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Crux Mathematicorum

Crux Mathematicorum with Mathematical Mayhem

EDITORIAL

This year marks my 10th year anniversary being the Editor-in-Chief of ***Crux***.

Ten years feels like a long time. Indeed, it's approximately a quarter of my life. Apart from my dog, my partner, parents, friends and math itself, almost nothing else has been constant in my life over the past 10 years. I was a PhD student when I stepped into this role; I've since defended my thesis, done a PostDoc, got a job, obtained a tenure. I moved 5 times. I had a child. I lost 3 grandparents.

On the other hand, these ten years flew by. My daughter is starting grade 2 this September and I don't quite recall where the years between the age of 1 and 6 went. My dog is now nearly 14 but it feels like I was complaining about his puppy energy last year. Maybe the recent pandemic distorted my time perception or maybe that's just how human brains perceive time. Either way, I cannot believe that ***Crux*** has been a major part of my life for ten whole years.

In these years, ***Crux*** has evolved from a subscription-based printed journal to an open access online publication. We have grown our audience and with it we have grown the Editorial Board to efficiently work with a big increase in submissions. We have embraced new technologies and processes. Amidst these changes, we kept producing high-quality mathematical content. We introduced a number of new sections. We developed journal components that target secondary students and teachers.

Thank you to our readers, contributors and my incredible team that supported me over the last decade. Here is to the next one.

Kseniya Garaschuk

MathemAttic

No. 47

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **October 30, 2023**.

MA231. A grocery store clerk wants to make a large triangular pyramid of oranges. The bottom level is an equilateral triangle made up 1275 oranges. Each orange above the first level rests in a pocket formed by three oranges below. The stack is completed at the final level with a single orange. How many oranges are in the stack?

MA232. Determine the number of integers of the form $\overline{abc}+\overline{cba}$, where $\overline{abc}$ and $\overline{cba}$ are three-digit numbers with $ac \neq 0$.

MA233. Determine a polynomial function $p(x)$ with the property that if a line is drawn and intersects the graph of $y = p(x)$ in two distinct points $(a, p(a))$ and $(b, p(b))$, then the y-intercept of the line is ab.

MA234. *Proposed by Ed Barbeau.*

Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are two distinct fractions of positive integers that are both less than $\frac{1}{2}$. Prove that the numerator of one of the fractions can be increased by 1 so that the sum of the two resulting fractions is less than 1.

MA235. *Proposed by Aravind Mahadevan.*

In ΔABC, $\cos A \cos B + \sin A \sin B \sin C = 1$. Find $a : b : c$, where a, b, and c are the lengths of sides BC, CA and AB respectively.

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 octobre 2023**.

MA231. Un commis d'épicerie veut faire une grande pyramide triangulaire avec des oranges. Le niveau inférieur de la pyramide est un triangle équilatéral formé de 1 275 oranges. Chaque orange au-dessus du premier niveau se situe dans la cavité formée par trois oranges du niveau précédent. La pile est complétée par une seule orange au dernier niveau. Combien y a-t-il d'oranges dans la pile ?

MA232. Déterminer le nombre d'entiers de la forme $\overline{abc} + \overline{cba}$, où $\overline{abc}$ et $\overline{cba}$ sont des nombres à trois chiffres tels que $ac \neq 0$.

MA233. Déterminer une fonction polynomiale $p(x)$ ayant la propriété suivante : si l'on trace une droite et que celle-ci coupe le graphe de $y = p(x)$ en deux points distincts $(a, p(a))$ et $(b, p(b))$, alors l'ordonnée à l'origine y de la droite est ab.

MA234. *Proposé par Ed Barbeau.*

Supposons que $\frac{a}{b}$ et $\frac{c}{d}$ sont deux fractions distinctes d'entiers positifs qui sont toutes deux inférieures à $\frac{1}{2}$. Prouvez que le numérateur de l'une de ces fractions peut être augmenté de 1 de sorte que, malgré cela, la somme des deux fractions résultantes demeure inférieure à 1.

MA235. *Proposé par Aravind Mahadevan.*

Pour triangle ABC, on a $\cos A \cos B + \sin A \sin B \sin C = 1$. Déterminer $a : b : c$, où a, b et c sont les longueurs des côtés BC, CA et AB respectivement.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2023: 49(2), p. 62–64.

MA206. Place algebraic operations $+$; $-$; $\div$; $\times$ between the numbers 1 to 9, in that order, so that the total equals 100. You may also freely use brackets before or after any of the digits in the expression and numbers may be placed together, such as 123 and 67. Two examples are given below:

$$123+45-67+8-9=100 \quad \text{and} \quad 1+((2+3)\times 4\times 5)-((6-7)\times(8-9))=100.$$

Originally from Mathematics Competitions Vol. 34, #1 (2021), A brief history of the South African Mathematics Olympiad, classic well-known problems with a twist, example 3.

We received 4 submissions. Each submission contained multiple answers. Here are 7 possible solutions provided by Giotas Orfeas-Dimitrios, Papafragkas Ioannis and Skias Antonios-Michail of 1st Model High School of Chalkida, Greece:

$12+3+4+5-6-7+89,$

$12+3-4+5+67+8+9,$

$1+2+3+4+5+6+7+8\times 9,$

$(1-2+3)\times(4+5)\times(-6+7\times 8)\div 9,$

$1\times(2+3+4-5+6)\times(-7+8+9),$

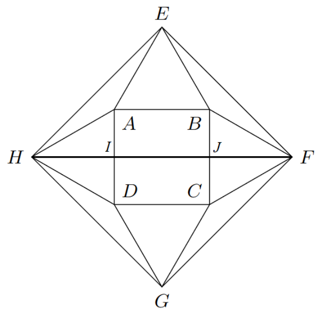
$1\times(2+3)\times 4(-5-6+7\times 8)\div 9,$

$1\times(-2+3+4+5)\times 6\times(7+8)\div 9.$

MA207. Suppose that the points E, F, G, H lie in the plane of the square $ABCD$ such that AEB, BFC, CGD, and DHA are equilateral triangles. If the area of $EFGH$ is 25, then find the area of $ABCD$.

Originally question 9 from the 36th University of Alabama High School Mathematics Tournament: Team Competition, 2017.

We received 12 submissions, 9 of them were complete and correct. We present 3 solutions by Alex Yang.



Let $AB=x$. Since $ABCD$ is a square, we have that $AB=DA=x$ and $\angle DAB=90^o$. The fact that triangles HAD and EAB are equilateral triangles implies $HA=AD$, $AE=AB$ and $\angle HAD=\angle EAB=60^\circ$.

So, $AH=AE=x$ and $\angle HAE=150^o$. Similarly, we can show that triangles BEF, CFG and DHG are isosceles triangles with legs of length x and vertex angle of 150° and each isosceles angle of 15°.

Therefore triangles AHE, BEF, CFG and DHG are congruent. Then, we get $HE = EF = FG = GH$. Meanwhile,

$$\angle HEF = \angle HEA + \angle AEB + \angle BEF = 15^\circ + 60^\circ + 15^\circ = 90^\circ.$$

Similarly, we can show that

$$\angle EFG = \angle FGH = \angle GHE = 90^\circ.$$

Thus $EFGH$ is a square.

Method 1. Connect H and F. Denote the intersection of HF with AD by I and that of HF with BC by J. Since the area of square $EFGH$ is 25, we get the length of side of $EFGH$ is 5. So, $HF = 5\sqrt{2}$.

Since $AB = x$, $AD = BC = IJ = x$. In equilateral triangle AHD, we get $HI = \frac{\sqrt{3}x}{2}$. Similarly, $FJ = \frac{\sqrt{3}x}{2}$. So, we obtain

$$5\sqrt{5} = \frac{2\sqrt{3}x}{2} + x.$$

Then, $x = \frac{5\sqrt{2}(\sqrt{3}-1)}{2}$. Thus, the area of $ABCD$ is $x^2 = 25(2-\sqrt{3})$.

Method 2. In triangle AHE, applying the sine law, we have

$$\frac{AH}{\sin\angle HEA} = \frac{HE}{\sin\angle HAE}.$$

Since the square $EFGH$ has an area of 25, we get $HE = 5$. This, together with the fact that $\angle HEA = 15^\circ$ and $\angle HAE = 150^\circ$, gives

$$AH = \frac{5\sin 15^\circ}{\sin 150^\circ}.$$

It follows from $\sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$ and $\sin 150^o = \frac{1}{2}$ that $AH = \frac{5(\sqrt{6}-\sqrt{2})}{2}$. Thus, the area of $ABCD$ is

$$AH^2 = 25(2-\sqrt{3}).$$

Method 3. The figure consists of three types: one small square, four congruent equilateral triangles, and four congruent isosceles triangles. The area of the square $ABCD$ is x^2. The triangle AHD has area $\frac{\sqrt{3}x^2}{4}$. The triangle AHE has area of

$$\frac{AH \times AE \times \sin(\angle HAE)}{2} = \frac{x^2}{4}.$$

So,

$$25 = x^2 + 4\left(\frac{\sqrt{3}x^2}{4} + \frac{x^2}{4}\right),$$

which gives $x^2 = 25(2-\sqrt{3})$.

MA208. Solve the following equation for $0 \leq x < 2\pi$:

$$2^{3\cos x+3} - 2^{2\cos x+2} - 2^{\cos x+1} + 1 = 0.$$

We received 14 submissions, of which 13 were correct and complete. We present the solution by Mingshen Zong, lightly edited.

Let $y = 2^{\cos x}$; note that, since $-1 \leq \cos x \leq 1$, we have $\frac{1}{2} \leq y \leq 2$. Substituting in the original equation we obtain

$$8y^3 - 4y^2 - 2y + 1 = 0.$$

We group the left hand side of the above equation as follows

$$\begin{aligned} 8y^3 - 4y^2 - 2y + 1 &= 4y^2(2y-1) - (2y-1) \\ &= (4y^2-1)(2y-1) \\ &= (2y+1)(2y-1)(2y-1) \\ &= (2y-1)^2(2y+1). \end{aligned}$$

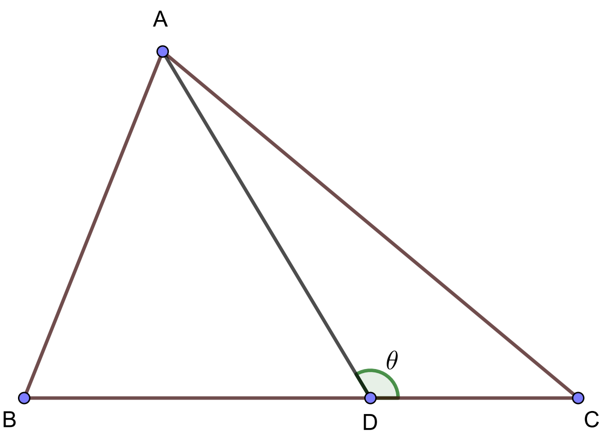
Therefore y must be one of $-\frac{1}{2}$ and $\frac{1}{2}$. Since $y \geq \frac{1}{2}$ the only option is $y = \frac{1}{2}$. This means $2^{\cos x} = \frac{1}{2} = 2^{-1}$, so $\cos x = -1$. Since $0 \leq x < 2\pi$, we obtain $x = \pi$.

MA209. *Proposed by Aravind Mahadevan.*

In ΔABC, D is on BC. If $\angle ADC = \theta$, prove that

$$BC\cot\theta = DC\cot B - BD\cot C.$$

We received 1 solution. We present the solution by Richard Hess, lightly edited.



From the sine law in $\triangle ADC$ we have

$$\frac{\sin(180^\circ - (\theta + C))}{DC} = \frac{\sin C}{AD} \Rightarrow AD = DC\frac{\sin C}{\sin(\theta + C)}.$$

From the sine law in $\triangle ABD$ we have

$$\frac{\sin(180^\circ - ((180^\circ - \theta) + B))}{BD} = \frac{\sin B}{AD} \Rightarrow AD = BD\frac{\sin B}{\sin(\theta - B)}.$$

Equating the two expressions for the length of AD we get

$$\begin{aligned} & DC\frac{\sin C}{\sin(\theta + C)} = BD\frac{\sin B}{\sin(\theta - B)} \\ \iff \quad & DC \sin C \sin(\theta - B) = BD \sin B \sin(\theta + C) \end{aligned}$$

Using the sum and difference formula for sine,

$$DC \sin C(\sin\theta \cos B - \cos\theta \sin B) = BD \sin B(\sin\theta \cos C + \cos\theta \sin C).$$

Dividing through by $\sin B \sin C \sin\theta$ we get

$$DC(\cot B - \cot\theta) = BD(\cot C + \cot\theta).$$

Finally, we rearrange to get

$$\begin{aligned} & DC \cot B - BD \cot C = BD \cot\theta + DC \cot\theta \\ \iff \quad & DC \cot B - BD \cot C = BC \cot\theta, \end{aligned}$$

which is what we needed to show.

MA210. *Proposed by Neculai Stanciu.*

Determine all triplets (x, y, z) of real numbers which satisfy:

$$2xy - (z + x - 1)^2 = 2xy - (x + y - 1)^2 = 2zx - (y + z - 1)^2 = 1.$$

We received 7 submissions, showing solution to two problems: one as defined in Vol. 49 (2) and one with slight modification of the second part of given equation. 4 of them were complete and correct. We present the solution by Ivan Hadinata for the original problem and Daniel Vacaru for modified.

Original problem. Determine all triplets (x, y, z) of real numbers which satisfy:

$$2xy - (z + x - 1)^2 = 2xy - (x + y - 1)^2 = 2zx - (y + z - 1)^2 = 1$$

Note that we have

$$2xy - (x + y - 1)^2 = 1 \quad \Longrightarrow \quad (x - 1)^2 + (y - 1)^2 = 0 \quad \Longrightarrow \quad x = y = 1.$$

Substituting $x = y = 1$ to $2xy - (z + x - 1)^2 = 2zx - (y + z - 1)^2 = 1$ yields $2 - z^2 = 2z - z^2 = 1$ and thus $z = 1$. The only triplet is $(x, y, z) = (1, 1, 1)$ which clearly satisfies (1).

Modified problem. Determine all triplets (x, y, z) of real numbers which satisfy:

$$2xy - (z + x - 1)^2 = \mathbf{2yz} - (x + y - 1)^2 = 2zx - (y + z - 1)^2 = 1$$

We have

$3 = 2xy - (z + x - 1)^2 + 2yz - (x + y - 1)^2 + 2zx - (y + z - 1)^2 \Leftrightarrow$

$3 = \cancel{2xy} - z^2 - x^2 - 1 - \cancel{2xz} + 2z + 2x + \cancel{2yz} - x^2 - y^2 - 1 - \cancel{2xy} + 2x + 2y + \cancel{2zx} - y^2 - z^2 - 1 - \cancel{2yz} + 2y + 2z \Leftrightarrow$

$6 + 2x^2 + 2y^2 + 2z^2 - 4x - 4y - 4z = 0 \Leftrightarrow (x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 0.$

It follows that $(x, y, z) = (1, 1, 1)$.

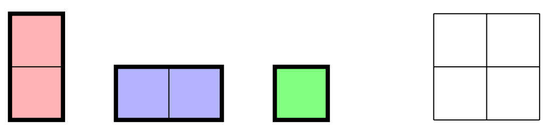
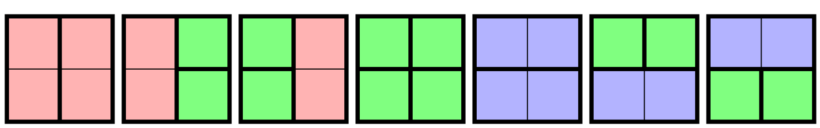
PROBLEM SOLVING VIGNETTES

No. 28

Shawn Godin

Tiling a Strip

In this issue, we will look at problem 15 from the 2022 *Canada Jay Mathematical Competition*. The *Canada Jay Mathematical Competition* is hosted by the Canadian Mathematical Society. The contest is aimed at students in grades 5 through 8 and is made up of 15 multiple choice questions. More information and past contests can be found on the contest website.

