Crux Mathematicorum

Volume/tome 49, issue/numéro 6
June/juin 2023

Canadian Mathematical Society
Société mathématique du Canada

Crux Mathematicorum is a problem-solving journal at the secondary and university undergraduate levels, published online by the Canadian Mathematical Society. Its aim is primarily educational; it is not a research journal. Online submission:

`https://publications.cms.math.ca/cruxbox/`

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire publiée par la Société mathématique du Canada. Principalement de nature éducative, le Crux n'est pas une revue scientifique. Soumission en ligne:

`https://publications.cms.math.ca/cruxbox/`

ISSN 1496-4309 (Online)

ISSN 1496-4309 (électronique)

Supported by / Soutenu par :

- Intact Financial Corporation
- University of the Fraser Valley

IN THIS ISSUE / DANS CE NUMÉRO

MathemAttic

No. 46

The problems featured in this section are intended for students at the secondary school level.

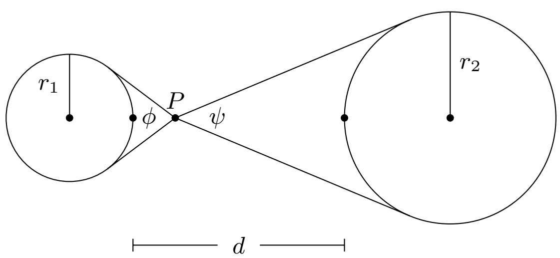
Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 15, 2023**.

MA226. The numbers a, b and c are in arithmetic sequence. The numbers b, c and d are in geometric sequence. If $a = 1.27$, $d = 3.68$ and c is positive, determine c.

MA227. Find and prove the general formula for the square root of the product of four consecutive integers plus 1.

MA228. Two circles with radii r_1 and of r_2 are a distance d apart from each other. A point P is to be placed on the line connecting the centers of the two circles so that the tangent lines to the circles go through the point forming angles ϕ and ψ as shown below. How far from the center of the left circle should one place the point P so that $\phi = \psi$? Write your answer in terms of r_1, r_2, and d.



MA229. Determine the largest real number t such that the two polynomials $x^4 + tx^2 + 1$ and $x^3 + tx + 1$ have a common root.

MA230. *Proposed by Titu Zvonaru, Comăneşti, Romania.*
Let ABC be an isosceles triangle with $AB = AC$ and AD, BE, and CF be its altitudes. A circle of diameter CE intersects the lines BC and CF at M and N, respectively. The line MN intersects the altitude AD at P. Prove that $DP = ME$.

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

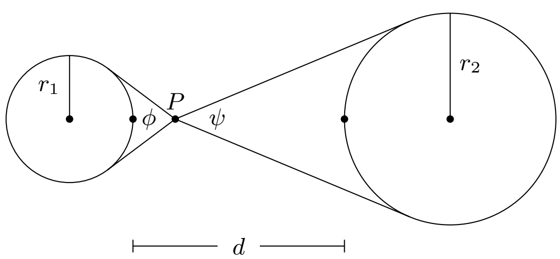
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2023**.

MA226. Les nombres a, b et c sont en progression arithmétique, tandis que les nombres b, c et d sont en progression géométrique. Si $a = 1.27$, $d = 3.68$ et c est positif, déterminer c.

MA227. Déterminer et démontrer comment, par une formule générale, calculer la racine carrée du produit de quatre entiers consécutifs, auquel on a ajouté 1.

MA228. Deux cercles de rayon r_1 et r_2 respectivement sont situés à une distance d l'un de l'autre. On place un point P sur la droite liant les centres des deux cercles de sorte que les droites tangentes aux cercles passant par le point forment des angles ϕ et ψ tel qu'illustré ci-dessous. À quelle distance du centre du cercle de droite doit-on placer le point P afin que $\phi = \psi$? Exprimez votre réponse en fonction de r_1, r_2 et d.



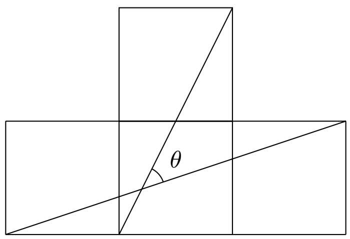
MA229. Déterminer le plus grand nombre réel t, tel que les deux polynômes $x^4 + tx^2 + 1$ et $x^3 + tx + 1$ ont une racine en commun.

MA230. *Proposé par Titu Zvonaru, Comăneşti, Romania.*
Soit ABC un triangle isocèle tel que $AB = AC$ et soient AD, BE et CF ses hauteurs. Un cercle de diamètre CE rencontre les lignes BC et CF en M et N respectivement. De plus, la ligne MN rencontre la hauteur AD en P. Démontrer que $DP = ME$.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2023: 49(1), p. 5–6.

MA201. The figure below consists of 4 congruent squares. Find the angle θ.



Originally question 10 from the 34th University of Alabama High School Mathematics Tournament: Team Competition, 2015.

We received 16 submissions of which 15 were correct and complete. We present the solution by Brian Bradie.

Because the squares are congruent, the steeper line has a slope of 2, which implies that the angle of inclination, θ_1, of the steeper line satisfies $\tan\theta_1 = 2$. On the other hand, the shallower line has a slope of $\frac{1}{3}$, so the angle of inclination, θ_2, of the shallower line satisfies $\tan\theta_2 = \frac{1}{3}$. Thus,

$$\theta = \theta_1 - \theta_2 = \tan^{-1} 2 - \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{2-\frac{1}{3}}{1+2\cdot\frac{1}{3}} = \tan^{-1} 1 = \frac{\pi}{4}.$$

MA202. Two players, A and B, play a game with a fair six-sided die. The goal is to roll a 2 or a 5: whoever does so first wins the game. The players take turns rolling the die, with player A going first. They keep rolling until someone rolls a 2 or a 5. What is the probability that player A wins the game?

Originally question 3 from the 2009 Fifth Annual Kansas Collegiate Mathematics Competition.

We received 8 solutions. We present the solution by Aravind Mahadevan, lightly edited.

Let P_A and P_B denote the probabilities of A and B winning the game respectively. If A misses getting a 2 or 5 on his first turn, then B has the same likelihood of winning the game that A had at the beginning of the game. Therefore $P_B = \frac{2}{3}P_A$. Since $P_A + P_B = 1$, we obtain

$$1 = P_A + \frac{2}{3}P_A = \frac{5}{3}P_A.$$

Thus the probability of A winning equals $\frac{3}{5}$.

MA203. *Proposed by Digby Smith, Calgary, AB.*

Suppose that a, b, c, d are positive real numbers such that

$$a^2 + b^2 = c^2 + d^2 \text{ with } a + b > c + d.$$

Show that

$$c^4 + d^4 > a^4 + b^4.$$

We received 10 submissions, all of them were complete and correct. We present the solution by Ivan Hadinata, modified by the editor.

Let $a, b, c, d \in \mathbb{R}^+$ satisfying $a^2 + b^2 = c^2 + d^2$ and $a + b > c + d$. Then,

$$a^{2k} + b^{2k} < c^{2k} + d^{2k}$$

for $k \in \{2, 3\}$.

To prove this, note that we are given that $a^2 + b^2 = c^2 + d^2$ and $a + b > c + d$. Thus,

$$2ab = (a + b)^2 - a^2 - b^2 > (c + d)^2 - c^2 - d^2 = 2cd \implies ab > cd.$$

Therefore $a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 < (c^2 + d^2)^2 - 2c^2d^2 = c^4 + d^4$, as the original problem wants to prove. And then,

$$a^6 + b^6 = (a^4 + b^4 - a^2b^2)(a^2 + b^2) < (c^4 + d^4 - c^2d^2)(c^2 + d^2) = c^6 + d^6.$$

MA204. *Proposed by Alaric Pow Ian-Jun, Singapore.*

Find the exact value of

$$\sqrt{28 + 12\sqrt{5}} + \sqrt{28 - 12\sqrt{5}} + 2\sqrt{43 - 30\sqrt{2}}.$$

We received 17 submissions, 16 of them were complete. We present the solution by Matteo Vitali.

It is well known that $\sqrt{(a + b)^2} = \sqrt{a^2 + 2ab + b^2} = |a + b|$. Thus, to get rid of the first square root, we can search for a and b such that:

$$\begin{cases} a^2 + b^2 = 28 \\ 2ab = 12\sqrt{5} \end{cases}$$

In particular, this is the simplest system to solve for a and b. All the others of the form:

$$\begin{cases} a^2 + b^2 = \alpha \\ 2ab = \beta \end{cases}$$

where $\alpha+\beta = 28+12\sqrt{5}$ and $\alpha \geq 0$ will lead (with more calculations) to the same value of $a+b$. So, we have:

$$\begin{cases} \dfrac{180}{b^2} + b^2 = 28 \\ a = \dfrac{6\sqrt{5}}{b} \end{cases} \iff \begin{cases} b^4 - 28b^2 + 180 = 0 \\ a = \dfrac{6\sqrt{5}}{b} \end{cases} \iff \begin{cases} (b^2-18)\cdot(b^2-10) = 0 \\ a = \dfrac{6\sqrt{5}}{b} \end{cases}$$

There are four solutions:

$$\{(\sqrt{10},\sqrt{18});(-\sqrt{10},-\sqrt{18});(\sqrt{18},\sqrt{10});(-\sqrt{18},-\sqrt{10})\}$$

Thus: $\sqrt{28+12\sqrt{5}} = |\sqrt{10}+\sqrt{18}| = \sqrt{10}+\sqrt{18}$.

In the second square root the double product is negative. So, a and b have different signs: $\sqrt{28-12\sqrt{5}} = |\sqrt{10}-\sqrt{18}| = \sqrt{18}-\sqrt{10}$.

We can repeat the same process for the third square root:

$$\begin{cases} a^2+b^2 = 43 \\ 2ab = -30\sqrt{2} \end{cases} \iff \begin{cases} \dfrac{450}{b^2} + b^2 = 43 \\ a = -\dfrac{15\sqrt{2}}{b} \end{cases} \iff \begin{cases} b^4 - 43b^2 + 450 = 0 \\ a = -\dfrac{15\sqrt{2}}{b} \end{cases}$$

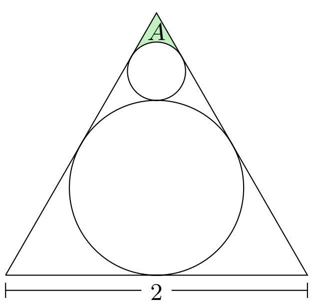
Again a and b are:

$$\{(5,-3\sqrt{2});(-5,3\sqrt{2});(3\sqrt{2},-5);(-3\sqrt{2},5)\} \Longrightarrow \sqrt{43-30\sqrt{2}} = 5-3\sqrt{2}$$

In the end, we can say:

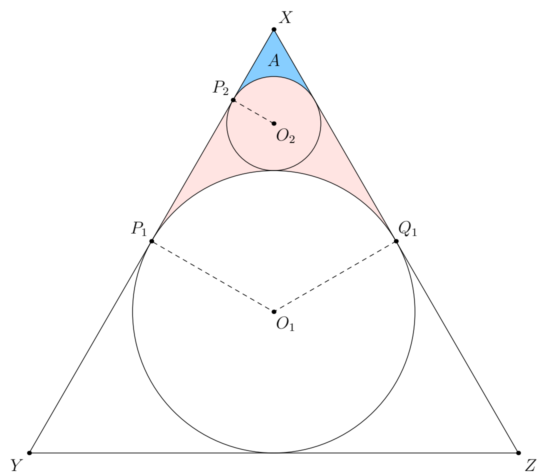
$$\begin{aligned} &\sqrt{28+12\sqrt{5}} + \sqrt{28-12\sqrt{5}} + 2\sqrt{43-30\sqrt{2}} \\ &= \sqrt{18}+\sqrt{10}+\sqrt{18}-\sqrt{10}+2\cdot(5-3\sqrt{2}) = 10. \end{aligned}$$

MA205. Suppose a circle is inscribed in an equilateral triangle with side length two metres. Another circle is inscribed in the upper corner as shown below. Find the area A between the smaller circle and the corner of the triangle.



Originally question 12 from the 35th University of Alabama High School Mathematics Tournament: Team Competition, 2016.

We received 9 submissions, 8 of which were correct. We present the solution by Ivan Hadinata.



Notice the above picture. Let ΔXYZ be the given equilateral triangle. Let us name Γ_1 and Γ_2 respectively the bigger and smaller circles as drawn in the figure. Suppose that Γ_1, Γ_2 are respectively centered at O_1, O_2 and touching side XY respectively at P_1 and P_2. Γ_1 touches XZ at Q_1. Also, let B be the region constrained by segment XP_1, segment XQ_1, and minor arc P_1Q_1 of Γ_1.

Observe that there exists a homothety $\mathcal{H}$ with center X and factor $\frac{P_2O_2}{P_1O_1}$ that sends Γ_1 to Γ_2, O_1 to O_2, and B to A. Now we have $XP_1 = 1$, and let $P_2O_2 = r$. By using trigonometric ratios, we get $XO_2 = 2r$, $P_1O_1 = \frac{1}{\sqrt{3}}$, and $XO_1 = \frac{2}{\sqrt{3}}$. We have that the area of B is $\frac{3\sqrt{3}-\pi}{9}$; and

$$\frac{2}{\sqrt{3}} = XO_2 + O_1O_2 = 3r + \frac{1}{\sqrt{3}} \quad \Longrightarrow \quad r = \frac{1}{3\sqrt{3}}.$$

Thus, the area of A equals $\left(\frac{P_2O_2}{P_1O_1}\right)^2$ times the area of B, that is

$$\left(\frac{P_2O_2}{P_1O_1}\right)^2 \cdot \frac{3\sqrt{3}-\pi}{9} = \left(\frac{\frac{1}{3\sqrt{3}}}{\frac{1}{\sqrt{3}}}\right)^2 \cdot \frac{3\sqrt{3}-\pi}{9} = \frac{3\sqrt{3}-\pi}{81}.$$

Editor's Comments. The scaling factor is immediate if you use $\frac{XP_2}{XP_1} = \frac{1/3}{1}$. The area of B was calculated by taking one third of the difference between the large triangle (area $\sqrt{3}$) and its incircle (area $\pi/3$).

