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Crux Mathematicorum with Mathematical Mayhem

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MathemAttic

No. 48

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 30, 2023**.

MA236. Let S be a set of real numbers that is closed under multiplication. Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T and U is closed under multiplication.

MA237. Determine the number of integral solutions of $|x| \cdot |y| \cdot |z| = 12$.

MA238. *Proposed by Aravind Mahadevan.*
In a right-angled triangle, if the median to the hypotenuse is the geometric mean of the sides forming the right angle, find the measures of the acute angles of the triangle.

MA239. *Proposed by Nguyen Viet Hung.*
Find all integers x, y and prime p satisfying the equation

$$x^4 + (x-1)(x^2 - 2x + 2) = p^y.$$

MA240. *Proposed by Pranav Milind Sawant.*
Let a and b be two positive rational numbers such that $\sqrt[3]{a} + \sqrt[3]{b}$ is also a rational number. Prove that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are both rational numbers.

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 novembre 2023**.

MA236. Soit S un ensemble de nombres réels, fermé par rapport à la multiplication, et soient T et U des sous ensembles disjoints de S dont la réunion est S. De plus, on suppose que le produit de trois éléments de T (pas nécessairement distincts) se trouve toujours dans T, et de même pour U. Démontrez qu'au moins un des ensembles T et U est fermé par rapport à la multiplication.

MA237. Déterminez le nombre de solutions entières à $|x| \cdot |y| \cdot |z| = 12$.

MA238. *Soumis par Aravind Mahadevan.*

Dans un triangle rectangle, si la mesure de la médiane s'abaissant à l'hypoténuse est la moyenne géométrique des mesures des côtés formant l'angle droit, alors trouvez la mesure des angles aigus du triangle.

MA239. *Soumis par Nguyen Viet Hung.*

Trouvez tous les entiers x et y ainsi que tous les nombres premiers p satisfaisant l'équation suivante :

$$x^4 + (x-1)(x^2 - 2x + 2) = p^y.$$

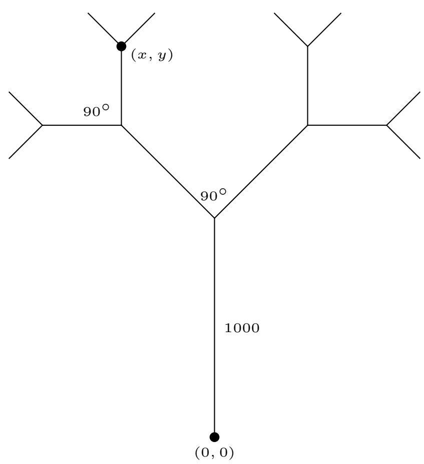
MA240. *Soumis par Pranav Milind Sawant.*

Soient a et b deux nombres rationnels positifs tels que $\sqrt[3]{a}+\sqrt[3]{b}$ soit aussi un nombre rationnel. Prouvez que $\sqrt[3]{a}$ et $\sqrt[3]{b}$ sont tous les deux des nombres rationnels.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2023: 49(3), p. 119–121.

MA211. Starting at coordinates $(0,0)$, a line 1000 units long is drawn as indicated. This line then branches into two separate lines (which form a 90° angle, as shown). Each of these lines is 60% the length of the previous segment. The process continues. Find the (x,y) coordinates of the indicated point.



Originally question 9 from the 2018 Kansas City Area Teachers of Mathematics High School Math Contest.

We received 3 submissions of which 2 were correct and complete. We present the solution by Daniel Văcaru.

We denote by O, A, B, C the points on the road to $C\,(x,y)$. We obtain that $A\,(0,1000)$. The length of AB is $\frac{60}{100}\cdot 1000 = 600$. Consider the line $y = y_B$, which is perpendicular to the line $x = 0$. We have

$$(x_B - x_A)^2 + (y_B - y_A)^2 = 600^2 \Leftrightarrow (x_B - 0)^2 + (y_B - 1000)^2 = 600^2.$$

Furthermore, if D is the intersection point of $y = y_B$ and $x = 0$, $AD = DB$ is the solution to the equation

$$\begin{aligned} AD^2 + DB^2 = AB^2 &\Leftrightarrow 2AD^2 = 600^2 \\ &\Leftrightarrow AD^2 = 600\cdot 300 \\ &\Leftrightarrow AD^2 = 9\cdot 2\cdot 100^2 \\ &\Leftrightarrow AD = BD = 300\sqrt{2}. \end{aligned}$$

It follows that

$$y_{\mathrm{B}} = 1000 + 300\sqrt{2} = 100\left(10 + 3\sqrt{2}\right).$$

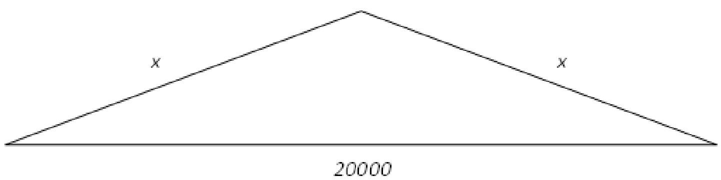
We obtain $x_{\mathrm{B}} = -300\sqrt{2}$. We have $B\left(-300\sqrt{2}, 100\left(10+3\sqrt{2}\right)\right)$. The length of BC is $\frac{60}{100} \cdot 600 = 360$. It follows that $C\left(-300\sqrt{2}, 1360 + 300\sqrt{2}\right)$.

MA212. On a distant planet, railway tracks are built using one solid railway bar. A railway is built between two towns 20 km apart on a big flat section of the planet. Unfortunately the bar was made one metre too long and the constructor decided to lift it in the middle to try to make the ends fit. *Approximately* how high does he have to lift it in the middle?

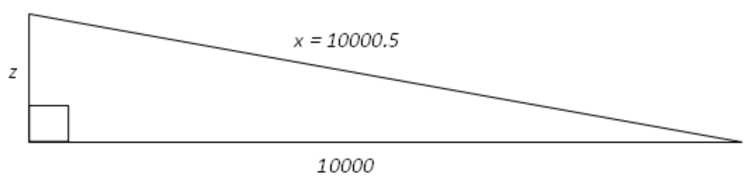
Originally from the Mathematics Competitions, Vol. 34, #1 (2021), A brief history of the South African Mathematics Olympiad*, "Surely that can't be" problems, example 1.*

We received 2 submissions, both of which were complete and correct. We present the solution by $\pi rat\varepsilon s$ of change.

A triangle can be constructed from the information given, where $2x$ is equal to 20,001 meters and the base is equal to 20,000 meters. That is,



where $x + x = 20001$. Next, the triangle can be divided into two to create a right triangle with a side x, a side of 10,000, and a side z, where x is equal to 10,000.5 and z is the unknown to be calculated. That is,



Next, the Pythagorean theorem can be used to determine the approximate value of z. That is,

$$z^2 + 10000^2 = 10000.5^2 \quad \Leftrightarrow \quad z^2 = 10000.5^2 - 10000^2 \quad \Leftrightarrow \quad z = \sqrt{10000.5^2 - 10000^2}.$$

Next, the radical can be simplified through the difference of two squares. That is,

$$z = \sqrt{(10000.5 - 10000) \cdot (10000.5 + 10000)} \quad \Leftrightarrow \quad z = \sqrt{0.5 \cdot 20000.5} \quad \Leftrightarrow \quad z = 0.5\sqrt{40001}.$$

The square root of 40,001 is not a perfect square, so the value can be approximated to the nearest perfect square, the square root of 40,000, which is 200. The numbers are so large that the difference in approximation is negligible. That is,

$$z \approx 0.5\sqrt{40000} \quad \Leftrightarrow \quad z \approx 0.5 \cdot 200 \quad \Leftrightarrow \quad z \approx 100.$$

Hence, the constructor must lift the bar approximately 100 meters in the middle for the bar to fit on both ends.

MA213. A shopkeeper orders marbles made up of 19 identical packets of a larger amount and 3 identical packets of a smaller amount. A total of 224 marbles arrive loosely tossed in a container. How would you repackage the marbles properly to satisfy the shopkeeper's order? Justify your answer and show that it is unique.

Originally from the J.I.R. McKnight Mathematics Scholarship Paper, Scarborough Board of Education, Question 5, 1982.

We received 5 submissions, of which 4 were correct and complete. We present the solution by Aravind Mahadevan.

Let the 19 larger packets contain x marbles in each and let the 3 smaller packets contain y marbles in each. Then

$$19x + 3y = 224 \qquad (1)$$

The integer value combinations of x and y that satisfy (1) are

- $x = 2$ and $y = 62$
- $x = 5$ and $y = 43$
- $x = 8$ and $y = 24$
- $x = 11$ and $y = 5$

But we are given that $x > y$. Therefore, $x = 11$ and $y = 5$ is the only possible solution. So we need to repackage the marbles so that there are 19 identical packets containing 11 marbles each and 3 identical packets containing 5 marbles each.

MA214. *Proposed by Neculai Stanciu.*

Determine all pairs (x, y) of real numbers which satisfy

$$\sqrt{x^2+2x+1} + \sqrt{x^2-4x+4} + \sqrt{y^2-6y+9} + \sqrt{x^2-2xy+y^2} = 4.$$

We received 5 submissions, of which only 1 was correct and complete as solvers either missed some cases or did not fully explain why some cases could be excluded. We present the solution by Ivan Hadinata, lightly edited.

Note that the given equation can be rewritten as follows:

$$|x+1| + |x-2| + |y-3| + |x-y| = 4. \qquad (1)$$

From (1) and the triangle inequality, we can deduce

$$4 \geq |x+1+x-2+y-3+x-y| = |3x-4|, \tag{2}$$

and also

$$4 \geq |x+1| + |x-2-y+3-x+y| = |x+1|+1. \tag{3}$$

Inequality (2) holds for $0 \leq x \leq \frac{8}{3}$ and (3) holds for $-4 \leq x \leq 2$; combining the two yields $0 \leq x \leq 2$. Hence $x+1>0$ and $x-2 \leq 0$, so equation (1) becomes

$$|y-3| + |x-y| = 1.$$

Applying the triangle inequality again,

$$1 = |3-y| + |y-x| \geq 3-y+y-x = 3-x, \tag{4}$$

so $x \geq 2$. Therefore it must be the case that $x=2$. Note that when $x=2$ equality holds in (4), which happens if and only if both of $3-y$ and $y-2$ are nonnegative, in the other words $2 \leq y \leq 3$.

Hence the solutions are $(x,y) = (2,m)$ for all real $m \in [2,3]$. Note: It is easy to check that these solutions satisfy (1).

MA215. *Proposed by Aravind Mahadevan, Hong Kong.*

In ΔABC, $\angle B = 2\angle A$ and $\angle C = 4\angle A$. Prove that $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ where, a, b and c denote the lengths of BC, CA, and AB respectively.

We received 11 correct solutions. The following is the solution by Ricard Peiró and Manescu-Avram Corneliu (done independently).

It is not difficult to see that A, B, C are three vertices of a regular heptagon. Consider the regular heptagon $A_1A_2\ldots A_7$ inscribed in the circumcircle of the triangle ΔABC, with vertices

$$A_1 = A,\ A_3 = C,\ A_4 = B,\ A_5 = D.$$

The quadrilateral $ABCD$ is cyclic and $BC = BD = a$, $AC = CD = b$ and $AB = AD = c$. By Ptolemy's theorem $bc = ac + ab$, which gives the required equality if it is divided by abc.

Remark. We leave as an exercise to check that $b^2 = a^2 + ac$ and $c^2 = b^2 + ab$.

TEACHING PROBLEMS

No. 23

Shawn Godin

Exploring a Fryer Contest Problem

Math contests are great places to find problems for your classroom. They are designed to be challenging and are usually quite different from the types of questions you will find in textbooks. One set of contests in particular is full of problems that can quite easily be turned into classroom explorations. These contests are the *Fryer*, *Galois*, and *Hypatia* contests hosted by The Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo, aimed at students in grades 9, 10 and 11, respectively.

I have been on the problems committee for the Fryer, Galois and Hypatia contests for about fifteen years. Each of the contests consists of four multi-part questions worth 10 marks each for which students provide written solutions. The parts of each of the problems are related to each other and in many cases earlier parts are used to nudge the students towards a particular result. The problems committee actually refers to some problems as *teaching problems* when we are presenting an idea that is outside the curriculum which is, however, within the reach of the students. Let's explore problem 3 from the 2022 Fryer contest.

> *3. If an integer n is written as a product of prime numbers, this product (known as its* prime factorization*) can be used to determine the number of positive factors of n. For example, the prime factorization of* $28 = 2 \times 2 \times 7 = 2^2 \times 7^1$*. The positive factors of 28 are:*
>
> $$28 = 2^2\times 7^1,\ 14 = 2^1\times 7^1,\ 7 = 2^0\times 7^1,\ 4 = 2^2\times 7^0,\ 2 = 2^1\times 7^0,\ 1 = 2^0\times 7^0.$$
>
> *Each positive factor includes 2, 1 or 0 twos, 1 or 0 sevens, and no other prime numbers. Since there are 3 choices for the number of twos, and 2 choices for the number of sevens, there are* $3 \times 2 = 6$ *positive factors of 28.*
>
> *(a) How many positive factors does 675 have?*
> *(b) A positive integer n has the positive factors 9, 11, 15, and 25 and exactly fourteen other positive factors. Determine the value of n.*
> *(c) Determine the number of positive integers less than 500 that have the positive factors 2 and 9 and exactly ten other positive factors.*

The ideas in the problem are straight from the elementary curriculum: factoring numbers into primes, counting and multiplication. However, the exploration is really centred on a function used in number theory and its properties. The *divisor function*, $d(n)$, returns the number of positive integer divisors (factors) of a positive integer n. For example, from the problem $d(28) = 6$. The example and preamble of the problem lead the students to discover the following number theory theorem.

Theorem 1. *If the prime factorization of n is*

$$n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times p_3^{\alpha_3} \times \cdots \times p_k^{\alpha_k}$$

for primes $p_1 < p_2 < p_3 < \cdots < p_k$ and integers $\alpha_i \geq 1$, for $1 \leq i \leq k$, then

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \cdots (\alpha_k + 1).$$

Thus for the example, since $28 = 2^2 \times 7^1$, then the theorem gives us

$$d(28) = (2 + 1)(1 + 1) = 3 \times 2 = 6$$

as indicated in the problem.

Having given the students the theorem, without explicitly stating it, they are now open to exploring questions regarding the number of factors of a positive integer. We will look at the solution to the problem, discuss how to implement it in your classroom, and present some ideas for further exploration.

Part (a) is just getting the students used to using their new idea. They need to realize that they have to factor the numbers involved into primes, $675 = 3^3 \times 5^2$ and hence we have

$$d(675) = (3 + 1)(2 + 1) = 4 \times 3 = 12.$$

In part (b), we get the students thinking a bit. Since we are interested in the factors of n, we should also factor the factors! So writing

$$9 = 3^2, \quad 11, \quad 15 = 3 \times 5, \quad 25 = 5^2$$

we can tell that n is of the form

$$n = 3^{2+\alpha_1} \times 5^{2+\alpha_2} \times 11^{1+\alpha_3} \times k$$

where $\alpha_1, \alpha_2, \alpha_3 \geq 0$ and k is a positive integer that is relatively prime with 3, 5, and 11 and hence

$$d(n) = (3 + \alpha_1) \times (3 + \alpha_2) \times (2 + \alpha_3) \times d(k)$$

From the problem statement we know that $d(n) = 18$. The only way to obtain this is if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $k = 1$, so that $d(k) = 1$ and hence $n = 2475$.

. . . .