> *You have a number of* 2×1, 1×2, *and* 1×1 *tiles and want to tile a* 2×2 *square.*
>
> 
>
> *You notice that there are* 7 *different ways you could do the tiling as shown below.*
>
> 
>
> *How many different ways could you tile the* 2×4 *rectangle, pictured below, using the tiles?*
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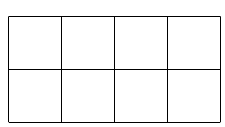
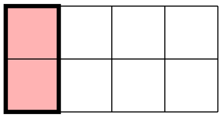
> 
>
> **(A)** *49* **(B)** *55* **(C)** *63* **(D)** *71* **(E)** *81*

It may be hard to see how to proceed with this problem. In cases like this, just jumping in and playing with the problem can give us some inspiration. First, to make things easier, we will refer to the 2×1 tile (red) as a *vertical tile*, the 1×2 tile (blue) as a *horizontal tile*, and the 1×1 tile (green) as a *square tile*.

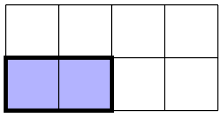
Proceeding systematically, we will look at things from the point of view of which tile is covering the bottom left square on our rectangle. We then have three cases.

Case 1: A vertical tile covers the bottom left square.

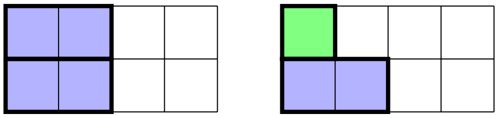


In this case we are left with a 2×3 rectangle to tile.

Case 2: A horizontal tile covers the bottom left square.



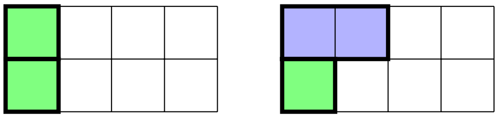
This leads us to two subcases where we consider which tile covers the top left square, either another horizontal tile or a square tile.



In the first subcase we are left with a 2×2 square, which we know from the problem can be tiled in 7 ways.

Case 3: A square tile covers the bottom left square.

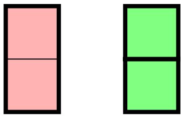
As in case 2, there will be two subcases: one where another square tile covers the top left square and one where a horizontal tile covers the top left square.



In the first case we are left with a 2×3 rectangle to cover, as in the first case. In the second case we are left with a shape similar to the second subcase in case 2.

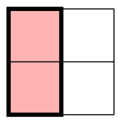
From this playing, we see a possible strategy. We may be able to build $2 \times n$ rectangles from smaller rectangles.

Let's begin with a 2×1 rectangle. There are only two ways to cover the lower left square: either with a vertical tile or by a square. That gives us only $\boxed{\mathbf{2}}$ ways of covering a 2×1 rectangle, shown below.

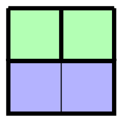


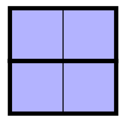
Proceeding to a 2×2 rectangle, there will be (as always from this point on) three ways to cover the bottom left square.

1. When the bottom left square is covered with a vertical tile, we are left with a 2×1 rectangle which we know can be covered in 2 ways.

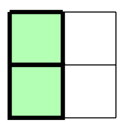


2. When the bottom left square is covered with a horizontal tile, we are left with a 1×2 rectangle which, like the 2×1 rectangle, can be covered in 2 ways.

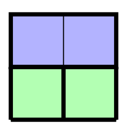




3. When the bottom left square is covered with a square tile, there are two possible ways to cover the top left square. When the top left square is covered by another square, we are left with a 2×1 rectangle which can be covered in 2 ways.

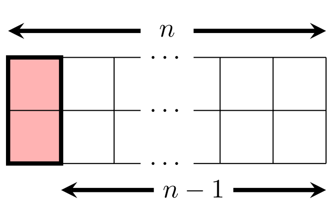


When the top left square is covered by a horizontal tile there is only one way to finish covering the original rectangle.

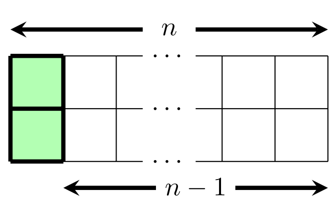


Thus we see that there are $2+2+2+1 = \boxed{\mathbf{7}}$ ways to cover a 2×2 rectangle (which agrees with what we were told in the problem statement).

Moving on to the 2×3 rectangle, the first case with the vertical tile leaves us with a 2×2 rectangle, which can be covered in 7 ways. We can see that in general, case 1 reduces a $2 \times n$ rectangle to a $2 \times (n-1)$ rectangle, which we would have calculated in the previous step. Similarly, case 3, subcase 1 will also reduce a $2 \times n$ rectangle to a $2 \times (n-1)$ rectangle. For our 2×3 rectangle, this gives us $7 + 7 = 14$ tilings.

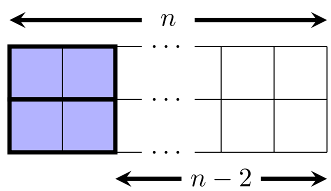


Case 1



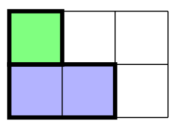
Case 3, subcase 1

Looking at case 2 subcase 1, in general, we see it reduces a $2 \times n$ rectangle to a $2 \times (n-2)$ rectangle. For our 2×3 rectangle this corresponds to 2 more tilings, bringing us up to 16, so far.

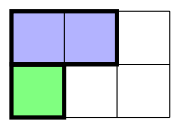


Case 2, subcase 1

That leaves us with case 2, subcase 2 and case 3 subcase 2. Both of these cases look like a rectangle with an extra square on the leading edge. For our 2×3 case these look like letter L's in different configurations, each of which can be covered in 3 ways (I will leave that to the reader to verify).



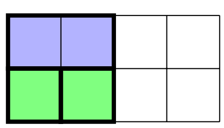
Case 2, subcase 2

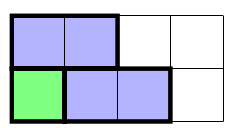


Case 3, subcase 2

Thus a 2×3 rectangle can be tiled in $2 \times 7 + 2 + 2 \times 3 = 22$ ways.

We can proceed to the desired 2×4 rectangle. Case 1 and case 3 subcase 1 yield the 2×3 rectangle with 22 tilings, while case 2 subcase 1 yields a 2×2 rectangle with 7 tilings. We are only left with case 2 subcase 2 and case 3 subcase 2. By symmetry, each of these should yield the same number of tilings so we will deal with case 3 subcase 2. There are only two possible choices for the tile in the bottom left: a square which leaves a 2×2 rectangle, with 7 tilings; or a horizontal tile which leaves an "L" shape, with 3 tilings.

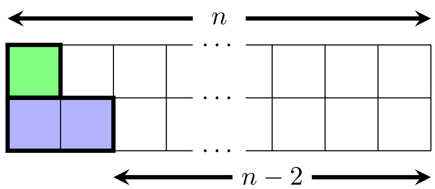




Thus the total number of tilings for a 2×4 rectangle is

$$2 \times 22 + 7 + 2 \times (7 + 3) = 71 \qquad \boxed{\textbf{D}}$$

At this point we get the feeling that the process can be generalized. Let R_n count the number of tilings of a $2 \times n$ rectangle and let B_n count the number of tilings of a $2 \times n$ rectangle with a square "bump" on the end. Thus, starting with a $2 \times n$ rectangle, case 2 subcase 2 yields the figure below, which can be tiled in B_{n-2} ways. Similarly, the figure resulting from case 3 subcase 2 can also be tiled in B_{n-2} ways.



Case 2, subcase 1

To summarize, starting with a $2 \times n$ rectangle:

- case 1 yields a $2 \times (n-1)$ rectangle, which can be tiled in R_{n-1} ways;
- case 2, subcase 1 yields a $2 \times (n-2)$ rectangle, which can be tiled in R_{n-2} ways;
- case 2, subcase 2 yields a $2 \times (n-2)$ rectangle with a bump, which can be tiled in B_{n-2} ways;
- case 3, subcase 1 yields a $2 \times (n-1)$ rectangle, which can be tiled in R_{n-1} ways; and
- case 3, subcase 2 yields a $2 \times (n-2)$ rectangle with a bump, which can be tiled in B_{n-2} ways;

which we can write as

$$R_n = 2R_{n-1} + R_{n-2} + 2B_{n-2} \tag{1}$$

However, if we have a $2 \times n$ rectangle with a bump that we want to tile – starting from the left – there are only two possible cases:

- fill the bump with a square, leaving a $2 \times n$ rectangle that can be tiled in R_n ways; or
- fill the bump with a horizontal tile, leaving a $2 \times (n-1)$ rectangle with a bump that can be tiled in B_{n-1} ways

which gives us

$$B_n = R_n + B_{n-1} \tag{2}$$

Note that the "L" shapes we talked about earlier can be tiled in $B_1 = 3$ ways. However, recursion equation (2) tells us

$$\begin{aligned} B_1 &= R_1 + B_0 \\ 3 &= 2 + B_0 \end{aligned}$$

which imply that $B_0 = 1$, which makes sense as this would be just a "bump" which could only be covered by the square. Thus substituting (2) back into itself repeatedly gives us

$$B_n = R_n + R_{n-1} + R_{n-2} + \cdots + R_2 + R_1 + 1 \tag{3}$$

and hence, (1) becomes

$$R_n = 2R_{n-1} + R_{n-2} + 2(R_{n-2} + R_{n-3} + \cdots + R_2 + R_1 + 1)$$
$$R_n = 2R_{n-1} + 3R_{n-2} + 2R_{n-3} + \cdots + 2R_2 + 2R_1 + 2 \tag{4}$$

Which gives us a nice way to calculate our tilings. Nevertheless, we can simplify this further. Using (4) for R_{n-1} gives

$$R_{n-1} = 2R_{n-2} + 3R_{n-3} + 2R_{n-4} + \cdots + 2R_2 + 2R_1 + 2$$

so

$$R_n - R_{n-1} = 2R_{n-1} + R_{n-2} - R_{n-3}$$

which can be rearranged to give

$$R_n = 3R_{n-1} + R_{n-2} - R_{n-3} \tag{5}$$

Combining (5) with $R_2 = 7$ (from the original problem), $R_1 = 2$ (from earlier in our solution), and $R_0 = 1$ (trivially), we can quickly verify our solution by calculating

$$R_3 = 3R_2 + R_1 - R_0 = 3 \times 7 + 2 - 1 = 22,$$
$$R_4 = 3R_3 + R_2 - R_1 = 3 \times 22 + 7 - 2 = 71.$$

Readers may continue to enjoy playing with the problem. Here are some ideas for further exploration.

1. Derive the value of R_5 from "first principles" like earlier in the column and show that it corresponds to the value given by the recursion.
2. One of the official solutions to the problem breaks the tilings into two cases: those where a vertical line can be drawn down the centre without crossing a tile, and those that cannot. See if you can use this strategy to determine R_4 more quickly than our original "first principles" solution.
3. Another method to attack the problem is to define an "unbreakable" tiling of a $2 \times n$ rectangle to be one where *no* vertical line can be drawn through the rectangle without crossing a tile. All tilings of a $2 \times n$ rectangle can be worked out by looking at its unbreakable parts.

Have fun playing with these, or other tiling problems.

Prime-Producing Pairs of Dice

Doddy Kastanya

In this short article, I would like you to consider special pairs of dice. The traditional die we are all familiar with has six faces numbered 1 through 6. For the first part of this article, let's consider a pair of four-faced dice. Instead of being cubes, these dice are tetrahedrons, another one of the Platonic solids. In addition, we are not going to simply number the faces of our dice 1 through 4. Each face of these two dice will have a unique number such that when we roll them, the sum of the numbers at the bottom of the dice will always be a prime number.

The key to building up these dice is to consider the answer – that is, all 16 possible prime numbers produced when rolling these dice – and then going backward to determine the number to be painted on each face. Symbolically, the answer is illustrated in the figure below.

		Die B			
		$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	$\boldsymbol{\beta_4}$
Die A	$\boldsymbol{\alpha_1}$	$\pi_{1,1}$	$\pi_{1,2}$	$\pi_{1,3}$	$\pi_{1,4}$
	$\boldsymbol{\alpha_2}$	$\pi_{2,1}$	$\pi_{2,2}$	$\pi_{2,3}$	$\pi_{2,4}$
	$\boldsymbol{\alpha_3}$	$\pi_{3,1}$	$\pi_{3,2}$	$\pi_{3,3}$	$\pi_{3,4}$
	$\boldsymbol{\alpha_4}$	$\pi_{4,1}$	$\pi_{4,2}$	$\pi_{4,3}$	$\pi_{4,4}$

The following are requirements that need to be satisfied:

- α_1 through α_4 as well as β_1 through β_4 are unique and non-negative integers.
- $\pi_{i,j}$ for $i = 1, \ldots, 4$ and $j = 1, \ldots, 4$ are unique and prime.
- For a given j, $\pi_{i,j} - \pi_{i-1,j} = \alpha_i - \alpha_{i-1}$ for $i \geq 2$.
- Similarly, for a given i, $\pi_{i,j} - \pi_{i,j-1} = \beta_j - \beta_{j-1}$ for $j \geq 2$.

So, the next step is to find four sets of prime numbers, each containing four prime numbers which satisfy the desired spacing requirement above. The following are some examples of the four sets of prime numbers which could be used as the basis for building up the two dice:

- $\{7, 11, 13, 17\}, \{37, 41, 43, 47\}, \{457, 461, 463, 467\}, \{1423, 1427, 1429, 1433\}$
- $\{251, 257, 263, 269\}, \{1741, 1747, 1753, 1759\}, \{3301, 3307, 3313, 3319\}, \{5101, 5107, 5113, 5119\}$
- $\{11, 19, 29, 31\}, \{23, 31, 41, 43\}, \{53, 61, 71, 73\}, \{89, 97, 107, 109\}$
- $\{23, 29, 31, 37\}, \{53, 59, 61, 67\}, \{563, 569, 571, 577\}, \{4643, 4649, 4651, 4657\}$
- $\{47, 53, 59, 61\}, \{167, 173, 179, 181\}, \{257, 263, 269, 277\}, \{557, 563, 569, 571\}$

Finally, to determine what numbers should be painted on each face of the dice, we need to work with the smallest prime number in the four available sets. The sum of α_1 and β_1 will be this number. We need to work with the smallest number since using other numbers could lead to a violation of the first requirement (i.e., α_i's and β_j's being non-negative integers) due to the last two requirements (i.e., the spacing requirements). As an illustration, let's work with the third example shown above. The smallest number in the four sets of primes is 11 in this case. So, we can have 12 combinations of (α_1, β_1) pairs namely $(0,11), (1,10), (2,9), (3,8), \ldots, (10,1), (11,0)$. The values for the others α_i's and β_j's could be easily calculated since they will follow the same "spacing" as the resulting prime numbers. If we pick $(3,8)$ as an example, the numbers to be painted on the faces of the two dice along with the results from rolling the two dice are shown in the figure below. Die A will have 3, 15, 45, and 81 painted on its faces while Die B will have 8, 16, 26, and 28.

		Die B			
		8	**16**	**26**	**28**
Die A	**3**	11	19	29	31
	15	23	31	41	43
	45	53	61	71	73
	81	89	97	107	109

Now, we are going to use the dice that we are more familiar with – the ones with six faces. The challenge is still the same. We would like to have two dice with the following conditions: each face of these two dice will have a unique number such that when we roll the two dice, the sum of the numbers shown must be a prime number. Now that we have regular dice, the number we are interested in is on top of the die when it settles after the roll. The same approach as in the previous case will work. However, this time around you would need to find six sets of prime numbers, each of which contains six prime numbers which satisfy the spacing requirement above. This problem is significantly more challenging than the tetrahedral dice one. Shown below is one solution for this challenge. Could you find another one that will give a different set of resulting prime numbers? Otherwise, there are 11 other solutions readily available. Good luck and have fun!

