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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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MathemAttic

No. 45

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 30, 2023**.

MA221. Without a calculator, determine which is larger $29\sqrt{14} + 4\sqrt{15}$ or 124.

MA222. A solid sphere is perfectly embedded in a cube where each side is 6 cm long. What is the amount of unoccupied space within the cube?

MA223. The *standard form* of a quadratic equation is $f(x) = ax^2 + bx + c$ while the *vertex form* of a quadratic equation is $f(x) = a(x - h)^2 + k$. Derive the vertex (h, k) in terms of a, b, and c.

MA224. *Proposed by Aravind Mahadevan, Hong Kong.*

The area of a triangle is $10\sqrt{3}$ and its perimeter is 20. If one of the angles is 60°, find the lengths of the sides of the triangle.

MA225. How many ways are there to assign the labels A, B, C, D, E, F to the vertices of a hexagon so that none of the pairs AB, CD, or EF form an edge?

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 juillet 2023**.

MA221. Sans calculatrice, déterminer lequel de $29\sqrt{14}+4\sqrt{15}$ et 124 est le plus élevé.

MA222. Une sphère solide est parfaitement emboîtée dans un cube dont les côtés mesurent 6 cm de long. Quelle est la quantité d'espace inoccupé dans ce cube?

MA223. La *forme générale* d'une équation quadratique est $f(x) = ax^2+bx+c$ tandis que la *forme canonique* d'une équation quadratique est $f(x) = a(x-h)^2+k$. Exprimez le sommet (h, k) en fonction des paramètres a, b et c.

MA224. *Proposé par Aravind Mahadevan, Hong Kong.*

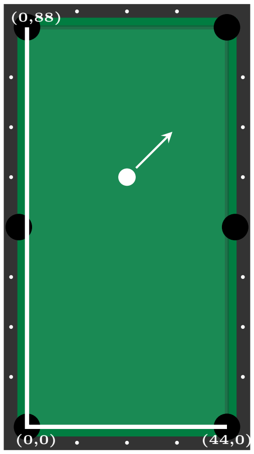
La surface d'un certain triangle est $10\sqrt{3}$ et son périmètre est 20. Si un des angles égale 60°, déterminer les longueurs des côtés du triangle.

MA225. Combien y a-t-il de façons d'assigner les étiquettes A, B, C, D, E et F à chacun des sommets d'un hexagone de sorte qu'aucune des paires AB, CD ou EF ne forme une arête ?

MathemAttic
SOLUTIONS

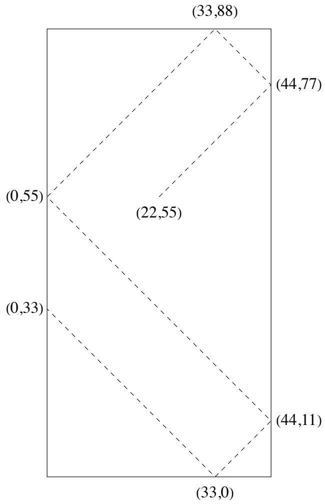
Statements of the problems in this section originally appear in 2022: 48(10), p. 584–586.

MA196. If a cue ball placed at the coordinates $(22, 55)$ was evenly struck so it hit the 1^{st} wall at the point $(44, 77)$, and bounced off with no spin, what are the coordinates when the ball strikes the 6^{th} wall?



Originally question 10 from the 2019 Kansas City Area Teachers of Mathematics High School Math Contest.

We received 5 submissions of which 5 were correct and complete. We present the solution by Brian Bradie.



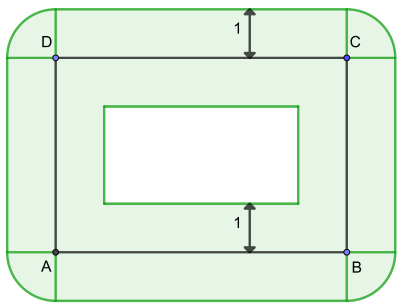
Assuming the cue ball has been struck sufficiently hard to make repeated contact with the walls of the table and that each time the cue ball bounces off the wall it does so with no spin so that the angle of incidence is equal to the angle of reflection, the cue ball will follow the path indicated in the figure above. The sixth contact with the wall will occur at coordinates $(0, 33)$. Interestingly, were the ball to continue bouncing off the walls, it would cycle through the points $(44, 77)$, $(33, 88)$, $(0, 55)$, $(44, 11)$, $(33, 0)$, and $(0, 33)$.

MA197. A rectangle has length 4 and width 6. A new shape is formed by taking the set of all points that lie within one unit of a point on the boundary of the rectangle. Compute the area of this new shape.

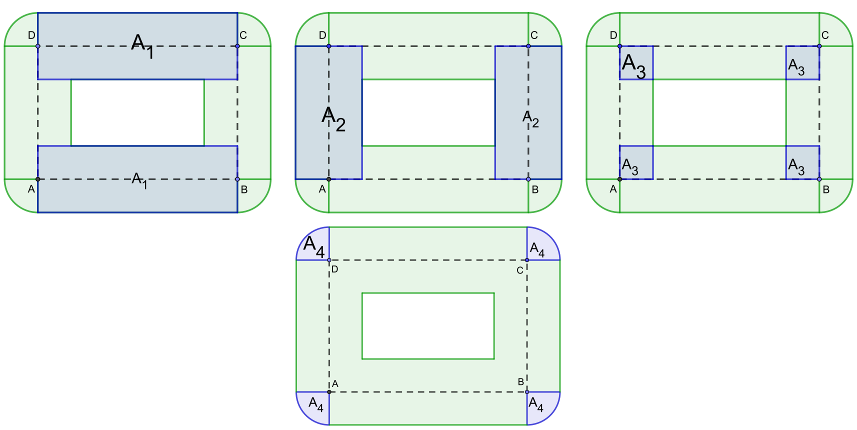
Originally question 8 from the 34th University of Alabama High School Mathematics Tournament: Team Competition, 2015.

We received 4 submissions of which 2 were correct and complete. We present the solution by Vishwesh Ravi Shrimali, edited.

In the figure below we show the 4×6 rectangle $ABCD$, and shade in the points which are distance at most 1 unit from a point on the boundary of $ABCD$. This is the shape whose area we want to calculate.



We use rectangles and circle quarters as indicated by the diagrams below:



The shaded area is equal to

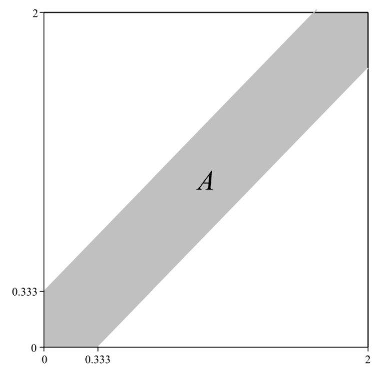
$$\begin{aligned} 2A_1 + 2A_2 - 4A_3 + 4A_4 &= 2(2 \cdot 6) + 2(2 \cdot 4) - 4(1^2) + 4\left(\frac{\pi \cdot 1^2}{4}\right) \\ &= 36 + \pi \quad \text{square units.} \end{aligned}$$

MA198. Two points P and Q are randomly selected in the interval $[0, 2]$. What is the probability that P and Q are within a distance of $\frac{1}{3}$ from each other?

Originally question 2 from the 2016 Kansas MAA Undergraduate Mathematics Competition.

We received 7 submissions, of which 5 are correct and 2 were incomplete. We present the solution by Henry Ricardo.

We see that P and Q are independent random variables. Furthermore, the square of side 2 given by $\{(P, Q) : 0 \le P \le 2,\ 0 \le Q \le 2\}$ represents all the equally likely possibilities of the values of P and Q.



The area A is bounded by the two lines $y = x + \frac{1}{3}$ and $y = x - \frac{1}{3}$, so that inside A we have $|P - Q| \le \frac{1}{3}$. It follows that the distance between the points will be less than or equal to $1/3$ only if the point (P, Q) lies in region A. Thus the probability we seek is given by the ratio of the area A to the area of the square:

$$\frac{4 \ - \ 2\left(\frac{25}{18}\right)}{4} \ = \ \frac{11}{36}.$$

MA199. *Proposed by Aravind Mahadevan.*

If a, b, and c are the roots of the equation $x^3 + 6x^2 - 52x + 8 = 0$, find the value of

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

We received 13 submissions, of which 11 were correct and complete. We present the solution provided by Bing Jian.

Note that equation $x^3 + 6x^2 - 52x + 8 = 0$ can be rearranged as

$$(x + 2)^3 = x^3 + 6x^2 + 12x + 8 = 64x$$

or equivalently

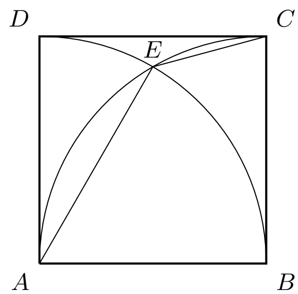
$$\sqrt[3]{x} = \frac{x + 2}{4}.$$

Also note that by Viéta's formula, we have $a+b+c=-6$, therefore

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}=\frac{(a+2)+(b+2)+(c+2)}{4}=\frac{a+b+c+6}{4}=0.$$

MA200. *Proposed by K. S. Bijesh.*

In the figure below $ABCD$ is a square. Arcs BD and AC intersect at E. Determine the exact value of $\dfrac{AE}{EC}$.



We received 11 correct solutions. The following is the solution by Brian Bradie.