Parts (a) and (b) have led the students to thinking about the connections between the prime factorization of a number n and the number of positive integer factors (divisors in number theory) it has. Part(c) is the *real* question we want to ask. Let's refresh our memory:

> *(c) Determine the number of positive integers less than 500 that have the positive factors 2 and 9 and exactly ten other positive factors.*

The other two parts were leading and lending some support to the student. We can use our method of part (b) to say if $n < 500$ is one of the desired numbers, then $d(n) = 12$ and $n = 2^{1+\alpha_1} \times 3^{2+\alpha_2} \times k$, where k is relatively prime to 2 and 3. Hence

$$(2 + \alpha_1) \times (3 + \alpha_2) \times d(k) = 12.$$

We will look at two cases:

Case 1: $k = 1$
Since $(2 + \alpha_1) \times (3 + \alpha_2) = 12$ and

$$12 = 1 \times 12 = 2 \times 6 = 3 \times 4$$

the only possibilities are $\alpha_1 = 0$ and $\alpha_2 = 3$, which yield $n = 2 \times 3^5 = 486$; $\alpha_1 = 1$ and $\alpha_2 = 1$, which yield $n = 2^2 \times 3^3 = 108$; and $\alpha_1 = 2$ and $\alpha_2 = 0$, which yield $n = 2^3 \times 3^2 = 72$.

Case 2: $k > 1$
In this case $d(k) > 1$. However, since $d\left(2^{1+\alpha_1} \times 3^{2+\alpha_2}\right) \geq 6$, the only possibility is $\alpha_1 = \alpha_2 = 0$ and $d(k) = 2$. What numbers have only two positive factors? That would be the primes, as any prime p has only itself and 1 as factors. Thus any number of the form $n = 2 \times 3^2 \times p$, for a prime $p > 3$ satisfies $d(n) = 12$. Therefore, since we want $n < 500$ we need

$$\begin{aligned} 2 \times 3^2 \times p &< 500 \\ p &< 27\tfrac{7}{9} \\ p &\leq 27 \end{aligned}$$

Thus we can choose p to be 5, 7, 11, 13, 17, 19, or 23 giving us, when we add in the three solutions from case 1, ten possible numbers with the desired property.

. . . .

OK, so we have answered the questions. Is there anything else that we can wrangle out of this for our students? We can always try to dive deeper, expanding on the richness of the problem and deepening our understanding of number theory along the way. The problem focused on numbers with a specific number of factors. Can we describe *all* numbers n such that $d(n) = k$, that is for each k, which numbers have *exactly* k factors?

We will start with a systematic search. If $k = 1$, we get the only possible number with one factor is 1. As such, 1 becomes its own "group" when grouping positive integers into primes, composites and *units* (1). As mathematicians explore more complex "numbers" this idea of a unit becomes more important. When discussing all of the integers, we have primes, composites and units 1 and -1 (0 is something else entirely). Mathematicians say two numbers are *associates* if we can write one as the product of the other and a unit. So 6 and -6 are associates. Thus the primes 7 and -7 are associates, so they belong together and don't really constitute two different primes. Moving on to the complex numbers or other number fields we get even more associates, so we have to keep track of them by understanding what our units are.

When we go to $k = 2$ we get the primes. That is, $d(n) = 2$ if and only if n is prime, as we surmised in the solution to the problem. This makes sense from our definition of a prime. If p is prime, then its only factors are 1 and p.

Next we are on to our first non-trivial case, $k = 3$. Depending on the level of sophistication of the students, this might not jump out at them. In many cases when we are problem solving, doing some computation and then looking for patterns is never a bad strategy. If we go and factor a few numbers we will get

$d(1) = 1$	$d(2) = 2$	$d(3) = 2$	$d(4) = 3$	$d(5) = 2$
$d(6) = 4$	$d(7) = 2$	$d(8) = 4$	$d(9) = 3$	$d(10) = 4$
$d(11) = 2$	$d(12) = 6$	$d(13) = 2$	$d(14) = 4$	$d(15) = 4$
$d(16) = 5$	$d(17) = 2$	$d(18) = 6$	$d(19) = 2$	$d(20) = 6$
$d(21) = 4$	$d(22) = 4$	$d(23) = 2$	$d(24) = 8$	$d(25) = 3$

Colour coding cases with the same result we see, in red, our primes, p, with $d(p) = 2$ as expected. Looking at the case of $k = 3$, in blue, we get $n = 4, 9, 25$, which are squares of prime numbers. If we go back to our discovery in the problem, if $n = p^2$ for some prime p, then $d(n) = 2 + 1 = 3$. At this point we may come to the conclusion, if $n = p^x$ for some prime p, then $d(n) = x + 1$. Checking our list, it works for $8 = 2^3$, with $d(8) = 4$ and $16 = 2^4$, with $d(16) = 5$. If we wanted a bit more evidence we could check the next few prime powers: $27 = 3^3$, with factors 1, 3, 9, and 27 (3^0, 3^1, 3^2, 3^3), so $d(27) = 4$; and $32 = 2^5$ with factors 1, 2, 4, 8, 16, and 32 (2^0, 2^1, 2^2, 2^3, 2^4, 2^5), so $d(32) = 6$.

Checking theorem 1, we see that it makes sense that $d(n) = 3$ only for squares of primes. Since 3 is itself prime, then from $d(n) = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\cdots(\alpha_k+1)$, one of the $\alpha_i + 1$ terms must be 3 and the others must be 1. That means that $\alpha_i = 2$ and $\alpha_j = 0$ for $j \neq i$, or $n = p_i^2$. Thus, for $d(n) = k$, we are interested not only in the factorization of n, but also in the factorization of k!

Moving to the case of $k = 4$ (brown in our list) illustrates this. As $4 = 1\times 4 = 2\times 2$ we get two possibilities for $d(n) = 4$. We could have $n = p^3$ for some prime p, as illustrated by $n = 8$ and $n = 27$. We could also have $n = p \times q$ for distinct primes p and q. We can see this for the remaining cases in our list: $6 = 2 \times 3$, $10 = 2 \times 5$, $14 = 2 \times 7$, $15 = 3 \times 5$, $21 = 3 \times 7$, and $22 = 2 \times 11$.

We can now make some generalizations. For example, if $k = q$ is a prime, then $d(n) = q$ only for $n = p^{q-1}$, for some prime p. So $d(n) = 5$ only for fourth powers of primes, $d(n) = 7$ for sixth powers of primes, $d(n) = 11$ for tenth powers of primes, and so on (trivially, $d(n) = 2$ only for the primes themselves).

Returning to part (c) of the original problem, we were interested in numbers with twelve factors. Since $12 = 2^2 \times 3$ we can write 12 as a product as

$$12 = 1 \times 12 = 2 \times 6 = 3 \times 4$$

which suggests numbers of the form $n_1 = p^{11}$, $n_2 = p \times q^5$, and $n_3 = p^2 \times q^3$, for primes p and q. However, we are interested in *all* ways that we can form a number with 12 factors and since $12 = 2 \times 2 \times 3$, we get a fourth class $n_4 = p \times q \times r^2$, where r is also prime.

From here the sky is the limit. We could explore all the *classes* of numbers with a certain number of factors, we could determine the smallest number n that satisfies $d(n) = k$ for each k, how many numbers are in a particular range (similar to (c)), and so on.

We could even go in another direction and explore functions with properties similar to those of $d(n)$. The function $d(n)$ is called a *multiplicative* function, since if a and b are relatively prime, then $d(a \times b) = d(a) \times d(b)$. You may enjoy verifying (or proving) this and seeing that if a and b have a common factor greater than one then $d(a \times b) \neq d(a) \times d(b)$ (a function that satisfies $f(a \times b) = f(a) \times f(b)$ for all a and b is called *completely multiplicative*). There are some functions that are very familiar to high school students that are (completely) multiplicative, can you find some?

There are many functions used in number theory, called *arithmetic functions*, which are multiplicative. You may enjoy exploring the properties of some of these functions with your students. *Euler's totient function*, $\phi(n)$, is a nice example. The function $\phi(n)$ returns the number of numbers less than n which are relatively prime to n. So, $\phi(12) = 4$ since 1, 5, 7, and 11 are the only numbers less than 12 which are relatively prime to it. The nice thing with multiplicative functions is we can focus on what $f(p^n)$ is for prime p. Then, since the function is multiplicative, if $n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times p_3^{\alpha_3} \times \cdots \times p_k^{\alpha_k}$, then $f(n) = f\left(p_1^{\alpha_1}\right) \times f\left(p_2^{\alpha_2}\right) \times f\left(p_3^{\alpha_3}\right) \times \cdots \times f\left(p_k^{\alpha_k}\right)$.

I hope you will have some fun exploring some of the ideas in this article. Check out the Fryer, Galois and Hypatia contests for other candidates for classroom exploration.

MATHEMAGICAL PUZZLES

No. 6

Tyler Somer

Rep-Tiles, Part 1

In this article and the next, I wish to explore some tiles which can replicate to larger, yet similar, two-dimensional shapes when multiple copies of the basic tile are used. In the world of mathematical puzzles, such replicating tiles are known simply as *rep-tiles*. I will limit this introduction of rep-tiles to relatively simple tiles, especially those which can be made with wood in a work-shop.

If not using wood from a work-shop, acrylic pieces can be laser-cut from sheets at an engraver or similar professional service. If your school has the equipment, the students in the shop classes can create kits for the math classes. Otherwise, inexpensive classroom kits can be cut from paper, card-stock, or foam-board.

Using discrete tiles, one will need either $4, 9, 16, 25, \ldots,$ or n^2 tiles to replicate a given shape in some larger size. Four copies of a square will replicate, trivially, to a 2×2 square, and this can be extended to any set of n^2 equal square tiles. A triangle, being half of a parallelogram, can similarly replicate to larger copies of itself. Figure 1 shows this for both 4 and 9 copies of an acute-scalene triangle.

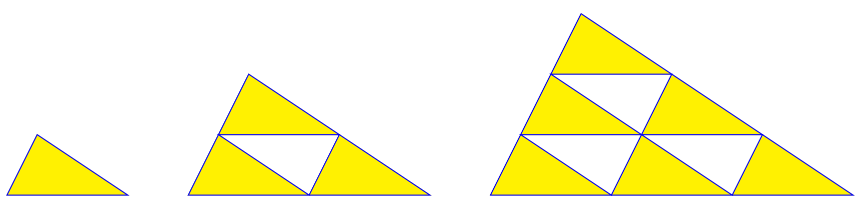


Figure 1: Four and nine copies of an acute scalene triangle rep-tile

A domino has dimensions 1×2. Four dominoes can create a 2×4 tiling which replicates the domino shape. It is interesting, however, that there are four distinct solutions, up to rotation and reflection, for the four dominoes rep-tile. This is unlike the unique – but still trivial – solutions for squares and triangles.

To avoid trivial cases, rep-tiles must move beyond triangles, squares, and rectangles. Three relatively simple rep-tiles are given in Figure 2. They are composed of squares, but they are interesting that they avoid trivial solutions that the earlier shapes provided. The tromino-L, tetromino-L, and pentomino-P have areas 3, 4, and 5, respectively. The letter suffixes (-L, -L, -P) are assigned as the shapes approximate the given letter of the alphabet.

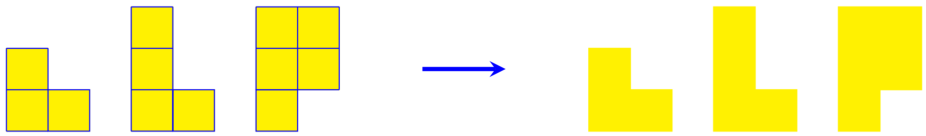


Figure 2: the tromino-L, the tetromino-L, and the pentomino-P

For readers not familiar with *polyominoes*, here is a brief introduction: First presented by Solomon Golomb in 1953, polyominoes are plane geometric figures made by joining equal squares along common edges. A single square is thus named a *monomino*. Two squares form the common *domino*. Three squares form a *tromino*, and there are two distinct tromino shapes. Continuing the construction, there are five *tetrominoes*, then twelve *pentominoes*. Most puzzle designers like the "practical" set size of twelve pentominoes, with its total area of 60; although some designers do extend to include some of the 35 *hexominoes* and 108 *heptominoes*. (Interestingly, my spell-checker recognizes domino and pentomino, but none of the other -omino names. It would seem that the set of pentominoes is now common enough for word processing software spell checkers.) The number of higher-order polyominoes does become quite large, so the introduction shall end here [*Ed.*: Check out sequence A000105 from The On-Line Encyclopedia of Integer Sequences]. Interested readers can find more information online with a search of either Solomon Golomb or Polyomino.

Let us first consider the tetromino-L. In the wood-shop, waste of material can be reduced by taking rectangular stock and cutting it as dominoes. Two such dominoes can be glued as a butt joint, creating the tetromino-L, as in Figure 3a.

A set of four such tiles creates the simplest of puzzles for any young child. For the adult, it is – it should be – trivial to set two tetromino-L tiles as a larger domino, as in Figure 3b; then two such larger dominoes form the replicated 4-unit tile. Owing to reflections of the two-piece sub-structure, there are four solutions to this rep-tile puzzle.

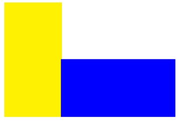


Figure 3a: gluing two dominoes as a tetromino-L

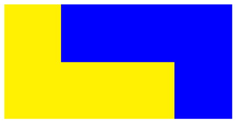


Figure 3b: setting two tetromino-L pieces as a larger domino sub-structure

The puzzle becomes interesting when it is increased to a set of nine basic tetromino-L rep-tile pieces. There are fourteen solutions, but none of them include the 4-piece rep-tile sub-structure. The solutions do split among four families – three families of four solutions, and one family of two solutions – as a result of the domino-like two-piece sub-structure. The one family of two solutions is the closest we get to a unique solution. Nine tetromino-L pieces also occupy an area of 36, but it is remarkable that they do not form any rectangle with area 36.

With 16 tetromino-L rep-tiles, triviality returns. The simplest solution is to replicate the 4-piece solution four times over, with its many sub-reflections. Thousands of other solutions exist, no less than 59 009 in total, for 16 tetromino-L rep-tiles. Increasing to 25 tetromino-L rep-tiles, the size-5 shape can be replicated in over 17.5 million of ways. For this and larger sizes, determining the number of solutions becomes an exercise for computer programmers. As a practical matter, with a large set of pieces and working on a table or desk, it becomes almost impossible NOT to find a solution. We can conclude that the set of nine pieces is not only interesting, but it also provides a practical limit for puzzle enjoyment.

While the tetromino-L rep-tile is suited for youngsters and novice puzzle solvers, the next three cases are a bit more sophisticated. Teachers can develop their discovery activities to suit the grades they teach. Independent students can pursue the puzzles to suit their level of curiosity, and determine various relationships among the solutions.

The next rep-tile puzzle to consider is based on the pentomino-P piece. The puzzle is interesting with a set of four pieces, and two solutions exist. At nine pieces, the triple-size pentomino-P has five solutions. Four of these five solutions are related: a four-piece pentomino-P sub-structure exists, and another two pieces form a 2×5 rectangle which can be reflected. The fifth solution is unique unto itself, and the reader is encouraged to find it without computer assistance.

Similar as with the tetromino-L rep-tile, a set of 16 pentomino-P rep-tiles can be solved as a trivial extension of the 4-piece puzzle. Computer analysis finds that 16 pentomino-P rep-tiles will solve the size-4 puzzle in 3451 ways, and 25 pentomino-P rep-tiles will solve the size-5 puzzle in over 5 million ways.

Moving to the smaller tromino-L rep-tile, this puzzle is also interesting with a set of four pieces. It is the first non-trivial result that is unique. It is closely related to one of the two solutions of the pentomino-P rep-tile.

The reader is encouraged to make the pieces, solve the puzzles, and determine the relationship between the four-piece pentomino-P and tromino-L rep-tile solutions.

At nine pieces, the tromino-L rep-tile remains practical, with the size-3 result having 3 related solutions. It is left as a challenge to the reader to determine the related solutions.

As before, the tromino-L rep-tiles of size-4, -5, and higher become impractical for table-top work. For the computer programmers among the readers, it becomes an exercise to determine the number of solutions. Some numbers are given in a table at the end of this article.

Bisecting the tromino-L, we get a right-angled trapezoid that also includes a $45°$ angle, as shown in Figure 4.