		Die B					
		6	**12**	**66**	**78**	**132**	**168**
Die A	**5**	11	17	71	83	137	173
	31	37	43	97	109	163	199
	61	67	73	127	139	193	229
	101	107	113	167	179	233	269
	145	151	157	211	223	277	313
	185	191	197	251	263	317	353

. . . .

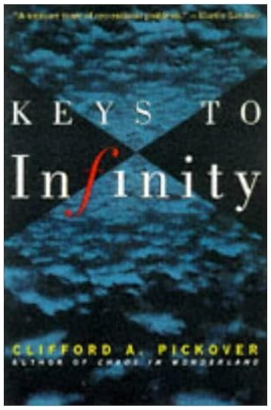
Doddy is a math enthusiast working as a nuclear engineer. The love of math and physics was the reason for him to choose this field. Starting in June 2023, Doddy serves a one-year term as the President of the Canadian Nuclear Society. In his spare time, among other things he likes to solve math puzzles and problems. In addition to ***Crux***, the Project Euler and MATLAB Cody have provided him with enough challenges and enjoyment in this area. Doddy and his family share their Oakville home with their four cats: Luke, Lorelai, Lincoln, and Lilian. Communications can be shared with the author via email: `kastanya@yahoo.com`.

From the bookshelf of ...

Shawn Godin

This MathemAttic *feature brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to* `MathemAttic@cms.math.ca`*. Publishers are also welcome to send along books for possible review.*

Keys To Infinity
by Clifford A. Pickover
ISBN 0-471-11857-5, 352 pages
Published by John Wiley & Sons, 1995.



I own – at the time I am writing this – 14 books by Cliff Pickover. He has written many wonderful books on mathematics, puzzles, computer science and more (I have yet to read any of his fiction). *Keys To Infinity* was the first of his books that I purchased. After reading it, I always kept my eyes open for more of his titles.

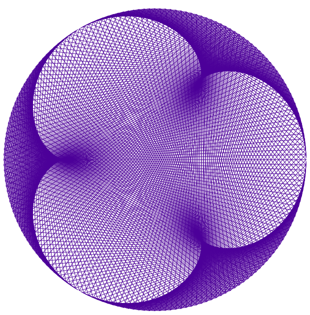
This book has a little bit of everything. Many of the 31 chapters are on some sort of mathematical puzzle. For example, chapter 28 *Chaos in Ontario* talks about the following problem the author encountered while visiting the Ontario Science Centre in Toronto:

In the ten boxes below, write a ten-digit number. The digit in the first box (of row 2, Ed.) indicates the total number of zeros in the entire number. The box marked 1 indicates the total number of 1's in the entire number. The box marked 2 indicates the total number of 2's in the entire number, and so on. For example, the 3 in the box labelled 0 would indicate that there must be exactly three 0's in the ten-digit number.

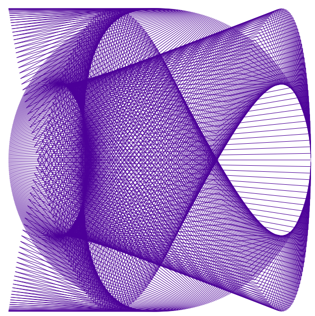
0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*Row 1*
3										*Row 2*

The chapter includes a discussion of the problem and the solution. As well, Pickover discusses a possible extension to the problem where the "solution" to the puzzle becomes the description of a new number and the process is repeated. It turns out that the sequence of numbers derived by this method eventually cycles. The author then created a "contest" for colleagues to find the longest sequence of such numbers before cycling occurs. Being a computer scientist, the author includes code for many of the problems explored in the book and Appendix 2 contains the program for one of the solutions to the "contest".

The author is also well-known in the area of computer graphics, and several of the chapters are on mathematically based graphics. In chapter 9, *The Loom of Creation*, he discusses shapes created by a hypothetical spider. The spider spins webs made up of line segments with starting points at $(r\cos\theta, r\sin\theta)$ and the end points at $(r\cos a\theta, r\sin b\theta)$ while θ ranges from 0° to 360° in equal sized steps. The variable a and b are positive integer parameters, each pair defining a new figure. Below are two diagrams created using the author's method.



$a = 4$, $b = 4$



$a = 2$, $b = 5$

Pickover also loves to play with numbers that have peculiar properties, especially ones that can be investigated with computer programs. Chapter 22, *The Loneliness of the Factorions* introduces two similarly defined families of numbers: factorions and narcissistic numbers, while chapter 30, *Vampire Numbers*, introduces the titular family. Factorions are numbers that can be written as a sum of the factorials of their digits, such as

$$145 = 1! + 4! + 5!.$$

Similarly, narcissistic numbers are n-digit numbers that are equal to the sum of the nth powers of its digits, such as

$$153 = 1^3 + 5^3 + 3^3.$$

Vampire numbers are numbers with an even number of digits that can be factored into two factors, each with half as many digits as the original, in such a way that all the digits of the original number are contained in the factors. For example,

$$1395 = 15 \times 93$$

where the original number can be formed by writing out the two factors and rearranging the digits. Pickover discusses each family, some of their properties as well as looking at some programs to explore them.

Keys To Infinity contains much of interest to *MathemAttic* readers. Pick it up – or any other of Pickover's books for that matter – and you will not be disappointed!

. . . .

This book is a recommendation from the bookshelf of Shawn Godin. Shawn, a retired high school math teacher, is a co-editor of *MathemAttic* and has been involved with ***Crux*** in one form or another for over 20 years. Shawn continues to be involved in mathematical activities in his retirement: helping with mathematics contest creation and marking, writing columns and doing the occasional presentation. He lives in Carleton Place, Ontario with his wife, Julie, and their dog, Daisy.

OLYMPIAD CORNER

No. 415

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **October 30, 2023**.

OC641. Let a, b, c be integers. Prove that there exists a positive integer n such that the number $n^3 + an^2 + bn + c$ is not a perfect square.

OC642. Determine if there exist positive integers n and k such that

$$\frac{n}{11^k - n}$$

is the square of an integer.

OC643. Find the smallest natural number n such that for every 3-colouring of the numbers $1, 2, \ldots, n$ there are two (different) numbers of the same colour such that their positive difference is a perfect square.

OC644. In a 2018×2018 chessboard, some of the cells are painted white, the rest are black. It is known that from this chessboard one can cut out a 10×10 square, all cells of which are white, and a 10×10 square, all cells of which are black. What is the smallest d for which it can be guaranteed that a 10×10 square can be cut out of it, in which the number of black and white cells differs by no more than d?

OC645. Prove that if $a, b, c \geq 0$ and $a + b + c = 3$, then

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}.$$

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 octobre 2023**.

OC641. Soient a, b, c des entiers. Démontrer qu'il existe un entier positif n tel que le nombre $n^3 + an^2 + bn + c$ n'est pas un carré parfait.

OC642. Déterminer s'il existe des entiers positifs n et k tels que

$$\frac{n}{11^k - n}$$

est le carré d'un entier.

OC643. Déterminer le plus petit nombre naturel n tel que pour tout 3-colorage des nombres $1, 2, \ldots, n$ il existe deux nombres différents mais de même couleur dont la différence positive est un carré parfait.

OC644. Sur un échiquier de taille 2018×2018, certaines cellules sont colorées blanc, tandis que les autres sont colorées noir. On sait que cet échiquier spécifique comporte une section 10×10 entièrement blanche et une autre section 10×10 entièrement noire. Quelle est la plus petite valeur d possible assurant l'existence d'une section 10×10 dont les nombres de cellules blanches et noires diffèrent par au plus d?

OC645. Démontrer que si $a, b, c \geq 0$ et $a + b + c = 3$, alors

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}.$$

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2023: 49(2), p. 80–81.

OC616. Let a, b, c be integer side-lengths of a triangle, $\gcd(a, b, c) = 1$ and all the values

$$\frac{a^2+b^2-c^2}{a+b-c}, \quad \frac{b^2+c^2-a^2}{b+c-a}, \quad \frac{c^2+a^2-b^2}{c+a-b}$$

are integers as well. Prove that

$$(a+b-c)(b+c-a)(c+a-b) \quad \text{or} \quad 2(a+b-c)(b+c-a)(c+a-b)$$

is a perfect square.

Originally problem 3 of the 2018 Czech Mathematical Olympiad.

We received 6 solutions. We present the solution by Ivan Hadinata.

We first prove the following lemma:

Lemma. For any positive integers p, q, r with $\gcd(p, q, r) = 1$ and $r \mid pq$, $q \mid pr$, $p \mid qr$ then pqr is a perfect square.

0.7Proof. Let $d, m, n \in \mathbb{N}$ be such that $d = \gcd(p, q)$, $p = dm$, $q = dn$, and $\gcd(m, n) = 1$. Since $\gcd(p, q, r) = 1$ then $\gcd(d, r) = 1$. Therefore

$$r \mid pq \implies r \mid d^2mn \implies r \mid mn \qquad (1)$$

$$q \mid pr \implies n \mid mr \implies n \mid r \qquad (2)$$

$$p \mid qr \implies m \mid nr \implies m \mid r \qquad (3)$$

By (2) and (3), $\operatorname{lcm}(m, n) = mn$ can divide r. Combining it with (1) implies that $r = mn$. Hence $pqr = d^2r^2$, that is a perfect square. □

Back to the problem, let $x, y, z \in \mathbb{N}$ where

$$x = a+b-c, \ y = a-b+c, \ z = -a+b+c.$$

Then $2a = x + y$, $2b = y + z$, $2c = z + x$. Clearly the parity of x, y, z must be the same. Let $\alpha = \gcd(x, y, z)$. Then, α divides all of $2a$, $2b$, and $2c$. So $\alpha \mid 2 \cdot \gcd(a, b, c) = 2$, which implies that $\alpha = 1$ or 2.

Note that

$$\frac{a^2+b^2-c^2}{a+b-c} = \frac{1}{2}\left(x+y+z-\frac{xy}{z}\right) \in \mathbb{Z}$$

implies that $z \mid xy$. Similarly, $y \mid xz$ and $x \mid yz$.

If $\alpha = 1$, by the Lemma it implies that $(a+b-c)(b+c-a)(c+a-b) = xyz$ is a perfect square.

If $\alpha = 2$, let $x = 2x_0,\ y = 2y_0,\ z = 2z_0,\ x_0, y_0, z_0 \in \mathbb{N}$ and $\gcd(x_0, y_0, z_0) = 1$. Then

$$\frac{a^2 + b^2 - c^2}{a + b - c} = x_0 + y_0 + z_0 - \frac{x_0 y_0}{z_0}$$

gives that $z_0 \mid x_0 y_0$. Similarly, $y_0 \mid x_0 z_0$ and $x_0 \mid y_0 z_0$. By the Lemma,

$$2(a + b - c)(b + c - a)(c + a - b) = 16x_0 y_0 z_0$$

is a perfect square. The proof is complete.

OC617. Consider a positive integer n, a circle of circumference $6n$ and $3n$ points on the circle that divide it into $3n$ small arcs so that n of these arcs have a length of 1, another n of these arcs have a length of 2, and the remaining arcs have a length of 3. Show that among the considered points there are two that are diametrically opposite.

Originally 4th Problem of Grade 9, Final Round of the 2018 Romania Mathematical Olympiad.

We received 2 solutions. We present the solution by UCLan Cyprus Problem Solving Group.

Given a circle of circumference $4m + 2n$ and $2m + n$ points on it dividing it into $2m + n$ arcs where m of them have length 1, n of them have length 2, and another m have length 3, then two of the points are diametrically opposite.

Assume that this is not true and pick a counterexample where m is minimal. If $m = 0$ then n is even, say $n = 2k$ and we have $2k$ arcs of length 2. Then the k-th and $2k$-th point are diametrically opposite, a contradiction.

So $m > 0$. Let $x_1, x_2, \ldots, x_{4m+2n}$ be equally spaced points on the circle containing the $2m{+}n$ chosen points and assume for contradiction that no two are diametrically opposite. By rotation, we may assume that x_1, x_2 are two of the chosen points. So x_{m+2n+1}, x_{m+2n+2} are not chosen. This implies that x_{m+2n} and x_{m+2n+3} are chosen.

We delete the arc x_1x_2 merging the chosen points x_1, x_2 into a new chosen point, say a. We also delete the middle third of the arc $x_{m+2n}x_{m+2n+3}$ merging the unchosen points x_{m+2n+1}, x_{m+2n+2} into a new unchosen point, say b. So the new points are $a, x_3, \ldots, x_{m+2n}, b, x_{m+2n+3}, \ldots, x_{4m+2n}$.

So we removed an arc of length 1 and changed an arc of length 3 into an arc of length 2. Thus we now have $m - 1$ arcs of length 1, $n + 1$ arcs of length 2 and $m - 1$ arcs of length 3. Because the deleted arc and part of arc were diametrically opposite we still don't have chosen points which are diametrically opposite. So we have a counterexample with $m - 1$ arcs of length 1, a contradiction.

OC618. Let $n \in \mathbb{N}, n \geq 2$. For all real numbers $a_1, a_2, \ldots, a_n$ denote $S_0 = 1$ and

$$S_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} a_{i_1} a_{i_2} \cdot \ldots \cdot a_{i_k}$$

the sum of all the products of k numbers chosen among $a_1, a_2, \ldots, a_n$, $k \in \{1, 2, \ldots, n\}$. Find the number of n-tuples $(a_1, a_2, \ldots, a_n)$ such that

$$(S_n - S_{n-2} + S_{n-4} - \ldots)^2 + (S_{n-1} - S_{n-3} + S_{n-5} - \ldots)^2 = 2^n S_n.$$

Originally 4th Problem of Grade 10, Final Round of the 2018 Romania Mathematical Olympiad.

We received 3 solutions and we will present 2 of them.

Solution 1, by Oliver Geupel.

We prove that the required number is 2^{n-1}.

Consider the polynomials $S_0\,(x_1, \ldots, n_n) = 1$ and, for $k \in \{1, 2, \ldots, n\}$,

$$S_k\,(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \cdot \ldots \cdot x_{i_k}.$$

Then, for every n-tuple $E = (e_1, \ldots, e_n) \in \{0, 1, 2\}^n$, there is an integer coefficient c_E such that it holds, with the notation $X = (x_1, \ldots, x_n)$ and $X^E = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$, that

$$(S_n(X) - S_{n-2}(X) + - \ldots)^2 + (S_{n-1}(X) - S_{n-3}(X) + - \ldots)^2 = \sum_{E \in \{0,1,2\}^n} c_E X^E. \tag{1}$$

By inspection, if $\sum_{i=1}^n e_i$ is odd, then $c_E = 0$.

Next consider

$$E = (\underbrace{2, 2, \ldots, 2}_{\ell \text{ times}}, \underbrace{1, 1, \ldots, 1}_{2m \text{ times}}, 0, 0, \ldots, 0),$$

where $\ell \geq 0$ and $m \geq 0$. With the notation $L = \{1, 2, \ldots, \ell\}$ and $M = \{\ell + 1, \ell + 2, \ldots, \ell + 2m\}$, we have

$$c_E X^E = \sum_{A \subseteq M} (-1)^{m - |A|} \left(\prod_{k \in L} x_k \cdot \prod_{i \in A} x_i \right) \left(\prod_{k \in L} x_k \cdot \prod_{j \in M \setminus A} x_j \right).$$

If $m = 0$ then it follows that $c_E = 1$. If $m > 0$ then

$$c_E = \sum_{i=0}^{2m} (-1)^{m-i} \binom{2m}{i} = (-1)^{-m} \sum_{i=0}^{2m} (-1)^{2m-i} \binom{2m}{i} = (-1)^{-m} (1 - 1)^{2m} = 0.$$

By symmetry, we may deduce $c_E = 0$ whenever there is an index $i \in \{1, \ldots, n\}$ such that $e_i = 1$. Thus, the polynomial (1) is equal to

$$\sum_{E \in \{0,2\}^n} X^E = \prod_{i=1}^{n} \left(x_i^2 + 1\right).$$

Therefore,

$$(S_n - S_{n-2} + S_{n-4} - \ldots)^2 + (S_{n-1} - S_{n-3} + S_{n-5} - \ldots)^2 = \prod_{i=1}^{n} \left(a_i^2 + 1\right),$$

which is greater than or equal to $2^n \left|\prod_{i=1}^n a_i\right| = 2^n |S_n|$ by the AM–GM inequality. By the equality condition of the AM–GM inequality, the equality holds if and only if $a_i = \pm 1$ for $1 \leq i \leq n$, where an even number of the a_i is equal to -1. The required number is therefore equal to

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} = 2^{n-1},$$

which completes the proof.