Let s denote the length of the side of the square. Then $AE=s$, and by symmetry,

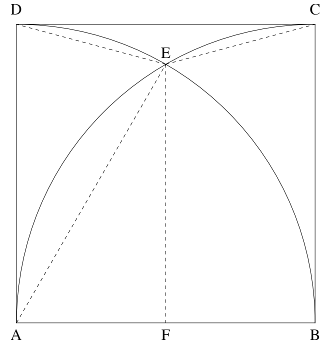
$$AF=\frac{s}{2} \quad \text{and} \quad EC=ED$$

(see the figure below). Thus, $\angle EAF=60^\circ$ and $\angle DAE=30^\circ$. By the Law of Cosines,

$$ED^2=2s^2-2s^2\cos 30^\circ=s^2(2-\sqrt{3}),$$

or $ED=s\sqrt{2-\sqrt{3}}$. Finally,

$$\frac{AE}{EC}=\frac{AE}{ED}=\frac{1}{\sqrt{2-\sqrt{3}}}=\sqrt{2+\sqrt{3}}=\frac{1+\sqrt{3}}{\sqrt{2}}=2\cos 15^\circ.$$



PROBLEM SOLVING VIGNETTES

No. 27

Shawn Godin

Some Square Roots

In the last column, we looked at three problems that had appeared on 2022 Canadian mathematics competitions that I had come across while preparing a talk for a professional development day. There is never a short supply of nice contest problems, so in this issue we continue the same theme and look at another contest problem from 2022.

The problem in question was problem 1 from the 2022 *W.J. Blundon Mathematics Contest*. The *W.J. Blundon Mathematics Contest* is a 10 question, full solution contest written in February and hosted by Memorial University. More information and copies of past contests can be found on their website:
https://www.mun.ca/math/community/wj-blundon-contest/

> *Let*
> $$A = \sqrt{19} + \sqrt{99} \text{ and } B = \sqrt{20} + \sqrt{98}.$$
> *Determine which number is larger and justify your conclusion.*

With a calculator, the problem is trivial. However, not surprisingly, calculators are not allowed on this competition, so we need some insight in order to solve this problem. Since $A, B > 0$, then if $A > B$, so must $A^2 > B^2$ and vice versa. Squaring yields

$$A^2 = 19 + 99 + 2\sqrt{19 \times 99} = 118 + 2\sqrt{1881}$$
$$B^2 = 20 + 98 + 2\sqrt{20 \times 98} = 118 + 2\sqrt{1960}$$

and since $1960 > 1881$, $B^2 > A^2$ and therefore $B > A$.

We can make sense of this result by observing that perfect squares get more "spread out" as numbers get larger. That is,

$$4^2 = 16 < 19 < 20 < 25 = 5^2$$

and

$$9^2 = 81 < 98 < 99 < 100 = 10^2$$

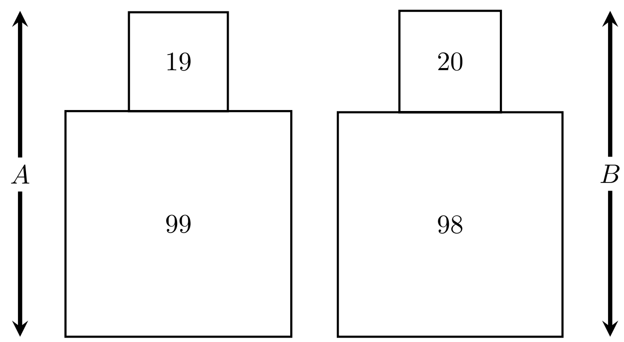
Consider the function $f(x) = \sqrt{x}$. For $16 \le x \le 25$, the function changes, on average, by $\frac{1}{9}$ each time x increases by 1. However, when $81 \le x \le 100$, the function changes, on average, by $\frac{1}{19}$ for the same increase in x. Thus

$$\frac{1}{19} \approx \sqrt{99} - \sqrt{98} < \sqrt{20} - \sqrt{19} \approx \frac{1}{9}$$

and therefore
$$A = \sqrt{19} + \sqrt{99} < \sqrt{20} + \sqrt{98} = B.$$

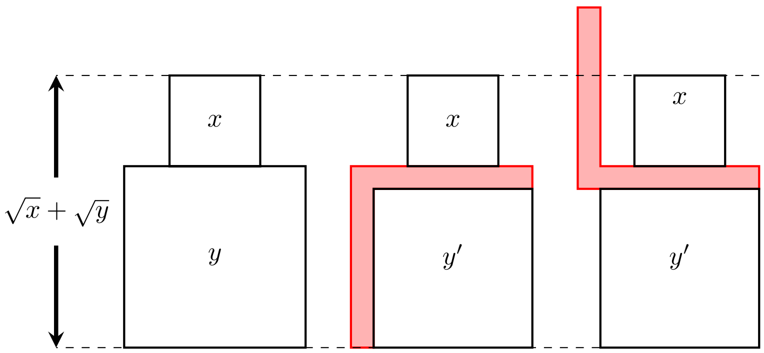
The first thing that pops to my mind is, *is there a way to generalize this?* If we set
$$A = \sqrt{a} + \sqrt{b} \text{ and } B = \sqrt{c} + \sqrt{d},$$
where $a + b = c + d$ and the four values are distinct, can we decide which of A and B is larger? The original problem corresponds to $a = 19$, $b = 99$, $c = 20$, and $d = 98$.

The condition $a + b = c + d$ forces one set of numbers to be *between* the other two (why?). Thus, we can assume without loss of generality that $a < c < d < b$. If we imagine a, b, c, and d are areas of squares, then $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, and $\sqrt{d}$ are the side lengths of the squares. Thus if we think of the situation as two sets of two squares. In each set the total area is the same, however the squares are stacked on top of each other, so we are deciding which stack is taller. For example, the original problem in the form yields the diagram below.

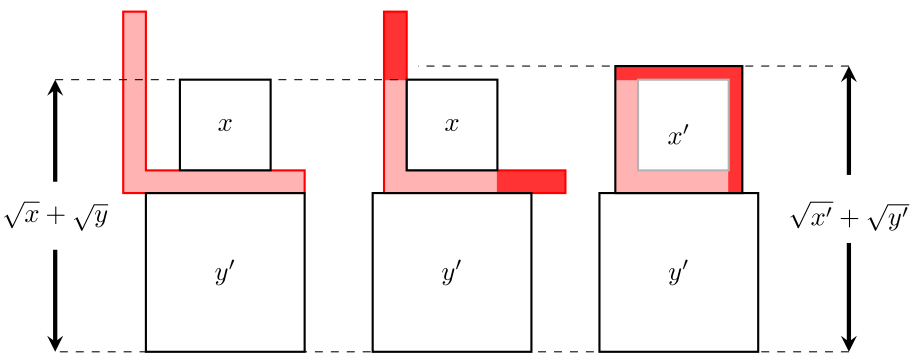


In this case, it is difficult to see which stack is taller.

If we consider all expressions of the form $\sqrt{x} + \sqrt{y}$, with $x \le y$, where
$$x + y = a + b = c + d = k$$
for some constant k, then let's observe what happens when we transform one expression of this form into another. Suppose we are interested in $\sqrt{x'} + \sqrt{y'}$, with $x' \le y'$,
$$x' + y' = x + y = k$$
and $y' < y$. If we take our original stack, we can imagine taking a strip of constant width off of the larger square of area $y - y'$, which would leave a square of area y'. However, to keep the total area constant, that area must be added to the smaller square to yield a square with area $x' = x + (y - y')$. Since $y > x$, the strip taken off of the larger square is "too long" to fit on the smaller square. Nonetheless, if the smaller square was placed on the strip on top of the "new" square, the height of the stack made up of the square with area y', the strip, and the square with area x would be the same as the original stack as shown in the diagram.



However, the area of the strip must be added to the smaller square to create the square with area $x' = x + (y - y')$. As the strip is too long, the extra area would be used to make the strip *wider*, and thus the new stack is taller.



Hence, if $x' + y' = x + y$ and $x < x' < y' < y$, then $\sqrt{x'} + \sqrt{y'} > \sqrt{x} + \sqrt{y}$. In other words, the closer in area the two squares are, the taller the stack (given the total area of the two squares is constant).

If we go back to our original analysis in the general form we get

$$A^2 = a + b + 2\sqrt{ab}$$
$$B^2 = c + d + 2\sqrt{cd}.$$

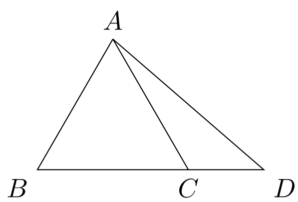
Since $a + b = c + d$, the question is resolved by deciding which of $\sqrt{ab}$ and $\sqrt{cd}$ is larger, which is equivalent to deciding which of ab and cd is larger. If we now consider $a + b = c + d = s$, where s is the semiperimeter of a rectangle, we are trying to decide which of two rectangles with the same perimeter has the largest area. Most students will have seen this type of problem early in their high school career when studying quadratics as well as later on when studying calculus. We know that the area of a rectangle with fixed perimeter is maximized when the rectangle is a square. Therefore in our case, the set of numbers that are closest together will provide our maximum.

The problem could also have been resolved using Calculus or by looking at power series. I will leave further exploration of the problem to interested readers.

It is always interesting to dig deeper into problems. It is one thing to be able to answer a question, but it is another to understand *why* things are behaving as they are in the given situation. Looking at problems under a different light often reveals new insights. Although these are not things you would want to do *while* writing a math contest, exploring a problem deeper will help you make connections between concepts. In some cases, the extra time you spent on a problem *outside* of contest time will help you with a problem you are faced *during* a contest. For your amusement, below are a few more contest questions from 2022.

. . . .

1. (2022 *Euclid Contest*, #7 (b)) In the diagram, ΔABD has C on BD. Also, $BC = 2$, $CD = 1$, $\frac{AC}{AD} = \frac{3}{4}$, and $\cos(\angle ACD) = -\frac{3}{5}$. Determine the length of AB.



