# Canadian Mathematical Olympiad Qualifying Repêchage 2024

*A competition of the Canadian Mathematical Society.*

## Official Solutions

February 20, 2024

**1 [10 points]** Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the functional equation

$$f(x + f(xy)) = f(x)(1 + y).$$

**Solution:** Let $g(x, y)$ be the assertion $f(x + f(xy)) = f(x)(1 + y)$. We see that if $f(x) = 0$ for all $x \in \mathbb{R}$, we have LHS $= 0$ and RHS $= 0$. Thus this is a valid solution.

Then $g(0, y)$ gives $f(f(0)) = f(0)(1 + y)$. Note that both $f(f(0))$ and $f(0)$ are both constants, whereas $1 + y$ can take on any real value. This implies that $f(0)$ must be 0.

Assume $f(1) = 0$. Then $g(x, \frac{1}{x})$ implies $f(x) = f(x)(1 + \frac{1}{x})$ which also implies $f(x) = 0$ for all $x$.

Now, assume $f(1) \neq 0$. Assume there exists an $a \in \mathbb{R}$ such that $f(a) = b \neq 0$. Then $g(a, y)$ gives

$$f(a + f(ay)) = b(1 + y).$$

Since $b$ is nonzero, RHS can take on any real value. Therefore $f$ is surjective.

Now assume $f(c) = f(d)$ for some real numbers $c \neq d$. Then $g(1, c)$ and $g(1, d)$ give us

$$f(1)(1 + c) = f(1 + f(c)) = f(1 + f(d)) = f(1)(1 + d)$$

which implies that $c = d$ (since $f(1) \neq 0$). Therefore $f$ is injective. These insights together give that $f$ is bijective.

Now $f(x, -1)$ gives

$$f(x + f(-x)) = 0.$$

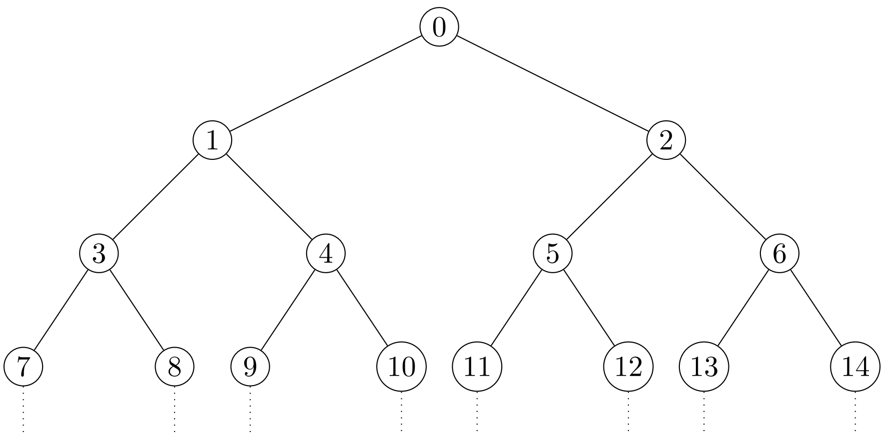
By bijectivity, since $f(0) = 0$, this implies $x + f(-x) = 0$, which then gives $f(x) = x$. Plugging this back into LHS and RHS both give $x + xy$, and thus this is a valid solution.

Therefore, $f(x) = x$ and $f(x) = 0$ are the only solutions.

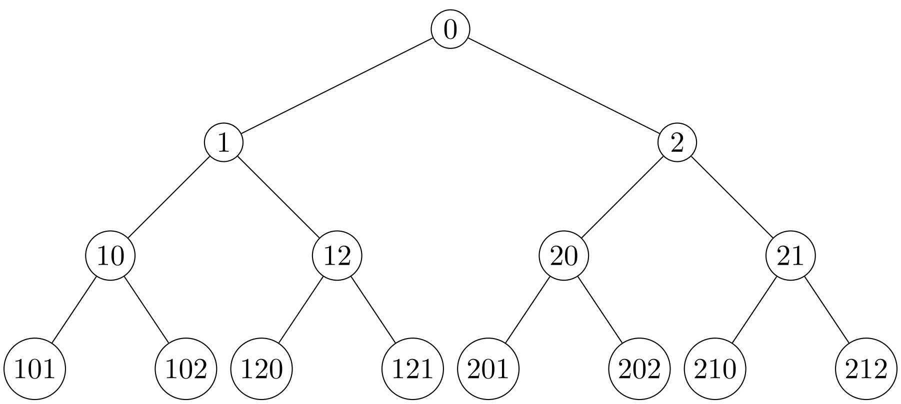
**2 [10 points]** Call a natural number $N$ *good* if its base 3 expansion has no consecutive digits that are the same. For example, 289 is good since its base 3 representation is $101201_3$. Find the $2024^{th}$ smallest good number (0 is *not* considered to be a natural number). Your answer should be in base 10.