Solution 2, by UCLan Cyprus Problem Solving Group.

Define $T_0 = 1$ and for $1 \leqslant k \leqslant n$ define T_k to be the sum of squares of all products of k numbers chosen among $a_1, a_2, \ldots, a_n$. We will prove by induction on n that

$$(S_n - S_{n-2} + \cdots)^2 + (S_{n-1} - S_{n-3} + \cdots)^2 = T_0 + T_1 + \cdots + T_n.$$

The case $n = 2$ is immediate as

$$\begin{aligned}(S_2 - S_0)^2 + S_1^2 &= (a_1a_2 - 1)^2 + (a_1 + a_2)^2 \\ &= (a_1a_2)^2 - 2a_1a_2 + 1 + a_1^2 + a_2^2 - 2a_1a_2 \\ &= (a_1a_2)^2 + a_1^2 + a_2^2 + 1 = T_2 + T_1 + T_0.\end{aligned}$$

For the inductive hypothesis assume the result is true for $n = k - 1$. As usual we write S_r for the sum of all products of r numbers chosen among $a_1, a_2, \ldots, a_k$. We also write S_r' for the sum of all products of r numbers chosen among $a_1, a_2, \ldots, a_{k-1}$. Note that a_k is not included in any of the sums of S_r', so essentially we have

$$S_r' = S_r(a_1, \ldots, a_{k-1}, 0).$$

We define T_r' analogously and observe that

$$S_r = S_r' + a_k S_{r-1}' \quad \text{and} \quad T_r = T_r' + a_k^2 T_{r-1}'$$

for each $r \geqslant 1$. We get

$$\begin{aligned}
&(S_k - S_{k-2} + \cdots)^2 + (S_{k-1} - S_{k-3} + \cdots)^2 \\
&= (S'_k - S'_{k-2} + \cdots + a_k\left(S'_{k-1} - S'_{k-3} + \cdots\right))^2 \\
&\quad + (S'_{k-1} - S'_{k-3} + \cdots + a_k(S'_{k-2} - S'_{k-4} + \cdots))^2 \\
&= (S'_k - S'_{k-2} + \cdots)^2 + (S'_{k-1} - S'_{k-3} + \cdots)^2 + 2a_k(S_k - S_{k-2} + \cdots)(S_{k-1} - S_{k-3} + \cdots) \\
&\quad + 2a_k(S_{k-2} - S_{k-4} + \cdots)(S_{k-1} - S_{k-3} + \cdots) \\
&\quad + a_k^2\left[\left(S'_{k-1} - S'_{k-3} + \cdots\right)^2 + (S'_{k-2} - S'_{k-4} + \cdots)^2\right]
\end{aligned}$$

Since $S'_k = 0$ and by the induction hypothesis we have

$$(S'_{k-1} - S'_{k-3} + \cdots)^2 + (S'_{k-2} - S'_{k-4} + \cdots)^2 = T'_0 + T'_1 + \cdots + T'_{k-1}$$

we deduce that

$$\begin{aligned}
&(S_k - S_{k-2} + \cdots)^2 + (S_{k-1} - S_{k-3} + \cdots)^2 \\
&= T'_0 + T'_1 + \cdots + T'_{k-1} + a_k^2(T'_0 + T'_1 + \cdots + T'_{k-1}) \\
&= T'_0 + T_1 + \cdots + T_{k-1} + a_k^2 T'_{k-1} \\
&= T_0 + T_1 + \cdots + T_k
\end{aligned}$$

as claimed.

Now observe that $T_0 + T_1 + \cdots + T_n$ has a total of 2^n monomials, one for each subset of $\{1, 2, \ldots, n\}$. Furthermore, each a_i^2 appears in 2^{n-1} monomials, one for each subset of $\{1, 2, \ldots, n\} \setminus \{i\}$. Thus by AM-GM we have

$$T_0 + T_1 + \cdots + T_n \geqslant 2^n \sqrt[2^n]{(a_1^2 \cdots a_n^2)^{2^{n-1}}} = 2^n |S_n| \geqslant 2^n S_n \,.$$

Equality occurs if and only if $S_n \geqslant 0$ and all of the monomials are equal. This happens if and only if $S_n \geqslant 0$ and each a_i is equal to ± 1.

So there are 2^{n-1} cases of equality. Once we fixed the signs of $a_1, \ldots, a_{n-1}$, then the sign of a_n is determined as the product has to be positive.

OC619. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy simultaneously the following conditions:

(a) $f(x) + f(y) \geq xy$ for all real numbers x and y;

(b) for every real number x there is a real number y such that $f(x) + f(y) = xy$.

Originally 1st Problem, Second Round of the 2018 Poland Mathematical Olympiad.

We received 9 solutions. We present the solution by Michel Bataille.

Let $f_0 : \mathbb{R} \to \mathbb{R}$ be defined by $f_0(x) = \frac{x^2}{2}$. Then f_0 is a solution (since $x^2 + y^2 \geq 2xy$ with equality if $y = x$). We show that there is no other solution.

Let f be a solution. From condition (a), we have $f(x)+f(x) \geq x^2$, hence $f(x) \geq \frac{x^2}{2}$ for all $x \in \mathbb{R}$.

Let x be a real number. From condition (b), there exists $y_x \in \mathbb{R}$ such that $f(x) + f(y_x) = xy_x$ and we deduce that $xy_x \geq \frac{x^2}{2} + \frac{y_x^2}{2}$.

It follows that $x^2 + y_x^2 - 2xy_x \leq 0$, that is $(x - y_x)^2 \leq 0$. Thus implies $y_x = x$ and therefore $2f(x) = x^2$. Thus, $f = f_0$ and the proof is complete.

OC620. Given a trapezoid $ABCD$ with bases AB and CD, with the circle of diameter BC tangent to the line AD, prove that the circle of diameter AD is tangent to the line BC.

Originally 4th Problem, Second Round of the 2018 Poland Mathematical Olympiad.

We received 9 solutions. We present 2 solutions.

Solution 1, by Theo Koupelis.

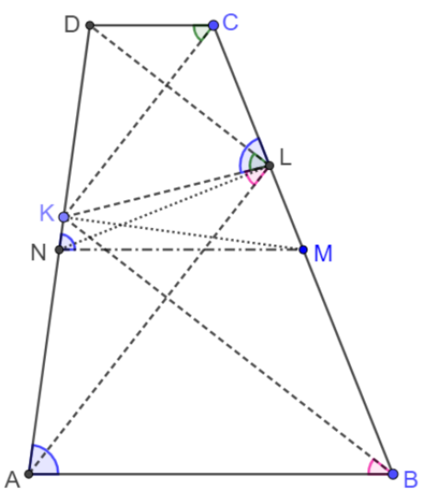
Let M be the midpoint of BC, and K the projection of M on AD. Similarly, let N be the midpoint of AD, and L the projection of N on BC. It is given that K is on the circle of diameter BC. Clearly $MN \parallel AB \parallel DC$, and the quadrilateral $NKLM$ is cyclic, because $\angle NKM = \angle NLM = 90°$. Then

$$180° - \angle KDC = \angle BAN = \angle MNK = \angle KLC,$$

and therefore $KLCD$ and $AKLB$ are cyclic. Therefore,

$$\angle ALD = \angle ALK + \angle KLD = \angle ABK + \angle DCK = 180° - (\angle KBL + \angle KCL) = 90°,$$

because $\angle BKC = 90°$. Thus, L is on the circle of diameter AD, and BC is tangent to this circle.



Solution 2, by UCLan Cyprus Problem Solving Group.

We can set the coordinates so that $A = (a, 0), B = (b, 0), C = (c, 1)$ and $D = (d, 1)$. The midpoint M of BC has coordinates $M = (\frac{b+c}{2}, \frac{1}{2})$. The line AD has equation $x - (d - a)y - a = 0$. So the distance from M to this line is

$$\frac{|1 \cdot \frac{b+c}{2} - (d-a) \cdot \frac{1}{2} - a|}{\sqrt{1^2 + (d-a)^2}} = \frac{|(b+c) - (a+d)|}{2\sqrt{1 + (d-a)^2}}\,.$$

Also,

$$MB = MC = \sqrt{\left(\frac{b+c}{2} - b\right)^2 + \frac{1}{4}} = \frac{1}{2}\sqrt{1 + (c-b)^2}\,.$$

So the circle of diameter BC is tangent to AD if and only if

$$\frac{|(b+c) - (a+d)|}{2\sqrt{1 + (d-a)^2}} = \frac{1}{2}\sqrt{1 + (c-b)^2} \iff |(b{+}c){-}(a{+}d)|\sqrt{1 + (c-b)^2}\sqrt{1 + (d-a)^2}\,.$$

By symmetry this is also the condition for the circle of diameter AD being tangent to BC.

FOCUS ON...

No. 57

Michel Bataille

Solutions to Exercises from Focus On... No. 52 - 56

From Focus On... No. 52

1. *Solve the following system in real numbers x, y, z:*

$$x+y+z=2, \qquad (x^2+y^2)^2+(y^2+z^2)^2+(z^2+x^2)^2=(x+y)(y+z)(z+x).$$

Let (x, y, z) be a solution. Then, x, y, z satisfy

$$2[(x^2+y^2)^2+(y^2+z^2)^2+(z^2+x^2)^2]-(x+y+z)(x+y)(y+z)(z+x)=0, \quad (1)$$

or after a straightforward calculation:

$$4(x^4+y^4+z^4)+2(x^2y^2+y^2z^2+z^2x^2)-x^3y-xy^3-y^3z-yz^3-z^3x-zx^3-4xyz(x+y+z)=0.$$

The left-hand side can be rewritten as $A+B+C$ where

$$A=x^4+y^4+z^4-(x^2y^2+y^2z^2+z^2x^2),$$
$$B=[x^4+y^4)-(x^3y+xy^3)]+[(y^4+z^4)-(y^3z+yz^3)]+[(z^4+x^4)-(z^3x+zx^3)],$$
$$C=x^4+y^4+z^4+3(x^2y^2+y^2z^2+z^2x^2)-4xyz(x+y+z).$$

Now $A\geq 0$ (since $a^2+b^2+c^2\geq ab+bc+ca$ for all real numbers a, b, c) and $B\geq 0$ since for all real numbers a, b, we have

$$(a^4+b^4)-(a^3b+ab^3)=(a-b)(a^3-b^3)=(a-b)^2(a^2+ab+b^2)\geq 0.$$

Using the arithmetic-geometric mean inequality, we have

$$x^4+x^2y^2+y^2z^2+z^2x^2\geq 4\sqrt[4]{x^8y^4z^4}=4x^2|y||z|=4x^2|xy|\geq 4x^2xy.$$

Similarly,

$$y^4+x^2y^2+y^2z^2+z^2x^2\geq 4xy^2z$$

and

$$z^4+x^2y^2+y^2z^2+z^2x^2\geq 4xyz^2$$

and by addition, $C\geq 0$. Thus $A+B+C\geq 0$ and (1) forces $A=B=C=0$. In particular, from $B=0$ we deduce $x=y=z$, hence $x=y=z=\frac{2}{3}$.

Conversely, it is readily checked that $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ is a solution, hence is the unique solution to the system.

2. *Solve for real numbers x, y, z:*

$$3^{x^2+3x} + 3^{y^2+3y} + 3^{z^2+3z} = 3^{x+y+z+1}.$$

Clearly, a solution (x, y, z) satisfies $3^{x^2+2x-y-z} + 3^{y^2-x+2y-z} + 3^{z^2-x-y+2z} = 3$.

Applying Jensen's inequality to the convex function $t \mapsto 3^t$, we are led to

$$\begin{aligned} 3 &= 3^{x^2+2x-y-z} + 3^{y^2-x+2y-z} + 3^{z^2-x-y+2z} \\ &\geq 3 \cdot 3^{\frac{(x^2+2x-y-z)+(y^2-x+2y-z)+(z^2-x-y+2z)}{3}} \\ &= 3 \cdot 3^{\frac{x^2+y^2+z^2}{3}} \end{aligned}$$

and therefore $3^{\frac{x^2+y^2+z^2}{3}} \leq 1$.

But we have $3^t \geq 1$ for $t \geq 0$. It follows that $3^{\frac{x^2+y^2+z^2}{3}} = 1$, which implies $x^2 + y^2 + z^2 = 0$, that is, $x = y = z = 0$.

Conversely, the equation is obviously satisfied if $x = y = z = 0$ and we conclude that $(0, 0, 0)$ is the unique solution.

3. *Find all real numbers x such that $\sqrt[3]{48-2x} + \sqrt[3]{9-3x} + \sqrt[3]{5x-30} = 3$. Here $\sqrt[3]{a}$ denotes the unique real u such $u^3 = a$.*

First, let us show that if u, v, w are real numbers such that $u + v + w = 3$ and $u^3 + v^3 + w^3 = 27$, then at least one of the numbers u, v, w equals 3. Indeed, u, v, w are the roots of the polynomial

$$p(X) = X^3 - 3X^2 + mX - p,$$

where $m = uv + vw + wu$ and $p = uvw$. Since

$$u^3 + v^3 + v^3 - 3uvw = (u + v + w)((u + v + w)^2 - 3(uv + vw + wu)),$$

we have $27 - 3p = 27 - 9m$, hence $p = 3m$. Thus,

$$p(X) = x^3 - 3X^2 + mX - 3m = (X - 3)(X^2 + m)$$

and 3 is a root of $p(X)$.

Returning to the problem, if x is a solution, then

$$u = \sqrt[3]{48-2x},\ v = \sqrt[3]{9-3x},\ w = \sqrt[3]{5x-30}$$

satisfy the above conditions, hence one of them must equal 3. Since $u = 3$ (resp. $v = 3$, resp. $w = 3$) only if $x = \frac{21}{2}$ (resp. $x = -6$, resp. $x = \frac{57}{5}$), we see that $x \in S = \{\frac{21}{2}, -6, \frac{57}{5}\}$.

Conversely, it is readily checked that if $x \in S$, then x is a solution to the equation. Thus, S is the set of all solutions.

From Focus On... No. 54

1. *Prove that* $\displaystyle\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} = \frac{3}{4}\pi.$

Recall that for $ab < 1$, the following formula holds

$$\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right).$$

Taking $a = \frac{1}{n-1}, \quad b = \frac{-1}{n+1}$ yields

$$\arctan\left(\frac{1}{n-1}\right) - \arctan\left(\frac{1}{n+1}\right) = \arctan\frac{2}{n^2}.$$

Thus, for all integer $N > 2$, we have

$$\begin{aligned}\sum_{n=1}^{N} \arctan\frac{2}{n^2} &= \arctan(2) + \sum_{n=2}^{N}\left(\arctan\left(\frac{1}{n-1}\right) - \arctan\left(\frac{1}{n+1}\right)\right) \\ &= \arctan(2) + \arctan(1) + \arctan\frac{1}{2} - \arctan\frac{1}{N} - \arctan\frac{1}{N+1}\end{aligned}$$

that is,

$$\sum_{n=1}^{N} \arctan\frac{2}{n^2} = \frac{3}{4}\pi - \arctan\frac{1}{N} - \arctan\frac{1}{N+1}$$

(since $\arctan(1) = \frac{\pi}{4}$ and $\arctan(2) + \arctan\frac{1}{2} = \frac{\pi}{2}$). Letting $N \to \infty$, we obtain

$$\sum_{n=1}^{\infty} \arctan\frac{2}{n^2} = \frac{3}{4}\pi$$

2. *Evaluate* $\displaystyle\sum_{n=1}^{\infty} \frac{2H_n}{n(n+1)(n+2)}.$

By coincidence, the problem has also appeared as problem **4826** in the March 2023 issue. So, we refer the reader to the forthcoming solution.