TEACHING PROBLEMS

No. 22

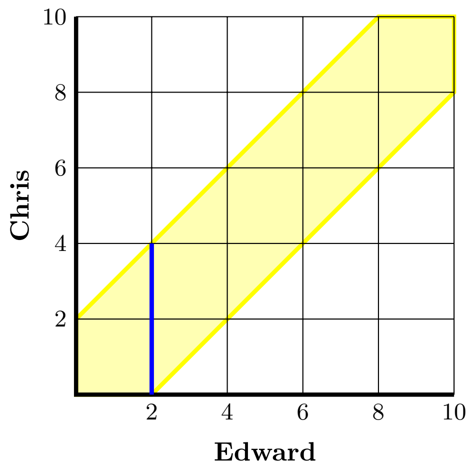
John Grant McLoughlin

Promoting Geometric Probability as a Method of Solution

Last week in a school I found myself with two middle school students who wanted a mathematical challenge. The following instruction was given to them. "Write down a number between 0 and 10. The number does not have to be an integer." The question then posed was *What is the probability that these two numbers will be within 2 of each other?*

This was the starting point for a discussion. We first made some predictions. Then Susan (the teacher) and I also wrote down numbers. This gave us six different possible pairings to consider with respect to how many would meet the requirement of not differing by more than two.

The idea of drawing a picture to represent the problem was then suggested by me. The intention here was to introduce geometric probability. That is, the probability could be found by visually representing all possibilities and determining the fraction of that area which would meet the requirements. My suggestion was to make a grid of some sort and then figure a way of working out the probability. Time was given to figure a plan. One of the students noted that the area is going to be found by using lines related to differences of 2. This person suggested a grid with spacing by 2, as shown below. That is, the axes would go from 0 to 10 with intervals of 2. Further it was observed that the values differing by at most 2 would be represented by the shaded region. The axes were labelled with the names of the two students. Hence, it can be seen that if Edward selected 2, the number Chris selected could be as small as 0 or as large as 4.



What would be the desired probability? Initially the number of squares was counted as 9 including the five full squares and four pairs of half-squares. Care then had to be taken to recognize that only 25 squares were in the grid rather than the 100 as may be suggested by 10×10. The desired probability was $\frac{9}{25}$ or 0.36. We considered this answer and revisited the predictions. A difference of 2 may suggest that there is a window of 4 for any guess as being 2 above or below would satisfy the requirement. However, numbers less than 2 or greater than 8 would not give such wide windows as the boundaries of 0 and 10 would limit the acceptable ranges. So an answer a bit less than 4 out of 10 or 0.40 would seem reasonable.

Alternative approach

A reasoned approach without a picture could be employed here. When a number is initially selected it would be expected to be between 2 and 8 in $\frac{6}{10}$ of the cases. Such selections would offer the full range of 4 or a $\frac{4}{10}$ chance of selecting the second number that differs by at most 2. In the remaining $\frac{4}{10}$ of the cases, there will be a restricted range of sorts to consider. For example, if a number is selected between 0 and 2 it will be limited in the lower direction by the 0. Likewise the numbers from 8 through 10 will be restricted on the high side. That is, 4 out of 10 numbers will be limited on average to 1 rather than 2 in one direction, thus, making for only a $\frac{3}{10}$ chance of getting a second number within 2.

The desired probability becomes

$$\frac{6}{10} \times \frac{4}{10} + \frac{4}{10} \times \frac{3}{10} = \frac{36}{100}.$$

Personally, as one who is comfortable with such explanations this is fine. However, as a teacher it has become clear to me that the visual representation of the probability has made much more sense to students generally. Geometric probability was not part of my school math experience. Rather it was first encountered by me in the 1980's through a piece that appeared in an NCTM publication entitled *Mathematical Challenges II Plus Six*. The problem discussed by Philip Smith is shared here.

> *Two witches enjoy meeting each other over a cauldron of tea. Both witches have serious shortcomings, however. First, each witch is poorly organized and arrives at the meeting place randomly between midnight and 1:00 a.m. Second, each is notoriously evil-tempered and becomes outraged on having to wait 15 minutes or longer for her companion. Thus, the following temper-saving arrangement has been agreed on: when either witch has waited fifteen minutes - or when one o'clock arrives and she is still alone - she disappears at once, not returning until the next night. Here is our problem: On a given night, what is the probability that the two witches meet?*

It is the so called "witches problem" that has been adapted by me as a teaching problem using two people arriving at a coffee shop with similar time constraints to those of the witches.

Concluding comments

Visual representations can be used in other ways to solve probability problems. It remains a helpful idea for solving probability problems if one can represent a sample space, or all possibilities, in an area model and then identify the portion that satisfies the requirements as the desired probability.

Readers are encouraged to solve the problem with the witches using both geometric probability and a reasoned approach resulting in a sum of products, as shown earlier. Further, one may wish to play with the initial problem by considering different selection boundaries than 0 to 10 for the numbers or considering a difference of no more than 1 or 3, for example. Facility with visual representations of probabilistic situations will broaden one's mathematical problem solving repertoire.

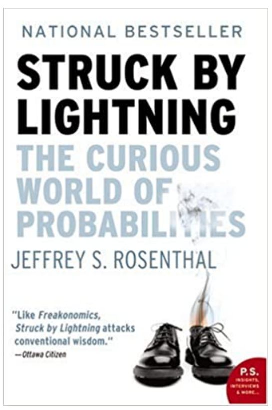
From the bookshelf of . . .

Ruwan C. Karunanayaka

This MathemAttic *feature brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to* `MathemAttic@cms.math.ca`*. Publishers are also welcome to send along books for possible review.*

Struck by Lightning: The Curious World of Probabilities
by Jeffrey S. Rosenthal
ISBN 978-0006394952, 288 pages
Published by Harper Perennial, Year 2006.

Struck by Lightning: The Curious World of Probabilities by Jeffrey S. Rosenthal is a captivating exploration of the world of probability and its profound influence on our daily lives. The book is an invaluable resource that can be used to enrich the learning experience in a classroom setting, particularly for secondary school teachers and students.



Rosenthal's book is a journey into the heart of probability, illustrating how it fills every aspect of our lives. He uses relatable examples to explain complex concepts, making the subject matter accessible to readers of all backgrounds. For instance, he discusses the role of probability in medical studies, weather forecasts, and even the spam emails we receive. These examples could serve as excellent teaching tools, providing real-world context to abstract mathematical concepts.

One of the most memorable parts of the book is when Rosenthal discusses the butterfly effect in the context of weather forecasting. He explains how weather, like coins and cards, is a chaotic system where very small changes today can cause significant differences tomorrow. This could be an excellent way to introduce students to the concept of chaos theory and its implications.

Rosenthal also examines the world of spam emails, explaining how probability theory helps computers separate spam from genuine messages. This example could be used to teach students about the practical applications of probability in technology and cybersecurity.

Rosenthal's book is filled with case studies that provide a deeper understanding of the role of probability in various aspects of life. For instance, he discusses the principle of ignoring the extremely improbable, explaining how events of extremely small probability should generally be ignored when making decisions. This principle is illustrated with an example of fearing an accident while standing under the CN Tower. This could be a great way to teach students about risk assessment and the role of probability in decision-making.

Another interesting case study in the book is the discussion on the correlation and causation. Rosenthal uses the example of a study that found that medical-school class presidents lived an average of 2.4 years less than other medical school graduates. He explains that just because being class president is correlated with shorter life expectancy does not mean that it causes shorter life expectancy. This could be a great way to teach students about the difference between correlation and causation, a concept that is often misunderstood.

In the chapter "*What Are The Odds Of That?*", Rosenthal discusses the concept of games of chance. He suggests that to win, one needs three ingredients: a well-studied strategy that wins on average, repetition of the strategy, and patience for the Law of Large Numbers to eventually lead to victory. This concept can be applied in the classroom to teach students about strategic thinking and the importance of patience and consistency in achieving long-term goals.

One of the most intriguing examples Rosenthal provides involves the game of poker. He explains the concept of "drawing to an Inside Straight" versus "drawing to an Outside Straight". If you're dealt a 5, 6, 8, and 9, you're drawing to an Inside Straight, and your chances of getting a 7 to complete the straight are quite small. However, if you're dealt a 5, 6, 7, and 8, you're drawing to an Outside Straight, and your chances of getting a 4 or 9 to complete the straight are twice as good. This example can be used to teach students about conditional probability and the importance of strategic thinking in games of chance.

Rosenthal also discusses the game of five-card stud poker. If you have three Queens in your first four cards, and your opponent has two 5's and one 4 showing, plus one secret face-down card, you might assume that your three Queens beat three 5's. However, your opponent's heavy betting might suggest that their face-down card could be another 5. This scenario can be used to teach students about the concept of uncertainty and the role of probability in decision-making.

Rosenthal also explains why casinos always win. The secret lies in the Law of Large Numbers. Even if a player has a strategy that wins on average, the casino relies on players playing repeatedly. Over time, the Law of Large Numbers ensures that the casino, which always has a slight edge, will come out on top. This can be a valuable lesson for students in understanding how seemingly fair games can be subtly biased and the long-term effects of probability.

Rosenthal also discusses the concept of true randomness, explaining how traditional science believed that randomness is caused purely by ignorance. He uses the example of predicting the movement of a baseball or a planet, explaining how physics can predict these movements precisely. This could be a great way to introduce students to the concept of randomness and its role in science.

The book also digs into the famous "birthday problem". The birthday problem is a well-known probability puzzle that goes like this: In a group of just 23 people, there's a 50% chance that at least two people have the same birthday. Despite there being 365 days in a year, it only takes a group of 23 for there to be a 50-50 chance of a shared birthday, and a group of 70 for there to be a 99.9% chance. This counter-intuitive problem can be a great way to spark students' interest in probability.

"*Struck by Lightning: The Curious World of Probabilities*" by Jeffrey S. Rosenthal is a treasure trove of real-world examples and engaging stories that make the abstract concept of probability accessible and interesting. It's a great resource for secondary school teachers and students alike, providing numerous opportunities for classroom discussions and activities.

This book is a recommendation from the bookshelf of Dr. Ruwan C. Karunanayaka. Dr. Karunanayaka is an Assistant Professor in Statistics at the University of the Fraser Valley (UFV). He brings a unique approach to teaching, employing practical examples to clarify theoretical ideas in statistics. Dr. Karunanayaka believes in the importance of understanding real-world applications to enhance learning in his field.

MATHEMAGICAL PUZZLES

No. 5

Tyler Somer

Molten Gold - V

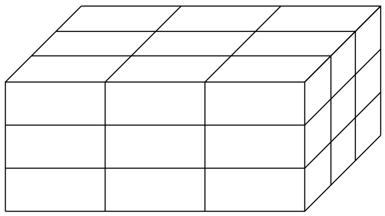
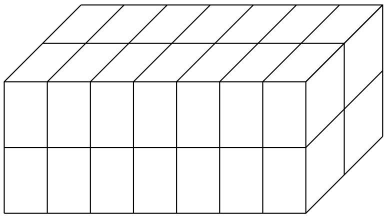
This is the concluding article in this series. We started the series by looking at seemingly-impossible packing puzzles. Many of the puzzles include pieces that have measurements which can be related to the numbers in the Fibonacci Sequence. Readers unfamiliar with the Fibonacci Sequence and its association to the Golden Ratio have a wonderful journey of mathematical discovery ahead of them.

More recently, we have looked at tray-packing puzzles in which a tile seemingly melts into the tray, but only after the pieces are rearranged. For rectangular rearrangements, the pieces are turned 90°, thus changing their horizontal and vertical components. Equivalently, turning over rectangular pieces by reflecting them through a 45° axis creates the same swap of their components. When Bill Cutler created the non-rectangular parallelogram Pentominoes-MB puzzle (see the previous article in this series [2023: 49(4), 192-194]), he gave the hint to turn the pieces over. Notice that to fill a non-rectangular parallelogram, the acute angles of the pieces must be oriented the same way. When these pieces are flipped, and the acute angles are properly realigned, the lateral components – we cannot call them horizontal and vertical components – are similarly swapped.

Modern puzzle designers continue to seek out innovative ways of hiding the trick of such melting-tile puzzles in two dimensions and, analogously, melting-block puzzles in three dimensions. In fact, Bill Cutler states that the "MB" suffix of his Pentominoes-MB puzzle is a direct reference to the "Melting Block" puzzle, first created by Thomas H. O'Beirne of Scotland. This class of puzzle apparently originated with O'Beirne's Melting Block. Let us take a closer look at the original.

Thomas H. O'Beirne was a professor of Computer Studies at the University of Glasgow. He wrote an ongoing column, *Puzzles and Paradoxes*, as a long-time contributor to the British magazine *The New Scientist*. Many of these columns were compiled in book form in 1965, with the unsurprising title: *Puzzles and Paradoxes* (New York, Oxford University Press). According to personal correspondence with Bill Cutler, the original Melting Block puzzle was designed by O'Beirne in the 1970s. There seems to be, however, very little literature about the puzzle. I am once again thankful to Bill Cutler for providing information on the original Melting Block puzzle, which I can happily share with readers here:

The Melting Block uses pieces which are based on a rectangular solid "brick" of dimensions $19 \times 29 \times 44$. When 27 of these bricks are stacked in a $3 \times 3 \times 3$ arrangement, with all the bricks oriented the same way, the result is a rectangular solid of size $57 \times 87 \times 132$. However, if you take 28 of these same bricks and stack them in a $2 \times 2 \times 7$ arrangement, with the smallest dimension being stacked 7 wide, then the resulting rectangular solid has size $58 \times 88 \times 133$.

Figures 1a and 1b: The original Melting Block puzzle

To determine the dimensions reported above, we can construct a system of three variables. Figure 1 provides a satisfactory visual aid. Consider that $a < b < c$, and notice that all three variables change attitude between the $3 \times 3 \times 3$ and $2 \times 2 \times 7$ arrangements. This provides the system:

$$7a = 3c + 1;$$
$$2b = 3a + 1;$$
$$2c = 3b + 1.$$

The reader can either solve the system as an exercise, or simply verify the above report.

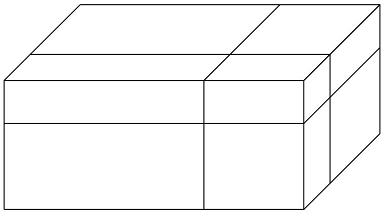


Figure 2: The $3 \times 3 \times 3$ dissection of the original Melting Block puzzle

Once the basic brick is defined, a meaningful puzzle must be designed. Merely using 28 unit bricks is neither interesting nor challenging. O'Beirne chose a simple dissection of the $3 \times 3 \times 3$ arrangement, as shown in Figure 2. By slicing the large block in a $2 : 1$ ratio in each orthogonal direction, eight unique pieces are formed. The simplicity of this dissection is quite elegant.