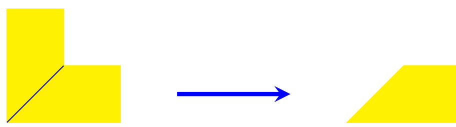


Figure 4: the 1-1-2-$\sqrt{2}$ right trapezoid

For this trapezoid, like the polyomino pieces, interesting replication puzzles do exist for both 4 and 9 pieces. With 4 pieces – size 2 – there is a singular solution, and it is indeed related to the comparable unique tromino-L solution. At size 3, the 9 trapezoid pieces replicate in 10 solutions, divided in 3 families, as described in the table at the end of this article. Similar to the earlier rep-tiles, 16 and 25 copies of this trapezoid also give a large number of solutions. The puzzle would seem to lose its appeal at these sizes, although computer programmers may enjoy the challenge of counting the number of solutions at these and larger sizes.

In summary, there are interesting puzzles for 4 and 9 physical copies of the non-trivial rep-tiles presented in this article. The small number of solutions that do exist – some which are unique – keep the puzzles engaging. Where multiple solutions exist, the challenges are to find all the families of related solutions, in general; and to find the families of fewer solutions, in particular.

Solutions are not given here. The joy of discovery is left to the reader. These puzzles are not very difficult, and so they may be used as classroom discoveries or as simple challenges for children.

Once we expand to 16 and 25 pieces, the solutions become numerous. Finding a solution with physical pieces is no longer a challenge. The exercise becomes better suited to computer programmers, with the goal to count the number of solutions that exist.

We conclude this article with the following table, which provides a summary of the results discussed earlier.

REP-TILES	**4 copies (size 2)**	**9 copies (size 3)**	**16 copies (size 4)**	**25 copies (size 5)**
Tetromino-L	4 solutions	14 solutions 4 families: - 3 families of 4 - 1 family of 2	59 009 solutions	over 17.5 million solutions
Pentomino-P	2 solutions	5 solutions - 1 family of 4 - 1 unrelated	3451 solutions	over 5 million solutions
Tromino-L	1 solution	3 solutions: all related	205 solutions	54 213 solutions
1-1-2-$\sqrt{2}$ right trapezoid	1 solution	10 solutions - 2 families of 4 - 1 family of 2	721 solutions	96 158 solutions

. . . .

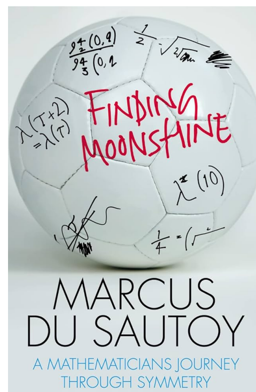
When he was teaching, Tyler often had mechanical puzzles in his classroom. As a freelancer, Tyler has worked with numerous inventors and co-designers to bring dozens of table-top solo-logic puzzle kits to market. He continues to design puzzles, and he spends a good deal of time in his woodshop, building his own and others' puzzle designs.

From the bookshelf of . . .

Frédéric Morneau-Guérin

This MathemAttic *feature brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to* `MathemAttic@cms.math.ca`.

Finding Moonshine : A Mathematician's Journey Through Symmetry
by Marcus du Sautoy
ISBN-13: 978-0007214617
Published by Fourth Estate, 2009.



Il est indubitable que la nature regorge de mystérieuses manifestations de symétries. À titre d'exemple, dans le monde de la chimie physique, il appert que le diamant tire sa force de l'arrangement hautement symétrique des atomes de carbone qui le composent. Autre exemple, issu celui-là du monde vivant : parmi l'infinité de structures que les abeilles pourraient exploiter pour emmagasiner une quantité donnée de miel, c'est le treillis hexagonal isométrique (un motif hautement symétrique) qu'elles emploient pour leurs alvéoles qui minimise la quantité de cire nécessaire; un fait que les abeilles semblent savoir d'instinct depuis la nuit des temps, mais qui n'a été confirmé par la science qu'à l'aube du vingt-et-unième siècle. Considérant ce qui précède, il n'est probablement pas fortuit si l'esprit humain semble aussi intrigué et attiré par tout ce qui incarne un aspect de symétrie. Notre cerveau semble en effet programmé pour non seulement remarquer, mais même rechercher l'ordre, la régularité et, par voie de conséquence, la symétrie. Notre inclinaison naturelle à apprécier, rechercher, voire générer de la symétrie s'exprime depuis des temps immémoriaux dans les arts visuels et vivants, l'architecture, la poésie et la musique. La redondance et la répétition de motifs sont, quant à eux, des éléments clés de l'apprentissage du langage humain de même que de la communication efficace et fluide.

Vu l'importance que revêt la symétrie tant dans l'univers matériel que dans l'univers abstrait des mathématiques, le professeur pour la compréhension de la science à l'Université d'Oxford Marcus Du Sautoy a jugé bon d'employer son immense talent de vulgarisateur à démystifier ce concept aussi fondamental et omniprésent que difficile à cerner. On lui en est reconnaissant, car l'ouvrage qui en a résulté, *Finding Moonshine : A Mathematician's Journey Through Symmetry*, est un splendide tourne-page regorgeant d'informations historiques et mathématiques captivantes rendues accessibles et compréhensibles pour un lectorat composé de non-spécialistes cultivés. Paru en 2008, ce livre unique en son genre nous fait parcourir plusieurs siècles de développement de théories mathématiques avancées avec, en filigrane, une réflexion personnelle sur la vie du mathématicien, de l'éveil à cette discipline jusqu'à l'espoir de conquête des plus hauts sommets avec tous les risques et les écueils auxquels cela expose.

Du Sautoy nous fait voir d'entrée de jeu que la notion de symétrie a longtemps semblé évanescente. Déjà chez les Grecs de l'Antiquité, tous semblaient en avoir une certaine conception intuitive, mais personne n'était en mesure de mettre le doigt sur une définition entièrement convenable. La difficulté à saisir ce dont il s'agit se reflète d'ailleurs jusque dans le mot lui-même. Le substantif "symétrie" tire en effet son origine du grec ancien. Il s'agit d'une synthèse des mots $\sigma\upsilon\nu$, qui signifie *même*, et $\mu\epsilon\tau\rho\omega\nu$, (metron), qui signifie *mesure*. Juxtaposés, ces deux mots expriment donc l'idée de différents éléments "de même mesure". Or, s'il semble conforme à l'intuition qu'un polygone est plus symétrique lorsque tous ses côtés sont de mêmes mesures, ce constat élude la question davantage qu'il n'y répond. Quand bien même on disposerait d'un critère universel nous permettant de dire que tel objet est *plus symétrique* que tel autre ou moins symétrique que tel autre encore (et rien n'indique que ce concept de *même mesure* soit applicable dans tous les contextes), cela ne nous aiderait en rien à répondre à la question quant à savoir ce qu'est au fond la symétrie.

Petit à petit, nous dit l'auteur, on finit par saisir que, d'une certaine manière, une symétrie est quelque chose d'actif et non de passif. En effet, on peut considérer une symétrie comme une action pouvant être effectuée sur un objet et qui le laisserait identique, plutôt que comme une propriété de l'objet lui-même. Un triangle équilatéral, par exemple, possède six symétries : trois symétries axiales et trois symétries de rotations. Si vous fermez vos yeux et que j'effectue une rotation de 120°, 240° ou 360° autour de l'orthocentre, ou encore une réflexion par rapport à l'une ou l'autre des médiatrices de ses côtés, voire un enchanement de l'une ou l'autre de ces actions, alors, lorsque j'aurai terminé et que je vous inviterai à rouvrir les yeux, vous seriez incapable de déterminer avec certitude si une action a été posée ou si on a plutôt laissé le triangle inchangé tant les portraits *ex ante* et *ex post* se ressemblent.

À la suite d'une épiphanie soudaine survenue dans la première moitié du 19e siècle chez un jeune révolutionnaire français à l'approche de sa mort aussi tragique que prématurée, la vraie nature de la notion de symétrie se révéla enfin. De ce moment Eureka, admirablement bien décrit par Du Sautoy, surgit un nouveau langage celui de la théorie des groupes permettant d'en capturer la véritable signification. Au cur de cette vision nouvelle et géniale se trouve la reconnaissance du fait qu'il ne faut pas se contenter de regarder les symétries individuelles d'un objet. Il faut plutôt aborder ces symétries comme une collection qu'il convient d'appeler un *groupe*. Ce sont en effet les interactions unissant les diverses symétries entrant dans la composition du *groupe de symétrie* d'un objet qui synthétisent les caractéristiques essentielles de la symétrie d'un objet.

Il n'avait pas échappé aux Grecs de l'Antiquité que tout nombre naturel peut être factorisé en produit de nombres dits *premiers* (à savoir des nombres indivisibles en ceci qu'ils n'admettent que deux diviseurs, soit 1 et eux-mêmes) et que ces nombres premiers sont en quelque sorte les éléments constitutifs de tous les autres nombres naturels. Le développement de la théorie des groupes qui offre un langage permettant de concevoir puis de nommer, et par conséquent, d'étudier la symétrie comme jamais auparavant elle ne l'avait été rendit possible la réalisation d'un fait étonnant d'un haut niveau de subtilité : les groupes de symétries, tout comme les nombres naturels, peuvent être factorisés encore et encore en groupes de symétries plus petits jusqu'à obtenir des groupes de symétries qui sont en quelque sorte, à l'image des nombres premiers, élémentaires puisqu'indivisibles. Ces groupes de symétries indivisibles, comme les nombres premiers, ont pour caractéristique essentielle d'être les éléments constitutifs à partir desquels tous les groupes de symétries peuvent être construits. Il fallut un temps considérable et des efforts intellectuels soutenus à la communauté mathématique pour appréhender ce concept d'indivisibilité s'appliquant en matière de symétrie. Mais lorsqu'ils y parvinrent enfin, l'opportunité de produire une sorte de *tableau périodique* de la symétrie se profila. De la même manière que le tableau périodique des éléments est une classification systématique rassemblant les constituants élémentaires de toute substance, un éventuel Atlas des groupes de symétrie indivisible listerait *tous* (sinon exhaustivement du moins en compréhension) les groupes de symétries primitifs entrant dans les compositions des groupes de symétries composés.

Nous l'avons dit, l'esprit humain semble en quête d'ordre et de régularité. Catégoriser, classifier et ordonner sont des processus mentaux auxquels nous nous prêtons tous avec une certaine aisance et avec un enthousiasme certain. Il arrive cependant que la nature refuse de se plier à nos dispositions naturelles vers l'ordre. C'est notamment le cas dans le domaine de la symétrie. Alors qu'au milieu du 19e siècle la classification des éléments constitutifs de la symétrie allait plutôt bon train, le mathématicien français mile Mathieu mit au jour, presque par inadvertance, cinq groupes de symétries indivisibles plutôt insolites. Alors que tous les groupes indivisibles découverts précédemment s'étaient avérés appartenir à une famille infinie de groupes partageant certaines propriétés communes, les cinq groupes dont l'existence fut révélée par Mathieu ne semblaient ni appartenir à une famille connue ni suggérer l'existence d'une nouvelle famille infinie ayant jusque-là échappé à l'attention. Tout indiquait plutôt qu'ils constituaient une sorte d'archipel isolé dans du mystérieux monde de la symétrie. Afin de souligner le caractère épisodique de ces groupes, on les qualifia de *sporadiques.* Plusieurs décennies s'écoulèrent sans que l'on puisse déterminer clairement où logeaient les groupes de Mathieu dans l'édifice mathématique. Puis, relate Du Sautoy, en 1954, à l'occasion du Congrès international des mathématiciens d'Amsterdam, un appel fut lancé pour dresser une liste exhaustive de ces groupes dits sporadiques. Ainsi s'amorça une aventure épique, unique dans l'histoire des mathématiques, qui devait durer trente ans.

Les explorateurs se divisèrent naturellement en deux camps reflétant des philosophies distinctes. Le premier camp, composé d'une troupe bigarrée de mathématiciens anticonformistes à la tête de laquelle se trouvait le coloré et charismatique John Horton Conway, se spécialisait, tels des pirates à la recherche de trésors, dans la découverte d'objets mathématiques de plus en plus exotiques dont les symétries sont modélisées par des groupes sporadiques. Quant au second camp, il était constitué d'un vaste réseau d'individus travaillant de façon organisée et disciplinée. Telle une force de pacification, cette équipe, désireuse d'enfin acquérir une connaissance complète, vérifiait chaque avenue en procédant méticuleusement et systéma-tiquement en vue d'exploiter les limites inhérentes à la symétrie pour expliquer pourquoi il n'y avait pas de nouvelles symétries indivisibles qui pourraient exister si l'on partait dans telle ou telle direction. Au grand dam de la horde de corsaires et de flibustiers s'accrochant à l'espoir de voir se poursuivre indéfiniment l'aventure exploratoire, au tournant des années 1980, à la suite de la découverte d'un vingt-sixième groupe sporadique, les deux équipes commencèrent à s'apercevoir l'une et l'autre aux confins de l'horizon. La circumnavigation du monde de la symétrie avait été complétée. Le point d'orgue de cette épopée enlevante fut la construction d'un objet arborant 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000 symétries et dont le groupe de symétrie fut surnommé à bon droit le *groupe Monstre*.

Le présent texte serait ne saurait être qualifié de complet sans une explication minimale du titre de l'ouvrage considéré. Bien que la traduction littérale de *Finding Moonshine* soit Chercher de l'alcool de contrebande, il va sans dire qu'il ne s'agit pas là de la signification voulue par l'auteur. Il faut savoir que, dans le registre familier de la langue de Shakespeare, le mot *moonshine* possède un second sens. En effet, il est parfois utilisé pour qualifier une idée et signifier qu'elle est saugrenue, voire complètement dingue. C'est ce mot, dans cette acception, qu'aurait employé John H. Conway, le capitaine des corsaires, lorsqu'on aurait porté à son attention que, par une étrange concidence, la valeur numérique de certains attributs du groupe Monstre concordait avec une suite de nombres occupant un rôle prépondérant dans une théorie mathématique en apparence sans rapport avec la théorie des groupes. Il n'en fallut pas plus pour que le nom *Monstrous Moonshine* (qu'on pourrait rendre dans la langue de Molière par la monstrueuse *idée saugrenue*) soit accolé à la conjecture suivant laquelle, loin d'être le fruit du hasard, cette concordance numérique était en fait la manifestation visible d'une interconnexion inattendue et insoupçonnée entre deux lointains continents du mystérieux monde des mathématiques. On l'aura compris, au cours du voyage au cur de la symétrie que nous propose Marcus Du Sautoy on glanera suffisamment d'informations historiques et mathématiques pour nous permettre de savoir et d'apprécier comment des mathématiciens, qui sont nos contemporains, ont statué sur la valeur de vérité de cette conjecture.

Diplômé de l'Université Laval et de l'Université de Cambridge, Frédéric Morneau-Guérin est professeur agrégé au département Éducation de l'université Téluq, à Québec. Ses recherches portent sur l'analyse harmonique, la théorie des opérateurs et la théorie des matrices.

OLYMPIAD CORNER

No. 416

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **November 30, 2023**.

OC646. Let ω be a plane in 3-space which passes through a vertex A of a unit cube. For the vertices B_1, B_2, B_3 of the cube which are adjacent to A, let F_1, F_2, F_3 in turn be the feet of the perpendiculars dropped from these vertices to ω. Determine the value of

$$(AF_1)^2 + (AF_2)^2 + (AF_3)^2.$$

OC647. In an infinite arithmetic progression of positive integers there are two integers with the same sum of digits. Will there necessarily be one more integer in the progression with the same sum of digits?

OC648. Consider two concentric circles Ω and ω. The chord AD of the circle Ω is tangent to ω. Inside the minor segment AD of the disc with the boundary Ω, an arbitrary point P is selected. The tangent lines drawn from the point P to the circle ω intersect the major arc AD of the circle Ω at points B and C. The segments BD and AC intersect at the point Q. Prove that the segment PQ divides the segment AD into two equal parts.

OC649. For each positive integer n, denote by $\omega(n)$ the number of distinct prime divisors of n (for example, $\omega(1) = 0$ and $\omega(12) = 2$). Find all polynomials $P(x)$ with integer coefficients, such that whenever n is a positive integer satisfying $\omega(n) > 2023^{2023}$, then $P(n)$ is also a positive integer with

$$\omega(n) \geq \omega(P(n)).$$

OC650. Determine all real values of x for which

$$\sqrt{\log_2 x \cdot \log_2(4x) + 1} + \sqrt{\log_2 x \cdot \log_2\left(\frac{x}{64}\right) + 9} = 4.$$

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 novembre 2023**.

