, this lower bound is not achievable. Let $f(s) = (n+1)/s + s/4$. Then

$$f'(s) = -(n+1)/s^2 + 1/4 \geq -4/s^2 + 1/4.$$

When $s \geq 4$, $f'(s) \geq 0$, so that $f(s) \geq f(4) = (n+5)/4$, so that $P \geq (n+5)/4$. We show that this is an attainable lower bound when $n \leq 3$ and $s \leq 4$.

When $s \leq 4$, then $n/s \geq n/4$ so that it is enough to prove that

$$\frac{s^2+4}{4s-p} \geq \frac{5}{4},$$

or $4s^2 - 20s + 16 + 5p \geq 0$.

By Schur's inequality, we have, for $x, y, z \geq 0$, that

$$x^3 + y^3 + z^3 + 3xyz \geq x^2(y+z) + y^2(z+x) + z^2(x+y),$$

with equality if and only if $(x, y, z) = (t, t, t), (0, t, t), (t, 0, t), (t, t, 0)$ for some nonnegative real t. Since

$$x^3 + y^3 + z^3 = (x+y+z)^3 - 3[x^2(y+z) + y^2(z+x) + z^2(x+y)] - 6xyz,$$

then

$$\begin{aligned} s^3 - 3p &\geq 4[x^2(y+z) + y^2(x+z) + z^2(x+y)] \\ &= 4[(x+y+z)(xy+yz+zx) - 3xyz] = 16s - 12p, \end{aligned}$$

and

$$p \geq \frac{16s - s^3}{9} = \frac{s(4-s)(4+s)}{9}.$$

Thus

$$\begin{aligned} 4s^2 - 20s + 16 + 5p &\geq 4(1-s)(4-s) + \frac{5s(4-s)(4+s)}{9} \\ &= \frac{(4-s)(5s^2 - 16s + 36)}{9} \\ &= \frac{(4-s)((2s-4)^2 + s^2 + 20)}{9} \geq 0. \end{aligned}$$

Therefore

$$P = n/s + (s^2+4)/(4s-p) \geq n/4 + (s^2+4)/(4s-p) \geq (n+5)/4.$$

For equality to hold, we require that $s = 4$, at least two of x, y, z to be nonzero and equal, and

$$\frac{5}{4} = \frac{s^2+4}{4s-p} = \frac{20}{16-p}.$$

In turn, this forces $p = 0$. Wolog, let $z = 0$ and $x = y$. Then $a^2 = b^2$ and $a^2 + b^2 = c^2 = x + y = s = 4$. Therefore the lower bound of $(n+5)/4$ is attained when $a = b = \sqrt{2}$ and $c = 2$.

Summary of the solution by M. Bello, M. Benito, Ó. Ciaurri and E. Fernández, jointly.

Without loss of generality, assume that $a \geq b \geq c$ and let $u = b^2 + c^2$ and $v = bc\cos A$. Then $bc = 2\csc A \in [2, 4/\sqrt{3}]$, $u \geq 2bc \geq 4$, $u - 2v = a^2 > 0$, and

$$v = bc\sqrt{1-\sin^2 A} = bc\sqrt{1 - \frac{4}{b^2c^2}} = \sqrt{b^2c^2 - 4} \in [0, 2/\sqrt{3}].$$

The quantity to be minimized is

$$P(u,v) = \frac{n}{u-v} + \frac{1}{u-2v} + \frac{u}{v^2+4}.$$

Since b^2 and c^2 are real roots of the quadratic $t^2 - ut + (v^2+4)$, we must have

$$\begin{aligned} u^2 - 4v^2 - 16 &\geq 0, \\ b^2 &= \frac{1}{2}(u + \sqrt{u^2-4v^2-16}), \\ c^2 &= \frac{1}{2}(u - \sqrt{u^2-4v^2-16}). \end{aligned}$$

Since $a^2 \geq b^2$, $u - 4v \geq \sqrt{u^2-4v^2-16}$, whence $5v^2 - 2uv + 4 \geq 0$.