**Solution:** The answer is 51575.

Consider the tree below; call it $T$. In this tree, each node has two children, and where the numbers are in increasing order by row, and in each row, the numbers increase from left to right.

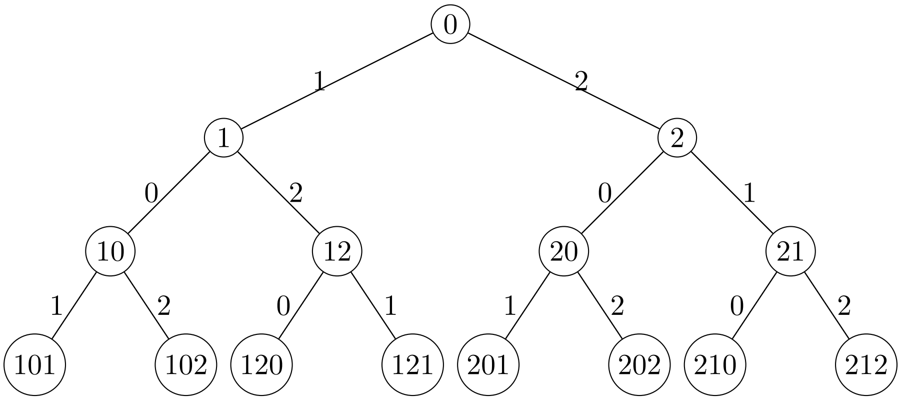



Now, consider replacing a circle that contains $i$ with the $i$'th good number written in base 3.




Note that, this diagram, each parent is a prefix of its child; each child has 1 more digit than its parent.




Consider the following process for building the tree. Going row by row, for each node:

1. If the node ends in 0, append 1 and make it the left child, and append 2 and make it the right child.
2. If the node ends in 1, append 0 and make it the left child, and append 2 and make it the right child.
3. If the node ends in 2, append 0 and make it the left child, and append 1 and make it the right child.

Claim: This process will produce all the valid good numbers. Furthermore, they are ordered by row number, and then from left to right.

Proof of claim: Proceed by induction on the row number (indexing from 0).

Base case: n=1 (since the cell with 0 is not a good number). We have 1 and 2 in this row. These are the smallest good numbers, and $1 < 2$

Induction Hypothesis: Assume the claim is true for rows $1, 2, 3, \dots i$ for some $i$.

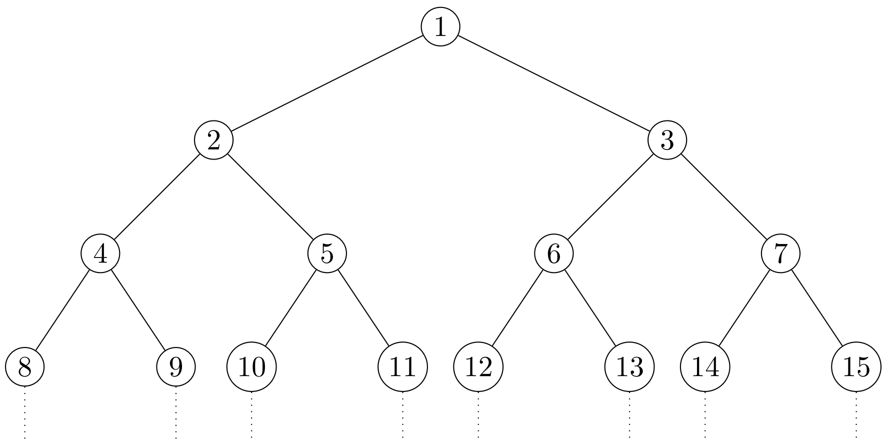
Induction Step: Perform the operation to the $i$th row to produce the $i+1$ row. Note that, by construction, all numbers have $i+1$ digits in the $i+1$ row. Consider two numbers in this row $A$ and $B$, where $A$ is to the left of $B$. Remove the rightmost digit from $A$, $B$ to get $A'$, $B'$ respectively. Let the removed digits be $a$, $b$, respectively.