OC646. Soit ω un plan dans l'espace trois dimensionnel, passant par le sommet A d'un cube unitaire. À partir des sommets B_1, B_2, B_3 du cube, adjacents à A, soient dans le même ordre F_1, F_2, F_3 les pieds des perpendiculaires vers ω. Déterminer la valeur de

$$(AF_1)^2 + (AF_2)^2 + (AF_3)^2.$$

OC647. Dans une progression arithmétique formée d'entiers positifs, il en existe deux avec la même somme de chiffres en représentation décimale. Y en aura-t-il un troisième avec cette même somme de chiffres?

OC648. Soient deux cercles concentriques Ω et ω. La corde AD du cercle Ω est tangente à ω. Dans le segment mineur AD du disque de frontière Ω est choisi un point P. Les tangentes de P au cercle ω rencontrent l'arc majeur du cercle Ω en B et C. De plus, les segments BD et AC se rencontrent en un point Q. Démontrer que le segment PQ divise le segment AD en deux parties égales.

OC649. Pour tout entier positif n, soit $\omega(n)$ le nombre de diviseurs premiers de n; par exemple, $\omega(1) = 0$ et $\omega(12) = 2$. Déterminer tous les polynômes $P(x)$ à coefficients entiers, tels que, pour tout n entier positif vérifiant $\omega(n) > 2023^{2023}$, $P(n)$ est un entier positif tel que

$$\omega(n) \geq \omega(P(n)).$$

OC650. Déterminer toutes les valeurs réelles x telles que

$$\sqrt{\log_2 x \cdot \log_2(4x) + 1} + \sqrt{\log_2 x \cdot \log_2\left(\frac{x}{64}\right) + 9} = 4.$$

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2023: 49(3), p. 140–142.

OC621. Find all natural numbers n for which the number $n^n + 1$ is divisible by $n + 1$.

Originally 2020 All-Ukranian Mathematical Olympiad (virtual), Grade 10, Day 1, problem 1.

We received 19 submissions, of which 16 were correct and complete. We present a typical solution.

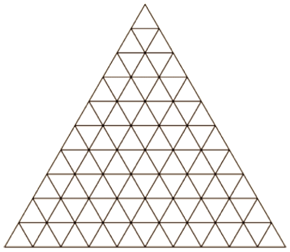
Let $m = n + 1$. Then $n \equiv -1 \pmod{m}$ and $n^n + 1 \equiv (-1)^n + 1 \pmod{m}$.

If n is odd, then $n^n + 1 \equiv -1 + 1 \pmod{m} \equiv 0 \pmod{m}$ and so $n + 1$ divides $n^n + 1$.

If n is even, then $n^n + 1 \equiv 1 + 1 \pmod{m} \equiv 2 \pmod{m}$. Since $m = n + 1 \neq 1, 2$, then $n + 1$ does not divide $n^n + 1$.

Thus $n + 1$ divides $n^n + 1$ if and only if n is odd.

OC622. An equilateral triangle with side length n is divided into n^2 small equilateral triangles of side length 1 (as in the picture for $n = 10$). At the start, one small internal triangle (with no points in common with external sides of the large triangle) is painted in blue, and the rest are painted yellow. In one move, you can choose any of the n^2 small triangles and swap its colour and the colours of the triangles adjacent to it along its sides. Using such moves, is it possible to make the entire board one colour?



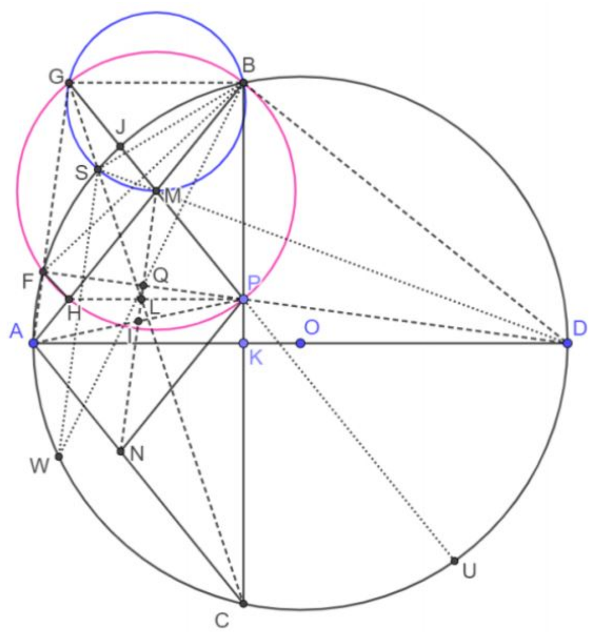
Originally 2020 All-Ukranian Mathematical Olympiad (virtual), Grade 10, Day 1, problem 2.

We received 2 submissions, but neither of them were correct and complete. We leave the problem as an exercise to the reader.

OC623. Let B and C be two points on the circumference of a circle with diameter AD such that $AB = AC$. Let P be a point on line segment BC and let M, N be points on line segments AB and AC, respectively, such that $PMAN$ is a parallelogram. Suppose PL is an angle bisector of triangle MPN with L lying on the line segment MN. If the line PD intersects MN in point Q, show that the points B, Q, L and C lie on the same circle.

Originally from the 2019 All-Ukranian Mathematical Olympiad, Grade 11, day 1, problem 4.

We received 5 submissions of which 4 were correct and complete. We present the solution by Theo Koupelis.



Let O be the centre of the circle with diameter AD. Let DP intersect this circle again at F, let I be the intersection point of the diagonals of parallelogram $PMAN$, let H be the intersection of PL with AB, let CL intersect the circle again at S and the line PM at G, let the segment PG intersect the circle at J, let the ray MP intersect the circle at U, and let BQ intersect the circle again at W.

We have

$$\angle MPB = \angle ACB = \angle ABC = \angle NPC,$$

so triangles MBP and NPC are isosceles and similar, and $\angle LPB = \angle LPC = 90^\circ$. But then

$$\angle MHP = \angle BAO = 90^\circ - \angle ABC = 90^\circ - \angle MPB = \angle MPH,$$

and thus M is the midpoint of the hypotenuse of the right triangle HPB. Because

PL is the angle bisector of $\angle MPN$ and $AC \parallel PG$ we have

$$GL/LC = ML/LN = MP/PN = BP/PC,$$

and thus $BG \parallel PL$. Therefore, $BGHP$ is a rectangle inscribed in the circle (M, MP).

But

$$\angle FBH = \angle FBA = \angle FDA = \angle FPH,$$

and thus F is on the circle (M, MP), and therefore $\angle GFP = \angle AFD = 90^\circ$ and A, F, G are collinear. Also, because M, I are midpoints of PG, PA, respectively, we have $MN \parallel AG$ and thus $DF \perp MN$. Now

$$2\angle SGM = 2\angle CGU = \overparen{CU} - \overparen{SJ} = \overparen{AJ} - \overparen{SJ} = \overparen{AS} = 2\angle SBA.$$

Thus, $GBMS$ is cyclic and $\angle MSB = \angle MGB = \angle PHB = \angle DAB = \angle DSB$, and therefore S, M, D are collinear.

Now, the quadrilateral $QMBD$ is cyclic because $\angle MQD = \angle MBD = 90^\circ$, so

$$\angle FDS = \angle QDM = \angle MBQ = \angle ABW.$$

Thus, $\overparen{AW} = \overparen{FS}$ and $SW \parallel AF \parallel MN$. Therefore, $\angle WBC = \angle WSC = \angle NLC$, and the quadrilateral $BQLC$ is cyclic.

OC624. A series contains 51 not necessarily different natural numbers which add up to 100. A natural number k is called *representable* if it can be represented as the sum of several consecutively written numbers in this series (perhaps one number). Prove that at least one of the two numbers k and $100-k$ is representable, where $1 \leq k \leq 100$.

Originally 2023 Kharkiv Regional Mathematical Olympiad, Grade 11, Final Round, problem 2.

We received 2 correct and complete submissions. We present both solutions.

Solution 1, by Oliver Geupel.

Let k be any integer such that $50 \leq k \leq 100$. We are going to show that at least one of the two numbers k and $100 - k$ is representable.

Suppose the given series of positive integers is $a_1, a_2, \ldots, a_{51}$. We extend it by integers $a_{52}, a_{53}, \ldots, a_{51+k}$, defined by the relation $a_i = a_{i-51}$ for $52 \leq i \leq 51 + k$. For $1 \leq j \leq 51$, consider the sums of $k + 1$ consecutively written numbers

$$s_j = a_j + a_{j+1} + \cdots + a_{j+k}.$$

We have

$$s_1 + s_2 + \cdots + s_{51} = (a_1 + a_2 + \cdots + a_{51})\,(k + 1) = 100(k + 1).$$

At least one of the 51 sums s_j is not greater than

$$\frac{1}{51}\left(s_1+s_2+\cdots+s_{51}\right)=\frac{100}{51}(k+1)<2k+1.$$

Let ℓ be an index such that $1\le \ell\le 51$ and

$$a_\ell+a_{\ell+1}+\cdots+a_{\ell+k}=s_\ell\le 2k.$$

Consider the $k+1$ sums

$$t_i=a_\ell+a_{\ell+1}+\cdots+a_{\ell+i}$$

for $0\le i\le k$. By the pigeonhole principle, at least two of them are congruent to each other modulo k. Suppose that

$$t_m\equiv t_{m+d}\pmod k$$

where $0\le m<m+d\le k$. It follows that

$$a_{m+1}+a_{m+2}+\cdots+a_{m+d}=t_{m+d}-t_m=k.$$

By the hypothesis $k\le 100=a_1+a_2+\cdots+a_{51}$, we have $d\le 51$.

Let $m+1\equiv r\pmod{51}$ where $1\le r\le 51$. If $r+d\le 51$, then k is representable by the sum

$$k=a_r+a_{r+1}+\cdots+a_{r+d}.$$

Otherwise, we have $r+d>51$. Then $100-k$ is representable by

$$\begin{aligned}100-k&=100-(a_r+a_{r+1}+\cdots+a_{51}+a_{52}+a_{53}+\cdots+a_{r+d})\\&=100-(a_r+a_{r+1}+\cdots+a_{51}+a_1+a_2+\cdots+a_{r+d-51})\\&=a_{r+d-50}+a_{r+d-49}+\cdots+a_{r-1}.\end{aligned}$$

Hence the result.

Solution 2, by UCLan Cyprus Problem Solving Group.

Let G_k be the graph with vertex set $V=\{0,1,2,\ldots,100\}$ where i is joined to j if and only if $|i-j|\in\{k,100-k\}$.

For $k\neq 50$ every vertex in G_k has degree 2. Indeed m is joined to exactly one out of $m-k, m+(100-k)$ and exactly one out of $m-(100-k), m+k$. Furthermore, for $k\neq 50$, these two numbers are distinct. For $k=50$ a similar argument shows that every vertex has degree 1.

Thus G_k is a union of cycles if $k\neq 50$ and a perfect matching if $k=50$.

Let $x_1,x_2,\ldots,x_{51}$ be the given sequence of numbers and for $i=1,2,\ldots,51$ define $s_i=x_1+\cdots+x_i$. Consider the 51 vertices $s_1,s_2,\ldots,s_{51}$. Two of them, say s_i

and s_j with $i < j$, must be adjacent in G_k. Then $x_{i+1} + \cdots + x_j$ is equal to k or $100 - k$.

OC625. Does there exist a convex 2021-gon with vertices at points with integer coordinates and such that the lengths of all its sides are equal?

Originally 2023 Kharkiv Regional Mathematical Olympiad, Grade 10, Final Round, problem 5.

We received 5 submissions of which 3 were correct and complete. We present the solution by Theo Koupelis.

This topic is discussed by D.G. Ball in "Constructibility of regular and equilateral polygons on a square pinboard," *Mathematical Gazette* 57 (1973) p. 119-122. If a planar n-gon whose vertices are at points with integer coordinates has all its sides equal, then n must be even; conversely, for n even and greater than 2, there exists an equilateral polygon of n sides on a square pinboard, in which case the n-gon can be taken to be convex. For the proof that n must be even we follow the argument found at `http://paulscottinfo.ipage.com/lattice-points/5regular.html#anchor2`.

If an n-gon $A_1A_2\dots A_n$ exists, where n is an odd natural number, such that its vertices have integer coordinates and its sides have equal lengths, then there is such a polygon with a side length ℓ that is minimal.

Let $\overrightarrow{u_{ij}} = \overrightarrow{A_iA_j} = (a_i, b_i)$, where $i = 1, 2, \dots, n$, $j = i+1$, and $A_{n+1} = A_1$. But $\ell^2 = a_i^2 + b_i^2$ and

$$\overrightarrow{u_{12}} + \overrightarrow{u_{23}} + \cdots + \overrightarrow{u_{n1}} = \vec{0}.$$

Thus, $\sum_i a_i = 0$ and $\sum b_i = 0$. Squaring and adding we have

$$n\ell^2 + 2S = 0,$$

where

$$S = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_i a_j + b_i b_j).$$

But n, ℓ^2, S are integers and n is odd, and thus $2 \mid \ell^2$. Therefore, a_i and b_i are either both even or both odd for each $i = 1, 2, \dots, n$. In either case, however, S is even because for each term $a_i a_j$ there is a corresponding term $b_i b_j$. Therefore, $4 \mid \ell^2$ and thus both a_i and b_i are even for each i. But this contradicts the assumption of a minimal ℓ because another polygon of half this side length can be constructed. In other words, for an n-gon with the given properties to exist, n must be even.

How Less is More (With Origami)

Medha Ravi

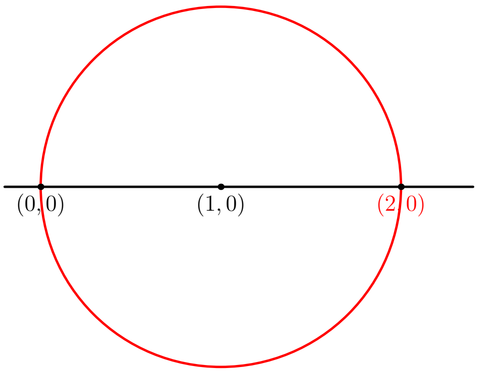
It is 478BC ancient Greece and a plague sent by Apollo has devastated the small island of Delos. The oracle has presented a problem that, once solved, will end the plague. Apollo must have his cubic altar's volume doubled. Everyone has turned to you, an avid member of the social club of mathematicians, to salvage the country and bring an end to the turmoil. In a world that consists solely of lines and circles, you may only use a straightedge and a compass.

Your feverish work on the problem of doubling the cube soon gives way to the problem of squaring the circle: given a circle of area A, you attempt to construct a square of the same area.

Your intellectual faculties having thus been stimulated, multiple new problems arise, including the construction of a regular heptagon and the trisection of an angle. As you and countless others toil over these problems over the course of 2000 years, there comes a breakthrough: none among these four problems is possible with just a straightedge and compass. Some other more powerful tool is necessary.

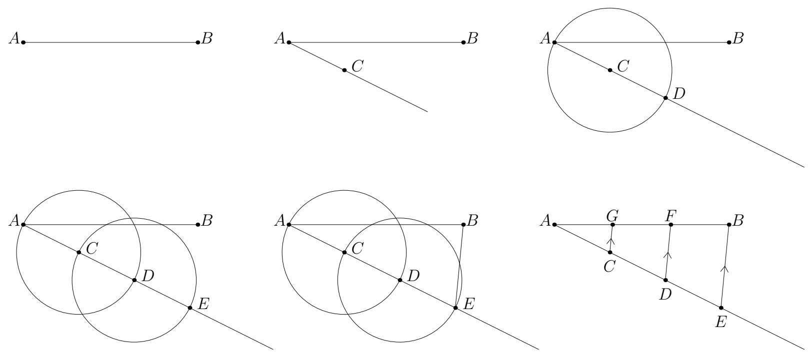
1 Setting the stage

Let us explore what we can construct with the straightedge and compass, given 2 points in the Cartesian coordinate plane.