The problem is to minimize the function $P(u,v)$ subject to the constraints $u \geq 4$, $0 \leq v \leq 2/\sqrt{3}$, $u^2 - 4v^2 - 16 \geq 0$ and $5v^2 - 2uv + 4 \geq 0$.

The equations $u^2 - 4v^2 - 16 = 0$ and $5v^2 - 2uv + 4 = 0$, with $u, v \geq 0$, describe branches, H_1 and H_2 respectively, of two hyperbolae in the positive quadrant, tangent at the point $(8/\sqrt{3}, 2/\sqrt{3})$, with H_2 to the right of H_1 and tangent to the line $u = 2\sqrt{5}$ at the point $(2\sqrt{5}, 2/\sqrt{5})$. The constraint region is bounded by the curves with equations $v = 0$, $5u^2 - 2uv + 4 = 0$ and $u^2 - 4v^2 = 16$.

Since the partial derivatives of $G(u,v)$ both vanish only when $5u^2 - 2uv + 4 = 0$, the minimizing point must lies on one of the bounding curves. When $v = 0$,

$$P(u,0) = \frac{n+1}{u} + \frac{u}{4} \geq \sqrt{n+1},$$

with equality only if $u = 2\sqrt{n+1}$. Since $u \geq 4$, this applies only when $n \geq 3$. When $n < \sqrt{3}$, the derivative of $P(u,0)$ is positive when $u > 4$, so

$$P(u,0) \geq P(4,0) = (n+5)/4.$$

On the arc of H_1 between $(4,0)$ and $(8/\sqrt{3}, 2/\sqrt{3})$, we have $4 \leq u \leq 8/\sqrt{3}$ and also $v = \frac{1}{2}\sqrt{u^2-16}$. The function $G(u,v)$ along this arc is increasing so its minimum value of $(n+5)/4$ is assumed at $(4,0)$.

The analysis of $P(u,v)$ on the arc of H_2 is complicated; it can be shown that on this arc $P(u,v) > (n+5)/4$. When $u > 2\sqrt{5}$ and (u,v) lies below H_2, we have

$$0 \leq [u - \sqrt{u^2-20}]/5 = 4/(u + \sqrt{u^2-20} < 4/u.$$

Therefore, subject to the constraint, $\lim_{u\to\infty} P(u,v) = \infty$. The answer follows.

Solution for $n \geq 3$, by Theo Koupelis.

Let the triangle be ABC and let AD be the altitude from A to BC. Let $x = BD$, $y = DC$ and $d = AD$. Then $a = x + y$, $ad = 2$, $b^2 = d^2 + y^2$, $c^2 = d^2 + x^2$ and

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \leq b^2 + c^2 = 2d^2 + x^2 + y^2 \\ &= 2d^2 + a^2 - 2xy = a^2 + \frac{8}{a^2} - 2xy. \end{aligned}$$

It follows that $xy \leq 4/a^2$, with equality if and only if $A = 90°$. Since $a^2 = (x+y)^2 \geq 4xy$, we also have that $xy \leq a^2/4$.

$$\begin{aligned} P &= \frac{2n}{2a^2 + \frac{8}{a^2} - 2xy} + \frac{1}{a^2} + \frac{a^2 + \frac{8}{a^2} - 2xy}{\frac{16}{a^4} + \frac{4}{a^2}(a^2 - 2xy) + x^2y^2} \\ &= \frac{(n+1)a^4 + (4 - xya^2)}{a^2[a^4 + (4 - xya^2)]} + \frac{a^6 + 2a^2(4 - xya^2)}{4a^4 + (4 - xya^2)^2}. \end{aligned}$$