2. (2022 *Pascal Contest*, #20) A pizza is cut into 10 pieces. Two of the pieces are each $\frac{1}{24}$ of the whole pizza, four are each $\frac{1}{12}$, two are each $\frac{1}{8}$, and two are each $\frac{1}{6}$. A group of n friends share the pizza by distributing all of these pieces. They do not cut any of these pieces. Each of the n friends receives, in total, an equal fraction of the whole pizza. The sum of the values of n with $2 \leq n \leq 10$ for which this is *not* possible is

   (A) 31 **(B)** 35 **(C)** 40 **(D)** 39 **(E)** 36

3. (2022 *Cayley Contest*, #24) A cube with edge length 8 is balanced on one of its vertices on a horizontal table such that the diagonal from this vertex through the interior of the cube to the farthest vertex is vertical. When the sun is directly above the top vertex, the shadow of the cube on the table is a regular hexagon. The area of this shadow can be written in the form $a\sqrt{b}$, where a and b are positive integers and b is not divisible by any perfect square larger than 1. What is the value of $a + b$?

The Third Marble

Geoffrey W. Brown and Adam C. Brown

A simple version of the following problem appears as Exercise 3.35 in Chapter 3 of [1]. We pose the following problem:

> *An urn contains n marbles, k of which are red and the remaining $n-k$ are black. You reach into the urn and draw out three marbles, one at a time. What is the probability that the third marble drawn is red?*

It is easy to calculate the probability that the first marble drawn is red. Thus:

$$P(\text{1st marble drawn is red}) = \frac{k}{n}.$$

We may also calculate the probability that the second marble drawn is red by observing that the possibilities are "Red, Red" or "Black, Red" as a disjoint union. Thus:

$$\begin{aligned} P(\text{2nd marble drawn is red}) = P(RR) + P(BR) &= \frac{k}{n}\cdot\frac{k-1}{n-1} + \frac{n-k}{n}\cdot\frac{k}{n-1} \\ &= \frac{k^2-k+nk-k^2}{n(n-1)} \\ &= \frac{k(n-1)}{n(n-1)} = \frac{k}{n}. \end{aligned}$$

We may similarly calculate the probability that the third marble drawn is red by summing over 4 possibilities as a disjoint union. Thus

$$P(\text{3rd marble drawn is red}) = P(RRR) + P(RBR) + P(BRR) + P(BBR)$$

can be written as

$$\begin{aligned} P(\text{3rd marble drawn is red}) = \frac{k}{n(n-1)(n-2)}&[(k-1)(k-2) + (n-k)(k-1) \\ &+ (n-k)(k-1) + (n-k)(n-k-1)] \qquad (1) \end{aligned}$$

Simplifying (1) involves a careful and thoughtful application of high school algebra.

$$\begin{aligned} (1) &= \frac{k}{n(n-1)(n-2)}\left[(k-1)\left[(k-2)+(n-k)\right] + (n-k)\left[(k-1)+(n-k-1)\right]\right] \\ &= \frac{k}{n(n-1)(n-2)}\left((k-1)(n-2) + (n-k)(n-2)\right) \\ &= \frac{k(n-2)}{n(n-1)(n-2)}\left((k-1)+(n-k)\right) \\ &= \frac{k(n-1)(n-2)}{n(n-1)(n-2)} \\ &= \frac{k}{n}, \end{aligned}$$

which is a rather remarkable result.

How do we explain this remarkable result? The basic idea is not to stop the drawing at the 3rd marble, but rather to draw out all n marbles from the urn. We now define some random variables related to drawing out all n marbles. For $1 \le i \le n$, let X_i represent the colour of the ith marble drawn by letting

$$X_i = \begin{cases} 1 \text{ if the } i\text{th marble drawn is red} \\ 0 \text{ if the } i\text{th marble drawn is black} \end{cases}$$

We are now interested in the probability $P(X_3 = 1)$. By the Law of Total Probability, we can write $P(X_3 = 1)$ as the following sum:

$$\begin{aligned} &P(X_3 = 1) \\ &= \sum_{i_1, i_2, i_4, i_5, \ldots, i_n} P[X_1 = i_1, X_2 = i_2, X_3 = 1, X_4 = i_4, X_5 = i_5, \ldots, X_n = i_n] \end{aligned} \tag{2}$$

where $i_k \in \{0, 1\}$ for $k = 1, 2, 4, 5, \ldots, n$. We can regard (2) as being the joint probabilities of $\{X_1, X_2, \ldots, X_n\}$, and therefore $P(X_3 = 1)$ is seen to be a marginal probability.

We now ask the following fundamental question: Suppose we draw out all n marbles from the urn. What is the probability that the first k marbles drawn are red, and the next $n - k$ marbles drawn are black? The answer is given by the following product:

$$\begin{aligned} \left\{\frac{k}{n} \cdot \frac{k-1}{n-1} \cdot \frac{k-2}{n-2} \cdots \frac{1}{n-(k-1)}\right\} \cdot \left\{\frac{n-k}{n-k} \cdot \frac{n-k-1}{n-k-1} \cdots \frac{1}{1}\right\} &= \frac{k!(n-k)!}{n!} \\ &= \frac{1}{\dbinom{n}{k}} \end{aligned} \tag{3}$$

Now suppose we were to draw out all n marbles in a different order. We can ask for the probability of drawing any *permutation* of k red and $n - k$ black marbles. If we inspect (3) closely, we see that the probability of any sequence would be given by permuting the *numerators* of the factors that appear in (3). Thus we have established the following result:

Theorem: The probability of drawing any sequence of k red marbles and $n - k$ black marbles is the same, and is therefore equal to $\dfrac{1}{\binom{n}{k}}$.

Thus our sample space consists of $\binom{n}{k}$ equally likely points (or sequences), so we may now compute $P(X_3 = 1)$ by counting the number of sequences (of k Rs and $(n - k)$ Bs) having an R in the third position, and then dividing this count by the total number of sequences, which is $\binom{n}{k}$.

Now the number of sequences of k Rs and $(n-k)$ Bs having an R in the third position is just the number of permutations of $(k-1)$ Rs and $(n-k)$ Bs. This is given by $\binom{n-1}{k-1}$. Therefore, we compute $P(X_3 = 1)$ as the following ratio:

$$P(X_3 = 1) = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{n}{k}\binom{n-1}{k-1}}{\frac{n}{k}\binom{n}{k}} = \frac{\binom{n}{k}}{\frac{n}{k}\binom{n}{k}} = \frac{1}{\frac{n}{k}} = \frac{k}{n},$$

the desired result.

We note from (2) that there is nothing special about the third position of the sequence of length n, and we conclude that $P(X_i = 1) = \frac{k}{n}$ for any i, $1 \le i \le n$. This completes our proof.

References

[1] *Mathematical Statistics with Applications*, 7th Edition, Dennis D. Wackerly, William Mendenhall III, Richard L. Scheaffer, Brooks/Cole Cengage Learning, 2008.

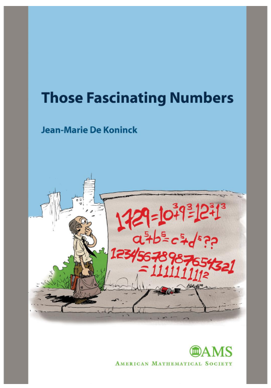
Geoffrey has a BA in Economics from the University of Toronto and an MSc in Mathematics from Queen's University. Geoffrey has been tutoring mathematics to high school students in Toronto since 2009 and also currently works as a mathematics tutor at the Toronto Metropolitan University since 2018. Geoffrey's pastimes include research in mathematics and tennis.

Adam studied math at the University of Waterloo (B.Math) and Queen's (M.Sc.) He is now in his 24th year of teaching high school mathematics. Most of this has been at UTS in Toronto. Away from the classroom, Adam enjoys opera, tennis and cooking.

From the bookshelf of . . .

John Grant McLoughlin

This MathemAttic *feature brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to* `MathemAttic@cms.math.ca`*. Publishers are also welcome to send along books for possible review.*



Those Fascinating Numbers
by Jean-Marie De Koninck
ISBN 978-0-8218-4807-4, 426 + xvii pages
Published by American Mathematical Society, 2009.

Prior to beginning commentary on the book, it is noted here that the original 2009 English edition is a translation of *Ces nombres qui nous fascinent* that was published by Ellipses in 2008. Relevant information concerning a more recent 2018 edition of the French publication is provided here.

Ces nombres qui nous fascinent
by Jean-Marie De Koninck
ISBN 978-2340025143, 448 pages
Ellipses, 2018.

This book more closely resembles a dictionary than a novel. That is, wherever you open the book to read an excerpt there is something to be learned. Let me put that to work here as I simply open the book somewhere and find myself reading pages 156 and 157. The number 3465 is identified as the fifth odd abundant number with a note in parentheses to see the number 945. Such parenthetic references appear throughout the book as a means of offering the reader insight into a term or concept without having to be explaining terminology repeatedly. My familiarity with abundant numbers made the reference in this case less significant. In contrast, the number 3684 is identified as being the 13th Keith number (see the number 197) where the reference explained unfamiliar terminology. So why is 197 or later 3684 a Keith number?

Begin a sequence of terms with the k separate digits that make up the number where $k \geq 2$. All subsequent terms represent the sum of the k preceding terms. For example, with 197 the sequence begins 1, 9, 7. Taking sums of terms in sets of three produces the sequence $1, 9, 7, 17, 33, 57, 107, 197, \ldots$. The appearance of the original number 197 in the sequence makes 197 a Keith number. You may verify that 3684 would also be a Keith number.

Minutes ago I did not know of Keith numbers. Rather the book was opened to a spot and a browse of the pair of visible pages brought a new idea forward into my mathematical lexicon. Primarily the book has served me in this manner by being a source of ideas and curiousities for a mathematician who enjoys playing with numbers. Occasionally the book has reminded me of topics that have not been attended to for periods of time whether Catalan numbers or Kaprekar's constant, for example. The latter, Kaprekar's constant, is one that has been integrated into outreach in schools from time to time as illustrated by this example. Suppose that we select four different digits like 2, 7, 4 and 9. Arrange these numbers to form the largest and smallest four-digit numbers, as in 9742 and 2479. Subtract to get the positive difference of 7263 and continue the process by evaluating $7632 - 2367$, and so on. The result will finally be 6174. That is the Kaprekar constant for four-digit numbers. It is reasonable to try to figure out whether there is a Kaprekar constant for three-digit numbers.