Case 1: $A$ and $B$ do not have the same parent. Since $A$ is to the left of $B$, $A$'s parent must be to the left of $B$'s parent, and therefore $A' < B'$ by the I.H. This implies that $A < B$.

Case 2: $A$ and $B$ have the same parent. Then $A' = B'$. By construction, since $A$ is to the left of $B$, this implies $a < b$. Thus $A < B$.

We conclude that if $A$ is to the left of $B$, then $A < B$. Thus the claim is proved.

Now, $T$. We wish to find where 2024 is in $T$. Construct $T'$, where we take $T$ and add 1 to every node.




Finding 2024 in $T$ is the same as finding 2025 in $T'$.

Note that every right child in $T'$ is odd, and ever left child in $T'$ is even.

Thus we can repeatedly divide 2025 by 2 and take the floor. If we get an odd number, we went right; if we get an even number, we went left. Then, we can retrace the number using the base 3 tree.

Here is the computation:

1. $\lfloor \frac{2025}{2} \rfloor = 1012 \longrightarrow$ since 2025 is odd, we went right from the 1012 circle to get 2025.
2. $\lfloor \frac{1012}{2} \rfloor = 506 \longrightarrow$ since 1012 is even, we went left from the 506 circle to get 2025.
3. $\lfloor \frac{506}{2} \rfloor = 253 \longrightarrow$ since 506 is even, we went left from the 253 circle to get 506.
4. $\lfloor \frac{253}{2} \rfloor = 126 \longrightarrow$ since 253 is odd, we went right from the 126 circle to get 253.
5. $\lfloor \frac{126}{2} \rfloor = 63 \longrightarrow$ since 126 is even, we went left from the 63 circle to get 2025.
6. $\lfloor \frac{63}{2} \rfloor = 31 \longrightarrow$ since 63 is odd, we went right from the 31 circle to get 2025.
7. $\lfloor \frac{31}{2} \rfloor = 15 \longrightarrow$ since 31 is odd, we went right from the 15 circle to get 2025.
8. $\lfloor \frac{15}{2} \rfloor = 7 \longrightarrow$ since 15 is odd, we went right from the 7 circle to get 2025.
9. $\lfloor \frac{7}{2} \rfloor = 3 \longrightarrow$ since 7 is odd, we went right from the 3 circle to get 2025.
10. $\lfloor \frac{3}{2} \rfloor = 1 \longrightarrow$ since 3 is odd, we went right from the 1 circle to get 2025.

Now, starting from right, we went right $\longrightarrow$ right $\longrightarrow$ right $\longrightarrow$ right $\longrightarrow$ right $\longrightarrow$ left $\longrightarrow$ right $\longrightarrow$ left $\longrightarrow$ left $\longrightarrow$ right.

By the rules of constructing the base 3 tree, this will be the number $2121202012_3 = 51575$.

**3 [10 points]** Let $\triangle ABC$ be an acute triangle with $AB < AC$. Let $H$ be its orthocentre and $M$ be the midpoint of arc $\widehat{BAC}$ on the circumcircle. It is given that $B, H, M$ are collinear, the length of the altitude from $M$ to $AB$ is 1, and the length of the altitude from $M$ to $BC$ is 6. Determine all possible areas for $\triangle ABC$.

**Solution:** Let $E$ be the feet of perpendicular from $B$ to $AC$. $H, M, B$ collinear implies that $E$ also lies on this line. Now let $N$ be the midpoint of $BC$, so $\angle MEC = \angle MNC = 90^\circ$, which implies $M, E, N, C$ are concyclic.

Now, we have that $\angle C = \angle ECN = \angle EMN = \frac{1}{2}\angle A$. Let $F$ be the foot of perpendicular from $M$ to $AB$, then we have that $M, F, A, E$ are concyclic. Moreover, we see that $\angle MAE = \angle MAC = 90^\circ - \frac{1}{2}\angle A$, which means $\triangle AEM \cong \triangle AFM$. Therefore, $MF = ME = 1$.