Let $t = (4 - xya^2)/4$. Then $0 \leq t \leq 1$, and thus

$$\begin{aligned} P &= \frac{(n+1)a^4 + 4t}{a^2(a^2 + 4t)} + \frac{a^6 + 8a^2t}{4(a^4 + 4t^2)} \geq \frac{(n+1)a^4 + 4t^2}{a^2(a^2 + 4t)} + \frac{a^6 + 8a^2t}{4(a^4 + 4t)} \\ &= \frac{1}{4a^2(a^4 + 4t)} \cdot [4(n+1)a^4 + (a^4 + 4t)^2] \\ &\geq \frac{1}{4a^2(a^4 + 4t)} \cdot 2 \cdot \sqrt{4(n+1)a^4} \cdot (a^4 + 4t) = \sqrt{n+1}. \end{aligned}$$

Equality occurs when $t = 0$ or $t = 1$ and

$$4(n+1)a^4 = (a^4 + 4t)^2$$

or

$$a^4 - (2\sqrt{n+1})a^2 + 4t = 0.$$

When $t = 1$, this quadratic has discriminant $4(n-3)$, and so the equation is solvable for $n \geq 3$. In this case, we find that $xy = 0$. Suppose $x = 0$, then

$$a^2 = \sqrt{n+1} \pm \sqrt{n-3};$$

$$b^2 = \frac{4}{a^2} = \sqrt{n+1} \mp \sqrt{n-3};$$

$$c^2 = d^2 + y^2 = b^2 + a^2 = 2\sqrt{n+1}.$$

When $t = 0$, then $a^2 = 2\sqrt{n+1} = b^2 + c^2$, $xy = 4/a^2$, $x^2 + y^2 = a^2 - (8/a^2)$. Therefore. say,

$$x^2 = \frac{a^4 - 8 \mp a^2\sqrt{a^4 - 16}}{2a^2} \qquad \text{and} \qquad y^2 = \frac{a^4 - 8 \pm a^2\sqrt{a^4 - 16}}{2a^2}.$$

For the solutions to be real, we require that $a^2 \geq 4$ and $n \geq 3$. In this case

$$\begin{aligned} a^2 &= 2\sqrt{n+1}; \\ b^2 &= \frac{a^2}{2} \pm \frac{\sqrt{a^4-16}}{2} = \sqrt{n+1} \pm \sqrt{n-3}; \\ c^2 &= \frac{a^2}{2} \mp \frac{\sqrt{a^4-16}}{2} = \sqrt{n+1} \mp \sqrt{n-3}, \end{aligned}$$

essentially as was the situation when $t = 0$.

4870*. *Proposed by Borui Wang.*

Define the series $\{a_n\}$ by the following recursion: $a_1 = 1$, $a_{n+1} = a_n + \dfrac{1}{q \cdot a_n}$ for $n > 0, q > 0$. Find the constant number $c(q)$ such that

$$\lim_{n\to\infty} (a_n - \sqrt{c(q) \cdot n}) = 0.$$

We received 9 submissions and they were all complete and correct. We present the solution by the majority of solvers.

We have

$$a_{n+1}^2 = \left(a_n + \frac{1}{q \cdot a_n}\right)^2 = \frac{2}{q} + a_n^2 + \frac{1}{q^2 \cdot a_n^2}.$$

Then, by induction, it follows that

$$a_n^2 = 1 + \frac{2}{q}(n-1) + \frac{1}{q^2}\sum_{k=1}^{n-1} \frac{1}{a_k^2}. \tag{1}$$

In particular, for all $n \in \mathbb{N}$, we have

$$a_n > \sqrt{2(n-1)/q}. \tag{2}$$

On the other hand, from equation (1) and inequality (2), we deduce that

$$a_n^2 = 1 + \frac{2}{q}(n-1) + \frac{1}{q^2}\sum_{k=1}^{n-1}\frac{1}{a_k^2} < 1 + \frac{2}{q}(n-1) + \frac{1}{q^2} + \frac{1}{2q}\sum_{k=2}^{n-1}\frac{1}{n-1}.$$

It is well-known that $H_n := \sum_{k=1}^{n} \frac{1}{n} \sim \log n$. Now it follows from the squeeze theorem that

$$\lim_{n\to\infty} (a_n - \sqrt{2n/q}) = 0.$$

We conclude that $c(q) = 2/q$.