The book is filled with intriguing facts about numbers and ideas that motivate wonder about numbers. The prints of a number theorist are all over the book as the mathematics is sophisticated in many places. Meanwhile many ideas are simple enough to encourage deeper examination through inviting the reader to explore further concepts at hand, or ask "What if ..."

The core 408 pages feature insights into numbers from 1 through to the Skewes number, as in

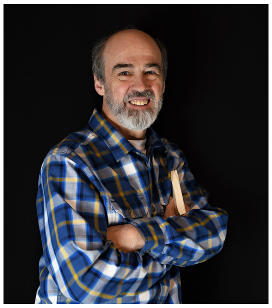
$$10^{10^{10^{34}}}.$$

Appendices follow with a list of prime numbers up to 9973, extensive references, and an index. It is the opening sections of the book though that made an impression upon this reader. Three separate sections, namely, *Notations*, *The Main Functions*, and *Frequently Used Theorems and Conjectures*, offer rich mathematical information while providing easily accessible guide posts that support the reader in understanding the text as a whole. If one is to read just a few pages, the Preface merits attention. The flavour of the book is offered with some notable "less famous" numbers including the selection offered here as a take off point for interested readers.

- 37 is the median value of the second prime factor of an integer
- 480 is possibly the largest number n such that $n(n+1)\cdots(n+5)$ has exactly the same distinct prime factors as $(n+1)(n+2)\cdots(n+6)$
- 736 is the only three digit number abc such that $abc = a + b^c$
- 612 220 032 is the smallest number $n > 1$ for which the sum of its digits equals $\sqrt[7]{n}$

Those Fascinating Numbers is one of those go-to books whether it is for a quick bit of learning or an idea for a challenge to offer students. The book is lengthy though much of its length comes from two of its strengths. First, the book is thorough though no such book would ever be a complete version. Second, it is noteworthy how reader-friendly the book is in terms of both font and presentation. Many dictionary-style books have pages crammed with detail. A strength of this book in terms of its accessibility is the ample spacing with the numbers being discussed appearing in boxes and the subsequent details in easy-to-read bullet form. The mathematical expressions and notations are presented in an invitational spirit.

Fittingly it seems appropriate to close with a challenge that has been shared by me in many contexts with teachers and students. Here though the input of the book is incorporated with a note that 40 585 is the largest number which is equal to the sum of the factorials of its digits. That is, $40\,585 = 4!+0!+5!+8!+5!$. The challenge here is to find the only three-digit number that shares this property.



This book is a recommendation from the bookshelf of John Grant McLoughlin. John is a professor in the Faculty of Education with a cross-appointment to the Department of Mathematics and Statistics at University of New Brunswick. His professional interests extend into community outreach, recreational mathematics, and problem solving. John enjoys writing, birdwatching, outdoor recreation, and coffeeshop conversations.

MATHEMATICS FROM THE WEB

No.10

This column features short reviews of mathematical items from the internet that will be of interest to high school and elementary students and teachers. You can forward your own short reviews to `mathemattic@cms.math.ca`.

For the learning of mathematics
`https://flm-journal.org`
This journal is published under the auspices of the Canadian Mathematics Education Study Group/Groupe Canadien d'Étude en didactique des mathématiques (CMESG/GCEDM).

> *The journal aims to stimulate reflection on mathematics education at all levels, and promote study of its practices and its theories: to generate productive discussion; to encourage enquiry and research; to promote criticism and evaluation of ideas and procedures current in the field. It is intended for the mathematics educator who is aware that the learning and teaching of mathematics are complex enterprises about which much remains to be revealed and understood.*

All articles aside from those in more recent volumes of the past couple of years are fully accessible online. This issue of *Mathematics from the Web* draws attention to three articles, from 1991, 2001 and 1996 respectively, that are likely to interest the readership of *MathemAttic*.

Intuitively Misconceived Solutions to Problems
`https://flm-journal.org/Articles/5D943808FF5510771CF17DF5457BEF.pdf`
This article, by Shmuel Avital and Edward J. Barbeau, features 13 examples of problems that are commonly answered incorrectly. Specifically the authors provide analysis and consideration of the characteristics of these examples that make them intuitively misleading. The authors suggest five sources for "such misleading intuitive generators", namely, lack of analysis, unbalanced perception, improper analogy, improper generalizaton and false symmetry. Readers will gain insights both in terms of the mathematical content and pedagogy. It is likely that one or more of the problems will resonate as potential examples for future use in illustrating mathematical concepts.

Looking at a Painting with a Mathematical Eye
`https://flm-journal.org/Articles/556861A1941982BC67D75CA318A4C6.pdf`
This article, by Marion Walter, brings together the author's interests in problem posing and visual mathematics. The article focuses attention on Theo van Doesburg's painting *Arithmetic Composition 1*, a collection of squares with tilted squares nested inside of the larger squares. Marion Walter utilizes this painting

as a launching point for problem posing while articulating observations and commentary in this piece written in memory of David Wheeler, the founding editor of the journal. David's interest in visual mathematics motivated the authorship of this particular piece at the time.

In Fostering Communities Of Inquiry, Must It Matter That The Teacher Knows "The Answer"?

https://flm-journal.org/Articles/5226E38BEF5A8F9AA1A96FB4866A55.pdf

This piece by Alan H. Schoenfeld opens with gratitude to Sophie Haroutunian-Gordon for an ongoing conversation that led him to compare and contrast the nature and implications of teaching practices in Schoenfeld's undergraduate courses in mathematical problem solving and a graduate research group in mathematics education.

Specifically, Sophie raised the following question in their discussions. "Is there a fundamental difference in the character or results of instruction when the teacher is a co-explorer with students, covering new ground for him- or her-self, as opposed to an "expert" traversing very familiar territory?

The paper focuses on this question. Pertinent experiences, anecdotes and examples from the research group and problem solving courses are drawn into the article as means of comparison and discussion.

OLYMPIAD CORNER

No. 413

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **July 30, 2023**.

OC631. The point O is the circumcenter of the triangle ABC, and AH is its altitude. The point P is the foot of the perpendicular dropped from the point A to the line CO. Prove that the line HP passes through the midpoint of the segment AB.

OC632. Triangle ABC is inscribed in a circle $\mathcal{C}(O,1)$. Let G_1, G_2, G_3 be the centroids of triangles OBC, OAC and OAB, respectively. Prove that triangle ABC is equilateral if and only if $AG_1 + BG_2 + CG_3 = 4$.

OC633. Let $n \in \mathbb{N}, n \geq 2$. Prove that for all complex numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ the following statements are equivalent:

(a) $\sum_{k=1}^n |z-a_k|^2 \leq \sum_{k=1}^n |z-b_k|^2$ for all $z \in \mathbb{C}$;

(b) $\sum_{k=1}^n a_k = \sum_{k=1}^n b_k$ and $\sum_{k=1}^n |a_k|^2 \leq \sum_{k=1}^n |b_k|^2$.

OC634. Prove that for infinitely many integers $n > 1$ the equation

$$(x+1)^{n+1} - (x-1)^{n+1} = y^n$$

has no integer solutions.

OC635. Let $n \geq 3$ be an integer. Prove that for all positive real numbers $x_1, \dots, x_n$,

$$\frac{1+x_1^2}{x_2+x_3} + \frac{1+x_2^2}{x_3+x_4} + \cdots + \frac{1+x_{n-1}^2}{x_n+x_1} + \frac{1+x_n^2}{x_1+x_2} \geq n.$$

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 juillet 2023**.

OC631. Le point O est le centre du cercle circonscrit du triangle ABC, tandis que AH est une de ses hauteurs; aussi, le point P est le pied de la perpendiculaire de A vers la ligne CO. Démontrer que la ligne HP passe par le point milieu du segment AB.

OC632. Le triangle ABC est inscrit dans le cercle $\mathcal{C}(0,1)$. Soient alors G_1, G_2 et G_3 les centroïdes des triangles OBC, OAC et OAB, respectivement. Démontrer que le triangle ABC est équilatéral si et seulement si $AG_1 + BG_2 + CG_3 = 4$.

OC633. Soit $n \in \mathbb{N}, n \geq 2$. Démontrer que pour tous nombres complexes $a_1, a_2, \ldots, a_n$ et $b_1, b_2, \ldots, b_n$ les deux énoncés suivants sont équivalents:

(a) $\sum_{k=1}^{n} |z - a_k|^2 \leq \sum_{k=1}^{n} |z - b_k|^2$ pour tout $z \in \mathbb{C}$;

(b) $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k$ et $\sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |b_k|^2$.

OC634. Démontrer que l'équation

$$(x+1)^{n+1} - (x-1)^{n+1} = y^n$$

n'a aucune solution entière pour un nombre infini d'entiers $n > 1$.

OC635. Soit n un entier tel que $n \geq 3$ et soient $x_1, \ldots, x_n$ des nombres réels positifs. Démontrer que

$$\frac{1+x_1^2}{x_2+x_3} + \frac{1+x_2^2}{x_3+x_4} + \cdots + \frac{1+x_{n-1}^2}{x_n+x_1} + \frac{1+x_n^2}{x_1+x_2} \geq n.$$

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2022: 48(10), p. 611–612.

OC606. Determine the number of triples of positive integers (a, b, c) such that

$$a + ab + abc + ac + c = 2017.$$

Originally from 2018 Czech-Slovakia Mathematics Olympiad, 4th Problem, Category B.

We received 18 correct solutions. We present 2 solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

We have

$$(c + 1)(ab + a + 1) = abc + ac + c + ab + a + 1 = 2018 = 2 \cdot 1009\,.$$

Since $2, 1009$ are primes, $c + 1 > 1$ and $ab + a + 1 > 2$, then $c + 1 = 2$ and $ab + a + 1 = 1009$. So $c = 1$ and $a(b + 1) = 1008 = 2^4 \cdot 3^2 \cdot 7$. Thus a is a positive divisor of $n = 1008$.