(Aa, Bb, Cc)	Piece dimensions
(a, b, c)	$(19, 29, 44)$
$(a, b, 2c)$	$(19, 29, 88)$
$(a, 2b, c)$	$(19, 58, 44)$
$(a, 2b, 2c)$	$(19, 58, 88)$
$(2a, b, c)$	$(38, 29, 44)$
$(2a, b, 2c)$	$(38, 29, 88)$
$(2a, 2b, c)$	$(38, 58, 44)$
$(2a, 2b, 2c)$	$(38, 58, 88)$

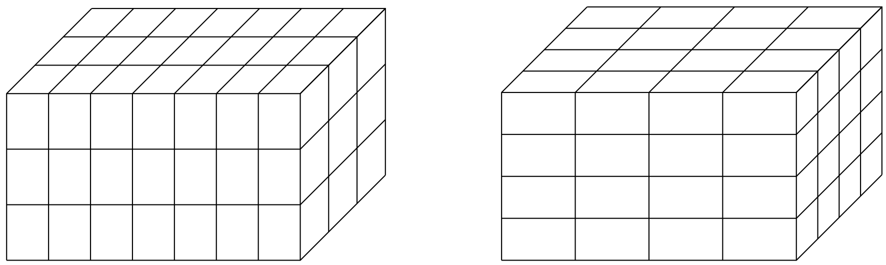
Table 1: The eight pieces of the original Melting Block.

Taking note of Figure 2, pairs of adjacent pieces can be swapped in a single layer. Alternatively, an entire layer can be rotated and/or reflected. By various combinations of these swaps, rotations, and reflections in the piece layers, some 93 "basic" solutions to the $3 \times 3 \times 3$ box packing are possible. The entire box packing can be subsequently rotated and reflected 8-fold, resulting in 744 apparent packing solutions. With a ninth piece added – necessarily a copy of the unit brick – the $2 \times 2 \times 7$ packing becomes the problem to solve. Cutler reassures us, dear reader, that this second packing is a little more interesting and challenging to solve. Indeed, there is but one fundamental solution to the $2 \times 2 \times 7$ packing. It is left to the reader to determine why there are 48 apparent packing solutions to this final challenge.

For the woodworkers among the readership, there are many construction options. One is to make 28 copies of the unit brick and create the larger pieces by gluing up the bits appropriately. Another is to make each piece from a single slab of wood, with no gluing. A single type of wood could be used for all the pieces, or up to nine species of wood can create an exotic look, with or without gluing. Taken to the extreme, gluing bricks made of 28 distinct wood species would create a countless variety of exotic combinations. A Red Stone version of the puzzle exists with the ninth piece painted red, and the other eight pieces a single species of wood. A suitable basic measurement must be chosen, perhaps 1 millimetre or some fraction of an inch, depending on the tools and equipment in your shop. A caliper is a necessity. And, of course, the box itself must be sized accordingly, to give enough play without being sloppy.

For the designers, there are four optional pieces that can be considered, each with $3a = 57$ as the measure in that one dimension. Note that the puzzle's design does limit both $2b = 58$ and $2c = 88$ as the maximum measures in those respective dimensions. As a practical matter, though, the pieces that are $57 \times 58 \times 44$ and $57 \times 58 \times 88$ should be removed from consideration. The almost-square profile of either piece can be problematic: even if the piece(s) and the box are very precise, the solver might not notice that such an almost-square piece could be in the wrong orientation. Removing these two pieces leaves us with only ten useful pieces in this universe of possibilities. The question arises: must there always be some piece – any piece, not just the unit brick – that would be repeated in the puzzle? Or, is it possible to have some set of unique non-unit pieces fill the $3 \times 3 \times 3$ arrangement, then have the addition of a single unit brick satisfy the $2 \times 2 \times 7$ arrangement?

The next pair of arrangements to try might be $4 \times 4 \times 4$ and $3 \times 3 \times 7$, since $64 = 63 + 1$. Indeed, Bill Cutler and colleague John Rausch did just that. Some of their early results were presented at a public gathering in February of 2016, while a formal compilation and summary of their extensive computer analysis was published in September of 2018. I include some highlights here.



Figures 3a and 3b: The Cutler-Rausch Melting Block 4

Notice that the $3 \times 3 \times 7$ arrangement uses 63 unit bricks, and so it is presented before the larger $4 \times 4 \times 4$ arrangement. Similar to the original Melting Block puzzle, we can construct a system of three variables in which $a < b < c$. Again, all three variables change attitude between the two arrangements, so providing the system:

$$\begin{aligned} 7a + 1 &= 4c; \\ 3b + 1 &= 4a; \\ 3c + 1 &= 4b. \end{aligned}$$

Solving, we get $a = 37$, $b = 49$, $c = 65$; this gives the measurements of the unit brick. The box must accommodate the $4\times4\times4$ arrangement as size $148\times196\times260$, as well as the smaller $147 \times 195 \times 259$ size of the $3 \times 3 \times 7$ arrangement.

To create possible puzzle pieces, Cutler and Rausch discounted zig-zag and L-shaped pieces, leaving only rectangular blocks. The limits on the measurements are from a to $4a$, b to $3b$, and c to $3c$ in these respective dimensions. The numerical values of these limits are, thus, one of $\{37, 74, 111, 148\}$ by one of $\{49, 98, 147\}$ by one of $\{65, 130, 195\}$. This provides 36 possible blocks to consider. For those who might consider duplicating the work of Cutler and Rausch, here are some computer time requirements to expect:

For the $3 \times 3 \times 7$ box, allowing the unit brick as a piece:

- 2.5 hours of run-time.
- Over 110 million basic solutions; this excludes rotations and reflections.
- Over 435 thousand piece combinations supplied the 110 million solutions.
- Just under 15 thousand piece combinations provide a unique solution.

For the $4 \times 4 \times 4$ box, allowing a second copy of the unit brick:

- 12 hours of run-time.
- Over 385 million basic solutions.
- Over 987 thousand piece combinations supplied the 385 million solutions.
- Just under 59 thousand piece combinations provide a unique solution.

Comparing the unique solutions among the above results:

- 204 sets of pieces solve both boxes uniquely.

These sets are defined as giving doubly-unique solutions, and they can be split in two categories:

- 106 sets of pieces use one unit brick in the $3 \times 3 \times 7$ box, and a second unit brick in the $4 \times 4 \times 4$ box.

- 98 sets of pieces use no unit brick in the $3 \times 3 \times 7$ box, but do use one in the $4 \times 4 \times 4$ box.

Cutler and Rausch concentrated the remainder of their analysis on these last 98 results, thus concluding that a single unit brick should appear only in the "$+\,1$" packing of the $4 \times 4 \times 4$ arrangement. They found that only 29 of the 36 pieces are used in these 98 piece sets.

They also found that a "duality" of the pieces – thus also a duality of the solutions – exists. This duality exists by virtue of swapping the b and c dimensions of some block. For example, the $111 \times 49 \times 130$ block can also be represented by its components $(3a, b, 2c)$. Its dual is formed by the swap $(3a, 2b, c)$, which is the $111 \times 98 \times 65$ block. This duality is possible since both the b and c dimensions can be at most 3 wide; whereas the a dimension can be up to 4 wide, so it cannot be considered for a duality. Some pieces are self-dual; $(4a, b, c) = 148 \times 49 \times 65$, for example, since it is only 1 wide in both the b and c dimensions. When considering piece duality, the 98 sets of pieces, above, form 49 pairs of dual sets.

Cutler and Rausch also chose to dismiss the $148 \times 147 \times 65$ piece, since its nearly-square cross-section is not desired. Note that the larger nearly-square pieces, $148 \times 147 \times 130$ and $148 \times 147 \times 195$, were among the seven unused pieces, previously removed from the analysis. Removing this eighth piece reduced the 98 sets with doubly-unique solutions to 76. The piece dual of $148 \times 147 \times 65$ is $148 \times 49 \times 195$, and this piece can be used in doubly-unique solutions. The 76 piece sets that remain after this sifting include, therefore, 22 piece sets that use the $148 \times 49 \times 195$ block (but not the 22 piece sets that use its nearly-square dual), and 54 piece sets that can be matched up as 27 dual pairs. The 152 unique packing problems range from $10 + 1$ to $13 + 1$ pieces.

As a conclusion to their analysis, Cutler and Rausch proposed a "multipuzzle" using the 28 pieces and the list of 76 desired piece sets with doubly-unique packing solutions. Among these, one piece set was selected as the Cutler Rausch Unique Melting Block – CRUMB – for a puzzle design competition. A few multipuzzle kits were produced in wood.

At the time of this writing, a metal version, named Melting Block 4, is available for a mere \$149 US from the Cubic Dissection online shop. It is not a multipuzzle kit. It is a single puzzle designed for collectors. The box is walnut. The first 11 pieces are polished aluminum, and the $+\,1$ unit is brass. Each CNC-machined and hand-finished block is accurate to within two one-thousandths of an inch. The polished blocks are stunning, with the appearance of a gold bar among silver bars. This is several steps beyond the painted, yet more affordable, Red Stone version available from a number of woodworkers.

. . . .

When he was teaching, Tyler often had mechanical puzzles in his classroom. As a freelancer, Tyler has worked with numerous inventors and co-designers to bring dozens of table-top solo-logic puzzle kits to market. He continues to design puzzles, and he spends a good deal of time in his woodshop, building his own and others' puzzle designs.

MATHEMATICS FROM THE WEB

No. 11

This column features short reviews of mathematical items from the internet that will be of interest to high school and elementary students and teachers. You can forward your own short reviews to mathemattic@cms.math.ca.

Magic Squares of Squares (are *PROBABLY* impossible)
https://youtu.be/Kdsj84UdeYg
This video from *Numberphile* looks at 3×3 magic squares whose entries are all perfect squares. This is a sequel to an earlier video called "The Parker Square", where mathematical author and comedian Matt Parker tries, unsuccessfully, to create said magic square. He shows how some deep mathematics was used on this problem and, even then, there still is no result (although it looks like there may not be any). Interestingly, magic squares of squares are known for higher order squares. The video shows one of order 4 created by Euler (who else?). This is an interesting video to watch for the number theoretical hobbist and expert alike.

The Beauty of Quadratic Equations by Suhrid Saha
https://media.pims.math.ca/pi_in_sky/pi22.pdf
This short note looks at the development of the quadratic formula from a historical perspective. The author uses Al-Khwarizmi's geometric method for solving quadratic equations and uses it to develop the quadratic formula visually. It is a nice piece of history focusing on a pivotal piece of mathematics in the high school curriculum.

Mathematical Concepts Illustrated by Hamid Naderi Yeganeh
http://www.ams.org/publicoutreach/math-imagery/yeganeh
This page from the *American Mathematical Society* website features the artwork of Hamid Naderi Yeganeh. Hamid uses mathematical functions to create stunning images and each image is accompanied by the functions that generated it. In the artists own words:

> *One of my goals is to create very beautiful images by using mathematical concepts such as trigonometric functions, exponential function, regular polygons, line segments, etc. I create images by running my program on a Linux operating system.*

OLYMPIAD CORNER

No. 414

The problems in this section appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **August 15, 2023**.

OC636. Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\frac{n^p+1}{p^n+1}$ is an integer.

OC637. For any positive integer x, we set

$$g(x) = \text{the largest odd divisor of } x,$$
$$f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)} & \text{if } x \text{ is even;} \\ 2^{\frac{x+1}{2}} & \text{if } x \text{ is odd.} \end{cases}$$

Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_1 = 1$, $x_{n+1} = f(x_n)$. Show that the integer 2018 appears in this sequence, determine the last integer n such that $x_n = 2018$, and determine whether n with the property $x_n = 2018$ is unique.

OC638. Find all the real numbers x such that

$$\frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3}$$

where $[x]$ denotes the integer part of x and $\{x\} = x - [x]$. For example $[2.5] = 2$, $\{2.5\} = 0.5$ and $[-1.7] = -2$, $\{-1.7\} = 0.3$.

OC639. For the curve $\sin(x) + \sin(y) = 1$ lying in the first quadrant, find the constant α such that

$$\lim_{x\to 0} x^{\alpha} \frac{d^2y}{dx^2}.$$

exists and is nonzero.

OC640. An equiangular hexagon has side lengths $1, 1, a, 1, 1, a$ in that order. Given that there exists a circle that intersects the hexagon at 12 distinct points, we have $M < a < N$ for some real numbers M and N. Determine the minimum possible value of the ratio $\frac{N}{M}$.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2023**.

OC636. Déterminer toutes les paires (p, n) telles que p est un nombre premier, n est un entier positif, et $\frac{n^p+1}{p^n+1}$ est entier.

OC637. Pour tout entier positif x, on pose

$$g(x) = \text{le plus grand diviseur impair de } x,$$
$$f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)} & \text{si } x \text{ est pair;} \\ 2^{\frac{x+1}{2}} & \text{si } x \text{ est impair.} \end{cases}$$

Soit alors la suite $(x_n)_{n\in\mathbb{N}}$ définie par $x_1 = 1$, $x_{n+1} = f(x_n)$. Démontrer que l'entier 2018 fait partie de cette suite, déterminer le plus grand entier n tel que $x_n = 2018$, puis déterminer si n tel que $x_n = 2018$ est unique ou non.

OC638. Soit $[x]$ la partie entière de x et $\{x\} = x - [x]$; par exemple, $[2.5] = 2$, $\{2.5\} = 0.5$ et $[-1.7] = -2$, $\{-1.7\} = 0.3$. Déterminer tous les nombres réels x tels que

$$\frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3}.$$

OC639. Pour la partie de la courbe $sin(x) + sin(y) = 1$ se situant dans le premier quadrant, déterminer la constante α telle que

$$\lim_{x\to 0} x^\alpha \frac{d^2y}{dx^2}$$

existe et est non nulle.

OC640. Les côtés d'un hexagone équiangulaire sont $1, 1, a, 1, 1, a$, dans cet ordre. Or, il existe un cercle qui rencontre cet hexagone en 12 points distincts, d'où il en découle que $M < a < N$ pour certains nombres réels M et N. Déterminer le minimum du ratio $\frac{N}{M}$.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2023: 49(1), p. 26–27.

OC611. Determine whether the following product is rational:

$$\prod_{k=0}^{\infty}\left(1-\frac{1}{2022^{k!}}\right).$$

Originally Problem 1 from the 2022, Harvard MIT Invitational Competition.

We received 5 submissions, all of which were correct and complete. We present the solution by Oliver Geupel.