3. *Use summation by parts to give a four-line solution to the following problem posed in The American Mathematical Monthly* in 2014:

> Let $H_{n,2} = \sum_{k=1}^{n} k^{-2}$, and let $D_n = n! \sum_{k=0}^{n} (-1)^k/k!$. (This is the *derangement number* of n, that is, the number of permutations of $\{1, \ldots, n\}$ that fix no element.) Prove that
>
> $$\sum_{n=1}^{\infty} H_{n,2}\frac{(-1)^n}{n!} = \frac{\pi^2}{6e} - \sum_{n=0}^{\infty} \frac{D_n}{n!(n+1)^2}.$$

We have
$$\sum_{n=1}^{N} H_{n,2}\frac{(-1)^n}{n!} = H_{N,2}\cdot\sum_{n=1}^{N}\frac{(-1)^n}{n!} - \sum_{n=1}^{N-1}(H_{n+1,2}-H_{n,2})\sum_{k=1}^{n}\frac{(-1)^k}{k!}$$
$$= H_{N,2}\cdot\left(-1+\sum_{n=0}^{N}\frac{(-1)^n}{n!}\right) - \sum_{n=1}^{N-1}\frac{1}{(n+1)^2}\left(\frac{D_n}{n!}-1\right)$$
$$= H_{N,2}\sum_{n=0}^{N}\frac{(-1)^n}{n!} - \sum_{n=0}^{N-1}\frac{D_n}{n!(n+1)^2}$$

and the result follows from $\lim_{N\to\infty} H_{N,2} = \frac{\pi^2}{6}$ and $\lim_{N\to\infty}\sum_{n=0}^{N}\frac{(-1)^n}{n!} = \frac{1}{e}$.

From Focus On... No. 55

1. *Calculate* $\sum_{n=0}^{+\infty}\frac{n!}{1\times 3\times 5\times\cdots\times(2n+1)}$ *via a differential equation satisfied by*

$$f(x) = \sum_{n=0}^{+\infty}\frac{n!}{1\times 3\times 5\times\cdots\times(2n+1)}\cdot x^{2n+1}.$$

For $|x| < \sqrt{2}$, denoting by $(2n+1)!!$ the product $1\times 3\times 5\times\cdots\times(2n+1)$, we successively obtain

$$xf(x) = \sum_{n=0}^{+\infty}\frac{n!}{(2n+1)!!}\,x^{2n+2},$$
$$-2f'(x) = -2 - \sum_{n=1}^{+\infty}\frac{2(n!)}{(2n-1)!!}\,x^{2n},$$
$$x^2f'(x) = x^2 + \sum_{n=1}^{+\infty}\frac{n!}{(2n-1)!!}\,x^{2n+2}.$$

By addition, $(x^2-2)f'(x)+xf(x) = -2$ [since on the left, the coefficient of x^{2n-1} (for $n\geq 1$) is 0, the coefficient of x^{2n} is -2 if $n=0$, $1-2+1=0$ if $n=1$ and if $n\geq 2$, is

$$\frac{(n-1)!}{(2n-1)!!} - \frac{2(n!)}{(2n-1)!!} + \frac{(n-1)!}{(2n-3)!!} = \frac{(n-1)!}{(2n-1)!!}\,(1-2n+(2n-1)) = 0.]$$

The function $f:(-\sqrt{2},\sqrt{2})\to\mathbb{R}$ satisfies $f(0)=0$ and the differential equation $(x^2-2)y'+xy=-2$. This equation can be solved by a classical method, which leads to
$$f(x) = \frac{2\text{Arcsin}(x/\sqrt{2})}{\sqrt{2-x^2}}.$$
Thus, the required sum is $f(1) = 2\times\frac{\pi}{4} = \frac{\pi}{2}$.

2. *Prove the following:*

$$\sum_{k=1}^{\infty} \frac{H_k}{k+1}\left(\frac{\pi^2}{6} - H_{k+1,2}\right) = \frac{\pi^4}{90},$$

where $H_k = \sum_{i=1}^{k} \frac{1}{i}$ *is the kth harmonic number and* $H_{k,2} = \sum_{i=1}^{k} \frac{1}{i^2}$ *is the kth generalized harmonic number.*

Recall that $\frac{\pi^4}{90} = \sum\limits_{n=1}^{\infty} \frac{1}{n^4}$. The result is obtained through the following calculation in which the interchanges $\sum / \int$ are possible, the involved functions being positive.

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{H_k}{k+1}\left(\frac{\pi^2}{6} - H_{k+1,2}\right)\\
&= \sum_{k=1}^{\infty} \frac{H_k}{k+1} \sum_{n=k+1}^{\infty} \frac{1}{(n+1)^2} = \sum_{k=1}^{\infty} \frac{H_k}{k+1} \sum_{n=k+1}^{\infty} \int_0^1 (-x^n \ln(x))\,dx\\
&= \sum_{k=1}^{\infty} \frac{H_k}{k+1} \int_0^1 \left(\sum_{n=k+1}^{\infty} (-x^n \ln(x))\right) dx = \sum_{k=1}^{\infty} \frac{H_k}{k+1} \int_0^1 (-\ln(x)) \frac{x^{k+1}}{1-x}\,dx\\
&= \int_0^1 \frac{-\ln(x)}{1-x} \cdot \left(\sum_{k=1}^{\infty} \frac{H_k x^{k+1}}{k+1}\right) dx = \frac{1}{2} \int_0^1 \frac{(-\ln(x))(\ln(1-x))^2}{1-x}\,dx\\
&= \frac{1}{2} \int_0^1 \frac{(-\ln(1-x))(\ln(x))^2}{x}\,dx = \frac{1}{2} \int_0^1 (\ln(x))^2 \left(\sum_{n=1}^{\infty} \frac{x^{n-1}}{n}\right) dx\\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} (\ln(x))^2\,dx = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.
\end{aligned}$$

Note that we have used some results met on p. 155 of Focus On... No 55.

From Focus On... No. 56

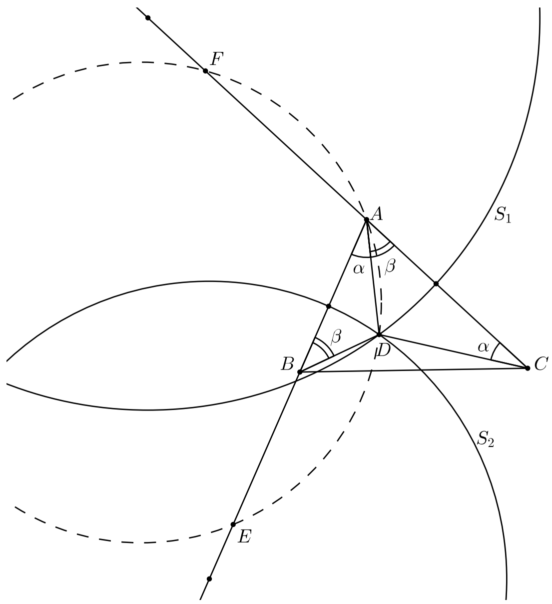
1. *Let* ABC *be a triangle for which there exists a point* D *in its interior such that* $\angle DAB = \angle DCA$ *and* $\angle DBA = \angle DAC$. *Let* E *and* F *be points on the lines* AB *and* CA, *respectively, such that* $AB = BE$ *and* $CA = AF$. *Prove that the points* A, E, D, *and* F *are concyclic. (First construct* D *satisfying the constraints.)*

Let $\alpha = \angle DAB = \angle DCA$, $\beta = \angle DBA = \angle DAC$ and $\theta = 180^\circ - (\alpha + \beta)$ so that $\angle ADB = \angle ADC = \theta$. From the law of sines,

$$\frac{DB}{\sin\alpha} = \frac{DA}{\sin\beta} = \frac{c}{\sin\theta} \quad \text{and} \quad \frac{DA}{\sin\alpha} = \frac{DC}{\sin\beta} = \frac{b}{\sin\theta}$$

where as usual, $c = AB$ and $b = CA$. It follows that $\frac{DA}{DC} = \frac{DB}{DA} = \frac{c}{b}$. Therefore, D is a point of the loci $S_1 = \{M : \frac{MA}{MC} = \frac{c}{b}\}$ and $S_2 = \{M : \frac{MB}{MA} = \frac{c}{b}\}$. If $b = c$, S_1

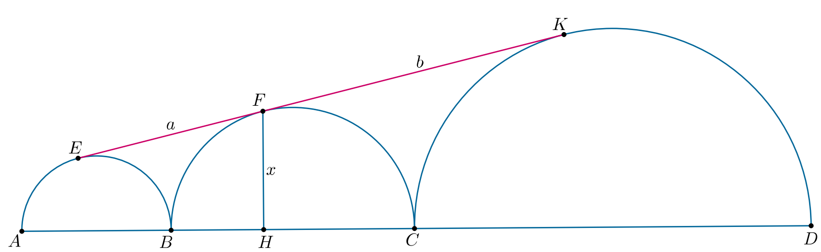
and S_2 are the perpendicular bisectors of AC and AB, respectively, and D must be the circumcenter of ΔABC (and the triangle ABC must be acute-angled for D to be interior to the triangle). Otherwise S_1 and S_2 are circles centered on AC and AB. S_1 for example, passes through the points of the line AC dividing the segment AC in the ratio $\frac{c}{b}$. Drawing S_1 and S_2 leads to D.



Moreover, we are prompted to consider the spiral similarity σ with center D which transforms C into A [its ratio is $\frac{c}{b}$ and its angle is θ oriented from DC towards DA].

We have $\sigma(C) = A$, $\sigma(A) = B$ from which it readily follows that $\sigma(F) = E$ (because $F' = \sigma(F)$ is on the ray $[AB)$ and $AF' = \frac{c}{b} \cdot CF = \frac{c}{b} \cdot 2b = 2c$, hence $F' = E$). As a result, $\angle EDF = \angle EAF \ (= \theta)$ and so A, E, D, F are concyclic.

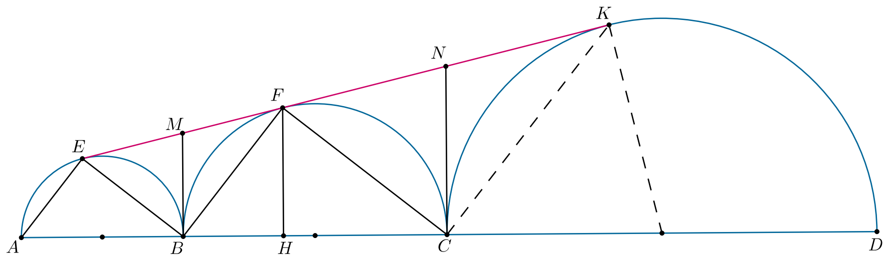
2. *Consider the semicircles in the configuration below:*



Prove that $\dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b}$.

(First construct the figure and then deduce a solution.)

We first draw the semi-circles with diameters AB and BC and the common tangent EF; to complete the figure, we use properties obtained in Focus On No 56: since the triangle FCK is right-angled at C, K is the point of intersection of the line EF and the perpendicular to CF through C. Then the center of the third semi-circle is the point of the line AC on the perpendicular to EF through K.



Let $AB = 2r_1, BC = 2r_2, CD = 2r_3$. As seen in the column, we have $a^2 = 4r_1r_2$ and $b^2 = 4r_2r_3$. Also AE is parallel to BF since both are perpendicular to BE. Similarly, CK is parallel to BF and it follows that

$$\frac{BC}{BA} = \frac{FK}{FE} = \frac{b}{a},$$

hence $\frac{r_2}{r_1} = \frac{b}{a} = \sqrt{\frac{r_3}{r_1}}$, that is, $r_2 = \sqrt{r_1 r_3}$ and therefore

$$ab = 2\sqrt{r_1 r_2} \cdot 2\sqrt{r_2 r_3} = 4r_2\sqrt{r_1 r_3} = 4r_2^2.$$

Now, M and N being the midpoints of EF and FK, we have

$$\frac{HC}{HB} = \frac{FN}{FM} = \frac{b}{a}$$

and $HB + HC = BC = 2r_2$, from which we deduce that

$$HB = \frac{2ar_2}{a+b} \quad \text{and} \quad HC = \frac{2br_2}{a+b}.$$

Finally, the altitude x of the right-angled triangle BFC satisfies

$$x^2 = HB \cdot HC = \frac{4abr_2^2}{(a+b)^2} = \frac{(ab)(ab)}{(a+b)^2}$$

and therefore $x = \dfrac{ab}{a+b}$, as desired.

PROBLEMS

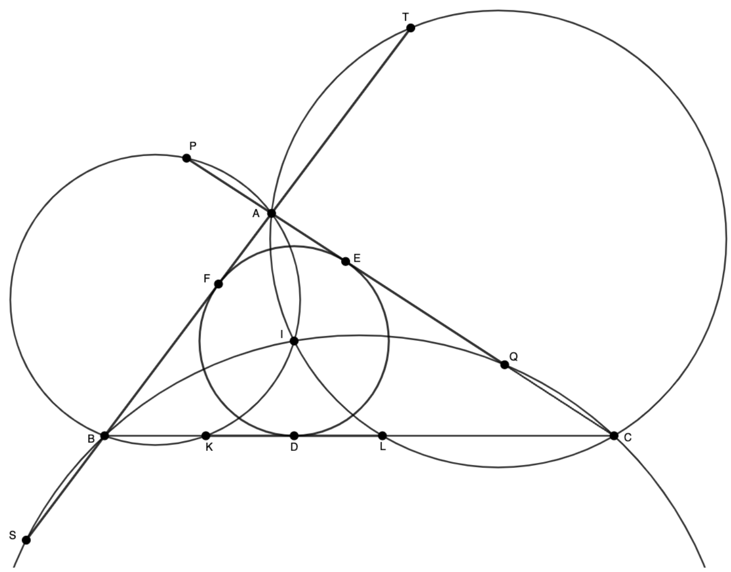
Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **October 30, 2023**.

4861. *Proposed by Pericles Papadopoulos.*

Let I be the incenter of a triangle ABC and let D, E and F be the points of contact of the incircle of the triangle with the side BC, AC and AB respectively. The circle AIB meets the sides BC and AC at points K and P respectively; the circle AIC meets the sides BC and AB at points L and T respectively; the circle BIC meets the sides AC and AB at points Q and S, respectively. Prove the following:

- $KL + PQ + ST = AB + BC + AC$
- Points D, E and F are the midpoints of KL, PQ and ST respectively.



4862. *Proposed by Michel Bataille.*

Let m be a nonnegative integer. Find

$$\lim_{n\to\infty} \frac{1}{2^n n^m} \sum_{k=0}^{n} \binom{m+k}{k} \binom{m+n+1}{n-k}.$$

4863. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

In a parallelogram $ABCD$, let E be the point where the diagonal BD is tangent to the incircle of ΔABD. If r_1 and r_2 are the inradii of the triangles DEC and BEC, prove that $\frac{r_1}{r_2} = \frac{DE}{EB}$.

4864. *Proposed by Goran Conar.*

Let a, b, c be side-lengths of an arbitrary three-dimensional box, and D the length of its main diagonal. Prove

$$\sqrt{1+a} + \sqrt{1+b} + \sqrt{1+c} \geq \frac{(a+b+c)^2}{D^2} \cdot \sqrt{1 + \frac{D^2}{a+b+c}}.$$

When does the equality occur?