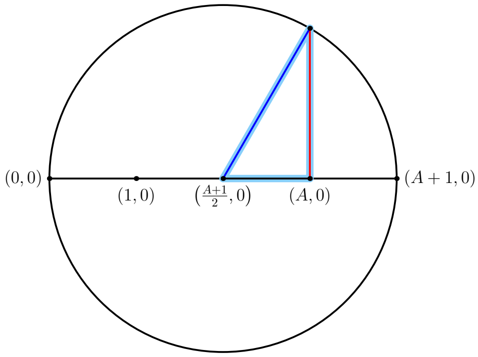


Given $(0, 0)$ and $(1, 0)$, we can construct the rest of the integers. We can create the x-axis that extends through both points, and create a circle centered at $(1, 0)$ that passes through the origin. The intersection point T of the circle and x-axis that isn't the origin is $(2, 0)$. In the same way, we can construct the rest of the integers.

We can also section any length into n equal length sections for any integer n. Below is a proof without words on such a construction. Thus, we can construct any rational number.



Additionally, we can see that the square root of any integer is constructible. Given a length A, the origin, and the length of 1 unit, we can construct the square root of the length A.



We begin by constructing a circle centered at $(\frac{A+1}{2},0)$ that passes through the origin, as shown in the diagram above. We know the radius, or length of the blue line, is $\frac{A+1}{2}$. Using the Pythagorean theorem on the triangle highlighted in blue confirms that the length of the red line is $\sqrt{A}$.

It turns out, beginning with the lengths 0 and 1, we can construct any integer number, any square root of a given number, and any iterative combination of the two with a field operation: addition, subtraction, multiplication, or division[1]. These numbers, which are composed solely of square roots, integers and operations that are included above, are called constructible numbers. Some examples of constructible numbers are $\frac{\sqrt{62}}{5}$, $\sqrt{1+\frac{\sqrt{3}}{8}}$, or $\sqrt{3+\frac{\sqrt{1+\frac{\sqrt{3}}{8}}}{6}}$.

Are any other numbers straight-edge and compass constructible? Surprisingly, no. These numbers that are a combination of integers, square roots, and field operations are the only numbers that can be constructed with a straightedge and a compass.

To understand this, let us take two points (a,b) and (c,d) in the coordinate plane with coordinates that are constructible numbers. The equation of such a line would be $(b-d)x+(c-a)y = bc-ad$. In this equation, note that all coefficients are constructible numbers. We can express y in terms of the linear expression x.

Similarly, let us take two points (a, b) and (c, d) in the coordinate plane that define a circle centered at (a, b) passing through (c, d), where a, b, c, and d are all constructible numbers. The equation of this circle is

$$(x-a)^2+(y-b)^2 = (c-a)^2+(d-b)^2.$$

Now consider the intersection of such a line with such a circle. By the equation of the straight line, we can express y as a linear function of x with constructible coefficients. If we plug y into this equation of our circle and solve, we get y as a quadratic function of x with constructible coefficients. This implies that the solution, x, must be constructible. Therefore, y must also be constructible, since y is a linear function of x with constructible coefficients.

We can also draw a similar conclusion for the intersection of two circles. The proof is done in a similar spirit to that of the circle and line. When we subtract one equation from the other and simplify, we obtain a linear equation in x and y. We can solve for y and then plug our value for y back into one of the original equations of the circles. This gives us, after some simplifying, a quadratic equation in terms of x with constructible coefficients.

The last link of this visualization is to understand what occurs when we construct points and lengths with a straightedge and compass. When we construct segments, we are creating lines and circles, taking their intersection points, creating more lines and circles with those points, so on and so forth.

Thus, if we begin with two points of constructible coordinates, and construct either a line or a circle, and solve for intersections that are constructed from the 2 points, we will arrive at intersections with constructible coordinates. This follows from the fact that solving linear and quadratic equations with constructible coefficients results in constructible numbers.

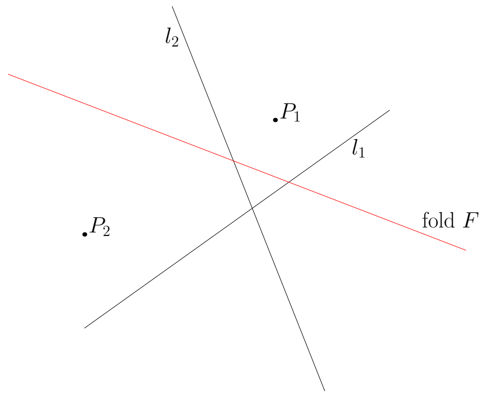
Summarizing, starting with the numbers 0 and 1 on the $x-$axis, we can construct any constructible number, but we can construct *only* constructible numbers.

What if, instead, we could use origami to fold points and lines? Let us provisionally call the numbers that we can fold in this manner "foldable numbers." Would the set of foldable numbers be larger or smaller than the set of constructible numbers? How are the constructions of origami different from the constructible numbers?

The crucial fold that sets origami apart from straightedge and compass constructions is the Beloch Fold[4,6,9].

Beloch Fold

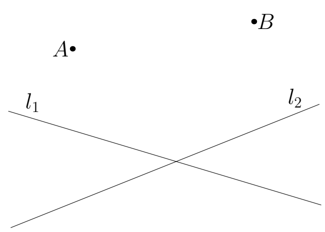
Given two points P_1 and P_2 and two lines l_1 and l_2, we can create, where possible, a fold F placing P_1 onto l_1 and P_2 onto l_2 simultaneously. Note that the Beloch fold does not exist for every set of two point line pairs. The Beloch Fold exists if l_1 and l_2 are nonparallel. If l_1 and l_2 are parallel, the Beloch Fold exists if and only if the distance between P_1 and P_2 is greater than or equal to the distance between l_1 and l_2[15].



Beloch Square

Given two points A and B and two lines l_1 and l_2, a Beloch square is a square $XYWZ$ such that X and Y lie on l_1 and l_2 respectively, A lies on line XZ and B lies on line YW.

1)



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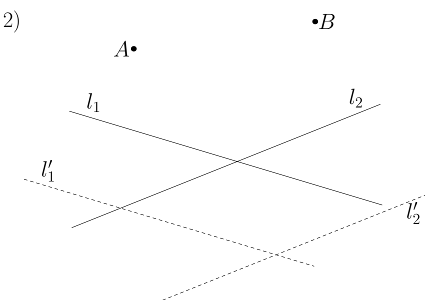

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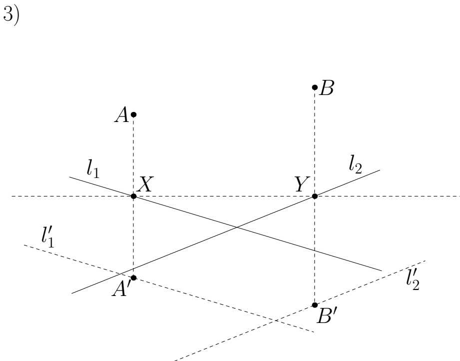
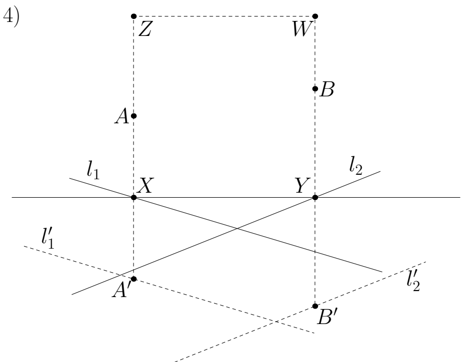


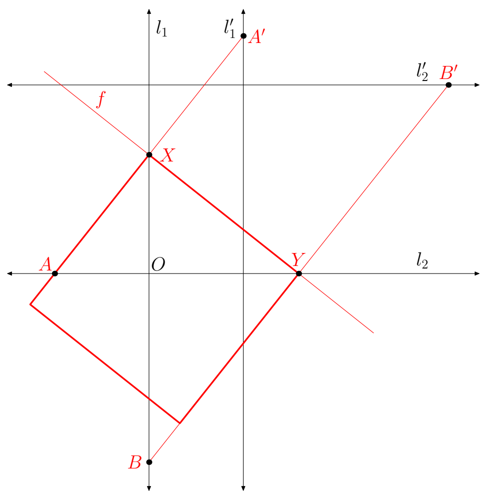
4)



We now fold the Beloch square[5, 8, 10]: to create a square, we must have 2 sets of parallel sides set at 90° to adjacent sides. Let x denote the perpendicular distance between A and line l_1 and let l_1' be a line parallel to line l_1 set a distance of x away from line l_1 such that l_1 lies between A and l_1'. Likewise, we do the same for point B and l_2 to create l_2'. (Figure 2)

We then employ the Beloch fold, folding A onto l_1' and B onto l_2' to create A' and B', respectively. This crease is the perpendicular bisector of AA' and BB'. If we find the midpoints of AA' and BB', X and Y respectively, we know that X lies on l_1 and Y on l_2 using the definitions of l_1 and l_2. (Figures 3) We then fold vertices W and Z of the square by extending lines XA and YB past X and Y, respectively, by a distance of the length XY. (Figure 4)

The Beloch square also provides us with the foldable length of the cube root of any number.



Let us define l_1 as the y-axis, l_2 as the x-axis, $A = (-1, 0)$, and $B = (0, -k)$. We fold lines l_1' and l_2' as $x = 1$ and $y = k$ respectively. We use the Beloch fold to fold A and B onto l_1' and l_2', respectively, to find X and Y. X is the intersection of the crease and l_1 and Y is the intersection of the crease and l_2. Let O be the origin. We then get that:

$$\triangle OAX \sim \triangle OXY \sim \triangle OYB.$$

From that, we have:

$$\frac{OX}{OA} = \frac{OY}{OX} = \frac{OB}{OY}.$$

We know that $OA = 1$ and $OB = k$. Plugging in, we get $OX = \frac{OY}{OX} = \frac{k}{OY}$. Using this, we can solve for OX:

$$OX^3 = OX \cdot \frac{OY}{OX} \cdot \frac{k}{OY} = k.$$

We get that OX is $\sqrt[3]{k}$. Therefore, folding the Beloch square results in the folded length of the cube root of any foldable number.

2 The Impossibility (and Possiblity) of Doubling the Cube

It is surprising to see that, when compared to straightedge and compass constructions, the origami constructions[†] are actually more versatile and can accomplish more than the straightedge and compass constructions. To put it differently, origami, which requires no tools whatsoever, can accomplish much more than using the straightedge and compass.

This was demonstrated by showing that origami can be used to take not only addition, subtraction, multiplication, division, and the square root, but also the cube root of a number as well. Similar to when we did straightedge and compass constructions, we must ask ourselves: Are there any additional folded lengths that are possible to obtain with origami? The answer would, again, be no. The numbers that are a combination of square roots, cube roots, and integer numbers using field operations are the only numbers that can be folded with origami. Such numbers make up the foldable set.

Since constructing a cube of side length 1 is doable with a straightedge and compass, doubling the cube ultimately boils down to constructing $\sqrt[3]{2}$. Countless people attempted to construct $\sqrt[3]{2}$ with a straightedge and compass for over 2000 years until, in 1837, Pierre Wantzel proved that the construction is impossible[12].

We attempt an induction-like approach[3] to prove that $\sqrt[3]{2}$ cannot be written in as a field operation combination of integers and square roots ($\sqrt[3]{2}$ is not a constructible number). We can use a proof by contradiction.

Key Point 1: $\sqrt[3]{2}$ is not of the form $\frac{a}{b}$.
Since it is known that $\sqrt[3]{2}$ is not a rational number, $\sqrt[3]{2}$is also not of the form $\frac{a}{b}$.

Key Point 2: $\sqrt[3]{2}$ is not of the form $a + b\sqrt{c}$, where c is a rational number but $\sqrt{c}$ is irrational.

We proceed by contradiction: assume that $\sqrt[3]{2}$ is of the form $a + b\sqrt{c}$ where a, b, and c are rational numbers, but $\sqrt{c}$ is not rational.

Then $a + b\sqrt{c}$ solves the equation $x^3 = 2$. If we plug in $a + b\sqrt{c}$ for x and expand, we get:

$$(a^3 + 3ab^2c - 2) + (3a^2b + b^3c)\sqrt{c} = 0$$

Now, we know that both of the expressions in the parenthesis (above) are in terms of only rational numbers, and, therefore, we can simplify the above equation in

[†] It is important to specify that we will be working only with single-folds: we only create one fold at a time. It is possible to form two folds at once, and this increases the range of origami-constructible numbers.

terms of 2 rational numbers:

$$Q_1 + Q_2\sqrt{c} = 0$$

From this, we know that Q_2 must be 0 because, if not, solving for $\sqrt{c}$ would result in the quotient of 2 rational numbers: $-Q_1/Q_2$, which is impossible, because it would imply that $\sqrt{c}$ is rational. Now that we have that Q_2 must be 0, we have the equation:

$$Q_1 = 0$$

This gives us that Q_1 must be 0 and Q_2 must be 0.

Hence, $Q_1 - Q_2\sqrt{c}$ must also be 0.

Plugging back our values for Q_1 and Q_2, we have

$$(a^3 + 3ab^2c - 2) - (3a^2b + b^3c)\sqrt{c} = 0.$$

Note that this is equivalent to, after some factorization, $(a - b\sqrt{c})^3 - 2 = 0$. In other words, $a - b\sqrt{c}$ is a solution to $x^3 = 2$.

Let us take a closer look at $x^3 = 2$. Restricting our attention to the real numbers, the equation has only one solution: $\sqrt[3]{2}$.[14] This implies that our $a + b\sqrt{c}$ and $a - b\sqrt{c}$ must be equal. Setting them equal gives us that $b = 0$. Therefore, a solves the equation $x^3 = 2$ and is equal to $\sqrt[3]{2}$. But we had stated previously that a is a rational number. We settled in key point 1 that $\sqrt[3]{2}$ is not a rational number. This is a contradiction. Thus, we have that $\sqrt[3]{2}$ is not of the form $a + b\sqrt{c}$.

Let us summarize the key points so far:

1. We have that $\sqrt[3]{2}$ is not rational. In other words, $\sqrt[3]{2}$ is not of the form $\frac{a}{b}$
2. While c is rational, $\sqrt{c}$ is not rational

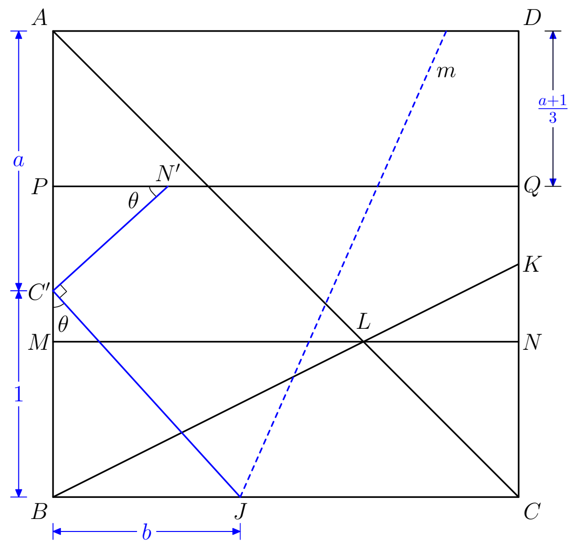
Now, we begin our iteration. Below, the iteration is presented first and then the walk through of each step. We begin with some rational number c.

1. c is a rational number
2. $\sqrt[3]{2}$ is not of the form of c
3. $a + b\sqrt{c}$ is constructible for all rational numbers a, b, c.
4. $\sqrt[3]{2}$ is not of the form $a + b\sqrt{c}$
5. $d + e\sqrt{f}$ is constructible for all d, e, and f of the form $a + b\sqrt{c}$ where a, b, and c are rational.
6. $\sqrt[3]{2}$ is not of the form $d + e\sqrt{f}$

We know that steps 1-3 are facts. Using the equation $x^3 = 2$, we showed that $\sqrt[3]{2}$ is not of the form $a + b\sqrt{c}$ (Step 4). Let us call such numbers constructible numbers of the first step. We know that step 5 is true, as it is explicitly written in terms of square roots and rational numbers. By the same logic used in steps 3-4, we have that $\sqrt[3]{2}$ is not of the form $d + e\sqrt{f}$ where d, e, and f are constructible numbers of the first step. Let us call such numbers constructible numbers of the second step. We can continue to do this in order to form an exhaustive set of constructible numbers with a straightedge and compass.