The answer is no. To prove this, it is enough to show that, for every integer $b \geq 2$, the number

$$\begin{aligned} p =& 1/(b-1) + \prod_{k=1}^{\infty}\left(1-b^{-k!}\right) \\ =& \left(b^{-1}+b^{-2}+b^{-3}+\dots\right) \\ &+ \left(1 - b^{-1!} - b^{-2!} + b^{-1!-2!} - b^{-3!} + b^{-1!-3!} + b^{-2!-3!} - b^{-1!-2!-3!} - b^{-4!} \pm \dots\right) \end{aligned}$$

is irrational.

Let S be the set of positive integers that can be written as a sum of one or more factorials of distinct positive integers, that is

$$S = \{1!,\ 2!,\ 1!+2!,\ 3!,\ 1!+3!,\ 2!+3!,\ 1!+2!+3!,\ 4!,\ 1!+4!,\ 2!+4!, 1!+2!+4!,\ \dots\}.$$

For every positive integer n, we have

$$\sum_{k=1}^{n} k! \leq n \cdot n! < (n+1)!.$$

Hence, the sum representations of the members of S are unique up to the order of terms. Therefore, the digits in base-b notation $B = (1.d_1d_2d_3d_4\dots)$ of p are either 0 or 1 or 2. Furthermore, the digit d_n at position of b^{-n} is 1 if and only if $n \in \mathbb{N} \setminus S$.

Since the expansion B contains infinitely many occurrences of the digit 2, it is infinite. For every positive number n, the consecutive integers

$$1 + \sum_{k=1}^{n} k!, \dots, (n+1)! - 1$$

are not elements of S. Thus, B contains arbitrarily long (finite) segments that consist of consecutive digits 1. Consequently, the infinite representation B is not periodic. This shows that p is irrational.

OC612. Find all real numbers x such that

$$x^9 + \frac{9}{8}x^6 + \frac{27}{64}x^3 - x + \frac{219}{512} = 0.$$

Originally Problem 32 from the 2011 Harvard-MIT November Tournament, Guts Round.

We received 12 submissions, of which 10 were correct and complete. We present a typical solution.

We multiply both sides of the equation by 512 to eliminate the denominators

$$512x^9 + (9 \times 64)x^6 + (27 \times 8)x^3 - 512x + 219 = 0,$$

and arrange the terms to get

$$\begin{aligned} (8x^3+3)^3 &= 512x - 192, \\ 8^3\left(x^3 + \frac{3}{8}\right)^3 &= 8^3\left(x - \frac{3}{8}\right), \\ \left(x^3 + \frac{3}{8}\right)^3 &= x - \frac{3}{8}. \end{aligned}$$

The real solutions of the equation above are the same as the real solutions of the equation

$$x^3 + \frac{3}{8} = \left(x - \frac{3}{8}\right)^{\frac{1}{3}}.$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + \frac{3}{8}$ then $f^{-1}(x) = \left(x - \frac{3}{8}\right)^{\frac{1}{3}}$. We have to solve the equation $f(x) = f^{-1}(x)$. Since $f(x)$ and $f^{-1}(x)$ are symmetrical about the x axis, we get that $f(x) = x$ gives us the required solutions. Note that $f(x) = x$ is equivalent to

$$x^3 - x - \frac{3}{8} = 0 \quad \text{or} \quad (2x-1)(4x^2+2x-3) = 0.$$

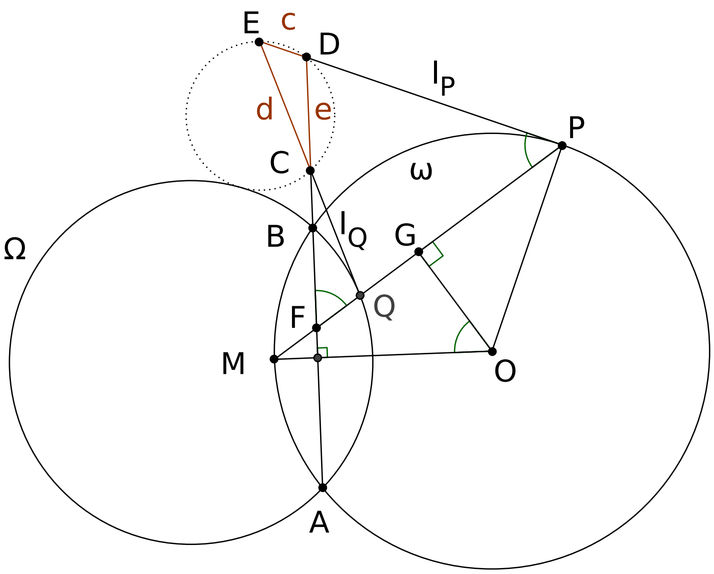
Hence the solutions are $x = \frac{1}{2}$, $x = \dfrac{-1+\sqrt{13}}{4}$, and $x = \dfrac{-1-\sqrt{13}}{4}$.

OC613. Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let l_P be the tangent line to ω at P, and let l_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines l_P , l_Q, and AB is tangent to Ω.

Originally Problem 5 from the 2014 Asian Pacific Mathematics Olympiad.

We received 5 submissions, of which 3 were correct and complete. We present the solution by Oliver Geupel.

Let $C = \ell_Q \cap AB$, $D = AB \cap \ell_P$, $E = \ell_P \cap \ell_Q$, $F = AB \cap MP$, let G be the midpoint of the line segment MP, and let O be the centre of ω. Also let $c = DE$, $d = EC$, $e = CD$, and let t_C, t_D, and t_E denote the lengths of the tangent segment from C, D, and E, respectively, to Ω.



By the observation that $\angle DPF = \angle GOM = \angle PFD$, we have $FD = PD$. By the power of point D with respect to circles Ω and ω, we obtain

$$t_D = \sqrt{AD \cdot BD} = PD = FD.$$

For the triangle CDE and the transversal MG it holds by Menelaus's theorem, with unsigned distances, that $\frac{FC}{FD} \cdot \frac{PD}{PE} \cdot \frac{QE}{QC} = 1$. Hence,

$$\begin{aligned} dt_D - et_E - ct_C &= CE \cdot FD - CD \cdot QE - DE \cdot QC \\ &= CE(FC + CD) - CD(QC + CE) - DE \cdot QC \\ &= CE \cdot FC - (CD + DE)QC \\ &= FC(QC + CE) - (FC + CD + DE)QC \\ &= FC \cdot QE - PE \cdot QC = 0. \end{aligned}$$

A variation of the Power Theorem of Casey states that the circle through C, D, and E is tangent to the circle Ω if and only if $dt_D \pm et_E \pm ct_C = 0$ Hence the result.

The variation of the Power Theorem of Casey is discussed together with a proof in Johnson, Roger A. *Advanced Euclidean Geometry*, Dover Publications, Inc., 1960, pp. 89-90, Theorem 117.

OC614. In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colours. Each clown is required to use at least five different colours. One day, the ringmaster of the circus orders that no two clowns have exactly the same set of colours and no more than 20 clowns may use any one particular colour. Find the largest number n of clowns so as to make the ringmaster's order possible.

Originally Problem 5 from the 2006 Asian Pacific Mathematics Olympiad.

We received 5 submissions, all of which were correct and complete. We present the solution by Theo Koupelis.

The maximum number of times a distinct colour is used is 12×20, which must be greater than the minimum number of colours used, which is $5 \times n$. Therefore, $n \leq 48$. It is possible to construct 48 distinct sets, each with 5 colours. For example, if we number the colours as $1, 2, 3, \ldots, 12$, each of the 8 sets

$$\{1, 2, 3, 4, 5, 6\}, \{3, 4, 5, 6, 7, 8, \}, \{5, 6, 7, 8, 9, 10\}, \{7, 8, 9, 10, 11, 12\},$$

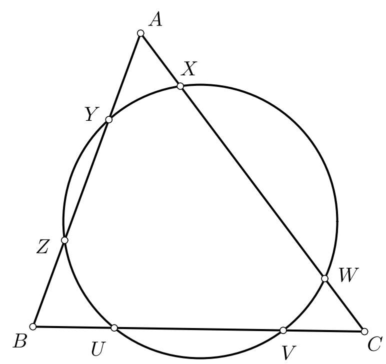
$$\{9, 10, 11, 12, 1, 2\}, \{11, 12, 1, 2, 3, 4\}, \{1, 2, 3, 4, 7, 8\}, \{3, 4, 7, 8, 9, 10\}$$

has 6 distinct subsets, each subset consisting of 5 colours; all 48 subsets are clearly distinct. Therefore, $n_{\max} = 48$.

OC615. Let ABC be a triangle. A circle intersects side BC at points U and V, side CA at points W and X, and side AB at points Y and Z. The points U, V, W, X, Y, Z lie on the circle in that order. Suppose that $AY = BZ$ and $BU = CV$. Prove that $CW = AX$.

Originally Problem 2 from the 2016 Australian Mathematical Olympiad.

We received 11 submissions, all of which were correct and complete. We present a typical solution.



Using the given equalities and the power of point with respect to the circle we have

$$(AX)(AW) = (AY)(AZ) = (BZ)(BY) = (BU)(BV) = (CV)(CU) = (CW)(CX).$$

Thus

$$(AX)^2 + (AX)(XW) = (CW)^2 + (CW)(XW),$$

which implies $(AX - CW)(AX + CW + XW) = 0$ and $AX = CW$.

Hales-Jewett theorem through examples and exercises: Part II

Veselin Jungić

"The ambition should always be to play an elegant game."

Edson Arantes do Nascimento Pelé, 1940-2022

1 Introduction

In the first part of this two-part note we introduced the so-called combinatorial lines.

Definition 1. *Let A be an alphabet, let $n \in \mathbb{N}$, and let $\tau \in A_*^n$ be a root. A* combinatorial line *in A^n rooted in τ is the set of words $L_\tau = \{\tau_a : a \in A\}$.*

For reader's convenience here are definitions of terms used in the definition of a combinatorial line. In the rest of this note, for a natural number n we will denote the set $\{1, 2, \ldots, n\}$ by $[1, n]$.

1. For $m \in \mathbb{N}$, any set A such that $|A| = m$ is called an *alphabet* on m symbols.

2. Let A be an alphabet on m symbols. For $n \in \mathbb{N}$, any function $w : [1, n] \to A$ is called a *word of length n on the alphabet A*. If $w(i) = a_i$, $i \in [1, n]$, then we write $w = a_1\ a_2\ \cdots\ a_n$.

3. The set of all words of length n on the alphabet A is denoted by A^n and called *the n-dimensional cube on the alphabet A*.

4. Let A be an alphabet (on m symbols) and let $*$ be a symbol such that $* \notin A$. We consider the alphabet $A_* = A \cup \{*\}$. Any word on the alphabet A_*, i.e., any element of $(A_*)^n = A_*^n$, for some $n \in \mathbb{N}$, that contains the symbol $*$ is called a *root*.

5. For a root $\tau \in A_*^n$ and a symbol $a \in A$ we define the word $\tau_a \in A^n$ in the following way. For $i \in [1, n]$

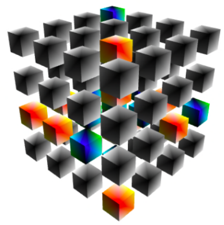
$$\tau_a(i) = \begin{cases} \tau(i) & \text{if} \quad \tau(i) \neq *, \\ a & \text{if} \quad \tau(i) = *. \end{cases}$$

In this note our goal is to motivate and state the claim of the Hales-Jewett theorem and to demonstrate a few of its applications.

2 Generalized Tic-Tac-Toe Game

Recall the game of Tic-Tac-Toe: two players take turns claiming the spaces in a 3×3 grid with the goal to claim a row, a column, or a diagonal.

A group of undergraduate students created a web game called *Quad-Tac-Toe* [1]. This is a game of a player vs. AI on a $4 \times 4 \times 4$ cube $Q(4,3) = \{(x,y,z) : x,y,z \in [1,4]\}$. The player and AI are given two different colours and tasked to colour one point in $Q(4,3)$ at each turn. Who first completes a monochromatic line wins.



Here, *a line* means a set of four collinear points, either horizontally, vertically, or diagonally.

We illustrate the fact that each of the combinatorial lines in $[1,4]^3$ corresponds to a winning position in the Quad-Tac-Toe[†] game by the following example.

Example 1. *For $a, b \in [1,4]$, consider the root $\tau = * \ a \ b \in [1,4]^3_*$ and the Euclidean line ℓ_τ in $\mathbb{R}^3$ given by its parametric equations $x = t, y = a, z = b$, $t \in \mathbb{R}$. Hence, ℓ_τ is the line that passes through the point $(0,a,b)$ and is parallel to the x-axis. Now, the combinatorial line $L_\tau = \{1 \ a \ b, 2 \ a \ b, 3 \ a \ b, 4 \ a \ b\}$ corresponds to the winning position*

$$\{(1,a,b),(2,a,b),(3,a,b),(4,a,b)\} = \{(t,a,b) : t \in [1,4]\} \subseteq \ell_\tau \cap Q(4,3).$$

Observe that the line $x = t, y = 5 - t, z = 1$, $t \in \mathbb{R}$, contains the winning position $\{(1,4,1),(2,3,1),(3,2,1),(4,1,1)\} = \{(t, 5-t, 1) : t \in [1,4]\}$ that does not correspond to any of the combinatorial lines in $[1,4]^3$.

Actually, there are 76 different Euclidean lines containing four points from $Q(4,3)$, but only 61 of them correspond to combinatorial lines in the 3-dimensional cube on alphabet $[1,4]$. Hence, not every winning position in the Quad-Tac-Toe game corresponds to a (monochromatic) combinatorial line. See Exercises 5 and 6 in Part I.

But what if one considers a k-player game that is played on the "board" given by $Q(m,n) = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in [1,m]\}$?[‡] Similar to the Tic-Tac-Toe game and the Quad-Tac-Toe game, each player is given one of k different colours and tasked to colour one point in $Q(m,n)$ at each turn. The player who first completes a monochromatic line wins. Here "a line" means a set of m collinear points, i.e., a set of m points in $Q(m,n)$ that lie on a line in $\mathbb{R}^n$ given by its parametric equations $x_1 = a_1 + \alpha_1 \cdot t, x_2 = a_2 + \alpha_2 \cdot t, \ldots, x_n = a_n + \alpha_n \cdot t$, $t \in \mathbb{R}$, for some fixed real numbers a_i, α_i, $i \in [1,n]$.

† In 1980, this variant of Tic-Tac-Toe was in detail analyzed by Oren Patashnik [8].