Now, we are ready to finish: let the length of $BC$ be $2x$, by power of a point we know that $BN \cdot BC = BE \cdot BM$, which implies

$$\begin{aligned}
x \cdot 2x &= (\sqrt{x^2 + 6^2} - 1)(\sqrt{x^2 + 6^2}) \\
2x^2 &= x^2 + 36 - \sqrt{x^2 + 36} \\
x^2 - 36 &= -\sqrt{x^2 + 36} \\
(x^2 - 36)^2 &= x^2 + 36 \\
x^4 - 73x^2 + 36 \cdot 35 &= 0 \\
(x^2 - 28)(x^2 - 45) &= 0
\end{aligned}$$

Therefore, $x = 2\sqrt{7}$ or $x = 3\sqrt{5}$. However, when $x = 3\sqrt{5}$, we do not have $BN \cdot BC = BE \cdot BM$. So we can only have $x = 2\sqrt{7}$.

Finally, we see that $EC = \sqrt{28 + 36 - 1} = 3\sqrt{7}$. Again by power of a point, we have $AE \cdot EC = ME \cdot EB$. So $AE = \frac{1}{3}\sqrt{7}$ and $AC = \frac{10}{3}\sqrt{7}$.

It follows that $[\triangle ABC] = \frac{1}{2}AC \cdot BE = \frac{35\sqrt{7}}{3}$.

**4 [10 points]** A sequence $\{a_i\}$ is given such that $a_1 = \frac{1}{3}$ and for all positive integers $n$

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}.$$

Prove that

$$\frac{1}{2} - \frac{1}{3^{2^{n-1}}} < a_1 + a_2 + \cdots + a_n < \frac{1}{2} - \frac{1}{3^{2^n}},$$

for all positive integers $n$.

**Solution:** We first claim that $s_n = \frac{1}{2} - \frac{a_{n+1}}{1-a_{n+1}}$. Indeed, we can proceed by induction to show this with the base case $n = 1$ being trivial.

$$\begin{aligned} s_{n+1} &= s_n + a_{n+1} \\ &= \frac{1}{2} - \frac{a_{n+1}}{1 - a_{n+1}} + a_{n+1} \\ &= \frac{1}{2} - \frac{a_{n+1}^2}{1 - a_{n+1}} \\ &= \frac{1}{2} - \frac{a_{n+1}^2}{a_{n+1}^2 - a_{n+1} + 1} \cdot \frac{a_{n+1}^2 - a_{n+1} + 1}{1 - a_{n+1}} \\ &= \frac{1}{2} - \frac{a_{n+2}}{1 - a_{n+2}} \end{aligned}$$

Now it suffices to show that

$$1 + 3^{2^{n-1}} < \frac{1}{a_{n+1}} < 1 + 3^{2^n}$$

Once again induction finishes here.

**5 [10 points]** Let $S$ be the set of 25 points $(x, y)$ with $0 \le x, y \le 4$. A triangle whose three vertices are in $S$ is chosen at random. What is the expected value of the square of its area?

**Solution:** We first count the number of triangles that we can choose with vertices in $S$. There are $\binom{25}{3} = 2300$ possible ways to choose three distinct points in $S$. These form a triangle if and only if the three points are not collinear; it is easier to count sets of three points that are collinear, as there are 12 lines with 5 points on them, 4 lines with 4 points on them, and 16 lines with 3 points on them. This gives $12 \cdot \binom{5}{3} + 4 \cdot \binom{4}{3} + 16 = 152$ possible ways to choose three collinear points in $S$, leaving 2148 ways to choose a triangle whose three vertices are in $S$.

Next, in order to sum up the areas of the possible triangles, it will be convenient to instead choose three points $A = (a_1 + 2, a_2 + 2), B = (b_1 + 2, b_2 + 2)$, and $C = (c_1 + 2, c_2 + 2)$ independently among the vertices in $S$. We add two to the coordinates of each of $A, B, C$ in order for our variables to range between $-2$ and 2 instead, and thus have mean zero. In this way, we count each triangle six times, and we also include degenerate triangles that do not contribute to the area. By the Shoelace Formula, triangle $ABC$ has area

$$\frac{1}{2}\left(a_1b_2 + b_1c_2 + c_1a_2 - a_2b_1 - b_2c_1 - c_2a_1\right).$$

Its square consists of many monomials of degree 4 in the variables $a_i, b_i, c_i$; when one variable occurs exactly once in the factorization of a monomial, then the monomial sums to zero: for example, the monomial $-2a_1^2 b_2 c_2$ appearing in the square of the area occurring from multiplying the first and last term sums to zero since we have

$$\sum_{a_1=-2}^{2}\sum_{a_2=-2}^{2}\sum_{b_1=-2}^{2}\sum_{b_2=-2}^{2}\sum_{c_1=-2}^{2}\sum_{c_2=-2}^{2} -2a_1^2 b_2 c_2$$

$$-250 \sum_{a_1=-2}^{2} a_1^2 \sum_{b_2=-2}^{2} b_2 \sum_{c_2=-2}^{2} c_2 = -250 \cdot 10 \cdot 0 \cdot 0 = 0.$$