Given any number composed of square roots and integers, by iterating the extension of the constructible numbers set, we can find, in a finite number of steps, a set containing the given number. And so, by the proof in the previous paragraph, that number, or any number that can be written in terms of square roots and integers, cannot be $\sqrt[3]{2}$. In other words, by iterating this process of proving that $\sqrt[3]{2}$ is not of the form of the constructible number of the n^{th} step, we show that $\sqrt[3]{2}$ cannot be constructed with a straightedge and a compass.

However, we could use origami — in square $ABCD$, we attempt to fold $\sqrt[3]{2}$.



1. Find the midpoint K of side CD by finding the intersection of BC and the fold placing C onto D.
2. Find the intersection L of lines AC and BK.
3. Create a fold MN parallel to line BC through L.
4. Create a fold PQ parallel to line MN halfway between MN and AD.
   We have now trisected side AB and DC into thirds with points M and P and N and Q, respectively.
5. Let fold m be the Beloch fold that places C on AB at C' and N on PQ at N'.
6. Point J is the intersection point of the fold m with BC.
7. AC' is $\sqrt[3]{2}$.

Proof. (inspired by [4]) Let us denote the length of the side of the square as $a+1$. We aim to show that $a = \sqrt[3]{2}$. After step 6, we label BC' of length 1 and BJ of length b. We know that $JC' = JC = a + 1 - b$.

We solve for the value of PC'. This is $\frac{2}{3}(a+1) - 1 = \dfrac{2a-1}{3}$. Additionally, we can solve for the value of b using the Pythagorean theorem on triangle $C'BJ$ to get $b = \dfrac{a(a+2)}{2(a+1)}$.

We have that angle $JC'N'$ is a right angle as it is just a projection of, and therefore congruent to, the angle JNC. From this, we can angle chase in triangles $BC'J$ and $C'PN'$ to get that $BC'J \sim C'PN'$ by Angle-Angle Theorem.

Using the similarity, we have:

$$\frac{b}{a+1-b} = \frac{\frac{2a-1}{3}}{\frac{a+1}{3}}.$$

Once we substitute for b, we get:

$$\frac{a^2+2a}{a^2+2a+2} = \frac{2a-1}{a+1}.$$

Once we simplify and solve for a, we get $a^3 = 2$ so $a = \sqrt[3]{2}$. $\square$

3 The Impossibility of Squaring the Circle

The objective of squaring the circle is, given a circle of area πr^2, to fold a square of the same area. In order to square the circle, $\sqrt{\pi}$ must be constructed. We have previously demonstrated that, given a number we can obtain its square root, and vice versa.

We established that all straightedge and compass constructible lengths must be obtained from $\mathbb{Q}$ by a finite process of creating and solving polynomials, where any number created at one stage can be used as a coefficient in later stages. In other words, these straightedge and compass lengths must be a solution to a polynomial with constructible coefficients. Note that any solution to a polynomial with constructible coefficients is also a solution to a polynomial with rational coefficients[11]. In 1882, Ferdinand von Lindemann proved that π is transcendental [2] i.e. not a solution to any polynomial with rational coefficients, yielding the impossibility of this construction.

Recall that in order to prove that $\sqrt[3]{2}$ is not constructible, we proved that $\sqrt[3]{2}$ can't be obtained from the set of constructible numbers by a finite process of creating and solving polynomials, where any number created at one stage can be used as a coefficient in later stages.

Since we know π can't be obtained from $\mathbb{Q}$ by a finite process of creating and solving polynomials, we know that π cannot be in the set of foldable numbers. This proves that folding a square with the same area of a circle is impossible.

4 Further Reading

There are some other intriguing problems not included in the paper that can be solved with origami, including the trisection of an angle, solution of the cubic, or construction of a regular heptagon, to name a few. This paper covers straightedge and compass constructible and origami "foldable" numbers with proofs in logic, but extensions of the topic would usually dabble in field theory.[11]

In fact, the proofs presented in this paper actually parallel aspects of fields, including field extensions and Galois theory. An understanding of field theory provides a rigorous way to prove the set of constructible numbers in both origami and straightedge and compass. Perhaps one of Galois theory's more fascinating applications is understanding – at a deeper level – the sets of constructible numbers defined through different methods in order to see exactly how powerful origami is compared to other tools. Hence the title:"How Less is More (With Origami)".

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5 Acknowledgements

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Medha Ravi is an Origami enthusiast from Saratoga, California. She is a junior in high school. She enjoys breaking down complex topics to their fundamentals and explaining the topics in an entertaining, interactive, and captivating way.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 30, 2023**.

4871. *Proposed by Mihaela Berindeanu.*

For ΔABC, let AM the A-symmedian with $M \in BC$. Moreover, let $P \in AC$ and $N \in AB$ such that $MP||AB$ and $MN||AC$. The tangent at P to the circumcircle Γ of ΔMNP cuts BC in Q. If $NQ \cap MP = \{X\}$, prove that $\dfrac{MX}{XP} = \left(\dfrac{AB}{BC}\right)^2$.

4872. *Proposed by Phan Ngoc Chau.*

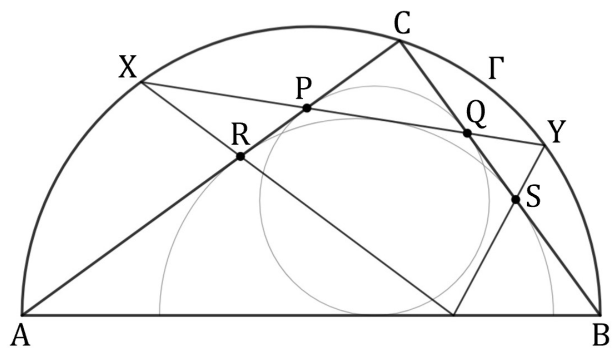
Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$2(a+b+c)\left(1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq 3(a+b)(b+c)(c+a).$$

When does equality occur?

4873. *Proposed by Byungjun Lee.*

Let Γ be a semicircle with diameter AB. For a point C on Γ, the incircle of $\triangle ABC$ touches AC and BC at points P and Q, respectively, and PQ intersects Γ at points X and Y $(XP < XQ)$. Another semicircle is inscribed in $\triangle ABC$ so that its diameter lies on AB, and it touches AC and BC at points R and S, respectively. Prove that the lines XR, YS, and AB are coincident.



4874. *Proposed by Michel Bataille.*

Let I_a, I_b, I_c be the excenters of a triangle ABC with circumradius R. Prove that the inradius of the triangle $I_aI_bI_c$ is equal to

$$2R\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1\right).$$

4875. *Proposed by Toyesh Prakash Sharma.*

For $n \in \mathbb{N}$, show that

$$\sqrt{\frac{1^1}{1!}+\sqrt{\frac{2^2}{2!}+\sqrt{\frac{3^3}{3!}+\cdots+\sqrt{\frac{n^n}{n!}}}}}<\frac{\sqrt{5e}}{2}.$$

4876. *Proposed by George Apostolopoulos, modified by the Editorial Board.*

For a triangle ABC, let R and r be the radii of its circumcircle and its incircle, respectively. What is the largest value of $\frac{r}{R}$ when the angle at A of the triangle ABC is fixed?

4877. *Proposed by Florentin Visescu.*

Let P be a point in the interior of triangle ABC. Show that

$$\cos(\angle PAB)\cos(\angle PAC)+\cos(\angle PBC)\cos(\angle PBA)+\cos(\angle PCA)\cos(\angle PCB)\le\frac{9}{4}.$$

When does equality hold?

4878. *Proposed by Paul Bracken.*

Evaluate the following infinite series in closed form for $|x|<1$.

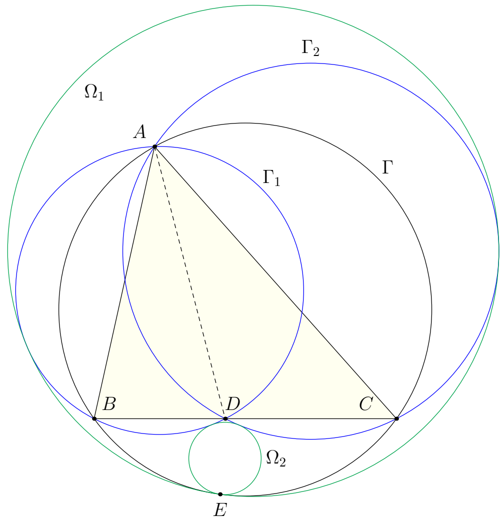
a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{(1-x)^2}-1-2x-\cdots-nx^{n-1}\right)$

b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{(1-x)^2}-1-2x-\cdots-nx^{n-1}\right)^2$

4879. *Proposed by Thanos Kalogerakis.*

Consider a triangle ABC and its angle bisector AD with D on BC. Let Γ, Γ_1 and Γ_2 denote the circumcircles of triangles ABC, ADB and ADC, respectively. Let Ω_1 denote the circle externally tangent to $\Gamma, \Gamma_1, \Gamma_2$. Let Ω_2 denote the circle externally tangent to Γ_1 and Γ_2 and internally tangent to Γ. Prove that

- Ω_1, Ω_2 and Γ have a common tangent point E;
- AE is the A-symmedian of triangle ABC.



4880. *Proposed by Austin Shapiro.*

Find all solutions to $a^3 + b^3 + c^3 - 3abc = 2023$, where a, b, c are integers and $a \leq b \leq c$.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 novembre 2023**.

4871. *Soumis par Mihaela Berindeanu.*

Pour ΔABC, soit AM la A-symédiane avec $M \in BC$. De plus, soient $P \in AC$ et $N \in AB$ tels que $MP||AB$ et $MN||AC$. La tangente en P au cercle circonscrit Γ de ΔMNP coupe BC en Q. Si $NQ \cap MP = \{X\}$, montrez que $\dfrac{MX}{XP} = \left(\dfrac{AB}{BC}\right)^2$.

4872. *Soumis par Phan Ngoc Chau.*

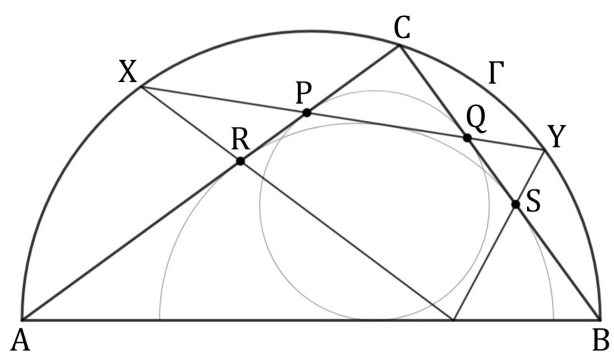
Soient a, b et c des nombres réels positifs tels que $abc = 1$. Montrez que

$$2(a+b+c)\left(1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq 3(a+b)(b+c)(c+a).$$

Quand a-t-on égalité?

4873. *Soumis par Byungjun Lee.*

Soit Γ un demi-cercle de diamètre AB. Étant donné un point C sur Γ, le cercle inscrit à $\triangle ABC$ rencontre AC et BC aux points P et Q, respectivement, et PQ rencontre Γ aux points X et Y $(XP < XQ)$. Un autre demi-cercle est inscrit dans $\triangle ABC$ de sorte que son diamètre est situé sur AB et il rencontre AC et BC aux points R et S, respectivement. Montrez que XR et YS se rencontrent sur le segment AB.



4874. *Soumis par Michel Bataille.*

Soient I_a, I_b et I_c les centres des cercles exinscrits au triangle ABC dont le rayon du cercle circonscrit est R. Montrez que le rayon du cercle inscrit au triangle $I_aI_bI_c$ est égal à

$$2R\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1\right).$$

4875. *Soumis par Toyesh Prakash Sharma.*

Pour $n \in \mathbb{N}$, montrez que

$$\sqrt{\frac{1^1}{1!}+\sqrt{\frac{2^2}{2!}+\sqrt{\frac{3^3}{3!}+\cdots+\sqrt{\frac{n^n}{n!}}}}}<\frac{\sqrt{5e}}{2}.$$

4876. *Soumis par George Apostolopoulos, modifié par le comité de rédaction.*

Étant donné un triangle ABC, soient R et r respectivement les rayons des cercles circonscrit et inscrit. Quelle est la plus grande valeur de $\frac{r}{R}$ lorsque l'angle A du triangle ABC est fixe ?

4877. *Soumis par Florentin Visescu.*

Soit P un point à l'intérieur du triangle ABC. Montrez que

$$\cos(\angle PAB)\cos(\angle PAC)+\cos(\angle PBC)\cos(\angle PBA)+\cos(\angle PCA)\cos(\angle PCB)\le\frac{9}{4}.$$

Quand a-t-on égalité ?

4878. *Soumis par Paul Bracken.*

Évaluez les séries infinies suivantes sous forme fermée pour $|x|<1$.

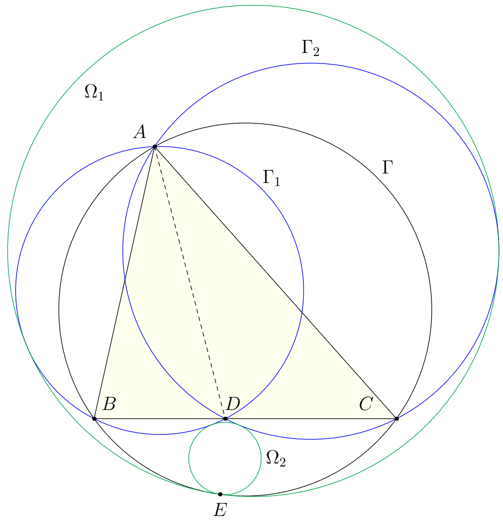
a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{(1-x)^2}-1-2x-\cdots-nx^{n-1}\right)$

b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{(1-x)^2}-1-2x-\cdots-nx^{n-1}\right)^2$

4879. *Soumis par Thanos Kalogerakis.*

Considérons un triangle ABC et sa bissectrice AD avec D sur BC. Γ, Γ_1 et Γ_2 désignent respectivement les cercles circonscrits des triangles ABC, ADB et ADC. Soit Ω_1 le cercle tangent extérieurement à $\Gamma, \Gamma_1, \Gamma_2$. Soit Ω_2 le cercle tangent extérieurement à Γ_1 and Γ_2 et tangent intérieurement à Γ. Montrer que

- Ω_1, Ω_2 et Γ ont un point tangent commun E;
- AE est la A-symédiane du triangle ABC.



4880. *Soumis par Austin Shapiro.*

Trouvez toutes les solutions de $a^3 + b^3 + c^3 - 3abc = 2023$, où a, b et c sont des entiers vérifiant $a \leq b \leq c$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2023: 49(3), p. 156–160.

4821. *Proposed by Corneliu Manescu-Avram.*

Let k be a positive integer, $n = 2^k+1$ and let N be the number of ordered solutions in n-tuples of positive integers to the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1.$$

Prove that $N - k$ is odd.

We received 5 submissions, all of which are correct, but one is slightly incomplete. We present the solution by Marie-Nicole Gras.

First we prove the following lemma:

Lemma. Let k be an integer, $k \geq 2$, and $n = 2^k + 1$. If $2 \leq m \leq n-2$, then the binomial coefficient $\binom{n}{m}$ is even.