‡ Observe that $(x_1, x_2, \ldots, x_n) \in Q(m,n)$ is a point in $\mathbb{R}^n$ with $x_1, x_2, \ldots, x_m \in [1,m]$. With $[1,m]^n = \{x_1 \ x_2 \ \cdots \ x_n : x_1, x_2, \ldots, x_n \in [1,m]\}$ we denote the n-cube on the alphabet $[1,m]$.

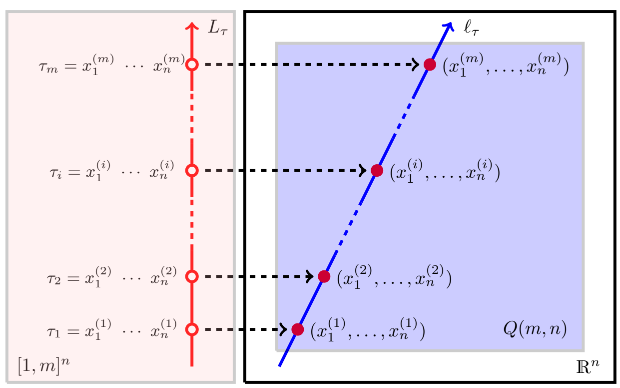
Graham, Rothschild, and Spencer called the above generalization of Tic-Tac-Toe a "k-person n dimensional Tic-Tac-Toe m-in a row" game [3].

Proposition 1. *Any combinatorial line in $[1,m]^n$ corresponds to a winning position in a k-person n dimensional Tic-Tac-Toe m-in a row game.*

Proof. For given $m, n \in \mathbb{N}$, we consider a root $\tau = a_1\ a_2\ \ldots\ a_n \in [1,m]^n_*$. Let the line ℓ_τ in $\mathbb{R}^n$ be given by its parametric equations

$$x_1 = b_1 + \alpha_1 \cdot t, x_2 = b_2 + \alpha_2 \cdot t, \ldots, x_n = b_n + \alpha_n \cdot t, t \in \mathbb{R},$$

where $b_i = 0$ and $\alpha_i = 1$ if $a_i = *$, and $b_i = a_i$ and $\alpha_i = 0$ if $a_i \in [1,m]$.



Recall that for $i \in [1,m]$, the word $\tau_i = x_1^{(i)}\ x_2^{(i)}\ \cdots\ x_n^{(i)} \in [1,m]^n$ is such that $x_j^{(i)} = i$ if $a_j = *$ and $x_j^{(i)} = a_j$ if $a_j \in [1,m]$.

It follows that, by taking $t = i$ in the parametric equation for ℓ_τ, the point $(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}) \in \ell_\tau \cap Q(m,n)$.

This establishes an injection between the combinatorial line $L_\tau = \{\tau_i : i \in [1,m]\}$ and the set $\ell_\tau \cap Q(m,n)$. Since $|L_\tau| = m$, the combinatorial line L_τ corresponds to "m-in a row" points in $Q(m,n)$, i.e., to a winning position.

□

3 The Hales-Jewett Theorem

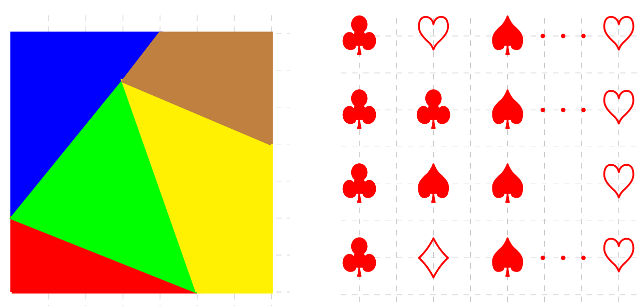
Suppose that a k-person n dimensional Tic-Tac-Toe m-in a row game ended up in a tie. Hence, the set $Q(m,n)$ was partitioned into k mutually disjunct parts, $Q(m,n) = P_1 \cup \ldots \cup P_k$, $P_i \cap P_j = \emptyset$ if $i \neq j$, in such a way that none of the parts contained a set of m collinear points.

Define a k-colouring C of the cube $[1,m]^n$ as following: for $a_1 \ \cdots \ a_n \in [1,m]^n$, $C(a_1 \ \cdots \ a_n) = i$ if and only if $(a_1, \ldots, a_k) \in P_i$. By Proposition 1 and our assumption that the game ended up in a tie, it follows that the colouring C of the n-dimensional cube on alphabet $[1,m]$ does not contain a monochromatic combinatorial line.

In the spirit of Ramsey theory, this observation leads us to the following question.

Question 1. *Let A be an alphabet on m symbols and let A^n be the n-dimensional cube on alphabet A, i.e., let $A^n = \{a_1 \ a_2 \ \cdots \ a_n : a_i \in A, i \in [1,n]\}$. If A^n is k-coloured, under which conditions can we be sure that A^n contains a monochromatic combinatorial line?*

More precisely, let $k, m \in \mathbb{N}$ and let A be an alphabet on m symbols. Does there exist an $n \in \mathbb{N}$ such that whenever A^n is k-coloured there exists a monochromatic line?



$\{\heartsuit, \clubsuit, \spadesuit, \diamondsuit\}^n$ **is** k**–coloured. . .** **. . . and a red combinatorial line.**

A nice answer to Question 1 would be that, for any $k, m \in \mathbb{N}$ and for a big enough natural number n, the k-person n dimensional Tic-Tac-Toe m-in a row game cannot end up in a tie.

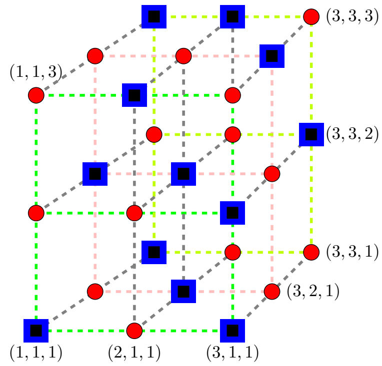
This is exactly what Hales and Jewett discussed in their paper *Regularity and position games*, published in 1963 [6]. Their famous Hales-Jewett theorem establishes that, if the dimension is sufficiently large, a generalized Tic-Tac-Toe game never ends up in a tie.

Theorem 1 ([6])**.** *Let $k, m \in \mathbb{N}$ and let A be an alphabet on m symbols. There exists an $n \in \mathbb{N}$ such that whenever A^n is k-coloured there exists a monochromatic combinatorial line.*

Definition 2. *The smallest n guaranteed by Theorem 1 is denoted by $H(k;m)$ and called a Hales-Jewett number.*

Exercise 1. *Let $A = \{a, b, c, d\}$. Can you find a 2-colouring of A^2 that does not contain a monochromatic combinatorial line? If yes, does this contradict the claim of the Hales-Jewett theorem?*

Exercise 2. *Let $A = [1, 3]$ be an alphabet. Check if the 2-colouring depicted in the figure below yields a monochromatic combinatorial line in A^3. Based on your observation, what can you tell about the Hales-Jewett number $H(2; 3)$?*



Exercise 3. *Prove that, for $r \geq 2$, $HJ(r; 2) = r$.*

Example 2. *Use the Hales-Jewett theorem to prove van der Waerden's theorem: If $k, l \in \mathbb{N}$ then any l-colouring of $\mathbb{N}$ contains a k-term monochromatic arithmetic progression.*

Solution. Let $k, l \in \mathbb{N}$ be given. Let $c : \mathbb{N} \to [1, l]$ be an l-colouring of the set of natural numbers. Let $N = HJ(l; k)$. We define an l-colouring of the N-cube $[1, k]^N$ as follows: if $x_1\ x_2\ \cdots\ x_N \in [1, k]^N$ then $c'(x_1\ x_2\ \cdots\ x_N) = c(x_1 + x_2 + \cdots + x_N)$.

By the Hales-Jewett theorem there is a c'-monochromatic combinatorial line L_τ rooted in a root $\tau = a_1\ a_2\ \cdots\ a_N \in [1, k]^N_*$. We observe that there is at least one $i \in [1, N]$ such that $a_i = *$.

Let S be the set of all $i \in [1, N]$ such that $a_i \in [1, k]$, i.e., the symbol $a_i \neq *$. Recall that the combinatorial line L_τ is the set of words of the form, for $j \in [1, k]$, $\tau_j = a_1^{(j)}\ a_2^{(j)}\ \cdots\ a_N^{(j)}$, with $a_i^{(j)} = a_i$ if $i \in S$ and $a_i^{(j)} = j$ if $i \notin S$.

Let $a = \sum_{i \in S} a_i$ and let $d = |[1, N] \setminus S|$, i.e., let $d \geq 1$ be the number of times that the symbol $*$ appears in the root τ. Note that, for $j \in [1, k]$,

$$\sum_{i=1}^{N} a_i^{(j)} = \sum_{i \in S} a_i^{(j)} + \sum_{i \in [1,N] \setminus S} a_i^{(j)} = a + dj.$$

On the other hand, $c'(\tau_1) = c'(\tau_2) = \cdots = c'(\tau_k)$ which together with, for each $j \in [1, k]$, $c'(\tau_j) = c\left(\sum_{i=1}^{N} a_i^{(j)}\right) = c(a + jd)$, implies

$$c(a + d) = c(a + 2d) = \cdots = c(a + kd).$$

Hence, the k-term arithmetic progression $a + d, a + 2d, \ldots, a + kd$ is c-monochromatic.

Exercise 4. *Use the Hales-Jewett theorem to prove that for any 2-colouring of natural numbers there is an ℓ-term arithmetic progression $a_1, a_2, \ldots, a_\ell$, not necessarily monochromatic, such that the set $A = \{2^{a_1}, 2^{a_2}, \ldots, 2^{a_\ell}\}$ is monochromatic.*

Exercise 5 (*Gallai's theorem for semigroups*)**.** *Tibor Gallai, a Hungarian mathematician, 1912-1992, was a lifelong friend and collaborator of Paul Erdős. In this exercise we use the Hales-Jewett theorem to prove Gallai's theorem for semigroups.*

Let $(A, \bullet)$ be a semigroup and let $\ell \in \mathbb{N}$ be given. For any r-colouring of the set A there are $a, b \in A$ such that the set $\{a, a \bullet b, a \bullet b^2, \ldots, a \bullet b^{\ell-1}\}$ is monochromatic.

Note: *A semigroup $(A, \bullet)$ is an algebraic structure consisting of a set A together with an associative binary operation $\bullet$. Commonly, $x \bullet y$, denotes the result of applying the semigroup operation to the ordered pair (x, y). For example, both $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$, i.e., the set of natural numbers together with the usual addition and multiplication, are semigroups. Associativity for all $x, y, z \in A$ is expressed as $(x \bullet y) \bullet z = x \bullet (y \bullet z)$. For $k \in \mathbb{N}$ and $x \in A$ we write*

$$x^k = \underbrace{x \bullet (x \bullet (x \bullet (\ldots (x \bullet x)}_{k} \ldots)) = \underbrace{x \bullet x \bullet \ldots x \bullet x}_{k}.$$

4 The Hales-Jewett Theorem and the Polymath Project

Similar to Ramsey's theorem and van der Waerden's theorem, the Hales-Jewett theorem was one of those significant mathematical results that became a source of inspiration for generations of mathematicians.

Here we briefly reflect on one of the developments, closely related to the Hales-Jewett theorem, that have enriched the whole 21st century's mathematical landscape.

In 1983, Ronald Graham offered $1,000 for a proof of what he called, a "density version for the Hales-Jewett theorem" (DHJ):

> For all finite A and $\varepsilon > 0$ there exists $N(A, \varepsilon)$ such that if $N \geq N(A, \varepsilon)$ and $R \subseteq A^N$ satisfies $|R| \geq \varepsilon |A^N|$ then R must contain a combinatorial line [4].

In simple terms, Graham's question was if any large enough subset of a cube of the appropriate dimension must contain a combinatorial line.

ln 1991, Hillel Furstenberg and Yitzhak Katznelson proved Graham's conjecture from 1983 by using the ergodic theory techniques [2].

In early 2009, in a series of blog posts, Timothy Gowers invited members of the mathematical community to jointly search for an elementary proof of DHJ. Gowers's initial blog titled *Is massively collaborative mathematics possible?* [5] marks the beginning of the Polymath Project, a still ongoing collaboration of mathematicians across the world on a variety of important mathematical problems.

For all of those who study, teach, and do mathematics, here are two of the "ground rules" of the Polymath Project that Gowers established in 2009 [5]:

> "3. When you do research, you are more likely to succeed if you try out lots of stupid ideas. Similarly, stupid comments are welcome here. (In the sense in which I am using "stupid," it means something completely different from "unintelligent." It just means not fully thought through.)
>
> 5. Don't *actually* use the word "stupid," except perhaps of yourself."

The first Polymath Project was a success. In 2012, the first elementary proof of DHJ, together with a quantitative bound on how large n needs to be, was published [9]. The collaborators fittingly attributed "D.H.J. Polymath" as the author.

5 Hints and solutions

Exercise 1. There is no monochromatic combinatorial line in this red/black colouring of the cube A^2:

$*a$	$*b$	$*c$	$*d$	$a*$	$b*$	$c*$	$d*$	$**$
a a	$a\ b$	$a\ c$	$a\ d$	**a a**	$b\ a$	$c\ a$	**d a**	**a a**
$b\ a$	**b b**	$b\ c$	$b\ d$	$a\ b$	**b b**	$c\ b$	$d\ b$	**b b**
$c\ a$	$c\ b$	**c c**	**c d**	$a\ c$	$b\ c$	**c c**	$d\ c$	**c c**
d a	$d\ b$	$d\ c$	$d\ d$	$a\ d$	$b\ d$	**c d**	$d\ d$	$d\ d$

The existence of such a colouring does not contradict the Hales-Jewett theorem. It just shows that $HJ(2;4) > 2$.

Exercise 2. There is no monochromatic combinatorial line in this colouring. Hence, $HJ(2;3) > 2$. *Note:* Observe that $\{(1,3,1),(2,2,1),(3,1,1)\}$ is a monochromatic set that contains three points that are collinear in $\mathbb{R}^3$. Does this set correspond to a combinatorial line in A^3?

Exercise 3. Let $r \geq 2$ and let $A = \{0,1\}$ be an alphabet. Let $2 \leq n < r$ and let $c : A^n \to [0, r-1]$ be an r-colouring defined in the following way: If $w = a_1\, a_2\, \cdots\, a_n$ then $c(w) = |\{i \in [1,n] : a_i = 0\}|$. Let $\tau \in A^n_*$ be any root. From $c(\tau_0) \neq c(\tau_1)$, it follows that the combinatorial line L_τ is not c-monochromatic. Hence, $HJ(r;2) \geq r$.