The number n has exactly $(4+1)(2+1)(1+1) = 30$ positive divisors. Each positive divisor d of n gives exactly one solution of $a(b + 1) = n$ in positive integers with $a = d$, namely $a = d, b + 1 = n/d$, except in the case that $d = n$.

So we have 29 triples of positive integers satisfying the equation.

Solution 2, by Missouri State University Problem Solving Group.

Solving for c gives

$$c = \frac{2017 - a - ab}{ab + a + 1} = \frac{2018}{ab + a + 1} - 1.$$

Therefore $ab + a + 1$ must be a divisor of 2018, i.e., $ab + a + 1 = 1, 2, 1009$, or 2018. Hence

$$a(b + 1) = 0, 1, 1008, \text{ or } 2017.$$

The first two possibilities cannot occur if $a, b > 0$. If $ab + a = 2017$, then $c = 0$. Thus, the only possibility is $a(b + 1) = 1008$, which gives $c = 1$. Now $b + 1$ can be any divisor of 1008, except 1 and there are no restrictions on a. Since 1008 has 30 positive divisors, this gives 29 solutions.

A similar argument shows that, more generally, if n is a positive integer and $f(n)$ denotes the number of solutions to

$$a + ab + abc + ac + c = n,$$

then

$$f(n) = \sum_{\substack{d|(n+1) \\ d\neq 1,2,n+1}} (\tau(d-1)-1),$$

where $\tau(m)$ is the number of positive divisors of m.

For example, if 2017 is replaced by 2023, we have 49 solutions.

OC607. Find the largest possible number of integers that can be selected from the set $\{1, 2, 3, \ldots, 100\}$ so that there are no two of them that differ by 2 or by 5.

Originally from 2018 Czech-Slovakia Mathematics Olympiad, 6th Problem, Category B, First Round.

We received 10 submissions, of which 7 were correct and complete. We present the solution by UCLan Cyprus Problem Solving Group.

Let S be the set of all $n \in \{1, 2, \ldots, 100\}$ with $n \equiv 1, 2, 5 \pmod 7$. Then

$$|S| = 3 \cdot 14 + 2 = 44$$

and no two elements of S differ by 2 or 5.

So we can have 44 integers with the required property. We will show that this is the maximum.

If we have 45 (or more integers) then at least $43 = 3 \cdot 14 + 1$ belong in $\{1, 2, \ldots, 98\}$ and so at least 4 belong in one of the 14 sets

$$\{1, 2, \ldots, 7\}, \{8, 9, \ldots, 14\}, \ldots, \{92, 93, \ldots, 98\}\,.$$

Suppose that we have four elements in the set $\{x+1, x+2, \ldots, x+7\}$ and write its elements in the order

$$x+1, x+3, x+5, x+7, x+2, x+4, x+6\,.$$

Given four out of these seven elements, either we have two consecutive, or we have the first and last one. In all cases, we have two elements with a difference of 2 or 5, a contradiction.

Editor's Comment. S. Muralidharan used a similar argument and gave the following generalization: let k be an odd positive integer, and let c, d be two numbers such that $c + d = k$. Then, any subset of $\{a+1, \ldots, a+k\}$ of size at least $\frac{k+1}{2}$ contains two elements that either differ by c or d. In our special case, $c = 2$, $d = 5$, $k = 7$.

OC608. Prove that $2^{-x} + 2^{-1/x} \leq 1$ for all real numbers $x > 0$.

Originally from 2018 Romania Mathematics Olympiad, 2nd Problem, Grade 11, Final Round.

We received 12 submissions, of which 10 were correct and complete. We present 2 solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = 2^{-x} + 2^{-1/x}$. Since $\lim_{x \to 0} f(x) = 1$, then f extends to a continuous function on $[0, \infty)$ which we will also call f. Since $f(x) = f(1/x)$, it is enough to show that $f(x) \leqslant 1$ for $x \in [0, 1]$. Since f is continuous on $[0, 1]$ then it is maximized either at the endpoints (where we have $f(0) = f(1) = 1$) or at an internal point. Since f is differentiable at $(0, 1)$, if it has a maximum at an internal point a, then $f'(a) = 0$. We have

$$0 = f'(a) = -\log 2 \left(\frac{1}{2^a} - \frac{1}{a^2 2^{1/a}} \right) \implies 2^a = a^2 2^{1/a} .$$

Then

$$f(a) = \frac{1}{2^a} + \frac{1}{2^{1/a}} = \frac{1}{2^a} + \frac{a^2}{2^a} = \frac{1 + a^2}{2^a} .$$

By Bernoulli inequality, since $0 < a < 1$ and $-1/2 \geqslant -1$ we have

$$\frac{1}{2^a} = \left(1 - \frac{1}{2} \right)^a \leqslant 1 - \frac{a}{2} = \frac{2 - a}{2} .$$

Thus

$$f(a) = \frac{1 + a^2}{2^a} \leqslant \frac{(1 + a^2)(2 - a)}{2} = \frac{2 + (2a^2 - a - a^3)}{2} = 1 - \frac{a(a - 1)^2}{2} \leqslant 1 .$$

This completes the proof.

Solution 2, by Paolo Perfetti.

The inequality is equivalent to

$$2^x \leq 2^{1/x}(2^x - 1) \iff 2^{x^2} \leq 2(2^x - 1)^x \iff \frac{1}{2} \leq \left(1 - \frac{1}{2^x} \right)^x$$

that is

$$-\ln 2 \leq x \ln(1 - 2^{-x})$$

Hence we have $\ln(1 - x) \geq -x$ and thus we prove

$$-\ln 2 \leq -x 2^{-x} \iff 2^x \geq x / \ln 2 \iff e^{x \ln 2} \geq x / \ln 2.$$

Now, $e^x \geq 1 + x + x^2/2$ for $x > 0$, so we prove

$$1 + x \ln 2 + \frac{x^2 \ln^2 2}{2} \geq \frac{x}{\ln 2} \iff x^2 \ln^3 2 + 2x(\ln^2 2 - 1) + 2 \ln 2 \geq 0.$$

The last one is a quadratic inequality in x and the discriminant condition $\Delta/4 < 0$ is equivalent to $\ln^4 2 + 2\ln^2 2 > 1$ and this is true because $\ln^4 2 + 2\ln^2 2 \sim 1.1$.

OC609. Let n be a positive integer, $n \equiv 4 \pmod 8$. The numbers

$$1 = k_1 < k_2 < \ldots < k_m = n$$

are all positive divisors of n. Prove that if the number $i \in \{1, 2, \ldots, m-1\}$ is not divisible by 3, then $k_{i+1} \leq 2k_i$.

Originally from 2018 Poland Mathematics Olympiad, 2nd Problem, Second Round.

We received 1 solution. We present the solution by UCLan Cyprus Problem Solving Group.

Suppose $n = 4n'$ where n' is odd and let $1 = d_1 < d_2 < \ldots < d_r = n'$ be the positive divisors of n'.

Then every divisor k_i of n is of the form d_j or $2d_j$ or $4d_j$ for some $1 \leqslant j \leqslant r$.

If $k_i = d_j$, then $k_{i+1} \leqslant 2d_j = 2k_i$. If $k_i = 2d_j$, then $k_{i+1} \leqslant 4d_j = 2k_i$.

So assume now that $k_i = 4d_j$. If $k_{i+1} = 2d_\ell$, then $d_\ell < 4d_j < 2d_\ell$ and so $k_{i+1} = 2d_\ell < 2 \cdot 4d_j = 2k_i$. Similarly, if $k_{i+1} = 4d_\ell$, then $2d_\ell < 4d_j < 4d_\ell$ and so $k_{i+1} = 4d_\ell < 2 \cdot 4d_j = 2k_i$.

So we may assume that $k_i = 4d_j$ and $k_{i+1} = d_\ell$. Since $4d_{j+1} > k_i$ and $4d_{j-1} < 4d_j$, then $\ell = j+1$. It follows that $k_1, \ldots, k_i$ are $d_1, 2d_1, 4d_1, \ldots, d_j, 2d_j, 4d_j$ in some order. But then i is divisible by 3.

OC610. The perpendicular bisector of side BC intersects the circumcircle of triangle ABC at points P and Q, with points A and P on the same part of side BC. Point R is the orthogonal projection of point P on the straight line AC. Point S is the midpoint of the segment AQ. Prove that points A, B, R and S lie on a circle.

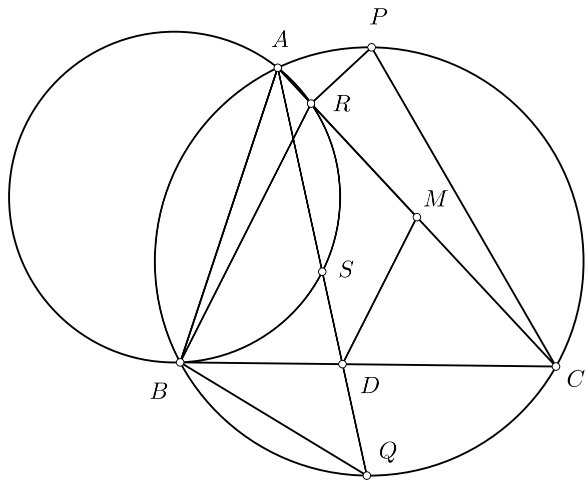
Originally from 2018 Poland Mathematics Olympiad, 3rd Problem, Second Round.

We received 6 correct solutions. We present 2 solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

Let D be the point of intersection of AQ with BC. It is well-known that AQ is the angle bisector of $\angle BAC$. It follows that the triangles ABQ and ADC are similar as $\angle BAQ = \angle DAC$ and $\angle AQB = \angle ACD$. So letting M be the midpoint of AC we have $\angle ASB = \angle AMD$. It is enough to show that $BR \parallel DM$ as then we would have $\angle AMD = \angle ARB$ and the result will follow.