Proof. We have $(1+x)^2 = 1+x^2+2x$, and we deduce by iteration that $(1+x)^{2^k} = 1 + x^{2^k} + 2P(x)$, $P(x) \in \mathbb{Z}[x]$. It follows, with $Q(x) = (1+x)\,P(x) \in \mathbb{Z}[x]$, that, since $k \geq 2$,

$$\begin{aligned}(1+x)^{2^k+1} =& (1+x)(1+x)^{2^k}\\ =& \binom{n}{0} + \binom{n}{1}x + 2Q(x) + \binom{n}{n-1}x^{2^k} + \binom{n}{n}x^{2^{k+1}},\end{aligned}$$

and the binomial coefficients $\binom{n}{m}$ are even, since $2 \leq m \leq n-2$. $\square$

We denote by ℓ the number of n-uples $(x_1, x_2, \ldots, x_n)$ solutions of the given equation with $x_i \leq x_{i+1}$, $1 \leq i \leq n-1$. The number N of ordered solutions is obtained considering all the distinct permutations of $\{x_1, x_2, \ldots, x_n\}$.

First, we consider the case $k = 1, n = 3$; there are $\ell = 3$ solutions:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1, \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1, \quad \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1,$$

which gives $N = 6 + 3 + 1 = 10$ ordered solutions, and $N - k = 9$ is odd.

Now, we consider the solutions $(x_1, x_2, \ldots, x_n)$ according to the number of terms x_i which are equal to each other.

- If we have a solution where all x_i are distinct, the corresponding number N_1 is equal to $n!$ and is even.

- If $n \geq 5$ and if we have a solution where j terms are equal, $2 \leq j \leq n-2$, the corresponding number N_2 is a multiple of $\binom{n}{j}$ and then is even (lemma).
- If we have a solution where $n-1$ terms are equal and distinct from the last, the corresponding number N_3 is equal to $\binom{n}{n-1} = n = 2^k+1, k \geq 1$ and N_3 is odd.
- There exists a unique solution with $x_1 = x_2 = \cdots = x_n = n$, and $N_4 = 1$.

Now, we compute the number of solutions such that, with $a, b \geq 2$ and $b \neq a$,

$$\frac{1}{a} + \frac{n-1}{b} = 1 \iff b(a-1) = (n-1)a = 2^k a.$$

The integers a and $a-1$ are coprime; it follows that $b = \lambda a, \lambda \geq 2$ $(b \neq a)$ and we deduce $\lambda(a-1) = 2^k$. Since $\lambda \geq 2$, there are exactly k solutions obtained with

$$a - 1 = 1, 2, 2^2, \ldots, 2^{k-1}.$$

The terms of the form N_1 and N_2 are even; there exist k solutions where $n-1$ terms are equal and distinct from the last, and, for each solution, $N_3 = 2^k+1$; it follows that

$$N \equiv k(2^k+1) + 1 \pmod 2 \equiv k + 1 \pmod 2$$

and $N - k$ is odd.

Note by the proposer. The case $k = 2$ is the problem 1, day 2, from the Team Selection Test, Moldova, 2013.

4822. *Proposed by Anton Mosunov.*

The n-th Chebyshev polynomial of the first kind is defined by means of the recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad \text{for } n \geq 2.$$

Prove that for all $n \geq 2$,

$$\frac{1}{3} < \int_1^{+\infty} \frac{dx}{T_n(x)^{2/n}} < \frac{1}{3}\sqrt[n]{4}.$$

We received 7 submissions and they were all complete and correct. We present the solution by the majority of solvers.

It is well known that

$$T_n(\cos\theta) = \cos n\theta.$$

Therefore,

$$T_n(\cosh\theta) = T_n(\cos i\theta) = \cos(ni\theta) = \cosh n\theta.$$

Hence, letting $x = \cosh\theta$, we have

$$I_n = \int_1^\infty \frac{dx}{T_n(x)^{2/n}} = \int_0^\infty \frac{\sinh\theta\, d\theta}{(\cosh n\theta)^{2/n}}.$$

Observe that for each $t > 0$, the following inequality holds:

$$\frac{e^t}{2} < \cosh t = \frac{e^t + e^{-t}}{2} < e^t.$$

Therefore

$$\int_0^\infty \frac{\sinh\theta\, d\theta}{(e^{n\theta})^{2/n}} < I_n < \int_0^\infty \frac{\sinh\theta\, d\theta}{(e^{n\theta}/2)^{2/n}} = \sqrt[n]{4}\int_0^\infty \frac{\sinh\theta\, d\theta}{(e^{n\theta})^{2/n}}.$$

Since

$$\int_0^\infty \frac{\sinh\theta\, d\theta}{(e^{n\theta})^{2/n}} = \int_0^\infty \frac{\sinh\theta\, d\theta}{e^{2\theta}} = \frac{1}{2}\int_0^\infty \left(e^{-\theta} - e^{-3\theta}\right)\, d\theta = \frac{1}{3},$$

the result follows.

4823*. *Proposed by Michael Friday, modified by the editorial board.*

Given four points A, B, C, D on a circle, define the *Simson segment of A with respect to the triangle BCD* to be the smallest line segment containing the feet of all three perpendiculars dropped from A to the sides of the triangle. For any four points on a circle, prove that the Simson segments determined by each point with respect to the triangle formed by the other three all have the same length.

Not counting the proposer (whose proposal arrived without a solution), we received 7 correct submissions. We feature the solution by the UCLan Cyprus Problem Solving Group.

Assume that A, B, C, D appear in this order on the circle, and let X, Y, Z be the projections of D on AB, BC, CA respectively.

Note that D, X, B, Y are concyclic on a circle of diameter DB. Using the Law of Sines twice, we have

$$XY = DB\sin(\angle XBY) = DB\sin(\angle ABC) = \frac{DB\cdot AC}{2R}$$

where R is the radius of the circumcircle of $ABCD$. By symmetry, the lengths of the other Simson segments are equal to that. (Note that XY has to be the Simson segment as XZ and ZX have lengths equal to $\dfrac{DA\cdot BC}{2R}$ and $\dfrac{DC\cdot BA}{2R}$ which by Ptolemy's theorem sum to $\dfrac{DB\cdot AC}{2R}$.)

Bonus Fact. The above calculations shows that we can deduce Ptolemy's Theorem from the existence of the Simson line.

4824. *Proposed by George-Florin Şerban.*

Find all prime numbers p for which there are integers x and y that satisfy the conditions $11p = 8x^2 + 23$ and $p^2 = 2y^2 + 23$.

We received nine submissions, eight of which were correct and complete. Presented is the one by the UCLan Cyprus Problem Solving Group, modified by the editor.

We will show that the only solutions are $p = 5$ (with $x = 2$, $y = 1$) and $p = 61$ (with $x = 9$, $y = 43$).

We may assume that x, y are positive integers. Subtracting the two equations yields

$$p(p - 11) = 2(y - 2x)(y + 2x).$$

Since p is prime it divides one of the three factors. $y + 2x$ is the largest factor in absolute value and is positive, so we get $p \le y + 2x$ or

$$2x \ge p - y. \tag{1}$$

Furthermore, we have

$$p^2 = 2y^2 + 23 > 2y^2 \Rightarrow \sqrt{2}y < p \tag{2}$$

and

$$11p = 8x^2 + 23 > 8x^2 \Rightarrow \sqrt{11p} > 2\sqrt{2}x. \tag{3}$$

Combining inequalities (3), (1), and (2) we obtain

$$\sqrt{11p} > 2\sqrt{2}x \ge \sqrt{2}(p - y) > p(\sqrt{2} - 1).$$

Squaring and rearranging gives

$$p < \frac{11}{(\sqrt{2} - 1)^2} = \frac{11}{3 - 2\sqrt{2}} = 11(3 + 2\sqrt{2}) < 66.$$

We also observe that $8x^2 = 11p - 23 < 11 \cdot 66 - 23 = 703$ and therefore

$$x \le 9.$$

Finally, $8x^2 = 11p - 23$ gives

$$\begin{aligned} & 8x^2 \equiv -1 \pmod{11} \\ \iff & \ x^2 \equiv 4 \pmod{11} \\ \iff & \ x \equiv \pm 2 \pmod{11}. \end{aligned}$$

It remains to check $x = 2$ and $x = 9$, which yield the two solutions $p = 5$ and $p = 61$.

4825. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Let $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$, $n \geq 1$. Calculate

$$\sum_{n=1}^{\infty} \frac{O_n}{n(n+1)}.$$

We received 23 solutions from 22 contributors. One of these was incomplete and three were incorrect. We present 3 different approaches.

Solution 1 by Walther Janous, C.R. Pranesachar and Yuyong Zhang, done independently.

The answer is $\log 4 = 2\log 2$. For each positive integer n, $O_{n+1} = O_n + \frac{1}{2n+1}$ and

$$\begin{aligned}\sum_{n=1}^{m} \frac{O_n}{n(n+1)} &= \sum_{n=1}^{m} \frac{O_n}{n} - \sum_{n=1}^{m} \frac{O_n}{n+1} = \sum_{n=1}^{m} \frac{O_n}{n} - \sum_{n=1}^{m} \frac{O_{n+1}}{n+1} + \sum_{n=1}^{m} \frac{1}{(n+1)(2n+1)} \\ &= \frac{O_1}{1} - \frac{O_{m+1}}{m+1} + 2\sum_{n=1}^{m} \frac{1}{(2n+1)(2n+2)} \\ &= 2\left(1 - \frac{1}{2}\right) + 2\sum_{n=1}^{m}\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) - \frac{O_{m+1}}{m+1} \\ &= 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2m+1} - \frac{1}{2m+2}\right) - \frac{O_{m+1}}{m+1}.\end{aligned}$$

Let $m \to \infty$. Then, since $O_{m+1} < 1 + \log(2m+1)$,

$$\sum_{n=1}^{\infty} \frac{O_n}{n(n+1)} = 2\log 2 = \log 4.$$

Solution 2, by Brian Bradie, Rahima Karimova, Christopher Linhardt, Raymond Mortini & Rudolph Rupp, and UCLan Cyprus Problem Solving Group, done independently.

We have:

$$\begin{aligned}\sum_{n=1}^{m} \frac{O_n}{n(n+1)} &= \sum_{n=1}^{m} \frac{1}{n(n+1)} \sum_{k=1}^{n} \frac{1}{2k-1} = \sum_{k=1}^{m} \frac{1}{2k-1} \sum_{n=k}^{m} \frac{1}{n(n+1)} \\ &= \sum_{k=1}^{m} \frac{1}{2k-1}\left(\frac{1}{k} - \frac{1}{m+1}\right) \\ &= 2\sum_{k=1}^{m} \frac{1}{(2k-1)(2k)} - \frac{1}{m+1}\sum_{k=1}^{m} \frac{1}{2k-1}.\end{aligned}$$

Letting $m \to \infty$ yields the result.

Solution 3, by Devis Alvarado, G.C. Greubel and the Missouri State University Problem Solving Group, done independently.

Since

$$O_n = \int_0^1 \frac{1-x^{2n}}{1-x^2}\,dx,$$

we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{O_n}{n(n+1)} &= \int_0^1 \left[\frac{1}{1-x^2} \sum_{n=1}^{\infty} \frac{1-x^{2n}}{n(n+1)}\right] dx \\
&= \int_0^1 \left[\frac{1}{1-x^2}\left(1 - \sum_{n=1}^{\infty}\left(\frac{x^{2n}}{n} - \frac{x^{2n}}{n+1}\right)\right)\right] dx \\
&= \int_0^1 \left[\frac{1}{1-x^2}\left(1 - \left(-\log(1-x^2) + \frac{1}{x^2}\log(1-x^2) + 1\right)\right)\right] dx \\
&= -\int_0^1 \frac{\log(1-x^2)}{x^2}\,dx \\
&= \lim_{a\uparrow 1} \int_0^a \frac{-\log(1-x^2)}{x^2}\,dx.
\end{aligned}$$

Integrating by parts (with $u = -\log(1-x^2), dv = x^{-2}dx$), we find that

$$\begin{aligned}
\int_0^a \frac{-\log(1-x^2)}{x^2}\,dx &= \left[\frac{\log(1-x^2)}{x}\right]_0^a + \int_0^a \frac{2}{1-x^2}\,dx \\
&= \frac{\log(1-a^2)}{a} + \int_0^a \left(\frac{1}{1+x} + \frac{1}{1-x}\right) dx \\
&= \left(\frac{\log(1-a) + \log(1+a)}{a}\right) + \log(1+a) - \log(1-a) \\
&= \frac{(1-a)\log(1-a)}{a} + \frac{(1+a)\log(1+a)}{a}.
\end{aligned}$$

Letting a increase to 1 reveals that

$$-\int_0^1 \frac{\log(1-x^2)}{x^2}\,dx = 2\log 2.$$

Editor's Comments. Several solvers used $O_n = H_{2n} - \frac{1}{2}H_n$, where $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$, as the basis of their manipulations.

One of the incorrect solutions had a rather subtle flaw, which might interest analysis lecturers as a teaching tool. We have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{O_n}{n(n+1)} &= \sum_{n=1}^{\infty}\left(\frac{O_n}{n}-\frac{O_n}{n+1}\right)\\ &= O_1+\sum_{n=1}^{\infty}\frac{O_{n+1}-O_n}{n+1}\\ &= 1+\sum_{n=1}^{\infty}\frac{1}{(n+1)(2n+1)}\\ &= 1+\sum_{n=1}^{\infty}\left(\frac{2}{2n+1}-\frac{1}{n+1}\right)\\ &= 1+2\sum_{n=1}^{\infty}\frac{1}{2n+1}-\sum_{n=1}^{\infty}\frac{1}{2n}-\sum_{n=1}^{\infty}\frac{1}{2n+1}\\ &= 1-\sum_{n=1}^{\infty}\frac{1}{2n}+\sum_{n=1}^{\infty}\frac{1}{2n+1} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots = \log 2.\end{aligned}$$

Unfortunately, one of the logs fell off the truck during the journey. To see what went amiss, consider the partial sum up to index m:

$$\begin{aligned}\sum_{n=1}^{m}\frac{O_n}{n(n+1)} &= \sum_{n=1}^{m}\left(\frac{O_n}{n}-\frac{O_n}{n+1}\right)\\ &= O_1+\sum_{n=1}^{m-1}\left(\frac{O_{n+1}}{n+1}-\frac{O_n}{n+1}\right)-\frac{O_m}{m+1}\\ &= 1+\sum_{n=1}^{m-1}\frac{1}{(n+1)(2n+1)}-\frac{O_m}{m+1}\\ &= 1+\sum_{n=1}^{m-1}\left(\frac{2}{2n+1}-\frac{1}{n+1}\right)-\frac{O_m}{m+1}\\ &= 1+2\sum_{n=1}^{m-1}\frac{1}{2n+1}-\sum_{n=1}^{k(m)}\frac{1}{2n}-\sum_{n=1}^{k(m)}\frac{1}{2n+1}-\frac{O_m}{m+1}\\ &= 1+\sum_{n=k(m)+1}^{m}\frac{1}{2n+1}-\sum_{n=1}^{k(m)}\frac{1}{2n}+\sum_{n=1}^{k(m)}\frac{1}{2n+1}-\frac{O_m}{m+1}\\ &= \sum_{n=k(m)+1}^{m}\frac{1}{2n+1}+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+(-1)^{m-1}\frac{1}{m}\right)-\frac{O_m}{m+1},\end{aligned}$$

where $k(m) = \lfloor m/2 \rfloor$.

The first term along with the partial sum in parentheses each tend to $\log 2$ and the last term tends to 0, as m tends to infinity.

4826. *Proposed by Paul Bracken.*

Let H_n be the n-th harmonic number $H_n = \sum_{k=1}^{n} 1/k$. Evaluate the following sum in closed form
$$S = \sum_{k=1}^{\infty} \frac{H_k}{k(k+1)(k+2)}.$$

We received 22 submissions. We present the solution by Michel Bataille.