Thus, we have

$$\frac{1}{4}\sum_{a_1=-2}^{2}\sum_{a_2=-2}^{2}\sum_{b_1=-2}^{2}\sum_{b_2=-2}^{2}\sum_{c_1=-2}^{2}\sum_{c_2=-2}^{2} (a_1b_2 + b_1c_2 + c_1a_2 - a_2b_1 - b_2c_1 - c_2a_1)^2$$

$$= \frac{1}{4}\sum_{a_1=-2}^{2}\sum_{a_2=-2}^{2}\sum_{b_1=-2}^{2}\sum_{b_2=-2}^{2}\sum_{c_1=-2}^{2}\sum_{c_2=-2}^{2} \left(a_1^2b_2^2 + b_1^2c_2^2 + c_1^2a_2^2 + a_2^2b_1^2 + b_2^2c_1^2 + c_2^2a_1^2\right).$$

Each of these terms evaluates to $\frac{1}{4} \cdot 5^4 \cdot 10^2 = 5^6$, so the sum is $6 \cdot 5^6$, and so the sum of the areas of the triangles in $S$ is equal to $5^6$. The answer is thus $\frac{5^6}{2148} = \frac{15625}{2148} \approx 7.274$.

**6 [10 points]** For certain real constants $p, q, r$, we are given a system of equations

$$\begin{cases} a^2 + b + c = p \\ a + b^2 + c = q \\ a + b + c^2 = r \end{cases}$$

What is the maximum number of solutions of real triplets $(a, b, c)$ across all possible $p, q, r$? Give an example of the $p, q, r$ that achieves this maximum.

**Solution:** The answer is $2^3 = 8$. We first make some simplifications before proving this bound or giving a construction. Let $a + b + c = s$. Then, we get $a^2 - a = p - s$, so

$$a = \frac{1}{2} \pm \sqrt{p - s + 1/4},$$

and we get similar expressions for $b$ and $c$; plugging back into the equation $a + b + c = s$, we have

$$s - \frac{3}{2} \pm \sqrt{p - s + 1/4} \pm \sqrt{q - s + 1/4} \pm \sqrt{r - s + 1/4} = 0. \tag{1}$$

Now we multiply all eight possibilities together to get a polynomial in $s$ of degree 8. We separate this out to a general fact about multiplying by conjugates:

**Lemma:** The expression

$$\prod_{i,j,k \in \{-1,1\}} (a + ib + jc + kd)$$

is a polynomial in each of the variables $a^2, b^2, c^2, d^2$. It is of degree 8.

**Proof:** We note that

$$\prod_{i,j,k\in\{-1,1\}}(a+ib+jc+kd)=\prod_{i,j\in\{-1,1\}}(a+ib+jc+d)(a+ib+jc-d)=\prod_{i,j\in\{-1,1\}}((a+ib+jc)^2-d^2),$$

which is indeed a polynomial in $d^2$. Similar manipulations show that this is also a polynomial in $b^2$ and $c^2$. As for $a^2$, we use a slightly different manipulation as follows:

$$\prod_{i,j,k\in\{-1,1\}}(a+ib+jc+kd)=\prod_{\frac{j}{i},\frac{k}{i}\in\{-1,1\}}(a^2-(b+jc/i+kd/i)^2).$$

This concludes the proof of the lemma. □

From the lemma, we see that the product of all eight possibilities for Equation (1) is a polynomial in $s$ of degree 8, hence has at most 8 solutions for $s$. Now, for some given solution $s$, say there are $k$ choices of the $\pm$ signs that make Equation (1) true. Then, the product

$$\prod_{i,j,k\in\{-1,1\}}\left(s-\frac{3}{2}+i\sqrt{p-s+1/4}+j\sqrt{q-s+1/4}+k\sqrt{r-s+1/4}\right)$$

should have a root at this value of $s$ with multiplicity $k$. Thus, there are at most 8 solutions for $(s,i,j,k)$, which determine $(a,b,c)$ uniquely.