By the bisector theorem we have $BD/DC = c/b$ and since $MC = b/2$, it is enough to show that $RM = c/2$.



We have

$$PC = 2R' \sin(\angle PCB) = 2R' \sin\left(\frac{180 - A}{2}\right) = 2R' \cos\left(\frac{A}{2}\right),$$

where R' is the circumradius of triangle ABC. We also have

$$\angle PCA = \frac{180 - A}{2} - C = \frac{B - C}{2}.$$

Thus

$$\begin{aligned} CR &= 2R' \cos\left(\frac{A}{2}\right) \cos\left(\frac{B-C}{2}\right) \\ &= R' \left[\cos\left(\frac{A+B-C}{2}\right) + \cos\left(\frac{A-B+C}{2}\right)\right] \\ &= R' \sin(C) + R' \sin(B) = \frac{c+b}{2}. \end{aligned}$$

The result follows.

Solution 2, by Theo Koupelis.

Let O be the center of the circumcircle of triangle ABC. Then the line OS is the perpendicular bisector of the chord AQ.

Let E be the intersection point of the lines PR and OS; then the quadrilateral $AESR$ is cyclic because we have $\angle ERA = \angle ESA = 90°$. If K is the center of the circumcircle of $AESR$, then K is the midpoint of the segment AE because AE is a diameter of circle (K). Also, points P, O, Q are collinear because the perpendicular bisector of the chord BC passes through O. Thus, PQ is a diameter of (O), $OS \parallel AP$ (because $\angle QAP = 90°$), $OS = AP/2$ (because points S, O are the midpoints of AQ, PQ, respectively), and AQ is the angle bisector of $\angle BAC$

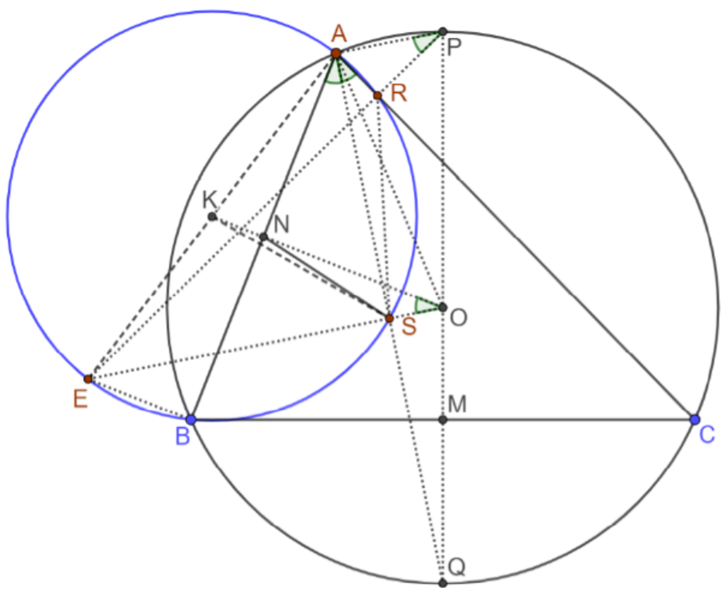
(because $\angle BOQ = \angle QOC$). Thus, triangles KSO and EAP are similar because $KS = EA/2$, $SO = AP/2$, and

$$\angle KSO = \angle KSA + \angle ASO = \angle KAS + 90^\circ = \angle KAS + \angle SAP = \angle EAP.$$

Therefore,

$$\angle SOK = \angle APE = \angle APR = \angle SAR,$$

because $PR \perp AR$ and $PA \perp AS$.



Let N be the intersection point of OK and AB. Then

$$\angle SON = \angle SOK = \angle SAR = \angle SAN,$$

because AS is the internal angle bisector of $\angle BAC$. Therefore, the quadrilateral $NAOS$ is cyclic and thus $\angle ONA = \angle OSA = 90^\circ$. Thus, OK is the perpendicular bisector of AB. Therefore, $EB \parallel KN$ because the points N, K are the midpoints of AB, AE, respectively, and thus $\angle EBA = \angle KNA = 90^\circ$. Therefore point B is on the circle (K), and thus points A, B, R, and S are concyclic.

FOCUS ON...

No. 56

Michel Bataille

Solving from the construction of the figure

Introduction

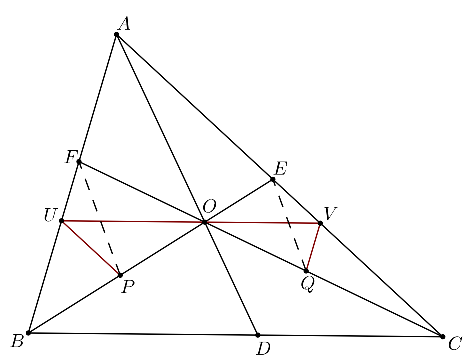
In most problems of plane geometry, the figure is constructed just by following the statement step by step. Yet, it may happen that some hypothesis blocks the process and the would-be solver is reduced to a freehand doodling instead of an exact corresponding diagram. And geometry software is of little help in that case! Actually, such a situation adds some spice to the problem and the efforts made to achieve the construction can be quite rewarding. Not only do they bring the satisfaction of solving a side-problem, but they often naturally lead to a solution of the posed problem. Various examples will help show this, also offering a good opportunity to review some classical constructions and theorems.

With triangles

To illustrate the topic of this number, we start with an easy problem, namely Problem **2857** [2003 : 317 ; 2004 : 311]:

> Let O be an interior point of $\triangle ABC$, and let D, E, F, be the intersections of AO, BO, CO with BC, CA, AB, respectively. Suppose that P and Q are points on the line segments BE and CF, respectively, such that $\frac{BP}{PE} = \frac{CQ}{QF} = \frac{DO}{OA}$. Prove that $PF \parallel QE$.

Choosing O inside $\triangle ABC$ and obtaining D, E, F is straightforward. But where are P and Q located? Clearly, the ratio $\frac{DO}{OA}$ needs to be carried to the line segments BE and CF. A short reflection brings the word "projection" to mind and the solution quickly appears: the parallel to BC through O projects O to U on AB and to V on AC. Thus, $\frac{DO}{OA} = \frac{BU}{UA} = \frac{CV}{VA}$ and we get P as the projection of U onto BE by means of the parallel to AC through U. The point Q is similarly obtained using V. See the figure below.



Thus, the intermediary points U and V enable an easy construction. In addition, they also give ideas for a solution to the problem. Two ideas in fact: first, we can remark that $\triangle PUF$ and $\triangle EVQ$ are perspective from the point O so that the intersections $PU \cap EV$, $UF \cap VQ$, $PF \cap QE$ are collinear (from Desargues' theorem). Since $PU \parallel EV$ and $UF \parallel VQ$, we must have $PF \parallel QE$. Alternatively, we can use the homothety h with centre O which transforms U into V. Since $UP \parallel VE$, we have $h(P) = E$ and since $UF \parallel VQ$, we have $h(F) = Q$. Thus, $PF \parallel QE$.

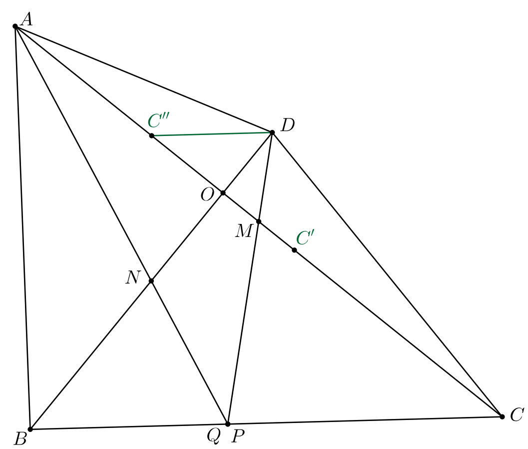
In our second example, problem **3014** [2005 : 105,108 ; 2006 : 119], a restricting condition on the areas of some of the involved triangles leaves us stuck at the very start.

> Given a convex quadrilateral $ABCD$, let O be the intersection of the diagonals AC and BD, and let M and N be the mid-points of AC and BD, respectively. Suppose that $[OAB] + [OCD] = [OBC]$, where $[PQR]$ denotes the area of triangle PQR. Prove that AN, DM, and BC are concurrent.

If we draw an arbitrary convex quadrilateral, chances to fulfill the condition on areas are weak. If $\theta = \angle AOB$, the condition writes as $|OA| \cdot |OB| \sin\theta + |OC| \cdot |OD| \sin\theta = |OB| \cdot |OC| \sin(180^\circ - \theta)$, that is,

$$\frac{AO}{OC} + \frac{DO}{OB} = 1. \tag{1}$$

(here and in the whole solution of this problem we use signed distances.)
We see that once we have drawn a triangle ABC and picked a point O on the side AC, we can obtain a suitable point D by $\frac{DO}{OB} = 1 - \frac{AO}{OC}$. Our side-problem is close to the previous one: the point C' such that $\overrightarrow{CC'} = \overrightarrow{OA}$ satisfies $\frac{OC'}{OC} = 1 - \frac{AO}{OC}$, hence if C'' denotes its reflection about O, we have $\frac{C''O}{OC} = \frac{DO}{OB}$ and the location of D follows by projecting C'' onto the line BO using a parallel to BC.