Let n be a positive integer. From
$$\frac{2}{k(k+1)(k+2)} = \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)},$$
we obtain
$$\sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)} = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$$
and, summing by parts, it follows that
$$\begin{aligned}\sum_{k=1}^{n} \frac{2H_k}{k(k+1)(k+2)} &= H_n\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) - \sum_{k=1}^{n-1}\left(\frac{1}{2} - \frac{1}{(k+1)(k+2)}\right)\frac{1}{k+1} \\ &= \frac{H_n}{2} - \frac{H_n}{(n+1)(n+2)} - \frac{1}{2}(H_n - 1) + \sum_{k=1}^{n-1} \frac{1}{(k+1)^2(k+2)} \\ &= \frac{1}{2} - \frac{H_n}{(n+1)(n+2)} + \sum_{k=1}^{n-1} \frac{1}{(k+1)^2(k+2)}.\end{aligned}$$

From
$$\frac{1}{(k+1)^2(k+2)} = \frac{1}{(k+1)^2} - \left(\frac{1}{k+1} - \frac{1}{k+2}\right),$$
we deduce that
$$\sum_{k=1}^{n-1} \frac{1}{(k+1)^2(k+2)} = \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} - \frac{1}{2} + \frac{1}{n+1}.$$

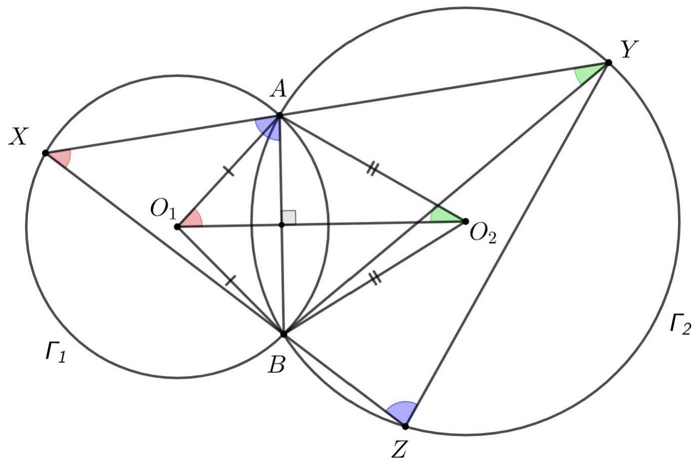
Since
$$\lim_{n\to\infty} \frac{H_n}{(n+1)(n+2)} = \lim_{n\to\infty} \frac{\ln n}{n^2} = 0,$$
we obtain
$$S = \frac{1}{2}\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2H_k}{k(k+1)(k+2)} = \frac{1}{2}\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} = \frac{1}{2}\left(\frac{\pi^2}{6} - 1\right) = \frac{\pi^2}{12} - \frac{1}{2}.$$

4827. *Proposed by Michel Bataille.*

In the plane, two circles Γ_1 and Γ_2, with respective centres O_1 and O_2, intersect at A and B. Let X be a point of Γ_1 with $X \neq A, B$. The lines XA and XB intersect Γ_2 again at Y and Z, respectively. Prove that

$$YZ = \frac{AB \cdot O_1O_2}{O_1A}.$$

We received 14 submissions. We present the solution by NUM Problem Solving Group.



Note that

$$\angle YAB + \angle BAX = 180^\circ.$$

Since $ABZY$ is cyclic, we have

$$\angle YAB + \angle BZY = 180^\circ.$$

It follows that $\angle BAX = \angle BZY$ and so triangle XYZ and XBA are similar. Hence

$$\frac{YZ}{AB} = \frac{XY}{XB} \iff YZ = AB \cdot \frac{XY}{XB}.$$

On the other hand, we have $\angle AXB = \angle AO_1O_2$ and $\angle BYA = \angle O_1O_2A$, so that triangles XYB and O_1O_2A are similar; hence

$$\frac{XY}{O_1O_2} = \frac{XB}{O_1A} \iff \frac{XY}{XB} = \frac{O_1O_2}{O_1A}.$$

From the above two identities, we get that

$$YZ = \frac{AB \cdot O_1O_2}{O_1A}.$$

4828. *Proposed by Narendra Bhandari.*

Prove
$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\frac{\sec(x+y)\sec(x-y)}{\sec x\sec y}dxdy=\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2.$$

We received 14 submissions, 13 of which are correct. We present here the solution by Michel Bataille.

Let I be the integral on the left.

Since $\cos x\cos y=\frac{1}{2}(\cos(x+y)+\cos(x-y))$, we have
$$I=\frac{1}{2}\int_0^{\pi/4}\int_0^{\pi/4}\left(\frac{1}{\cos(x+y)}+\frac{1}{\cos(x-y)}\right)dxdy.$$

It follows that
$$I=-\frac{1}{2}\int_0^{\pi/4}\left[\ln\left(\tan\left(\frac{\pi}{4}-\frac{x+y}{2}\right)\right)+\ln\left(\tan\left(\frac{\pi}{4}-\frac{x-y}{2}\right)\right)\right]_{x=0}^{x=\pi/4}dy.$$

Since $\ln\left(\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right)+\ln\left(\tan\left(\frac{\pi}{4}+\frac{y}{2}\right)\right)=\ln\left(\left(\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\left(\tan\left(\frac{\pi}{4}+\frac{y}{2}\right)\right)=\ln(1)=\right.\right.$ 0, the latter yields
$$\begin{aligned}I&=-\frac{1}{2}\int_0^{\pi/4}\left(\ln\left(\tan\left(\frac{\pi}{8}-\frac{y}{2}\right)\right)+\ln\left(\tan\left(\frac{\pi}{8}+\frac{y}{2}\right)\right)\right)dy\\&=-\frac{1}{2}\left(\int_{\pi/8}^{0}(\ln(\tan t))(-2dt)+\int_{\pi/8}^{\pi/4}(\ln(\tan t))(2dt)\right)\\&=-\int_0^{\pi/4}\ln(\tan t)\,dt.\end{aligned}$$

Now, integrating by parts, we readily obtain
$$I=\int_0^{\pi/4}t\cdot\frac{1+\tan^2 t}{\tan t}\,dt=2\int_0^{\pi/4}\frac{t}{\sin 2t}\,dt=\frac{1}{2}\int_0^{\pi/2}\frac{w}{\sin w}\,dw.$$

Finally the change of variables $w=2\arctan\alpha$ leads to
$$\begin{aligned}I&=\int_0^1\frac{\arctan\alpha}{\alpha}\,d\alpha=\int_0^1\left(\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^{2n}}{2n+1}\right)d\alpha=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1\alpha^{2n}\,d\alpha\\&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.\end{aligned}$$

Note that we can interchange $\int_0^1$ and $\sum_{n=0}^{\infty}$ because the series $\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^{2n}}{2n+1}$ is uniformly convergent on $[0,1]$ by Abel's Theorem, the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ being convergent.

Editor's Comment. The double integral expressed in terms of Catalan's constant G is given in the paper by S. W. Stewart, A Catalan constant inspired integral odyssey, *Mathematical Gazette* (2020), 449-459. This is integral Φ, found on page 458, where $\cos x \to 1/\sec(x)$.

4829. *Proposed by George Apostolopoulos.*

Let ABC be a triangle and K, L, M be interior points on the sides BC, CA, AB, respectively. Let $[XYZ]$ denote the area of a triangle XYZ.

a) Find the maximum value of the expression

$$\sqrt{\frac{[ALM]}{[ABC]}} + \sqrt{\frac{[BMK]}{[ABC]}} + \sqrt{\frac{[CKL]}{[ABC]}}.$$

b) Find the minimum value of the expression

$$\frac{[KLM]}{[ALM]} + \frac{[KLM]}{[BMK]} + \frac{[KLM]}{[CKL]}.$$

We received 8 correct solutions for this problem. The following is the solution by M. Bello, M. Benito, Ó. Ciaurri, and E. Fernández.

We will use absolute barycentric coordinates with respect to the triangle ABC. Let then be $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, $M = (x, 1-x, 0)$, $K = (0, y, 1-y)$, and $L = (1 - z, 0, z)$, with $0 < x, y, z < 1$.

a) The maximum value required is 3/2. We have

$$\frac{[ALM]}{[ABC]} = \begin{vmatrix} 1 & 0 & 0 \\ x & 1-x & 0 \\ 1-z & 0 & z \end{vmatrix} = (1-x)z, \quad \frac{[BMK]}{[ABC]} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & y & 1-y \\ x & 1-x & 0 \end{vmatrix} = (1-y)x,$$

$$\frac{[CKL]}{[ABC]} = \begin{vmatrix} 0 & 0 & 1 \\ 1-z & 0 & z \\ 0 & y & 1-y \end{vmatrix} = (1-z)y.$$

Let us define the function

$$M(x, y, z) := \sqrt{(1-x)z} + \sqrt{(1-y)x} + \sqrt{(1-z)y}.$$

It is enough to apply the AM-GM inequality in each term to obtain

$$M(x, y, z) \le \frac{1-x+z}{2} + \frac{1-y+x}{2} + \frac{1-z+y}{2} = \frac{3}{2},$$

with equality if and only if $1 - x = z$, $1 - y = x$ and $1 - z = y$, that is, if and only if $x = y = z = 1/2$.

Therefore, the maximum value of $M(x, y, z)$ in $(0, 1)^3$ is $3/2$. On the other hand, it can be verified that the maximum value on the border of the cube is $\sqrt{2} < 3/2$, and is reached at the points $(1,1/2,0)$, $(0,1,1/2)$ and $(1/2,0,1)$. Therefore, the maximum value of $M(x,y,z)$ in the compact cube $[0,1]^3$ is also $3/2$, the optimal configuration being when the points K, L, and M are the midpoints of the sides of triangle ABC.

b) The minimum value required is 3. With the same notation as above, we have

$$\frac{[KLM]}{[ABC]} = \begin{vmatrix} x & 1-x & 0 \\ 0 & y & 1-y \\ 1-z & 0 & z \end{vmatrix} = xyz + (1-x)(1-y)(1-z),$$

and

$$\frac{[KLM]}{[ALM]}+\frac{[KLM]}{[BMK]}+\frac{[KLM]}{[CKL]} = \frac{[KLM]/[ABC]}{[ALM]/[ABC]}+\frac{[KLM]/[ABC]}{[BMK]/[ABC]}+\frac{[KLM]/[ABC]}{[CKL]/[ABC]}$$

$$= \big(xyz+(1-x)(1-y)(1-z)\big)\left(\frac{1}{(1-x)z}+\frac{1}{(1-y)x}+\frac{1}{(1-z)y}\right) =: m(x,y,z),$$

say, where $0 < x,y,z < 1$. Operating, we get

$$m(x,y,z) = \frac{xy}{1-x}+\frac{yz}{1-y}+\frac{zx}{1-z}+\frac{(1-z)(1-y)}{z}+\frac{(1-x)(1-z)}{x}+\frac{(1-y)(1-x)}{y}.$$

But

$$\frac{xy}{1-x} = -y+\frac{y}{1-x}, \quad \frac{yz}{1-y} = -z+\frac{z}{1-y}, \quad \frac{zx}{1-z} - x + \frac{x}{1-z}$$

and, by the other hand,

$$\frac{(1-z)(1-y)}{z} = \frac{1-y}{z} - 1 + y,$$

and similarly for cyclic rotation of the variables x,y,z, so

$$\begin{aligned} m(x,y,z) &= \left(\frac{y}{1-x}+\frac{1-x}{y}\right)+\left(\frac{z}{1-y}+\frac{1-y}{z}\right)+\left(\frac{x}{1-z}+\frac{1-z}{x}\right)-3 \\ &\geq 2+2+2-3 = 3, \end{aligned}$$

as we have said. The equality is attained if and only if $1-x=y$, $1-y=z$ and $1-z=x$, that is, if and only if $x=y=z=1/2$. Thus, the optimal configuration for this inequality occurs again when the points K, L, and M are the midpoints of the sides of triangle ABC.

Remark. Several solvers mentioned that the answer to the part (b) follows from the well known inequality

$$1/[ALM] + 1/[BMK] + 1/[CKL] \geq 3/[KLM],$$

proved by W. Janous in [2]. The last inequality also generalizes Erdös and Debrunner inequality $[KLM] \geq min([ALM],[BMK],[CKL])$. See [1, p.80], [2] and

[3] for many generalizations and variations for perimeters, inradii, exradii, for n-gons, etc. Marie-Nicole Gras mentioned that part (b) was discussed in ***Crux*** 1413, March 1990, vol. 16(3), p. 95-96.

References:

[1] O. Bottema, et al.: *Geometric Inequalities*. Wolters and Noordhoff, Groningen 1969.

[2] W. Janous, A short note on the Erdös-Debrunner inequality, *Elemente der Mathematik*, 61 (2006) 32-35.

[3] Y.N. Aliyev, Inequalities involving reciprocals of triangle areas, *Crux Mathematicorum with Mathematical Mayhem* 36(8) (2010) 535-539.

4830. *Proposed by Goran Conar.*

Let $a_i \in (0, \frac{1}{2}),\ i \in \{1, 2, \ldots, n\}$ be real numbers such that $\sum_{i=1}^{n} a_i = 1$. Prove that the following inequalities hold:

$$n\sqrt{\frac{n-1}{n+1}} \leq \sum_{i=1}^{n} \sqrt{\frac{1-a_i}{1+a_i}} < (n+1)\sqrt{\frac{n-1}{n+1}}.$$

We received 11 submissions, all correct and complete. We present two solutions, slightly altered by the editor.

Solution 1, by Michel Bataille.

Let $f(x) = \sqrt{\dfrac{1-x}{1+x}}$, so

$$f''(x) = (1-x)^{-3/2}(1+x)^{-5/2}(1-2x).$$

Hence, $f''(x) > 0$ when $x \in (0, \frac{1}{2})$ and f is convex on $(0, \frac{1}{2})$. Jensen's inequality gives

$$\sum_{i=1}^{n} f(a_i) \geq nf\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) = nf\left(\frac{1}{n}\right) = n\sqrt{\frac{n-1}{n+1}},$$

proving the left inequality.

On the other hand, the Cauchy-Schwarz inequality yields

$$\sum_{i=1}^{n} f(a_i) \leq \left(\sum_{i=1}^{n}(\sqrt{1-a_i})^2\right)^{1/2}\left(\sum_{i=1}^{n}\left(\frac{1}{\sqrt{1+a_i}}\right)^2\right)^{1/2},$$

which is equivalent to,

$$\sum_{i=1}^{n} f(a_i) \leq \sqrt{n-1}\left(\sum_{i=1}^{n}\frac{1}{1+a_i}\right)^{1/2}.$$

Since $\dfrac{1}{1+a_i} < 1$ for $i = 1, \ldots, n$, we deduce that

$$\sum_{i=1}^{n} f(a_i) < \sqrt{n-1}\sqrt{n},$$

which is sharper than the required right inequality.

Solution 2, by Theo Koupelis.

From the given conditions we get that $n \geq 3$. Let $f(x) = \sqrt{\dfrac{1-x}{1+x}}$ with $x \in [0, \frac{1}{2}]$. Then

$$f'(x) = -\frac{1}{(1+x)^2}\sqrt{\frac{1+x}{1-x}} < 0,$$

and

$$f''(x) = -\sqrt{\frac{1+x}{1-x}} \cdot \frac{2x-1}{(1-x)(1+x)^3} \geq 0.$$

Thus, $f(x)$ is convex in $[0, \frac{1}{2}]$, and by Jensen's inequality we get

$$\sum_{i=1}^{n} \sqrt{\frac{1-a_i}{1+a_1}} = \sum_{i=1}^{n} f(a_i) \geq n \cdot f\left(\frac{\sum_{i=1}^{n} a_i}{n}\right) = n\sqrt{\frac{n-1}{n+1}}.$$

Equality occurs when $a_1 = a_2 = \ldots = a_n = \frac{1}{n}$.

Now let

$$g(x) = 1 - \frac{2}{3}(3-\sqrt{3})x.$$

Since, $g(0) = f(0) = 1$, $g(1/2) = f(1/2) = \sqrt{3}/3$, and $f(x)$ is strictly convex on $(0, \frac{1}{2})$ we have that $g(x) > f(x)$ in that open interval. Therefore, we obtain an improved upper bound:

$$\sum_{i=1}^{n} \sqrt{\frac{1-a_i}{1+a_1}} < \sum_{i=1}^{n} g(a_i) = n - \frac{2}{3}(3-\sqrt{3}) < (n+1)\sqrt{\frac{n-1}{n+1}} = \sqrt{n^2-1},$$

which holds for all $n \geq 3$.