4865. *Proposed by George Apostolopoulos.*

Let ABC be an acute triangle with inradius r and circumradius R. Prove that

$$\frac{(\sec A)^{\cos A} + (\sec B)^{\cos B} + (\sec C)^{\cos C}}{\sec A + \sec B + \sec C} < \frac{5R - r}{12r}.$$

4866. *Proposed by Ivan Hadinata.*

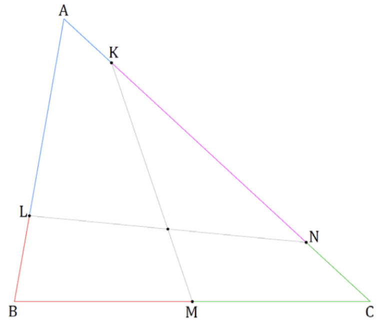
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equation

$$f(xy + f(f(y))) = xf(y) + y$$

holds for all real numbers x and y.

4867. *Proposed by Thanos Kalogerakis.*

Consider a triangle ABC with $|AC| > |BC| > |AB|$ and let M the midpoint of BC. Let K, L and N be points on the sides of ABC (see the figure) such that the points K, L, M, N divide the perimeter of ABC into 4 equal parts. Prove that KM bisects LN.



4868. *Proposed by Michel Bataille.*

Let $k \in [-1, 1]$ and let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 1$. Find the minimal and maximal values of $a^3 + b^3 + c^3 + kabc$.

4869. *Proposed by Leonard Giugiuc and Mohamed Amine Ben Ajiba.*

Let ABC be a non-obtuse triangle with area 1 and side-lengths a, b, c. Let n be a fixed non-negative real number. Find the minimum value of

$$\frac{2n}{a^2 + b^2 + c^2} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

4870*. *Proposed by Borui Wang.*

Define the series $\{a_n\}$ by the following recursion: $a_1 = 1$, $a_{n+1} = a_n + \dfrac{1}{q \cdot a_n}$ for $n > 0, q > 0$. Find the constant number $c(q)$ such that

$$\lim_{n\to\infty} (a_n - \sqrt{c(q) \cdot n}) = 0.$$

. . . .

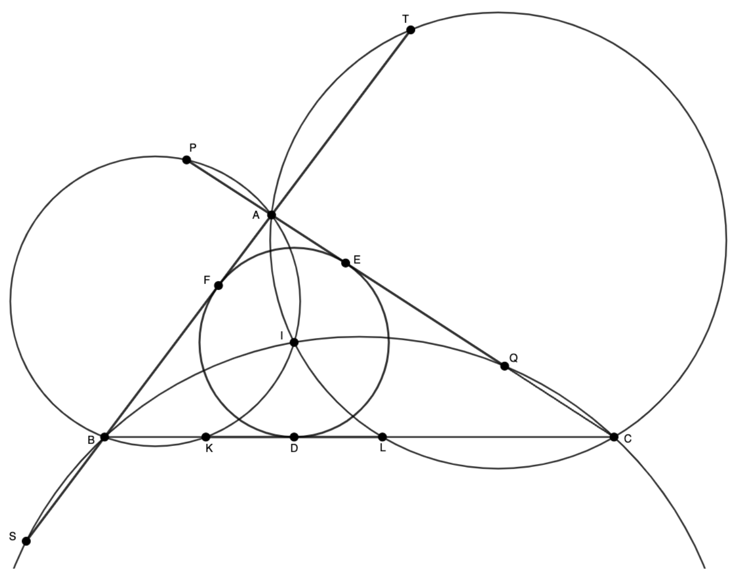
Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 octobre 2023**.

4861. *Soumis par Pericles Papadopoulos.*

Soit I le centre du cercle inscrit au triangle ABC et soit D, E et F les points de contact du cercle inscrit au triangle avec les côtés BC, AC et AB, respectivement. Le cercle AIB rencontre les côtés BC et AC aux points K et P, respectivement; le cercle AIC rencontre les côtés BC et AB aux points L et T, respectivement; le cercle BIC rencontre les côtés AC et AB aux points Q et S, respectivement. Montrez ce qui suit :

- $KL + PQ + ST = AB + BC + AC$
- Les points D, E et F sont les points milieux de KL, PQ et ST respectivement.



4862. *Soumis par Michel Bataille.*

Soit m un entier non négatif. Trouvez

$$\lim_{n\to\infty} \frac{1}{2^n n^m} \sum_{k=0}^{n} \binom{m+k}{k} \binom{m+n+1}{n-k}.$$

4863. *Soumis par Mihaela Berindeanu, modifié par le comité de rédaction.*

Dans un parallélogramme $ABCD$, soit E le point où la diagonale BD est tangente au cercle inscrit à ΔABD. Si r_1 et r_2 sont les rayons des cercles inscrits aux triangles DEC et BEC respectivement, montrez que $\frac{r_1}{r_2} = \frac{DE}{EB}$.

4864. *Soumis par Goran Conar.*

Soient a, b et c les longueurs des côtés d'un rectangle arbitraire à trois dimensions et soit D la longueur de sa diagonale principale. Montrez que

$$\sqrt{1+a} + \sqrt{1+b} + \sqrt{1+c} \geq \frac{(a+b+c)^2}{D^2} \cdot \sqrt{1 + \frac{D^2}{a+b+c}}\,.$$

Quand a-t-on égalité?

4865. *Soumis par George Apostolopoulos.*

Soit ABC un triangle acutangle. Soient r et R les rayons des cercles inscrit et circonscrit à ce triangle, respectivement. Montrez que

$$\frac{(\sec A)^{\cos A} + (\sec B)^{\cos B} + (\sec C)^{\cos C}}{\sec A + \sec B + \sec C} < \frac{5R - r}{12r}.$$

4866. *Soumis par Ivan Hadinata.*

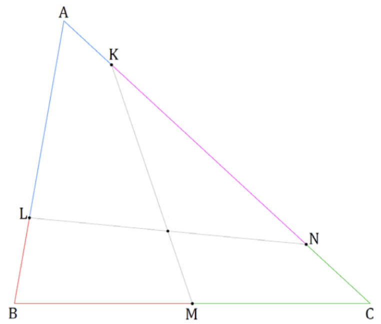
Trouvez toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ pour lesquelles l'équation

$$f(xy + f(f(y))) = xf(y) + y$$

est vérifiée pour tous les nombres réels x et y.

4867. *Soumis par Thanos Kalogerakis.*

Considérons un triangle ABC avec $|AC| > |BC| > |AB|$ et soit M le point milieu de BC. Si K, L et N sont des points sur les côtés de ABC (voir la figure) tels que K, L, M, N divisent le périmètre de ABC en 4 parties égales, montrez que KM bissecte LN.



4868. *Soumis par Michel Bataille.*

Soit $k \in [-1, 1]$ et soient a, b et c des nombres réels tels que $a^2 + b^2 + c^2 = 1$. Trouvez les valeurs minimales et maximales de $a^3 + b^3 + c^3 + kabc$.

4869. *Soumis par Leonard Giugiuc et Mohamed Amine Ben Ajiba.*

Soit ABC un triangle non obtus d'aire 1 avec les longueurs des côtés a, b et c. Soit n un nombre réel non négatif fixé. Quelle est la valeur minimale de

$$\frac{2n}{a^2 + b^2 + c^2} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}?$$

4870*. *Soumis par Borui Wang.*

Définissons la suite $\{a_n\}$ par récurrence comme suit : $a_1 = 1$, $a_{n+1} = a_n + \dfrac{1}{q \cdot a_n}$ pour $n > 0$ et $q > 0$. Trouvez le nombre constant $c(q)$ tel que

$$\lim_{n\to\infty} (a_n - \sqrt{c(q) \cdot n}) = 0.$$

BONUS PROBLEMS

These problems appear as a bonus. Their solutions will **not** *be considered for publication.*

B126. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with the circumcircle Γ_1, the incircle Γ_2, the circumcenter O, the incenter I, $\Gamma_2 \cap AC = \{E\}$, $\Gamma_2 \cap BC = \{F\}$ and $\cos A + \cos B + \cos C = \sqrt{2}$. If X respectively Y are the middle points of the segments OE, respectively OF, show that $IYOX$ has a circumcircle.

B127. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with the circumcenter O and the orthocenter H. Let $D \in BC$ be the foot of the altitude from A, Γ_1 be $\triangle ADC$ circumcircle and Γ_2 be $\triangle AHB$ circumcircle. If $\Gamma_1 \cap \Gamma_2 = \{X\}$ and $CX \cap BH = \{Y\}$, show that Y is the midpoint of BH.

B128. *Proposed by Adnan Ali, Salem Malikic, Nermin Hodzic.*

Do there exist positive integers a, b, c that satisfy the following equation:

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 6abc.$$

B129. *Proposed by Nguyen Viet Hung.*

Consider a triangle ABC with side lengths a, b, c opposite of angles A, B, C, respectively. Let r and R denote its inradius and circumradius. Prove that

$$\frac{(b+c)^2}{bc}\cos A + \frac{(c+a)^2}{ca}\cos B + \frac{(a+b)^2}{ab}\cos C = 5 + \frac{2r}{R}.$$

B130. *Proposed by Nguyen Viet Hung.*

Given a triangle ABC with incenter I. The ray AI intersects the side BC and the circumcircle at A_1, A_2, respectively. Pairs of points B_1, B_2 and C_1, C_2 are defined similarly. Prove that

$$\frac{a^4}{(b+c)^2(A_1A_2)^2} + \frac{b^4}{(c+a)^2(B_1B_2)^2} + \frac{c^4}{(a+b)^2(C_1C_2)^2} \leq 9.$$

B131. *Proposed by Daniel Sitaru.*

Evaluate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)}.$$

B132. *Proposed by Mihaela Berindeanu.*

In the right triangle ABC, $\measuredangle BAC = 90^\circ$, $D \in BC$, $AD \perp BC$ and $E \in AD$. The projections of E on AB and AC are F and K, respectively. If E is the center of gravity of $\triangle KFD$, find the value $\dfrac{AE}{ED}$.

B133. *Proposed by Nguyen Viet Hung.*

Find all triples (x, y, z) of positive integers satisfying the equation

$$(2x + 3y - 1)(2^{x-1} + x^2 - x + y) = 2^z + 1.$$

B134. *Proposed by Aravind Mahadevan.*

Given that

$$\tan^2 A \tan^2 B + \tan^2 B \tan^2 C + \tan^2 A \tan^2 C + 2\tan^2 A \tan^2 B \tan^2 C = 1,$$

find the value of $\sin^2 A + \sin^2 B + \sin^2 C$.

B135. *Proposed by Daniel Sitaru.*

Let $A, B, C \in M_2(\mathbb{R})$ such that $\det A > 0$, $\det B > 0$, $\det C > 0$ and $\det(ABC) = 64$. Show that

$$\det(A + B + C) + \det(-A + B + C) + \det(A - B + C) + \det(A + B - C) \geq 48.$$

B136. *Proposed by George Apostolopoulos.*

Let ABC be an arbitrary triangle and let x, y, z be positive real numbers. Prove or disprove that

$$\frac{x}{y+z}\csc^2 A + \frac{y}{x+z}\csc^2 B + \frac{z}{x+y}\csc^2 C \geq 2.$$

B137. *Proposed by Nguyen Viet Hung.*

Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\sqrt{\frac{bc}{(a+b)(a+c)}} + \sqrt{\frac{ca}{(b+c)(b+a)}} + \sqrt{\frac{ab}{(c+a)(c+b)}} \geq \sqrt{1 + \frac{10abc}{(a+b)(b+c)(c+a)}}.$$

When does the equality happen?

B138. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers with $a+b+c \leq 3$. Prove that

$$\frac{9}{abc} - \left(\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b}\right) \geq 6.$$

B139. *Proposed by Nguyen Viet Hung.*

Prove that for any positive real numbers a, b, c

$$\frac{ab}{a^2+ab+b^2} + \frac{bc}{b^2+bc+c^2} + \frac{ca}{c^2+ca+a^2} + 1 \geq \frac{2(ab+bc+ca)}{a^2+b^2+c^2}.$$

B140. *Proposed by George Apostolopoulos.*

Let ABC be an acute triangle. Prove that

$$\sqrt{\cot\frac{A}{2} \cdot \cot\frac{B}{2} \cdot \cot\frac{C}{2}} \geq \sqrt{\cot A} + \sqrt{\cot B} + \sqrt{\cot C}.$$

B141. *Proposed by George Apostolopoulos.*

Show that for any triangle ABC we have

$$\frac{\sqrt{2-\sin^2 A}}{1-\cos A} + \frac{\sqrt{2-\sin^2 B}}{1-\cos B} + \frac{\sqrt{2-\sin^2 C}}{1-\cos C} \geq 3\sqrt{5}.$$

B142. *Proposed by Goran Conar.*

Let $x_1, x_2, \ldots, x_n > 0$ such that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \frac{1}{1+x_3+x_3x_4} + \cdots + \frac{1}{1+x_n+x_nx_1} \geq \frac{n^2}{n+\sqrt{n}+1}.$$

In which cases does equality hold?

B143. *Proposed by George Apostopoulos.*

In an acute triangle ABC, point H is the intersection of the altitudes AA_1, BB_1 and CC_1. Prove that

$$\frac{AH}{HA_1} + \frac{BH}{HB_1} + \frac{CH}{HC_1} \geq \sec A + \sec B + \sec C.$$

B144. *Proposed by Goran Conar.*

Find the lowest constant $c > 0$ such that for all $a_1, a_2, \ldots, a_n \geq 1$ it is satisfied

$$(a_1 + a_2 + \cdots + a_n)^2 \leq c \left(\sum_{i=1}^{n} \sqrt{1 + a_i} \right) \left(\sum_{i=1}^{n} a_i^2 \sqrt{1 + a_i} \right).$$

B145. *Proposed by Florică Anastase.*

If $0 < a_1 \leq a_2 \leq ... \leq a_n,\ n \in \mathbb{N},\ n > 0$ then:

$$n + \frac{a_1 \cdot a_n}{n} \left(\sum_{k=1}^{n} \frac{1}{a_k} \right)^2 \leq (a_1 + a_n) \left(\sum \frac{1}{a_k} \right).$$

B146. *Proposed by Goran Conar.*

Let $u, v, w > 0$ be real numbers such that $uvw = 2 + u + v + w$. Prove that the following inequality holds:

$$4 \leq 1 + \frac{\sqrt{1 + u + v + uv}}{1 + w} + \frac{\sqrt{1 + u + w + uw}}{1 + v} + \frac{\sqrt{1 + v + w + vw}}{1 + u} \leq \frac{uvw}{2}.$$

When does the equality occur?

B147. *Proposed by Goran Conar.*

Let $a, b, c > 0$ be real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove

$$\frac{1}{b+c}\left(1 + \frac{c}{a}\right)\left(1 + \frac{b}{a}\right) + \frac{1}{c+a}\left(1 + \frac{a}{b}\right)\left(1 + \frac{c}{b}\right) + \frac{1}{a+b}\left(1 + \frac{b}{c}\right)\left(1 + \frac{a}{c}\right) \geq \frac{36}{abc - a - b - c}.$$

When does equality occur?

B148. *Proposed by Nguyen Viet Hung.*

Prove that for any positive real numbers a, b, c

$$\left(\frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^3 \geq \frac{8(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}{(a+b)^2(b+c)^2(c+a)^2}.$$

B149. *Proposed by Florentin Visescu.*

Let a, b, c be the side-lengths of a triangle such that $a + b + c = 1$. Show that

$$\frac{\pi}{2} \leq \arcsin \frac{a}{b+c} + \arcsin \frac{b}{c+a} + \arcsin \frac{c}{a+b} < \pi.$$

B150. *Proposed by George Apostolopoulos.*

Let R be the circumradius of triangle ABC. Let D, E, F be chosen on the sides BC, CA, AB respectively so that AD, BE, CF bisect the corresponding angles of ABC. Prove that

$$(AB + BC + CA) \cdot \left(\frac{DE^2}{AB} + \frac{EF^2}{BC} + \frac{FD^2}{CA} \right) \leq \frac{27}{4} R^2.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2023: 49(2), p. 101–104.

4811. *Proposed by Nguyen Viet Hung.*

Find all positive integers n such that $\sqrt{n^3+1}+\sqrt{n+2}$ is a positive integer.