The relation (1) also proves useful for a solution to the problem. Let AN meet BC at P. From Menelaus's theorem, we have

$$\frac{OA}{AC} \cdot \frac{CP}{PB} \cdot \frac{BN}{NO} = -1 \tag{2}$$

and similarly, if DM meets BC at Q,

$$\frac{OD}{DB} \cdot \frac{BQ}{QC} \cdot \frac{CM}{MO} = -1. \tag{3}$$

We are required to prove that $P = Q$ that is, $\frac{CP}{PB} = \frac{CQ}{QB}$ or (from (2) and (3)) $u = v$, with

$$u = AO \cdot DO \cdot MC \cdot NB, \qquad v = MO \cdot NO \cdot AC \cdot DB.$$

Now,

$$u = AO \cdot DO \cdot (MO + OC) \cdot (NO + OB)$$

$$\cdot OC \cdot \frac{1}{2}(BO + DO)$$

$$+ AO \cdot DO \cdot OB \cdot OC$$

$$= AO \cdot DO \cdot MO \cdot NO + \frac{1}{2} AO \cdot DO \cdot (AO \cdot OB + DO \cdot OC)$$

$$= AO \cdot DO \cdot MO \cdot NO + \frac{1}{2} AO \cdot DO \cdot OB \cdot OC \qquad \text{(with the help of (1).)}$$

In the same way, expanding $v = MO \cdot NO \cdot (AO + OC) \cdot (DO + OB)$ and using (1) lead to $v = AO \cdot DO \cdot MO \cdot NO + \frac{1}{2} AO \cdot DO \cdot OB \cdot OC$ and $u = v$ follows.

We conclude the section with a variant of solution to a problem proposed at the Vietnamese Olympiad 2006-7 [2009 : 439 ; 2010 : 500].

> Triangle ABC has two fixed vertices, B and C, while the third vertex A is allowed to vary. Let H and G be the orthocentre and the centroid of ABC, respectively. Find the locus of A such that the midpoint K of the segment HG lies on the line BC.

We first show how to construct a suitable vertex A and then deduce the desired locus from this construction.

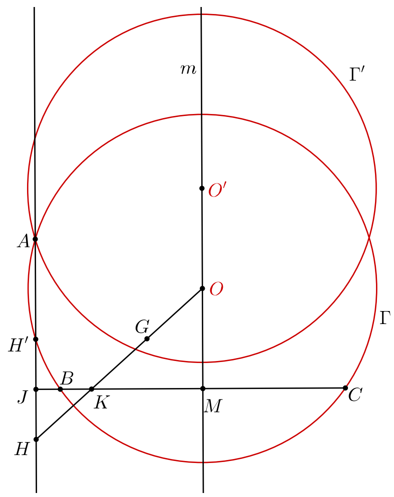
We denote the midpoint and perpendicular bisector of BC by M and m, respectively, and the circumcenter by O. We know that the reflection H' of H in BC is on the circumcircle of $\triangle ABC$. Let J be the midpoint of HH'. Recalling that $\overrightarrow{OH} = 3\overrightarrow{OG}$, we readily see that the midpoint K of GH satisfies $\overrightarrow{KH} = -\frac{1}{2}\overrightarrow{KO}$. Since J is on BC, it follows that K is on BC if and only if $\overrightarrow{HJ} = -\frac{1}{2}\overrightarrow{OM}$, that is, if and only if $\overrightarrow{HH'} = -\overrightarrow{OM}$.

Note that O must be distinct from M (if $O = M$, then $\triangle ABC$ is right-angled at A and $H' \neq H(= A)$).

Since $\overrightarrow{AH} = 2\overrightarrow{OM}$, we finally obtain that K is on BC if and only if

$$\overrightarrow{H'A} = \overrightarrow{MO} \neq \overrightarrow{0}.$$

Thus, a suitable point A can be constructed as follows: choose any point O on m, with $O \neq M$; draw the circle Γ with centre O and radius $R = OB$ and the image Γ' of Γ under the translation with vector $\overrightarrow{MO}$. Then A can be either of the common points of Γ and Γ' (these circles do intersect since $0 = R - R < OO' = OM < OB < 2R$ where O' is the centre of Γ').



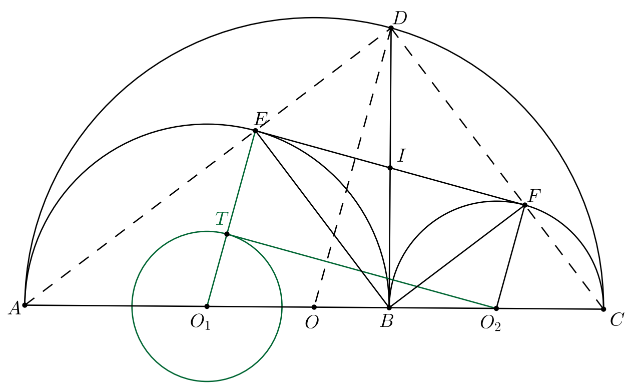
It is now easy to answer the problem. We introduce an orthonormal system of axes with origin at M such that $C(a, 0)$, $B(-a, 0)$ where $2a = BC$. If $O(0, t)$ with $t \neq 0$, then $\overrightarrow{OB}(-a, -t)$, $O'(0, 2t)$ and the circles Γ and Γ' have equations $x^2 + (y - t)^2 = a^2 + t^2$ and $x^2 + (y - 2t)^2 = a^2 + t^2$. They intersect at the points $(\sqrt{a^2 + \frac{3t^2}{4}}, \frac{3t}{2})$ and $(-\sqrt{a^2 + \frac{3t^2}{4}}, \frac{3t}{2})$ and the elimination of t yields the conditions $x^2 - \frac{y^2}{3} = a^2, \quad y \neq 0$ for the locus of these points. In conclusion, the desired locus is the hyperbola with centre M, focal axis BC, eccentricity 2, and vertices B and C (excluded).

With circles

We propose two examples involving the configuration called arbelos. The first one was proposed at a Croatian competition in 1997 [2001 : 90 ; 2003 : 163].

> Three points A, B, C are given on the same line, such that B is between A and C. Over the segments AB, BC, AC, as diameters, the semicircles are constructed on the same side of the line. The perpendicular from B to AC intersects the larger circle at point D.
>
> Prove the common tangent of the two smaller semicircles, different from BD, is parallel to the tangent on the largest semicircle through the point D.

To draw the figure, we have to construct the common tangent to two externally tangent circles. To this aim, we can use a classical construction: supposing $AB \geq BC$ and denoting by O, O_1, O_2 the centers and r, r_1, r_2 the radii of the semicircles with diameters AC, AB, BC, respectively, we draw the tangent O_2T from the center O_2 to the circle centered at O_1 with radius $r_1 - r_2$. The desired tangent EF is parallel to O_2T with E on O_1T.



Another construction is obtained in the wake of the following remark: the midpoint I of EF satisfies $IE = IF = IB$, hence the triangle EBF is right-angled at B. It follows that the triangle O_1IO_2 is right-angled at I (IO_1 and IO_2 are the perpendicular bisectors of BE and BF, respectively). Thus, this point I is also the intersection of the ray BD with the semicircle with diameter O_1O_2 and then E, F are on the circle with center I, radius IB.

Now, IB is the altitude from I in $\triangle O_1IO_2$, hence $IB^2 = BO_1 \cdot BO_2 = r_1r_2$ and DB is the altitude from D in right-angled triangle ADC, hence $DB^2 = BA \cdot BC = 4r_1r_2$. It follows that I is the midpoint of BD; therefore, $BEDF$ is a parallelogram, even a rectangle since $EB \perp BF$. We deduce that $\angle CFD = \angle AED = 180°$, meaning that E is on AD and F is on CD. Lastly, we have

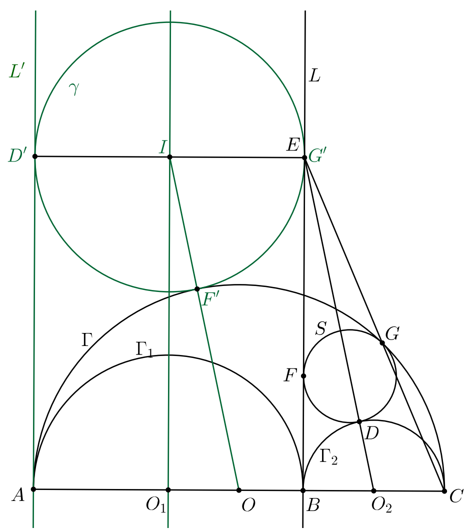
$$\angle ODA + \angle DEF = \angle OAD + \angle EFB = \angle OAD + \angle BCF = 90°,$$

so that OD is perpendicular to EF and the required result follows.

We conclude with another interesting problem about the arbelos: problem 10895 proposed in 2001 in *The American Mathematical Monthly*.

> Given a point B on the segment AC, erect semicircles on diameters AB, AC, and BC, all on the same side of AC. Let L be the line through B perpendicular to AC. Let S be the largest circle that fits into the region bounded by L and the semicircles on diameters AC and BC. Let D be the point of tangency between S and the semicircle on BC. Extend the diameter of S through D until it hits L at E. Prove that AB and DE have the same length.

Let $\Gamma, \Gamma_1, \Gamma_2$ be the circles on diameters AC, AB, BC and let r, r_1, r_2 be their respective radii. Here the difficulty is to draw the circle S which is tangent to L, externally to Γ_2 and internally to Γ. Inversion appears as the appropriate tool. Let L' be the parallel to L through A. It is easy to draw the circle γ tangent to L' (at D'), externally to Γ (at F') and to L (at G'). Note that its center I is on the perpendicular to AC at O_1 and $OI = r + r_1$. The inversion in the circle with center C and radius $2\sqrt{rr_2}$ exchanges A and B, Γ and L, Γ_2 and L', hence γ and S. This circle S is tangent to Γ_2, Γ, L at the inverses D, G, F of D', F', G'.