Let $n \geq r$. For $i \in [1, n+1]$, let the word $w_i = a_1^{(i)}\, a_2^{(i)}\, \ldots\, a_n^{(i)}$ be such that $a_j^{(i)} = 0$, if $j < i$, and $a_j^{(i)} = 1$, if $j \geq i$. Let c be any r-colouring of the cube A^n. By the pigeonhole principle, there are $p, q \in [1, n+1]$, $p < q$, such that $c(w_p) = c(w_q)$. We consider the root $\tau = \underbrace{0\,0\ldots\,0}_{p-1}\underbrace{*\ *\ldots\ *}_{q-p}\underbrace{1\,1\ldots\,1}_{n-q+1}$ and observe that the combinatorial line $L_\tau = \{\tau_0 = w_q, \tau_1 = w_p\}$ is c-monochromatic.

Note: Only in 2014, Neil Hindman and Eric Tressler established that $HJ(2;3)=4$ and obtained, what they called, "the first non-trivial Hales-Jewett number" [7].

Exercise 4. Let $N = HJ(2;\ell)$ and let c be a 2-colouring of the set of natural numbers. We define a 2-colouring of the N-cube $[1,\ell]^N$ as follows: for $x_1\ x_2\ \cdots\ x_N \in [1,\ell]^N$, $c'(x_1\ x_2\ \cdots\ x_N) = c\left(2^{x_1}\cdot 2^{x_2}\cdot\ldots\cdot 2^{x_N}\right)$. By the Hales-Jewett theorem, there is a c'-monochromatic line L_τ, determined by a root $\tau = a_1\ a_2\ \cdots\ a_N \in [1,\ell]^N_*$. Let $S = \{i\in[1,N] : a_i\in[1,\ell]\}$. Let $a=\sum_{i\in S} a_i$ and let $d = |[1,N]\backslash S|$.

Recall that $L_\tau = \{\tau_1,\tau_2,\ldots,\tau_\ell\}\subseteq[1,\ell]^N$, where, for $j\in[1,\ell]$, $\tau_j = a_1^{(j)}\ a_2^{(j)}\ \cdots\ a_N^{(j)}$, with $a_i^{(j)} = a_i$, if $i\in S$, and $a_i^{(j)} = j$, if $i\notin S$.

Note that, for any $j\in[1,\ell]$,

$$\sum_{i=1}^N a_i^{(j)} = \sum_{i\in S} a_i^{(j)} + \sum_{i\in[1,N]\backslash S} a_i^{(j)} = a + \sum_{i\in[1,N]\backslash S} j = a + jd.$$

On the other hand, $c'(\tau_1) = c'(\tau_2) = \cdots = c'(\tau_\ell)$, which together with $c'(\tau_j) = c\left(2^{\sum_{i=1}^N a_i^{(j)}}\right) = c\left(2^{a+jd}\right)$, for each $j\in[1,\ell]$, implies that the ℓ-term arithmetic progression $a_1 = a+d, a_2 = a+2d,\ldots,a_\ell = a+\ell d$ is with the required property.

Exercise 5. Let c be an r-colouring of A and let $N = HJ(r;\ell)$. Let $x\in A$ be fixed. We define an r-colouring of the N-cube $[1,\ell]^N$ as follows: for $n_1\ n_2\ \cdots\ n_N \in [1,\ell]^N$, $c'(n_1\ n_2\ \cdots\ n_N) = c\left(x^{n_1+n_2+\cdots+n_N}\right)$.

By the Hales-Jewett theorem, there is a c'-monochromatic line L_τ, determined by a root $\tau = a_1\ a_2\ \cdots\ a_N\in[1,\ell]^N_*$. Let $S=\{i\in[1,N] : a_i\in[1,\ell]\}$. Let $m = \sum_{i\in S} a_i$ and let $d = |[1,N]\backslash S|$.

Recall that $L_\tau = \{\tau_1,\tau_2,\ldots,\tau_\ell\}\subseteq[1,\ell]^N$, where, for $j\in[1,l]$, $\tau_j = a_1^{(j)}\ a_2^{(j)}\ \cdots\ a_N^{(j)}$, with $a_i^{(j)} = a_i$, if $i\in S$, and $a_i^{(j)} = j$, if $i\notin S$. Note that for any $j\in[1,\ell]$,

$$\sum_{i=1}^N a_i^{(j)} = \sum_{i\in S} a_i^{(j)} + \sum_{i\in[1,N]\backslash S} a_i^{(j)} = a + \sum_{i\in[1,N]\backslash S} j = a + jd.$$

On the other hand $c'(\tau_1) = c'(\tau_2) = \cdots = c'(\tau_\ell)$ which implies that, for each $j\in[1,\ell]$, $c'(\tau_j) = c\left(x^{\sum_{i=1}^N \tau_j(i)}\right) = c\left(x^{m+jd}\right)$.

Observe that, since binary operation $\bullet$ is associative, it follows, for each $j\in[1,\ell]$,

$$\begin{aligned} x^{m+jd} &= \underbrace{x\bullet x\bullet\ldots\bullet x}_{m+jd} = (\underbrace{x\bullet\ldots\bullet x}_{m+d})\bullet(\underbrace{x\bullet\ldots\bullet x}_{(j-1)d}) \\ &= (\underbrace{x\bullet\ldots\bullet x}_{m+d})\bullet\underbrace{((\underbrace{x\bullet\ldots\bullet x}_{d})\bullet\ldots\bullet(\underbrace{x\bullet\ldots\bullet x}_{d}))}_{j-1} = x^{m+d}\cdot\left(x^d\right)^{j-1}. \end{aligned}$$

Therefore, for $a = x^{m+d}$ and $b = x^d$, the set $\{a, a \bullet b, a \bullet b^2, \ldots, a \bullet b^{\ell-1}\}$ is c-monochromatic.

Note: Observe that, in the case of the semigroup $(\mathbb{N}, +)$, Gallai's theorem for semigroups implies van der Waerden's theorem. Also, observe that the set of all non-negative integer powers of 2, with the usual multiplication, is a semigroup.

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PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 15, 2023**.

4851. *Proposed by Mihaela Berindeanu.*

Let Γ be the circumcircle of $\triangle ABC$, with circumcenter O and radius R. Point X is diametrically opposed to A, $AX \cap BC = \{P\}$ and$AP = 2PX$. The tangent to the circle Γ through X cuts AB in M and AC in N. Show that

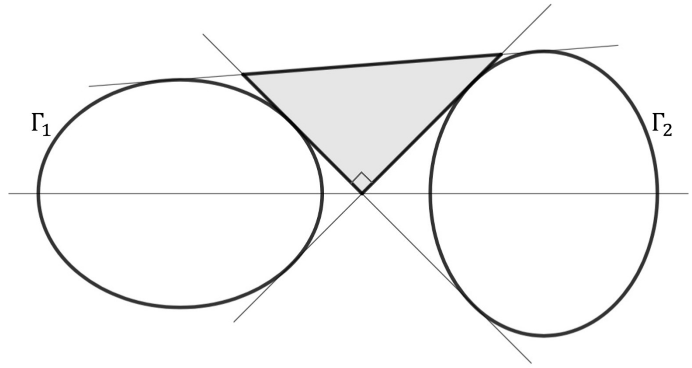
$$2R \cdot MN + OM \cdot CN + ON \cdot BM = MC \cdot ON + BN \cdot OM.$$

4852. *Proposed by Aravind Mahadevan.*

In triangle ABC, the bisectors of angles A, B and C meet the sides BC, CA and AB at D, E and F respectively. If $\angle ADC = x$, $\angle AEB = y$ and $\angle BFC = z$, prove that $a \sin 2x + b \sin 2y + c \sin 2z = 0$ where a, b and c are the lengths of BC, CA and AB respectively.

4853. *Proposed by Byungjun Lee.*

Two congruent ellipses Γ_1 and Γ_2 with semi-major axis a and semi-minor axis b are given. The major axis of Γ_1 and the minor axis of Γ_2 lie on the same line, and two common internal tangents of Γ_1 and Γ_2 are perpendicular. Find the area of the triangle formed by two common internal tangents and one common external tangent.



4854. *Proposed by Michel Bataille.*

Let n be a positive integer and let $\theta_k = \frac{k\pi}{n+1}$. For $r, s \in \{1, 2, \ldots, n\}$, evaluate

$$\sum_{j=1}^{n} (\sin\theta_{jr} + \sin\theta_{js})^2.$$

4855. *Proposed by Ivan Hadinata.*

Find all pairs of positive integers (a, b) such that $a^b - b^a = a - b$.

4856. *Proposed by Titu Zvonaru.*

Let ABC be a triangle with $\angle A = 30^\circ$ and $\angle B = 100^\circ$. Consider the points D and E on the sides AC and BC, respectively, such that $\angle ABD = \angle DBC$ and $DE || AB$. Find $\angle EAC$.

4857. *Proposed by Toyesh Prakash Sharma.*

Let a, b, c be positive real numbers such that $a + b + c = \dfrac{3}{2}$. Show that

$$a^a b^b + b^b c^c + c^c a^a \geq \frac{3}{2}.$$

4858. *Proposed by Anton Mosunov.*

Prove that for every positive integer n and for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, we have

$$\int_0^\pi \sqrt[n]{\prod_{k=1}^{n} \csc^2(x - \alpha_k)}dx \geq 4\pi.$$

4859. *Proposed by Trinh Quoc Khanh, modified by the Editorial Board.*

Given a triangle ABC, a point X on segment AB and a point Y on segment AC, such that B, X, Y, C are concyclic, let I, J, K be the incenters of triangles ABC, XBC, and YBC, respectively. Prove that AI is orthogonal to JK.

4860. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with $\angle A > 90^\circ$. Let $M_1, M_2, \ldots, M_n$ $(n \geq 1)$ be internal points on the side BC such that $BM_1 = M_1M_2 = \cdots = M_{n-1}M_n = M_nC$. Prove that

$$AM_1 + AM_2 + \cdots + AM_n < n\sqrt{\frac{2n+1}{6(n+1)}}BC.$$

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2023**.

4851. *Soumis par Mihaela Berindeanu.*

Soit Γ le cercle circonscrit à $\triangle ABC$, de centre O et de rayon R. Le point Xest diamétralement opposé à A, $AX \cap BC = \{P\}$ et $AP = 2PX$. La tangente au cercle Γ passant par X coupe AB en M et coupe AC en N. Montrez que

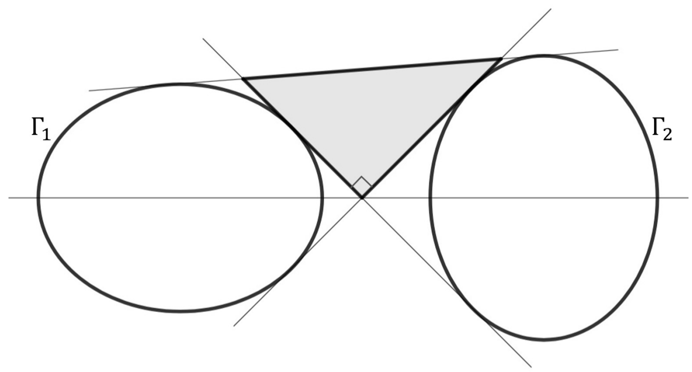
$$2R \cdot MN + OM \cdot CN + ON \cdot BM = MC \cdot ON + BN \cdot OM.$$

4852. *Soumis par Aravind Mahadevan.*

Dans le triangle ABC, les bissectrices des angles A, B et C rencontrent les côtés BC, CA et AB en D, E et F respectivement. Si $\angle ADC = x$, $\angle AEB = y$ et $\angle BFC = z$, montrez que $a \sin 2x + b \sin 2y + c \sin 2z = 0$, où a, b et c sont respectivement les longueurs de BC, CA et AB.

4853. *Soumis par Byungjun Lee.*

Deux ellipses congruentes Γ_1 et Γ_2 de demi-grand axe a et demi-petit axe b sont données. Le grand axe de Γ_1 et le petit axe de Γ_2 se trouvent sur la même ligne et deux tangentes internes communes de Γ_1 et Γ_2 sont perpendiculaires. Trouvez l'aire du triangle formé par deux tangentes internes communes et une tangente externe commune.



4854. *Soumis par Michel Bataille.*

Soit n un entier positif et soit $\theta_k = \frac{k\pi}{n+1}$. For $r, s \in \{1, 2, \ldots, n\}$. Évaluez

$$\sum_{j=1}^{n} (\sin\theta_{jr} + \sin\theta_{js})^2.$$

4855. *Soumis par Ivan Hadinata.*

Trouvez toutes les paires d'entiers positifs (a, b) telles que $a^b - b^a = a - b$.

4856. *Soumis par Titu Zvonaru.*

Soit ABC un triangle avec $\angle A = 30°$ et $\angle B = 100°$. Considérons les points D et E sur les côtés AC et BC, respectivement, tels que $\angle ABD = \angle DBC$ et $DE||AB$. Trouvez $\angle EAC$.

4857. *Soumis par Toyesh Prakash Sharma.*

Soient a, b et c des nombres réels positifs tels que $a + b + c = \dfrac{3}{2}$. Montrez que

$$a^a b^b + b^b c^c + c^c a^a \geq \frac{3}{2}.$$

4858. *Soumis par Anton Mosunov.*

Montrez que pour tout entier positif n et pour tout $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, on a

$$\int_0^\pi \sqrt[n]{\prod_{k=1}^{n} \csc^2(x - \alpha_k)}dx \geq 4\pi.$$

4859. *Proposée par Trinh Quoc Khanh, modifié par le comité de rédaction.*

Pour un triangle donné ABC, soient X un point situé sur le segment AB et Y un point situé sur le segment AC, de façon à ce que B, X, Y et C soient cocycliques; les centres des cercles inscrits des triangles ABC, XBC et YBC sont alors dénotés I, J et K, respectivement. Démontrer que AI est orthogonal à JK.

4860. *Proposée par George Apostolopoulos.*

Soit ABC un triangle tel que $\angle A > 90°$. Soient aussi $M_1, M_2, \ldots, M_n$ $(n \geq 1)$ des points internes du côté BC, tels que $BM_1 = M_1M_2 = \cdots = M_{n-1}M_n = M_nC$. Démontrer que

$$AM_1 + AM_2 + \cdots + AM_n < n\sqrt{\frac{2n+1}{6(n+1)}}BC.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2023: 49(1), p. 44–47.

4801. *Proposed by Michel Bataille.*

Find all functions $f : (0, \infty) \to \mathbb{R}$ such that

$$f\left(x + \frac{1}{y}\right) = yf(xy + y)$$

for all $x, y > 0$.