We received 20 submissions out of which 18 were correct and complete. We present the solution by C.R. Pranesachar, lightly edited.

Let $Q(n)=\sqrt{n^3+1}+\sqrt{n+2}$. We have

$$Q(n)^2 = n^3+1+2\sqrt{(n^3+1)(n+2)}+n+2.$$

Therefore if $Q(n)$ is a positive integer then $(n^3+1)(n+2) = n^4+2n^3+n+2$ must be a perfect square. Note that

$$(n^2+n-1)^2 = n^4+2n^3-n^2-2n+1 = (n^3+1)(n+2)-n^2-3n-1$$

and

$$(n^2+n)^2 = n^4+2n^3+n^2 = (n^3+1)(n+2)+n^2-n-2$$

Thus for $n>2$,

$$(n^2+n-1)^2 < (n^3+1)(n+2) < (n^2+n)^2.$$

Since $(n^3+1)(n+2)$ lies strictly between two consecutive perfect squares it cannot be a square itself. Finally $Q(1)=\sqrt{2}+\sqrt{3}\notin \mathbb{N}$ and $Q(2)=5$. Therefore $n=2$ is the only value for which $Q(n)$ is a positive integer.

4812. *Proposed by Michel Bataille.*

Let $ABCD$ be a tetrahedron. Prove that $a=BC^2+DA^2$, $b=CA^2+DB^2$, $c=AB^2+DC^2$ are the sides of a triangle. For which tetrahedra is this triangle equilateral?

All of the 12 submissions we received were correct, and we feature two of the various approaches.

Solution 1, by Oliver Geupel.

As usual, $\vec{A},\vec{B},\vec{C},\vec{D}$ denote the vectors from an arbitrary but fixed origin to the points A,B,C,D, respectively. Since the midpoints of BC and DA are distinct, it follows that $\vec{B}+\vec{C}\neq\vec{D}+\vec{A}$. Hence,

$$\begin{aligned} a &< \left(\vec{B}-\vec{C}\right)^2+\left(\vec{D}-\vec{A}\right)^2+\left((\vec{A}+\vec{D})-(\vec{B}+\vec{C})\right)^2 \\ &= \left(\vec{C}-\vec{A}\right)^2+\left(\vec{D}-\vec{B}\right)^2+\left(\vec{A}-\vec{B}\right)^2+\left(\vec{D}-\vec{C}\right)^2 = b+c. \end{aligned}$$

Similarly $b < c + a$ and $c < a + b$. By the converse of the triangle inequality it follows that a, b, and c are the side lengths of a nondegenerate triangle.

It turns out that a triangle with side lengths a, b, c is equilateral if and only if the four altitudes of the corresponding tetrahedron are concurrent. This was the subject of shortlisted problem 72-5 proposed by the German Democratic Republic for the 14th IMO, 1972. The problem called for a proof that the altitudes of $ABCD$ intersect in a point if and only if $BC^2 + DA^2 = CA^2 + DB^2 = AB^2 + DC^2$. A proof of this assertion is contained in D. Djukić, V. Janković, I Matić, N. Petrović, *The IMO Compendium. A Collection of Problems Suggested for The International Mathematical Olympiads: 1959-2009*, 2nd ed., 2011, Springer, page 397. Such a tetrahedron is called *orthocentric*.

Solution 2 is a composite of similar solutions by Vivek Mehra, Daniel Văcaru, and the Proposer.

Let M, N be the midpoints of BC, DA respectively. Since MN, AM, and DM are medians in the triangles AMD, BAC, and BDC, respectively, we have

$$\begin{aligned} 4MN^2 &= 2(AM^2 + DM^2) - DA^2, \\ 4AM^2 &= 2(AB^2 + AC^2) - BC^2, \\ 4DM^2 &= 2(DB^2 + DC^2) - BC^2, \end{aligned}$$

from which we readily deduce that

$$8MN^2 = 2 \cdot (AB^2 + DC^2 + AC^2 + DB^2 - BC^2 - DA^2).$$

Because M and N must be distinct points (assuming that the given tetrahedron is nondegenerate), it follows that $0 < 4MN^2 = b + c - a$. Thus, $a < b + c$. Similarly, we obtain $b < c + a,\ c < a + b$ and conclude that a, b, c are the sides of a triangle.

We now investigate the properties of a tetrahedron for which the quantities a, b, c are equal.

$$\begin{aligned} a = b \quad &\text{iff} \quad (\vec{B} - \vec{C})^2 + (\vec{D} - \vec{A})^2 = (\vec{C} - \vec{A})^2 + (\vec{D} - \vec{B})^2 \\ &\text{iff} \quad \vec{B} \cdot \vec{C} + \vec{D} \cdot \vec{A} = \vec{C} \cdot \vec{A} + \vec{D} \cdot \vec{B} \\ &\text{iff} \quad \vec{C} \cdot (\vec{B} - \vec{A}) - \vec{D} \cdot (\vec{B} - \vec{A}) = 0 \\ &\text{iff} \quad (\vec{C} - \vec{D}) \cdot (\vec{B} - \vec{A}) = 0. \end{aligned}$$

In words, $a = b$ if and only if the lines AB and CD point in perpendicular directions. A similar calculation for $a = c$ and $b = c$ leads to the conclusion that the triangle with sides a, b, and c is equilateral if and only if the corresponding tetrahedron has its three pairs of opposite sides pointing in perpendicular directions (in which case the tetrahedron is orthocentric).

Editor's comments. A tetrahedron $ABCD$ is defined to be *orthocentric* if the four altitudes concur in a point. References can be found in the *Wikipedia* article

"Orthocentric tetrahedron." The following six characterizations, mostly from the 19th century, are equivalent:

1. $ABCD$ is orthocentric.
2. Opposite edges are perpendicular.
3. $BC^2 + DA^2 = CA^2 + DB^2 = AB^2 + DC^2$.
4. The line segments joining the midpoints of opposite sides have equal lengths.
5. The feet of the altitudes of the tetrahedron are the orthocenters of the triangular faces.
6. For each of the quadrilaterals $ABCD, ABDC$, and $ACBD$, the midpoints of its sides are the vertices of a rectangle.

4813. *Proposed by Mihai Prunescu.*

Find all plane triangles ABC such that every side is equal with the opposed angle: $BC = \angle A$, $AC = \angle B$ and $AB = \angle C$.

We received 15 submissions: one was completely incorrect, and the others were somewhat flawed, as explained in the editor's comments. We feature the solution by the Eagle Problem Solvers of Georgia Southern University, which is typical of the submissions.

We claim that the only possible triangle is the equilateral triangle with sides of length $\frac{\pi}{3}$.

Suppose that ABC satisfies the given condition. Then by the Law of Sines,

$$\frac{\sin BC}{BC} = \frac{\sin AC}{AC} = \frac{\sin AB}{AB}. \tag{1}$$

Consider the function $f(x) = x\cos x - \sin x$. Since $f'(x) = -x\sin x < 0$ on the interval $(0, \pi)$, then f is decreasing on $(0, \pi)$. Since $f(0) = 0$, then $f(x) < 0$ on $(0, \pi)$. Then

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{f(x)}{x^2} < 0$$

on $(0, \pi)$, so $\dfrac{\sin x}{x}$ is decreasing, and hence injective, on $(0, \pi)$. The equations in (1) therefore imply that $BC = AC = AB$, whence ABC is equilateral, and

$$\angle A = \angle B = \angle C = \frac{\pi}{3} = BC = AC = AB. \tag{2}$$

Editor's comments. The proposer called for "all" triangles that satisfy the given conditions; note that the featured solution fails to include, among others, the equilateral triangle with sides of length 60. The source of the trouble lies in the statement of the problem: the problem is only meaningful if we agree on units. The display in (2) is not dimensionally consistent — it sets dimensionless radians equal to dimensioned measures of length. The problem can be salvaged by introducing an arbitrary but fixed unit of length L and agreeing to measure angles by radians. Then one can replace lengths everywhere by lengths divided by L, in which case (2) becomes

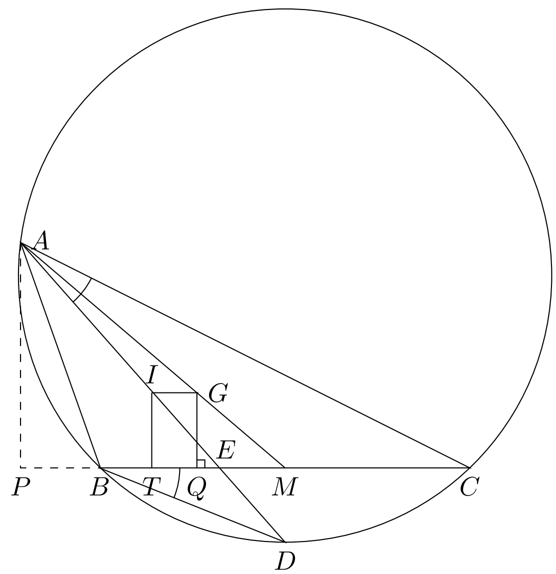
$$\angle A = \angle B = \angle C = \frac{\pi}{3} = \frac{BC}{L} = \frac{AC}{L} = \frac{AB}{L}.$$

4814. *Proposed by Mihaela Berindeanu.*

In triangle ABC, let G be the centroid and I be the incenter. Suppose that GI is parallel to BC, AI cuts BC in E and the circumcircle in D. Show that $BD = 2ED$.

We received 17 correct solutions. We present 7 solutions with a variety of approaches.

Solution 1, by Miguel Amengual Covas.



Let $h = AP$ be the height of triangle ABC from A. Then

$$\frac{h}{r} = \frac{AP}{IT} = \frac{AE}{IE} = \frac{AM}{GM} = 3,$$

so that

$$3r = h = \frac{2[ABC]}{a} = \frac{2rs}{a} \Rightarrow 2s = 3a \Rightarrow b + c = 2a.$$

Since $BE : EC = AB : AC$, it follows that the length of EC is $ab(b+c)^{-1} = b/2$. Since $\angle DBE = \angle DAC$ and $\angle BDE = \angle ECA$, triangles BDE and ACE are similar. Therefore

$$\frac{BD}{ED} = \frac{AC}{EC} = \frac{b}{b/2} = 2.$$

Solution 2, by Theo Koupelis.

Since triangles BDE and ACE are similar and since CI is an angle bisector in triangle ACE,

$$BD : DE = AC : EC = AI : IE = AG : GM = 2 : 1,$$

as desired.

Solution 3, by Ivan Hadinata.

We have that $\angle DIB = \frac{1}{2}(\angle BAC + \angle ABC)$ (exterior angle of triangle ABI) and so

$$\begin{aligned}\angle DBI &= \angle DBE + \angle EBI = \angle DBC + \angle CBI = \angle DAC + \angle CBI\\ &= \tfrac{1}{2}(\angle BAC + \angle ABC) = \angle DIB.\end{aligned}$$

Therefore $BD = ID$.

Since $AI : EI = AG : MG = 2 : 1$, $AI = 2EI$. Since triangles BED and ABD are similar, $DE : BD = BD : AD$, whereupon

$$ID^2 = BD^2 = DE \cdot AD = (ID - EI)(ID + AI).$$

Therefore

$$ID \cdot EI = ID(AI - EI) = AI \cdot EI = 2EI^2,$$

so that

$$BD = ID = AD - AI = 2ID - 2EI = 2ED.$$

Solution 4, by Cristbal Sanchez-Rubio.

Using the fact that BI and CI are angle bisectors of the respective triangles ABE and ACE, we find that

$$2 = \frac{AG}{GM} = \frac{AI}{EI} = \frac{AB}{BE} = \frac{AC}{CE} = \frac{AB + AC}{BE + CE} = \frac{b+c}{a},$$

whence $b + c = 2a$. Therefore $BD = ac(b+c)^{-1} = c/2$. Since triangles ABD and BED are similar,

$$\frac{BE}{ED} = \frac{AB}{BD} = \frac{c}{c/2} = 2.$$

Solution 5, by C.R. Pranesachar.

As before, $2a = b + c$.

$$\angle BED = \angle AEC = 180^\circ - C - \frac{A}{2} = 90^\circ + \frac{B-C}{2}$$

and $\angle DBE = \angle DAB = \frac{A}{2}$. By the Law of Sines,

$$\begin{aligned}\frac{BD}{ED} &= \frac{\sin(90^\circ - (B-C)/2)}{\sin A/2} = \frac{2\cos A/2 \cos(B-C)/2}{2\cos A/2 \sin A/2} \\ &= \frac{\sin B + \sin C}{\sin A} = \frac{b+c}{a} = 2.\end{aligned}$$

Solution 6, by UCLan Cyprus Problem Solving Group.

We have that $AE \cdot ED = BE \cdot EC = bc/4$. Since triangles ACE and ADB are similar, $b : AD = AE : c$, whence $AE(AE + ED) = bc$ and so $AE^2 + \frac{1}{4}bc = bc$. Therefore $AE^2 = \frac{3}{4}bc = 3AE \cdot ED$. Thus

$$ED = \frac{1}{3}AE = IE = \frac{1}{2}ID = \frac{1}{2}BD,$$

as desired.

Solution 7, by Michel Bataille.

We use familiar notations for the elements of triangle ABC and barycentric coordinates relative to (A, B, C). Since $3G = A + B + C$ and $2sI = aA + bB + cC$, we deduce that

$$6s\overrightarrow{IG} = 6s(G-I) = (2s-3a)A+(2s-3b)B+(2s-3c)C = (2s-3a)\overrightarrow{BA}+(2s-3c)\overrightarrow{BC}.$$

Since GI is parallel to BC, we have $2s - 3a = 0$ or $b + c = 2a$.

The equation of the circumcircle of triangle ABC is $a^2yz + b^2zx + c^2xy = 0$ and of the line AI is $cy - bz = 0$. Their point of intersection D is $(-a : 2b : 2c)$. Since $E = (0 : b : c)$,

$$3aD = (2b + 2c - a)D = -aA + 2(b + c)E = -aA + 4aE,$$

so that $3D = 4E - A$. Also, $3aI = 2sI = aA + 2aE$, whence $3I = A + 2E$. Therefore $D + I = 2E$ so that E is the midpoint of ID. Since $DB = DI = DC$, it follows that $BD = 2DE$.

Editor's Comment. Solution 2, in particular, proves that GI is parallel to BC if and only if $BD = 2ED$. This joins the many characterizations of triangles for which $b+c = 2a$ that were discussed in *Recurring Crux Configurations 2, triangles for which $b + c = 2a$* by J. Chris Fisher, *Crux Mathematicorum* 37:6 (October, 2011), pages 385 – 387.

4815. *Proposed by Aravind Mahadevan.*

In triangle ABC, let a, b, c denote the lengths of the sides BC, CA and AB, respectively. If $\tan A$, $\tan B$ and $\tan C$ are in harmonic progression, prove that a^2, b^2 and c^2 are in arithmetic progression. Does the converse hold?

We received 15 correct solutions from 14 respondents. Almost all solutions were as follows.

An exception to the converse is the right triangle with sides $(a, b, c) = (1, \sqrt{2}, \sqrt{3})$ on the technical grounds that $\tan C$ is undefined. We can avoid this complication by expressing the angle condition in terms of cotangents: $\cot A + \cot C = 2\cot B$.

Using the fact that $[ABC] = \frac{1}{2}bc\sin A$, *etc.* and $a^2 = b^2 + c^2 - 2bc\cos A$, *etc.*, we find that

$$\begin{aligned}\cot A + \cot C - 2\cot B &= \frac{b^2+c^2-a^2}{4[ABC]} + \frac{a^2+b^2-c^2}{4[ABC]} - 2\left(\frac{a^2+c^2-b^2}{4[ABC]}\right)\\ &= 2\left(\frac{2b^2-(a^2+c^2)}{4[ABC]}\right).\end{aligned}$$

Therefore $\cot A + \cot C = 2\cot B$ if and only if $a^2 + c^2 = 2b^2$.