We now let $(p,q,r)=(100,101,102)$. We see that for each choice of $i,j,k$, we have that the function

$$F(s)=s-\frac{3}{2}+i\sqrt{p-s+1/4}+j\sqrt{q-s+1/4}+k\sqrt{r-s+1/4}$$

satisfies $F(100)>0$ and $F(-100)<0$, so there must be at least one solution for $s$, and hence $(a,b,c)$.

**7 [20 points]**

1. In triangle $ABC$, let $I$ be the incentre. Let $H$ be the orthocentre of triangle $BIC$. Show that $AH$ is parallel to $BC$ if and only if $H$ lies on the circle with diameter $AI$.
2. In triangle $ABC$, let $I$ be the incentre, $O$ be the circumcentre, and $H$ be the orthocentre. It is given that $IO=IH$. Show that one of the angles of triangle $ABC$ must be equal to 60 degrees.

**Solution:**

1. We note that $IH$ is perpendicular to $BC$, so $AH$ is parallel to $BC$ if and only if $AH$ is perpendicular to $IH$. This is equivalent to $H$ lying on the circle with diameter $AI$.
2. Note that $IO=IH$, $AI=AI$, and the angles $HAI$ and $IAO$ are equal or supplementary, and the same property is true when replacing $A$ with any other vertex of the triangle. The angles $HAI$ and

$IAO$ are supplementary if and only if angle $A$ is obtuse. This is a general configuration whose properties we will extract into a lemma:

**Lemma:** Given triangles $XYZ$ and $X'Y'Z'$ such that $XY = X'Y', YZ = Y'Z'$, and such that the angles $YZX$ and $Y'Z'X'$ are either equal or supplementary, then the angles $YXZ$ and $Y'X'Z'$ are either equal or supplementary.

**Proof:** We can prove this with the Law of Sines. Note that

$$\frac{\sin \angle YXZ}{\sin \angle YZX} = \frac{YZ}{YX} = \frac{Y'Z'}{Y'X'} = \frac{\sin \angle Y'X'Z'}{\sin \angle Y'Z'X'}.$$

The denominators are equal by our angle assumption, so $\sin \angle YXZ = \sin \angle Y'X'Z'$, and we conclude that these angles are either equal or supplementary. $\square$

From the lemma, we get that the angles $AHI$ and $AOI$ are either equal or supplementary, as are the pairs $BHI$ and $BOI$, and $CHI$ and $COI$. Assume without loss of generality that angles $B$ and $C$ are acute.

We first deal with the case where angle $A$ is obtuse. Then, the angles $IAH$ and $IAO$ are supplementary, so the angles $AHI$ and $AOI$ must be equal. This shows that the points $H, A, I$, and $O$ lie on a circle. Now consider the angles $BHI$ and $BOI$. If they are equal, then $H$ and $O$ are reflections of each other across line $BI$, and if they are supplementary, then $B$ also lies on the circle containing $H, A, I$, and $O$. The same is true for $C$, so we may assume without loss of generality that $H$ and $O$ are reflections of each other about $BI$ and that $H, A, I, O, C$ all lie on a circle. But then $\angle AIC = \angle AOC$, and so we get $90^\circ + \frac{1}{2}\angle B = 2\angle B$, whence $\angle B = 60^\circ$, as desired.

Now we deal with the case where the triangle is acute. Assume that at least two pairs of $(\angle AHI, \angle AOI), (\angle BHI, \angle BOI), (\angle CHI, \angle COI)$ are equal, say the first two. Then triangles $AHI$ and $AOI$ are congruent, so $AH = AO$, and similarly $BH = BO$. This implies that triangles $ABH$ and $ABO$ are congruent. If $H$ and $O$ are identical, then triangle $ABC$ is equilateral, so the desired result holds. Otherwise, $H$ and $O$ are reflections of each other about line $AB$, which is not possible since we have assumed that triangle $ABC$ is acute.

Now assume that at least two pairs of $(\angle AHI, \angle AOI), (\angle BHI, \angle BOI), (\angle CHI, \angle COI)$ are supplementary, again, say the first two. Then the points $A, H, O, I, B$ all lie on a circle again, and by the same reasoning as above, the equality of angles $\angle AIB = \angle AOB$ implies that $\angle C = 60^\circ$.