We received 17 submissions, 10 of which are correct. We present 2 different solutions.

Solution 1, by the UCLan Cyprus Problem Solving Group.

For $t > 1$, letting $x = \frac{t-1}{2}$ and $y = \frac{2}{t+1}$ we have $xy + y = 1$ and so

$$f(t) = \frac{2f(1)}{t+1}.$$

Now for any $r > o$, letting $x = r/2, y = 2/r$ we have $xy + y = 1 + y > 1$ and so

$$f(r) = yf(1 + y) = \frac{2yf(1)}{y + 2} = \frac{2f(1)}{1 + 2/y} = \frac{2f(1)}{r + 1}.$$

It follows that there is a constant $k = 2f(1)$ such that $f(x) = k/(x + 1)$ for every $x > 0$.

It is easy to check that every such function satisfies the functional equation.

Solution 2, by Theo Koupelis.

Let $P(x, y)$ be the statement that

$$P(x, y) : \qquad f\left(x + \frac{1}{y}\right) = yf(xy + y), \quad \forall x, y > 0.$$

We have

$$P\left(\frac{1}{x}, \frac{x}{x+1}\right) : \qquad f\left(\frac{x+2}{x}\right) = \frac{x}{x+1} \cdot f(1),$$

$$P\left(\frac{x}{2}, \frac{2}{x}\right) : \qquad f(x) = \frac{2}{x} \cdot f\left(\frac{x+2}{x}\right).$$

From the above equations we get

$$f(x) = \frac{2}{x+1} \cdot f(1),$$

with $f(1) \in \mathbb{R}$, which clearly satisfies the given equation.

4802. *Proposed by Arkan Manva.*

Suppose there are $2n$ students in a class, each starting with a certain number of sweets, such that the total number is a multiple of n. On each day, the teacher chooses n students and gives $c \in \mathbb{N}$ sweets to each of them. Find the minimum number of days needed such that no matter how the sweets were distributed at the start, the teacher can leave all the students with an equal number of sweets. (Note: c can change on a daily basis).

On top of the proposer's solution, we only received the submission by UCLan Cyprus Problem Solving Group and it was correct. We present their solution, with some modifications and clarification by the editor based on the solution of the proposer.

We show that the minimum number of days needed is $2n - 1$.

(1) We first show that $2n - 1$ days are sufficient. It will be convenient to assume that the teacher can also subtract sweets from the students (so c is allowed to be negative). This will not have any impact on the minimum number of days needed because instead of subtracting sweets from a set of n students we can actually give the same number of sweets to the complementary set of n students. We pick a special child, say with y sweets. Let S be the total number of sweets at the start.

Label the $(2n - 1)$ non-special children by $1, \ldots, 2n - 1$. For each $1 \le i \le 2n - 1$, let a_i be the number of sweets the child i has. All indices in the following are taken modulo $2n - 1$. For each $1 \le j \le 2n - 1$, consider the interval $I_j = \{j, j + 1, \ldots, j + n - 1\}$. Our strategy is "cyclic", namely, give x_j sweets to each child with label in I_j on the j-th day, so that eventually all children have y sweets, the same as the special child. Using this strategy, the number of sweets that child i has will become

$$a'_i := a_i + \sum_{\substack{i \in I_j \\ 1 \le j \le 2n-1}} x_j = a_i + \sum_{j=i-n+1}^{i} x_j.$$

We need to guarantee that a'_i are all the same, that is, $a'_i = a'_{i+1}$ for each $1 \le i \le 2n - 1$. Given that $a'_i = a'_{i+1}$, we need

$$a_i + \sum_{j=i-n+1}^{i} x_j = a_{i+1} + \sum_{j=i-n+2}^{i+1} x_j$$

equivalently,

$$a_i + x_{i-n+1} = a_{i+1} + x_{i+1}. \tag{1}$$

Note that $\gcd(2n-1, n) = 1$. Thus, given x_0, we can use (1) to inductively determine x_n, x_{2n}, etc, and eventually $x_{n(2n-1)}$, so that $a'_1 = a'_2 = \cdots = a'_{2n-1}$. Note that if we sum over equation (1) over all $1 \le i \le 2n-1$, we get $0 = 0$. This shows that $x_0 = x_{n(2n-1)}$ is well-defined. Thus, it remains to determine x_0 so that $a'_0 = y$. We claim that choosing

$$x_0 = \frac{S}{n} - a_n - a_{2n-1}$$

would work. Note that

$$a'_n = a_n + \sum_{j=0}^{n-1} x_j, \quad a'_{2n-1} = a_{2n-1} + \sum_{j=n}^{2n-1} x_j.$$

Thus, if $a'_0 = z$, then we have

$$2z = a'_n + a'_{2n-1} = a_n + a_{2n-1} + x_0 + \sum_{j=1}^{2n-1} x_j = a_n + a_{2n-1} + x_0 + \frac{(2n-1)z - (S-y)}{n},$$

which implies that $z = y$. More explicitly, we can take

$$x_i = \frac{S}{n} - a_i - a_{i+n}.$$

(2) Next we show that $2n-2$ days are not enough using linear algebra. For the sake of contradiction, suppose otherwise that $2n-1$ days work.

Consider the vector space $V = \mathbb{Q}^{2n}$. Let $\mathbf{1}$ be the all-one vector in V. For each subset $A \subset \{1, 2, \dots, 2n\}$, let $\mathbf{1}_A$ be the indicator vector of A.

Let

$$U = \{u = (a_1, \dots, a_{2n}) \in \mathbb{Z}^{2n} : \quad a_i \ge 0, n \mid (a_1 + a_2 + \cdots + a_{2n})\},$$

that is, the collection of vectors representing the number of sweets the children have at the start. Note that $U \cup \{\mathbf{1}\}$ spans V. Indeed, if $v = (v_1, \dots, v_{2n}) \in V$, then there is a sufficiently large $\lambda \in \mathbb{Q}$ such that $v + \lambda \cdot \mathbf{1}$ consists of positive entries, and there is a sufficiently large integer M such that $M(v + \lambda \cdot \mathbf{1}) \in U$.

Let $u \in U$. By the assumption, there are integers $c_1, c_2, \dots, c_{2n-2}$ and n-subsets $A_1, A_2, \dots, A_{2n-2}$ of $\{1, 2, \dots, 2n\}$, such that if c_i sweets are given to children with label in A_i on the i-th day, then eventually all children obtain the same number of sweets. Alternatively, this means that

$$u + c_1 \cdot \mathbf{1}_{A_1} + \cdots + c_{2n-2} \cdot \mathbf{1}_{A_{2n-2}} = \lambda \cdot \mathbf{1}$$

for some integer λ.

Therefore,

$$U \subseteq \bigcup_{A_1, A_2, \dots, A_{2n-2}} \operatorname{span}\{\mathbf{1}_{A_1}, \mathbf{1}_{A_2}, \dots, \mathbf{1}_{A_{2n-2}}, \mathbf{1}\},$$

where the union is taken over all collections of n-subsets $A_1, A_2, \ldots, A_{2n-2}$ of $\{1, 2, \ldots, 2n\}$. It follows that

$$\mathbb{Q}^{2n} = \operatorname{span}(U \cup \{\mathbf{1}\}) \subseteq \bigcup_{A_1, A_2, \ldots, A_{2n-2}} \operatorname{span}\{\mathbf{1}_{A_1}, \mathbf{1}_{A_2}, \ldots, \mathbf{1}_{A_{2n-2}}, \mathbf{1}\},$$

that is, $\mathbb{Q}^{2n}$ can be covered by finitely many subspaces with dimension at most $2n-1$, which is absurd.

Editor's Comment. We refer to this Math Stackexchange post for a relevant discussion on covering a vector space by proper subspaces.

4803. *Proposed by Nguyen Viet Hung.*

Find all non-negative integers a, b, c and pairs (p, q) of prime numbers satisfying

$$p^{2a} + q^{2b} = (2c+1)^2.$$

We received 27 submissions, only 9 of which were completely correct. We present Oliver Geupel's solution.

By inspection, $(1, 2, 2, 3, 2)$ and $(2, 1, 2, 2, 3)$ are solutions for (a, b, c, p, q). We prove that there are no further solutions. Assume that a, b, c, p, and q satisfy the given conditions. By inspection, we see that a and b are positive. Since $(2c+1)^2$ is odd, exactly one of the primes p and q is even, that is, equal to 2. By the symmetry $(a, p) \leftrightarrow (b, q)$ it is enough to consider the case $q = 2$. We have

$$4^b = (2c + 1 - p^a)\,(2c + 1 + p^a)\,.$$

Hence there is an integer m where $1 \le m < b$ such that $2c + 1 - p^a = 2^m$ and $2c + 1 + p^a = 2^{2b-m}$. Then

$$2p^a = 2^{2b-m} - 2^m = 2^m\left(4^{b-m} - 1\right).$$

We obtain $m = 1$ and

$$p^a = \left(2^{b-1} - 1\right)\left(2^{b-1} + 1\right).$$

Therefore, p^a has two divisors that differ by 2. Thus, $p = 3$, $a = 1$, $b = 2$, and $c = 2$.

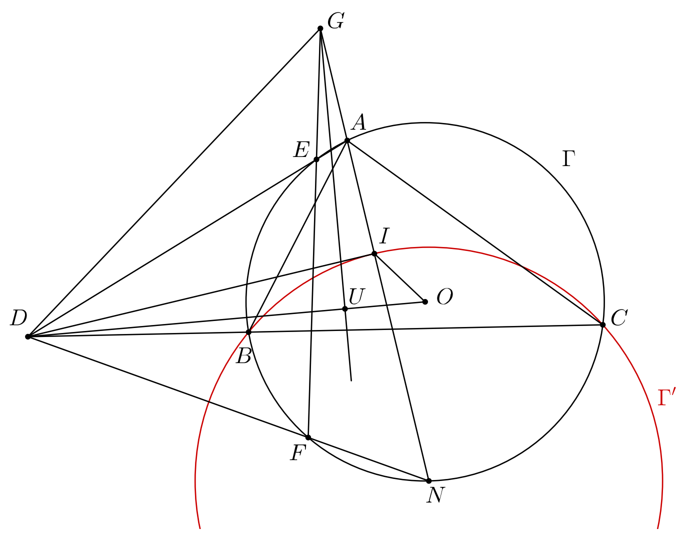
Editor's Comments. There were two unfortunately common errors: 10 solvers found only one solution, and 12 solvers overlooked the possibility that a or b is 0 when deducing p and q have opposite parity (consider $2^0 + 2^1$).

Most solvers gave some variation of the elementary argument above; some noted that $(p^a, q^b, 2c+1)$ is a Pythagorean triple. B. Roy was unique in reducing to Catalan's equation. S. Dutta pointed out this problem generalizes Problem 8 on the 1992 Indian National Mathematical Olympiad.

4804. *Proposed by Jimmy Zhao.*

Let ABC be an acute triangle with incenter I and circumcenter O. Let D be a point on BC such that $AI \perp ID$. Let N be the midpoint of minor arc BC. DA and DN meet the circumcircle of ABC at E, F respectively. Let EF meet AI at G. Show that $OI \perp DG$.

We received 8 correct solutions. We present the solution by Michel Bataille.



Let Γ and Γ' be the circumcircles of $\triangle ABC$ and $\triangle BIC$, respectively. We recall that N is on the line AI and that Γ' is centered at N. The point D, which is on the radical axis BC of Γ and Γ', has the same power $p = DB \cdot DC$ with respect to these two circles. Since $DI \perp IN$, the line DI is tangent to Γ' at I, hence $p = DI^2$. We also have $p = DE \cdot DA = DF \cdot DN$. Therefore, if $\mathbf{I}$ denotes the inversion in the circle with center D and radius DI, we have $\mathbf{I}(I) = I$, $\mathbf{I}(B) = C$, $\mathbf{I}(A) = E$, $\mathbf{I}(N) = F$.

Since the circle Γ is its own inverse, the point $U = \mathbf{I}(O)$ is the foot of the polar of D with respect to this circle. But the quadrilateral $AEFN$ is inscribed in Γ and AE,NF intersect at D, hence the lines AN and EF intersect on the polar of D. It follows that this polar is the perpendicular to DO through G and intersects DO at U.

Now, $\angle DUG = \angle DIG = 90°$, hence $U = \mathbf{I}(O)$ and $I = \mathbf{I}(I)$ are on the circle with diameter DG. Thus, this circle is the inverse of the line IO and therefore $DG \perp IO$, as desired.

4805. *Proposed by Goran Conar.*

Let $a, b, c > 0$ be real numbers such that $ab + bc + ca = 4abc$. Prove

$$\frac{1}{\sqrt[a]{a}} + \frac{1}{\sqrt[b]{b}} + \frac{1}{\sqrt[c]{c}} \geq 4\sqrt[3]{\frac{4}{3}}.$$

We received 20 submissions of which 17 were correct and complete. We present the solution by Oliver Geupel.

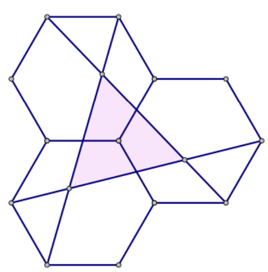
Let $x = 1/a$, $y = 1/b$, and $z = 1/c$. Then $x + y + z = (ab + bc + ca)/(abc) = 4$. Using the strict convexity of $t \mapsto e^{t\log t} = t^t$ for $t > 0$, we have

$$\frac{1}{\sqrt[a]{a}} + \frac{1}{\sqrt[b]{b}} + \frac{1}{\sqrt[c]{c}} = x^x + y^y + z^z \geq 3\left(\frac{x+y+z}{3}\right)^{(x+y+z)/3} = 3\left(\frac{4}{3}\right)^{4/3} = 4\sqrt[3]{\frac{4}{3}}.$$

By the equality condition of Jensen's inequality, here equality holds only if we have $a = b = c = \frac{3}{4}$.

4806. *Proposed by Arsalan Wares.*

The figure shows three congruent, non-overlapping regular hexagons with a vertex that is common to all three hexagons. Certain vertices of the hexagons are connected as shown and these line segments enclose the shaded triangle. If each side of each regular hexagon is 1, determine the exact area of the shaded triangle.



We received 26 solutions based on many different ideas and theorems. The following solution is by Brian D. Beasley.