Comment from the editor. Vivek Mehra pointed out that, when $a^2 + c^2 = 2b^2$, then the vertex B, respective midpoints E and D of AB and BC, and the centroid are vertices of a cyclic quadrilateral. This can be esablished from the equality

$$3a^2 + c^2 = 2(a^2 + b^2) = 4m^2 + c^2,$$

the latter a consequence of Apollonius' theorem for the length m of the median CE. This leads to the power result $CG\cdot CE = CD\cdot CB$. Our problem and Mehras observation describe two of the many interesting properties of triangles for which $2b^2 = a^2 + c^2$, the subject of *Recurring Crux Configurations 1* in this journal, volume 37:5, October 2011, pages 304 – 307.

4816. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Let $a, b, k \geq 0$. Calculate

$$\lim_{n\to\infty}\int_0^1 x^k\sqrt{\frac{a}{x} + bn^2x^{2n}}\,\mathrm{d}x.$$

We received 10 submissions, 7 of which are correct and complete. We present here the solution by Raymond Mortini and Rudolf Rupp.

We show that for $a, b, k \geq 0$ (k not necessarily an integer)

$$I_n := \int_0^1 x^k\sqrt{\frac{a}{x} + bn^2x^{2n}}\,dx \overset{n\to\infty}{\longrightarrow} \sqrt{b} + \frac{\sqrt{a}}{k+1/2}.$$

Let

$$f_n(x) = x^{k-1/2}\sqrt{a + bn^2x^{2n+1}}.$$

If $a = 0$, then

$$I_n = \int_0^1 \sqrt{b}nx^{n+k}dx = \frac{n\sqrt{b}}{n+k+1} \to \sqrt{b}.$$

For $a > 0$, let

$$d_n(x) := x^{k-1/2}\left(\sqrt{a + bn^2x^{2n+1}} - \sqrt{bn^2x^{2n+1}}\right).$$

Then

$$0 \le d_n(x) = x^{k-1/2}\,\frac{a}{\sqrt{a + bn^2x^{2n+1}} + \sqrt{b}nx^{n+1/2}} \le \frac{a}{\sqrt{a}}x^{k-1/2}.$$

Hence d_n is dominated by an $L^1[0,1]$ function and so, by using that $nx^n \to 0$ for $0 < x < 1$,

$$\lim_n \int_0^1 d_n(x)dx = \int_0^1 \lim_n d_n(x)dx = \int_0^1 \sqrt{a}x^{k-1/2} = \frac{\sqrt{a}}{k+1/2}.$$

Consequently,

$$\begin{aligned}\int_0^1 f_n(x)dx &= \int_0^1 d_n(x)dx + \sqrt{b}\int_0^1 nx^{k-1/2}x^{n+1/2}dx\\ &= \int_0^1 d_n(x)dx + \sqrt{b}\,\frac{n}{k+n+1}\\ &\underset{n\to\infty}{\longrightarrow} \frac{\sqrt{a}}{k+1/2} + \sqrt{b}.\end{aligned}$$

4817. *Proposed by Goran Conar.*

Let $a, b, c > 0$ be real numbers such that $abc = 1$. Prove that the following inequality holds

$$\frac{a^7 + a^3 + bc}{a + bc + 1} + \frac{b^7 + b^3 + ca}{b + ca + 1} + \frac{c^7 + c^3 + ab}{c + ab + 1} \ge 3\,.$$

When does equality occur?

We received 29 submissions, 28 of which were correct and complete. We present the solution by Ivan Hadinata, slightly altered by the editor.

Since $abc = 1$,

$$\sum_{cyc}\frac{a^7 + a^3 + bc}{a + bc + 1} = \sum_{cyc}\frac{a^8 + a^4 + abc}{a^2 + abc + a} = \sum_{cyc}\frac{a^8 + a^4 + 1}{a^2 + a + 1}.$$

By Hölder's inequality, for $x > 0$

$$(x^8 + x^4 + 1)(1 + 1 + 1)^3 \geq (x^2 + x + 1)^4$$

and so,

$$\frac{x^8 + x^4 + 1}{x^2 + x + 1} \geq \left(\frac{x^2 + x + 1}{3}\right)^3 \geq x^2 \cdot x \cdot 1 = x^3,$$

by the AM-GM inequality. Replacing x by a, b, c in succession and summing terms yields:

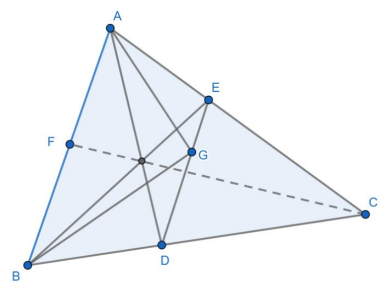
$$\sum_{cyc} \frac{a^8 + a^4 + 1}{a^2 + a + 1} \geq \sum_{cyc} a^3 \geq 3abc = 3.$$

Equality is achieved if and only if $a = b = c = 1$.

4818. *Proposed by Yagub Aliyev.*

In triangle ABC, let $E \in AC, D \in BC, F \in AB$ such that AD, BE, CF are concurrent. Let $G \in ED$. Prove that $\left(\frac{AF}{FB}\right)^2 = \frac{DG}{GE}$ if and only if

$$\frac{1}{[ADE]^2} + \frac{1}{[BDE]^2} = \frac{1}{[AEG]^2 + [BDG]^2}$$



We received 9 correct solutions. The following is the solution by Ivan Hadinata.

Clearly by Ceva's theorem we have $AE \cdot BF \cdot CD = AF \cdot BD \cdot CE$.
Note that

$$[ADE] = \frac{AE}{AC} \cdot \frac{CD}{BC} \cdot [ABC],$$
$$[BDE] = \frac{BD}{BC} \cdot \frac{CE}{CA} \cdot [ABC],$$
$$[AEG] = \frac{GE}{DE} \cdot \frac{AE}{AC} \cdot \frac{CD}{BC} \cdot [ABC],$$
$$[BDG] = \frac{DG}{DE} \cdot \frac{BD}{BC} \cdot \frac{CE}{AC} \cdot [ABC].$$

Substitute these values into the expression

$$\left([AEG]^2 + [BDG]^2\right)\left(\frac{1}{[ADE]^2} + \frac{1}{[BDE]^2}\right)$$

and observe that

$$\begin{aligned}
&([AEG]^2+[BDG]^2)\left(\frac{1}{[ADE]^2}+\frac{1}{[BDE]^2}\right)\\
&=\left(\frac{GE^2}{DE^2}\cdot\frac{AE^2}{AC^2}\cdot\frac{CD^2}{BC^2}+\frac{DG^2}{DE^2}\cdot\frac{BD^2}{BC^2}\cdot\frac{CE^2}{AC^2}\right)\left(\frac{AC^2}{AE^2}\cdot\frac{BC^2}{CD^2}+\frac{BC^2}{BD^2}\cdot\frac{AC^2}{CE^2}\right)\\
&=\frac{1}{DE^2}\left(DG^2+GE^2+\frac{GE^2\cdot AE^2\cdot CD^2}{BD^2\cdot CE^2}+\frac{DG^2\cdot BD^2\cdot CE^2}{AE^2\cdot CD^2}\right)\\
&=\frac{1}{DE^2}\left(DG^2+GE^2+DG\cdot GE\left(\frac{GE}{DG}\cdot\frac{AF^2}{FB^2}+\frac{DG}{GE}\cdot\frac{FB^2}{AF^2}\right)\right)\\
&=\frac{1}{DE^2}\left(DG^2+GE^2+DG\cdot GE\left(2+\left(\frac{AF}{FB}\cdot\sqrt{\frac{GE}{DG}}-\frac{FB}{AF}\cdot\sqrt{\frac{DG}{GE}}\right)^2\right)\right).
\end{aligned}$$

By this and the fact that $DE^2=DG^2+GE^2+2DG\cdot GE$, we have

$$\begin{aligned}
\frac{1}{[ADE]^2}+\frac{1}{[BDE]^2}=\frac{1}{[AEG]^2+[BDG]^2} &\iff \left(\frac{AF}{FB}\cdot\sqrt{\frac{GE}{DG}}-\frac{FB}{AF}\cdot\sqrt{\frac{DG}{GE}}\right)^2=0\\
&\iff \frac{AF}{FB}\cdot\sqrt{\frac{GE}{DG}}=\frac{FB}{AF}\cdot\sqrt{\frac{DG}{GE}}\\
&\iff \left(\frac{AF}{FB}\right)^2=\frac{DG}{GE}.
\end{aligned}$$

Editor's Comments. One of the solvers asked for more information about the geometry behind this very unusual area relation. The proposer informed us that the equality is a special case of a more general fact that

$$\frac{1}{[ADE]^2}+\frac{1}{[BDE]^2}\geq\frac{1}{[AEG]^2+[BDG]^2},$$

in which the equality case occurs if and only if $\left(\frac{AF}{FB}\right)^2=\frac{DG}{GE}$. We leave the proof of this inequality, which also follows from the above solution, as an exercise.

4819. *Proposed by Daniel Sitaru.*

Let $f:[0,1]\to[0,1]$ be a continuous function and $0<a\leq b<1$. Prove that:

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt\geq\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} f(t)dt\left(\int_0^{\frac{a+b}{2}} f(t)dt+\int_0^{\frac{2ab}{a+b}} f(t)dt\right)$$

We received 8 submissions and 7 of them were all complete and correct. We present the solution by the majority of solvers.

Let $x=2ab/(a+b)$ and $y=(a+b)/2$. Clearly $0<x\leq y<1$. The desired inequality is equivalent to

$$2\left(\int_0^y tf(t)\,dt-\int_0^x tf(t)\,dt\right)\geq\left(\int_0^y f(t)\,dt-\int_0^x f(t)\,dt\right)\left(\int_0^y f(t)\,dt+\int_0^x f(t)\,dt\right).$$

Thus, it suffices to show that

$$\left(\int_0^x f(t)\,dt\right)^2 - 2\int_0^x tf(t)\,dt \geq \left(\int_0^y f(t)\,dt\right)^2 - 2\int_0^y tf(t)\,dt.$$

Let

$$g(z) = \left(\int_0^z f(t)\,dt\right)^2 - 2\int_0^z tf(t)\,dt,$$

with $z \in (0,1)$. Since f is continuous, the Fundamental Theorem of Calculus gives

$$g'(z) = 2f(z)\left(\int_0^z f(t)\,dt - z\right) \leq 0$$

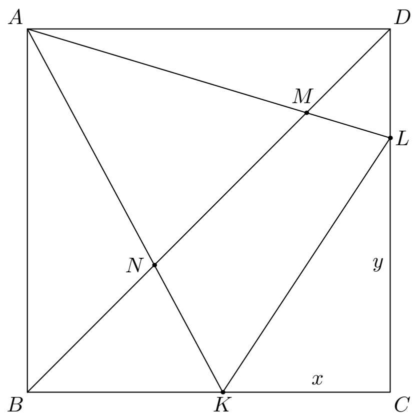
since $f(t) \in [0,1]$. Thus, $g(x) \geq g(y)$, as required.

4820. *Proposed by George Apostolopoulos.*

Let $ABCD$ be a square with side length a. Take interior points K, L on the sides BC and CD respectively so that the perimeter of triangle KCL equals $2a$. If the diagonal BD intersects the segments AK, AL in points N, M respectively, prove that the area of triangle AMN equals to the area of quadrilateral $KLMN$.

We received 17 submissions, all of which are correct, and we present here two solutions.

Solution 1, by C.R. Pranesachar.



Defining $x := KC$ and $y := CL$, we have $KL = \sqrt{x^2+y^2}$. As $BK = a - x$ and $DL = a - y$, since the perimeter of triangle KCL is $2a$, we have

$$x + y + \sqrt{x^2+y^2} = 2a,$$

that is

$$\sqrt{x^2+y^2} = 2a - x - y.$$

Squaring and simplifying, we have

$$2a^2 - 2a(x+y) + xy = 0.$$

Since BD bisects $\angle ABK$ as well as $\angle ADL$, we have by the angle bisector property that

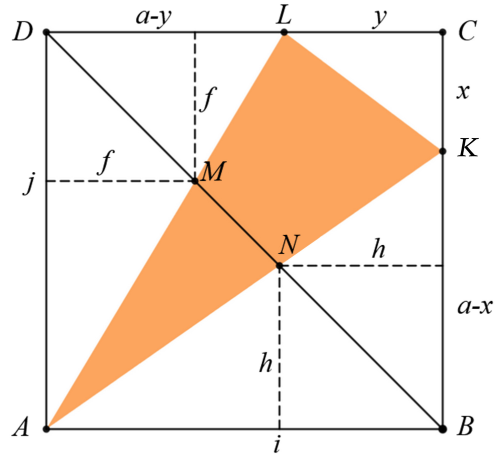
$$\frac{AN}{AK} = \frac{a}{2a-x} \quad \text{and} \quad \frac{AM}{AL} = \frac{a}{2a-y}.$$

Hence

$$\begin{aligned}\frac{[ANM]}{[AKL]} &= \frac{\frac{1}{2} \cdot AN \cdot AM \cdot \sin\angle NAM}{\frac{1}{2} \cdot AK \cdot AL \cdot \sin\angle KAL} \\ &= \frac{AN}{AK} \cdot \frac{AM}{AL} = \frac{a}{2a-x} \cdot \frac{a}{2a-y} \\ &= \frac{a^2}{2a^2 + (2a^2 - 2a(x+y) + xy)} = \frac{a^2}{2a^2+0} = \frac{1}{2}.\end{aligned}$$

This is enough to prove that $[ANM] = [KLMN]$, which completes the proof.

Solution 2, by Christopher Linhardt, Cal Poly Pomona Problem Solving Group, slightly modified.



We denote by x and y the lengths of segments CK and LC respectively and h and f as the heights of triangles NAB and MDA respectively. From similarity of right triangles ANi and AKB we find

$$\frac{h}{a-x} = \frac{a-h}{a} \Rightarrow h = \frac{a^2 - ax}{2a-x}.$$

Thus

$$Area_{NAB} = \frac{1}{2}\frac{a(a^2-ax)}{2a-x}.$$

The same can be done with triangle MDA to find

$$Area_{MDA} = \frac{1}{2}\frac{a(a^2-ay)}{(2a-y)}.$$

The remaining portion of half of the square is triangle ANM, thus

$$Area_{ANM} = \frac{a^2}{2} - \frac{a}{2}\left(\frac{a^2 - ax}{2a - x} + \frac{a^2 - ay}{2a - y}\right) = \frac{a^2}{2}\left(\frac{ay + ax - xy}{(2a - x)(2a - y)}\right). \quad (1)$$

Additionally from subtracting the areas of triangles ADL, LCK, ABK from the whole square we are left with the area of AKL

$$Area_{AKL} = a^2 - \frac{a}{2}(a - x) - \frac{a}{2}(a - y) - \frac{xy}{2} = \frac{a}{2}(x + y) - \frac{xy}{2} = \frac{1}{2}(ax + ay - xy). \quad (2)$$

It is given that the perimeter of KCL is $2a$.

$$x + y + \sqrt{x^2 + y^2} = 2a \Rightarrow 0 = 2a^2 - 2ax - 2ay + xy \Rightarrow (2a - x)(2a - y) = 2a^2.$$

We can substitute this into (1) to get

$$Area_{ANM} = \frac{a^2}{2}\left(\frac{ay + ax - xy}{2a^2}\right) = \frac{1}{4}(ay + ax - xy). \quad (3)$$

It can be seen from (3) and (2) that $Area_{AKL} = 2Area_{ANM}$. Since we have $Area_{AKL} = Area_{ANM} + Area_{KLMN}$, we obtain $Area_{ANM} = Area_{KLMN}$.

Editor's Comments. The given property (namely, the perimeter of ΔKCL equals $2a$) is equivalent to the condition that $\angle KAL = 45^\circ$ which, in turn, is equivalent to the condition that the line segment KL is tangent to the circle through B and D with the center at A. One of these variations appeared as problem M1895 in *Kvant* 2004(4), page 26.