**8 [20 points]** A sequence of $X$s and $O$s is given, such that no three consecutive characters in the sequence are all the same, and let $N$ be the number of characters in this sequence. Maia may swap two consecutive characters in the sequence. After each swap, any consecutive block of three or more of the same character will be erased (if there are multiple consecutive blocks of three or more characters after a swap, then they will be erased at the same time), until there are no more consecutive blocks of three or more of the same character. For example, if the original sequence were $XXOOXOXO$ and Maia swaps the fifth and sixth character, the end result will be $XXOOOXXO \to XXXXO \to O$. Find the maximum value $N$ for which Maia can't necessarily erase all the characters after a series of swaps. Partial credit will be awarded for correct proofs of lower and upper bounds on $N$.

**Solution:** Assume that in the initial configuration there are $a$ $X$s and $b$ $O$s. Assume, without loss of generality, that $a \geq b$. Because there are not three $X$s or three $O$s in a row, we also have $2b+2 \geq a$; between every two $O$s and to the left of the leftmost and right of the rightmost $O$, there are at most two $X$s. We start with the following lemma:

**Lemma 1:** We may make a finite number of moves moves from the initial configuration, such that no $X$s or $O$s are erased, and so that the resulting configuration is

$$\underbrace{XOXO\ldots XO}_{2b-a+2}\underbrace{XXOXXO\ldots XXO}_{a-2-b}XX$$

for $a \geq b+2$, and

$$\underbrace{XOXO\ldots XO}_{b}X$$

for $a = b+1$, and

$$\underbrace{XOXO\ldots XO}_{a}$$

for $a = b$.

**Proof:** We represent each configuration of $X$s and $O$s with a sequence of $b+1$ numbers, denoting the number of $X$s between consecutive pairs of $O$s, and including the number of $X$s before the first $O$ and after the last $O$. For example, represent the configuration $XOOXXOXOXXOXO$ with the sequence $(1,0,2,1,2,1,0)$. Note that the terms of this sequence must sum to $a$, no term in the sequence is greater than or equal to 3, and there do not exist two consecutive 0s in this sequence. Also note that the action of swapping an $X$ and $O$ that does not result in cancellation is equivalent to selecting a pair of consecutive terms of this sequence, adding 1 to one term, subtracting 1 from the other term, so that the conditions continue to hold.

We are now ready to proceed by induction. Our base cases will be $(a,b) = (2x+2,x), (1,0)$ and $(0,0)$ (for $x \geq 0$), and we will reduce every other case $(a,b) = (m,n)$ to the case $(a,b) = (m-1,n-1)$. When $(a,b) = (1,0)$ or $(0,0)$, it is easy to see that the lemma holds; there is exactly one configuration ($X$ and the empty string), which matches the string in the lemma. When $(a,b) = (2x+2,x)$, there is again only one possible initial configuration possible, which matches the one given in the lemma: it is

$$\underbrace{XXOXXO\ldots XXO}_{x}XX.$$

We now proceed with the inductive step. Assume that $(a,b)=(m,n)$, where $n\neq 0$ and $m\leq 2n+1$. Under these assumptions, the desired configuration starts with $XO\ldots$, which is equivalent to the first term of the corresponding sequence being equal to 1. Once we move the configuration to one where the first term is equal to 1, we may now apply the inductive hypothesis to arrange the remaining $n$ terms of the sequence, which sum to $m-1$, into the desired configuration.

Initially, the first term of the sequence may be equal to 0, 1, or 2. If it is equal to 1, we can already proceed with the inductive step.

If the first term of the sequence is initially equal to 0, we consider the leftmost term of the sequence that is 2, if it exists. We subtract 1 from it and add 1 to the term left of it, repeating until the net effect is that we subtracted 1 from the term that is 2 and added 1 to the first term. We can check that within this process, we never had a term in the sequence be greater than or equal to 3, since we chose the leftmost term that is 2. Further, we never created two consecutive 0s, since there were not initially two consecutive 0s. Thus we have created a series of steps to result in a configuration where the first term of the sequence is 1, and we can induct.

It is possible that no term of the sequence is 2. Since there are $b+1$ terms that sum to $a$, the sequence must then be $(0,1,1,\ldots,1)$, and $a=b$ must hold. In this case we simply transform to $(0,2,0,1,\ldots,1)$ and then to $(1,1,0,1,\ldots,1)$.