We model the hexagons with their common intersection point at the origin, placing the other vertices of the three sides common to that point at $(-1, 0)$, $(1/2, \sqrt{3}/2)$, and $(1/2, -\sqrt{3}/2)$. Then the sides of the shaded triangle lie on the following lines:

ℓ_1, connecting $(-1, \sqrt{3})$ and $\left(\frac{3}{2}, -\frac{\sqrt{3}}{2}\right)$; its equation is $y = -\frac{3\sqrt{3}}{5}(x+1) + \sqrt{3}$;

ℓ_2, connecting $\left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right)$ and $(2, 0)$; its equation is $y = \frac{\sqrt{3}}{7}(x - 2)$;

ℓ_3, connecting $(0, \sqrt{3})$ and $(-1, -\sqrt{3})$; its equation is $y = 2\sqrt{3}x + \sqrt{3}$.

Lines ℓ_1 and ℓ_2 intersect at $A = \left(\frac{12}{13}, -\frac{2\sqrt{3}}{13}\right)$;

lines ℓ_2 and ℓ_3 intersect at $B = \left(-\frac{9}{13}, -\frac{5\sqrt{3}}{13}\right)$;

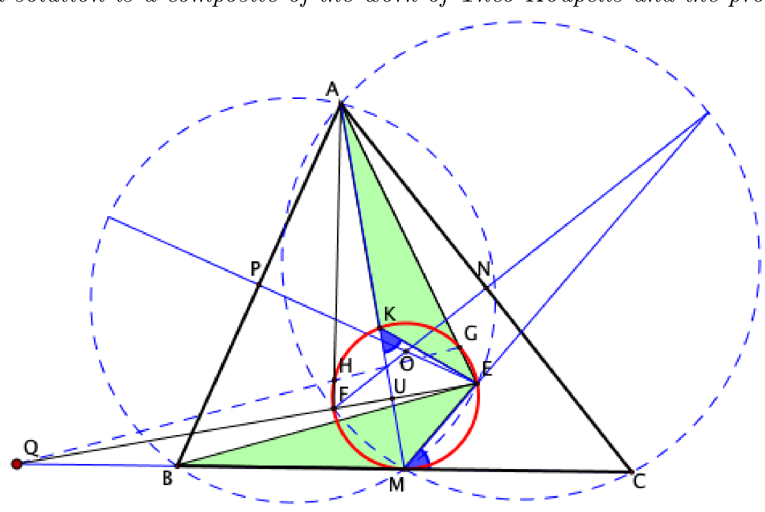
lines ℓ_3 and ℓ_1 intersect at $C = \left(-\frac{3}{13}, \frac{7\sqrt{3}}{13}\right)$.

Thus the side lengths of triangle ABC are $AB = BC = CA = \frac{6}{\sqrt{13}}$, so the triangle is equilateral. Hence its area is $\frac{\sqrt{3}}{4}\left(\frac{6}{\sqrt{13}}\right)^2 = \frac{9\sqrt{3}}{13}$.

4807. *Proposed by Dong Luu.*

Let an acute triangle ABC be inscribed in the circle (O). M, N, P are respectively the midpoints of BC, CA, AB. E is the intersection of ray PO and circle (ABM); F is the intersection of ray NO and circle (ACM). Suppose that AE, AF respectively intersect the circle (MEF) for the second time at the points G, H (G is different from E, H is different from F). Prove that GH, EF, BC are concurrent.

We received four submissions, of which two relied on computer calculations that were basically correct, but included a step where there was division by zero. Our featured solution is a composite of the work of Theo Koupelis and the proposer.



When $AB = AC$ the configuration is symmetric about the line AM, in which case the lines GH, EF, BC are parallel (and are concurrent in a point at infinity). So let us assume that $AB \neq AC$, so that EF intersects BC in a finite point, call it Q. Define K to be the point on the line segment AM for which $AK = BM$. We have

$\angle KAE = \angle MAE = \angle MBE$ (inscribed in the circle $ABME$), and $BE = AE$ (because PE is the perpendicular bisector of AB), whence the triangles BME and AKE are congruent. It follows that $EK = EM$; similarly, $FK = FM$. Consequently, EF is the perpendicular bisector of KM.

Furthermore, $\angle CME = \angle MKE$ (corresponding external angles at M and K), while $\angle MKE = \angle EMK$ (base angles of the isosceles triangle EMK), which implies that ME bisects $\angle CMK$. Similarly, MF bisects $\angle KMB$. It follows immediately that

- $\angle EMF = 90^\circ$,
- EF is a diameter of the circle (MEF),
- K also lies on this circle, and
- $EH \perp FH$ and $FG \perp EG$ (angles inscribed in a semicircle), so that EH and FG are altitudes of ΔAFE, and finally,
- QC is tangent to this circle at M (because the inscribed $\angle MKE$ equals the angle between QC and the chord ME).

We are now ready to apply Menelaus' theorem to the triangle AFE to prove that the line GH also passes through Q. First, we denote by U the point where AM intersects QE. Referring to the right triangles FEG, FEH, EAU, and FAU, we see that

$$\frac{GE}{HF} = \frac{FE \cdot \cos E}{FE \cdot \cos F} = \frac{EU/AE}{FU/AF} = \frac{EU}{FU} \cdot \frac{AF}{AE}. \tag{1}$$

Furthermore, in the right triangles AFG and AHE we have $\cos A = \frac{AG}{AF} = \frac{AH}{AE}$, or

$$\frac{AH}{GA} = \frac{AE}{AF}. \tag{2}$$

Finally, because MF and ME are the internal and external bisectors of $\angle UMQ$ in triangle UMQ, we have

$$\frac{EU}{FU} = \frac{QE}{FQ}. \tag{3}$$

Rewriting (1) then substituting (2) and (3) we find that

$$1 = \frac{GE}{HF} \cdot \frac{FU}{EU} \cdot \frac{AE}{AF} = \frac{AH}{HF} \cdot \frac{FQ}{QE} \cdot \frac{EG}{GA}.$$

Because Q is on the tangent BC to circle (MEF) (and therefore on the line EF external to the chord FE) while our assumption that the angles of ΔABC are acute insures that H and G lie in the interior of the sides AF and AE, respectively, of ΔAFE, Menelaus' theorem implies that Q, H, G are collinear. Consequently, the lines GH, EF, BC are concurrent at Q.

Editor's comment. Because neither computer argument explicitly distinguished the case in which $AB = AC$ (and the lines GH, EF, BC are not concurrent in a finite point), it is clear that there was an implicit division by zero somewhere in the intricate calculations.

4808. *Proposed by George Stoica.*

Find all $x, y, z \in \mathbb{N} \setminus \{0\}$ with $(x, y, z) = 1$ and $x + y + z \mid x^k + y^k + z^k$ for $k \in \{2, 4, 6\}$.

We received six solutions. We present the one by C. R. Pranesachar, lightly edited.

We have

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + xz + yz)$$

and

$$x^6 + y^6 + z^6 = (x^2 + y^2 + z^2)(x^4 + y^4 + z^4 - x^2y^2 - x^2z^2 - x^2y^2) + 3x^2y^2z^2.$$

Since $x + y + z$ divides both $x^2 + y^2 + z^2$ and $x^6 + y^6 + z^6$, we find

$$(x + y + z)|2(xy + xz + yz) \tag{1}$$

and

$$(x + y + z)|3x^2y^2z^2. \tag{2}$$

If a prime $p \geq 5$ divides $x + y + z$, then (2) implies that p divides $x^2y^2z^z$. Then p divides one of x, y, or z, say p divides x. From (1) we then get that p divides yz and thus p divides one of y or z, say y. Since p divides $x + y + z$ it then also divides z, a contradiction to $\gcd(x, y, z) = 1$. A similar argument show that 4 and 9 cannot divide $x + y + z$. Since $x, y, z \geq 1$, we find $x + y + z \in \{3, 6\}$.

Assuming $x \leq y \leq z$ the only options for (x, y, z) are then $(1, 1, 1)$, $(1, 2, 3)$, and $(1, 1, 4)$. Clearly $(1, 1, 1)$ is a solution, whereas $(1, 2, 3)$ is not, since $1 + 2 + 3$ does not divide $1^2 + 2^2 + 3^2 = 14$. Finally $1^2 + 1^2 + 4^2 = 18$, $1^4 + 1^4 + 4^4 = 258$, and $1^6 + 1^6 + 4^6 = 4098$ are all multiples of 6, thus $(1, 1, 4)$ is a solution as well.

4809. *Proposed by Daniel Sitaru.*

Let $a, b > 0$. Find

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \left(\int_0^1 \frac{x^k}{ax + b} dx \right)^{-1} \left(\int_0^1 \frac{x^k}{bx + a} dx \right)^{-1}.$$

We received 9 solutions, one of which was incorrect. We present the solution by Yunyong Zhang.

For $x \in [0, 1]$,

$$(a + b)x \leq (ax + b) \leq (a + b), \quad (a + b)x \leq (bx + a) \leq (a + b),$$

implying that

$$\frac{1}{(a+b)(k+1)} = \int_0^1 \frac{x^k}{a+b}\mathrm{d}x \leq \int_0^1 \frac{x^k}{ax+b}\mathrm{d}x \leq \int_0^1 \frac{x^{k-1}}{a+b}\mathrm{d}x = \frac{1}{(a+b)k}$$

and

$$\frac{1}{(a+b)(k+1)} = \int_0^1 \frac{x^k}{a+b}\mathrm{d}x \leq \int_0^1 \frac{x^k}{bx+a}\mathrm{d}x \leq \int_0^1 \frac{x^{k-1}}{a+b}\mathrm{d}x = \frac{1}{(a+b)k}.$$

Multiplying gives

$$(a+b)^2k^2 \leq \frac{1}{\left(\int_0^1 \frac{x^k}{ax+b}\mathrm{d}x\right)\left(\int_0^1 \frac{x^k}{bx+a}\mathrm{d}x\right)} \leq (a+b)^2(k+1)^2,$$

and summing then gives

$$\frac{(a+b)^2}{6}(2n^3+3n^2+n) < \sum_{k=1}^n \frac{1}{\left(\int_0^1 \frac{x^k}{ax+b}\mathrm{d}x\right)\left(\int_0^1 \frac{x^k}{bx+a}\mathrm{d}x\right)} < \frac{(a+b)^2}{6}(2n^3+9n^2+13n).$$

We thus have

$$\lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^n \left(\int_0^1 \frac{x^k}{ax+b}\mathrm{d}x\right)^{-1}\left(\int_0^1 \frac{x^k}{bx+a}\mathrm{d}x\right)^{-1} = \frac{(a+b)^2}{3}.$$

4810. *Proposed by Goran Conar.*

Let $a_1, a_2, \ldots, a_n > 0$ be real numbers such that $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$, $n > 1$. Prove that

$$\frac{a_2^2+a_3^2+\cdots+a_n^2}{(a_2+a_3+\cdots+a_n)^3} + \frac{a_1^2+a_3^2+\cdots+a_n^2}{(a_1+a_3+\cdots+a_n)^3} + \cdots + \frac{a_1^2+a_2^2+\cdots+a_{n-1}^2}{(a_1+a_2+\cdots+a_{n-1})^3} \geq \frac{n\sqrt{n}}{(n-1)^2}\,.$$

We received 18 submissions and 17 of them were all complete and correct. We present 2 different solutions by the majority of solvers.

Solution 1.

By Cauchy-Schwarz inequality, we have

$$(n-1)(a_2^2+\cdots+a_n^2) \geqslant (a_2+\cdots+a_n)^2 \implies \frac{a_2^2+\cdots+a_n^2}{(a_2+\cdots+a_n)^3} \geqslant \frac{1}{(n-1)(a_2+\cdots+a_n)}\,.$$

Letting

$$S = \frac{a_2^2+\cdots+a_n^2}{(a_2+\cdots+a_n)^3} + \cdots + \frac{a_1^2+\cdots+a_{n-1}^2}{(a_1+\cdots+a_{n-1})^3},$$

we have

$$S \geqslant \frac{1}{n-1}\left(\frac{1}{a_2+\cdots+a_n} + \cdots + \frac{1}{a_1+\cdots+a_{n-1}}\right).$$

By AM-HM inequality, we get

$$S \geqslant \frac{n^2}{(n-1)[(a_2+\cdots+a_n)+\cdots+(a_1+\cdots+a_{n-1})]} = \frac{n^2}{(n-1)^2(a_1+\cdots+a_n)}\,.$$

Again by Cauchy-Schwarz we have

$$n = n(a_1^2+\cdots+a_n^2) \geqslant (a_1+\cdots+a_n)^2 \implies \frac{1}{a_1+\cdots+a_n} \geqslant \frac{1}{\sqrt{n}}\,.$$

Thus

$$S \geqslant \frac{n\sqrt{n}}{(n-1)^2}$$

as required.

Solution 2.

By Cauchy-Schwarz inequality, we have

$$(a_2+a_3+\cdots+a_n)^2 \le (n-1)(a_2^2+a_3^2+\cdots+a_n^2).$$

It follows from the given condition $a_1^2+a_2^2+\cdots+a_n^2=1$ that

$$\frac{a_2^2+a_3^2+\cdots+a_n^2}{(a_2+a_3+\cdots+a_n)^3} \ge \frac{1-a_1^2}{[(n-1)(1-a_1^2)]^{3/2}} = \frac{1}{(n-1)^{3/2}}\cdot\frac{1}{\sqrt{1-a_1^2}},$$

with similar expressions for the other terms. Thus, it is sufficient to show that

$$\frac{1}{(n-1)^{3/2}}\sum_{i=1}^n \frac{1}{\sqrt{1-a_i^2}} \ge \frac{n\sqrt{n}}{(n-1)^2} \quad \text{or} \quad \sum_{i=1}^n \frac{1}{\sqrt{1-a_i^2}} \ge \frac{n\sqrt{n}}{\sqrt{n-1}}.$$

Let $f(t)=\frac{1}{\sqrt{1-t}}$, where $0<t<1$. Then $f'(t)=\frac{1}{2(1-t)^{3/2}}>0$, and $f''(t)=\frac{3}{4(1-t)^{5/2}}>0$. Thus, $f(t)$ is an increasing, convex function in $(0,1)$. Using Jensen's inequality for $t_i=a_i^2,\ i=1,2,\ldots,n$, we get

$$\sum_{i=1}^n \frac{1}{\sqrt{1-a_i^2}} = \sum_{i=1}^n f(t_i) \ge nf\left(\frac{t_1+t_2+\cdots+t_n}{n}\right) = nf\left(\frac{1}{n}\right) = \frac{n\sqrt{n}}{\sqrt{n-1}},$$

which is the desired inequality. Equality occurs when $a_1=a_2=\cdots=a_n=1/\sqrt{n}$.