On the other hand, if the first term of the sequence is initially equal to 2, we look for the leftmost term of the sequence that is not equal to 2. This term must exist since $m=2n+2$ if the sequence is entirely comprised of 2s. Now, add 1 to this term and subtract 1 from the term to its left, repeating until the first term is equal to 1. Once again we can check that in this process we never create a term in the sequence greater than or equal to 3 or two consecutive terms in the sequence that are both equal to 0.

This completes the induction step and proves the lemma. □

**Corollary 1:** Assume that in the initial configuration there are $a$ $X$s and $b$ $O$s, with $2b+2\geq a\geq b$. Then, for any other configuration of $a$ $X$s and $b$ $O$s with no three consecutive $X$s or $O$s, there is a sequence of moves resulting in this final configuration.

**Proof:** We use the lemma twice. We first get the initial configuration to the specified configuration in the lemma, and then do the moves backwards to get to the final configuration. □

We are now ready to understand which pairs of $(a,b)$ are clearable. The following corollary will help.

**Corollary 2:** Call a pair $(a,b)$ clearable if any (by Corollary 1, equivalently *all*) initial configurations can be cleared. Then, if $a\geq 4$ and $(a-3,b)$ is clearable, so is $(a,b)$.

**Proof:** First transform the configuration to the one given in Lemma 1. When $a\geq b+2$, we may move the last $O$ to the right and clear three $X$s, getting a configuration with $N-3$ letters. When $a=b+1$ or $a=b$ and $a\geq 4$, we replace the initial $XOXOXOX$ iteratively to

$$XOXOXOX\to OXXOXOX\to OXXOXXO\to OXXXOXO\to OOXO,$$

which again has $N-3$ letters with three fewer $X$s. □

We are now ready to solve the problem. We claim that the answer is $N=17$. We start with the configuration

$$XXOXXOXXOXXOXXOXX,$$

which has five $O$s, and show that it is not clearable. In fact, no configuration with exactly five $O$s is clearable; since we can only clear $O$s in groups of at least 3, we would have to clear all five $O$s at once. But this is impossible by the following general fact:

**Claim:** After a swap, several rounds of removals of three or more consecutive $X$s and $O$s occur. On the first round, either a block of three $X$s gets removed, a block of three $O$s gets removed, or both. If both occur then the blocks of $X$s and $O$s are next to each other. On future rounds, only one block of either 3 or 4 $X$s or $O$s gets removed.

**Proof:** The fact about the first round of removals is clear. In future rounds, the only possible removal is from the creation of a 3 or 4 in a row from the removed characters from the previous round, which are always contiguous by induction. There could not be 5 or more in a row since both parts that were combined together via the removal of characters in between have at most two characters, since they were not removable in the previous round. $\square$

We now need to verify that all configurations for $N \geq 18$ can be fully cleared. For this, we first demonstrate that certain pairs $(a, b)$ are clearable based on the residue class of $a$ and $b$ mod 3. These constructions, combined with Corollary 2, will allow us to show that every combination where $a + b = 18, 19$, and 20 is clearable. Recall that due to Corollary 1, we may start with any initial configuration.

1. **Configuration 1:** $(a, b) = (3, 3)$. We have
$$XX\underline{OX}OO \to \underline{XXXOOO}.$$
2. **Configuration 2:** $(a, b) = (4, 3)$. We have
$$XXOO\underline{XO}X \to XX\underline{OOO}XX \to \underline{XXXX}.$$
3. **Configuration 3:** $(a, b) = (4, 7)$. We have
$$OOXXOO\underline{XO}XOO \to OOXX\underline{OOO}XX \to OO\underline{XXXX}OO \to \underline{OOOO}.$$
4. **Configuration 4:** $(a, b) = (8, 6)$. We have
$$XXOO\underline{XO}XOXXOXOX \to XX\underline{OOO}XXOXXOXOX \to \underline{XXXX}OXXOXOX,$$
which reduces this to $(a, b) = (4, 3)$.
5. **Configuration 5:** $(a, b) = (8, 7)$. We use the same starting setup as Configuration 3, with $XX$ appended to both ends.
6. **Configuration 6:** $(a, b) = (8, 11)$. We use the same starting setup as Configuration 5, with $OO$ appended to both ends.

We can check that if $a + b \geq 18$, then it can be reduced to one of the above 6 configurations (possibly with swapping $a$ and $b$) by repeatedly using Corollary 2, so the configuration is always clearable. $\square$