From the known properties of inversion, S is the image of γ under the homothety h with center C and scale factor $\dfrac{4rr_2}{|CI^2 - r_1^2|} = \dfrac{r_2}{r}$ (note $CI^2 = CO_1^2 + OI^2 - OO_1^2$). We deduce that h also transforms Γ into Γ_2, hence F' into D. It follows that the parallel to OI through O_2 is the diameter of S through D as well as the line O_2G' (since $\overrightarrow{IG'} = \overrightarrow{OO_2}$). Therefore $G' = E$ and OO_2EI is a parallelogram. Thus, $O_2E = OI$ and

$$DE = OI - O_2D = r + r_1 - r_2 = 2r_1 = AB.$$

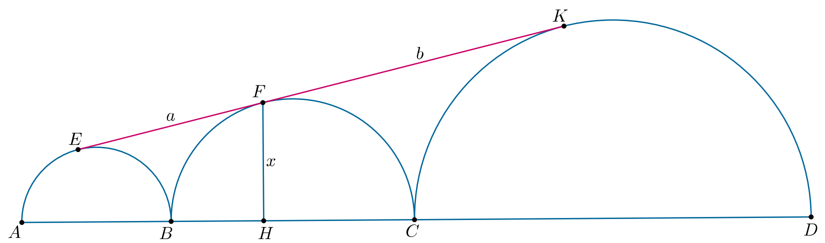
Exercises

1. (Problem **3289** [2007 : 485,487 ; 2008 : 490])

Let ABC be a triangle for which there exists a point D in its interior such that $\angle DAB = \angle DCA$ and $\angle DBA = \angle DAC$. Let E and F be points on the lines AB and CA, respectively, such that $AB = BE$ and $CA = AF$. Prove that the points A, E, D, and F are concyclic. (First construct D satisfying the constraints.)

2. (Problem **4620** [2021 : 98 ; 2021 : 368])

Consider the semicircles in the configuration below:



Prove that $\dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b}$.

(First construct the figure and then deduce a solution.)

Reading a Math Book

No. 2

Yagub Aliyev

M.P. Chernyaev, *Problems in synthetic geometry*

In each appearance of this column, we feature one math book by collecting, commenting, and solving problems from it. We focus on books and authors which are not well known in the English speaking world or have not been translated into English. There will also be some problems at the end for you to solve and send us.

. . . .

The book [1] that I am going to discuss here was first published in 1954 and then republished with significant extensions in 1961 by M.P. Chernyaev (other spellings of the name are Chernyayev and Tschernjaeff). Before this book, M.P. Chernyaev published many papers about geometry in the 1930s [4]. The book is intended as a workbook for a course in synthetic geometry, which is also referred to as *axiomatic geometry* or *pure geometry*. The term *synthetic geometry* was coined to differentiate it from other ways of doing geometry, such as analytic geometry using coordinates, differential geometry using calculus, algebraic geometry using group theory, etc. Synthetic geometry is based on the axiomatic method and uses traditional geometry tools such as straightedge-and-compass. Most of the problems in the book are elementary in nature. So, these problems can be interpreted and solved as usual Euclidean geometry problems. That is what we are going to do in the current paper. The problems in the book are divided into six chapters:

1. Cross-ratio
2. Some problems of elementary geometry
3. Conic sections, Theorems of Pascal and Brianchon
4. Poles and polars
5. Involutional correspondence
6. Solid geometry (stereometry) problems

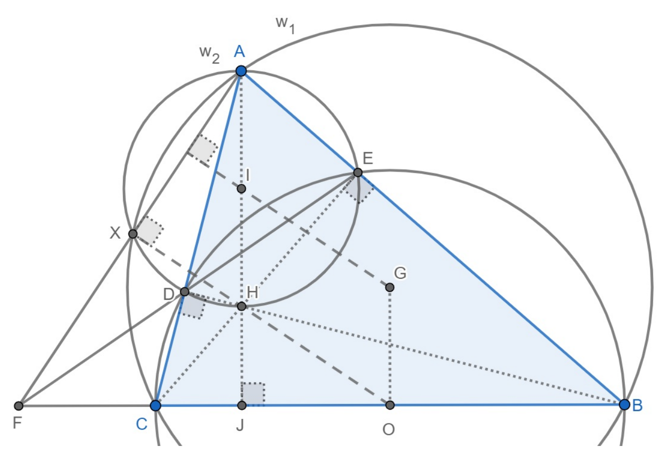
There is a short review of this book by V.G. Kopp [2]. Fun fact about the book is that its name even made it to the archives of CIA, which mentioned it among other mathematical works of the author: `https://www.cia.gov/readingroom/docs/CIA-RDP86-00513R000308620012-9.pdf`

I came across this book in my attempt to explore geometric inequalities. Some of the geometric inequalities that I noticed are given as exercises supplementing the solved problems below.

Solved Problems.

1. Let BD and CE be altitudes of a triangle ABC and let the line DE through the feet of these altitudes intersect the line BC at F. Prove that the line through the midpoint O of the side BC and the orthocenter H of $\triangle ABC$ is perpendicular to AF. (Problem 161 in [1])

Solution. The solution and the diagram below are given for the case when the orthocenter lies inside the triangle. We will use the well-known result which says that if G is the center of the circumscribed circle w_1 of $\triangle ABC$ then $|AH| = 2\cdot|GO|$ and $AH \parallel GO$ (see e.g. Exercise 1, Sect. 1.8 of [3]). So, if we denote the midpoint of the segment AH by I, then $IHOG$ is a parallelogram and therefore $GI \parallel OH$. On the other hand, if we draw the circle w_2 with diameter AH and denote its intersection with the circle w_1 by X, then $GI \perp AX$. Since $HX \perp AX$, the points X, H, and O are collinear. It remains to show that the points A, X, and F are collinear, too. But this immediately follows from the fact that F is the radical center of the circles w_1, w_2, and the circle with diameter BC.



The case when the orthocenter lies outside the triangle can be studied in a similar way.

Exercise. Show that $FH \perp AO$.

Exercise. Let J be the foot of the altitude of $\triangle ABC$ from vertex A. Show that if $\angle B$ and $\angle C$ of $\triangle ABC$ are acute then $4 \cdot |JH| \cdot |JA| \leq |BC|^2$.

2. Prove that the ratio of distances from two arbitrary points to the center of a circle is equal to the ratio of the corresponding distances from each one of these points to the polar of the other point with respect to the circle. (Problem 149 in [1].)

Solution. Denote the two given points by A and B. We will assume that both of them are outside of the circle. (The cases where one or both points lie inside the circle may be treated similarly.) Let the radius of the circle be R. Suppose that l_1 and l_2 are the polars of the points A and B, respectively, and let T be their intersection point. Let us draw the perpendiculars AY and BX to the lines l_1 and l_2, respectively. Let us also draw the perpendiculars OY_1 and OX_1 to the lines AY and BX, respectively. Denote

$$\angle YAO = \angle YTN = \angle MTX = \angle OBX = \alpha.$$

Then $|AY_1| = |OA| \cdot \cos\alpha$ and $|BX_1| = |OB| \cdot \cos\alpha$. On the other hand

$$|AY_1| = |AY| + |YY_1| = |AY| + |OM| = |AY| + \frac{R^2}{|OB|}.$$

Similarly,

$$|BX_1| = |BX| + |XX_1| = |BX| + |ON| = |BX| + \frac{R^2}{|OA|}.$$

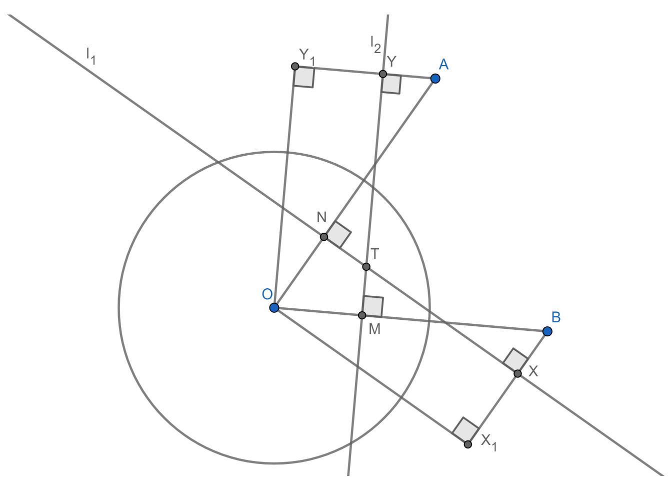
We obtain that

$$|OA| \cdot |OB| \cdot \cos\alpha = |OA| \cdot |BX| + R^2$$

and

$$|OA| \cdot |OB| \cdot \cos\alpha = |OB| \cdot |AY| + R^2.$$

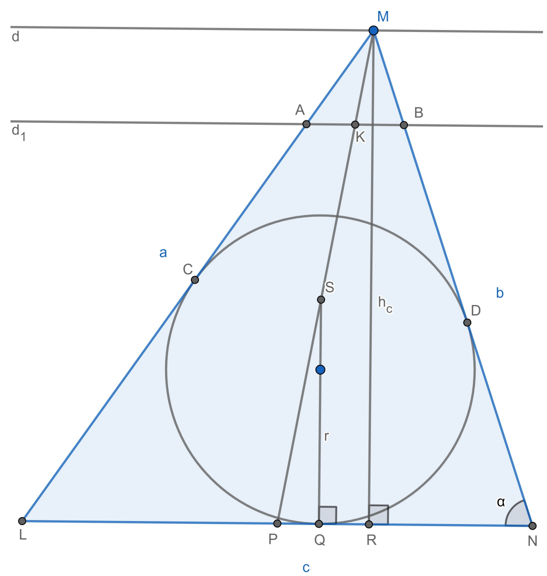
So, $|OA| \cdot |BX| = |OB| \cdot |AY|$ or $\frac{|OA|}{|OB|} = \dfrac{|AY|}{|BX|}$.



Exercise. Show that the second intersection point of the circles with diameters OA and OB is on the line OT.

3. Given are a circle and two parallel lines d and d_1. An arbitrary point M is chosen on the line d. Let the tangents MC and MD of the circle intersect the line d_1 at points A and B, respectively. Prove that as the point M changes, the line joining point M with the midpoint K of the line segment AB passes through a fixed point of the plane. (Problem 160 in [1])

Solution. Let us choose points L and N on the lines MC and MD, respectively, so that the given circle is the incircle of $\triangle LMN$ and the side LN is parallel to d. By similarity of $\triangle MAB$ and $\triangle MLN$, the line MK bisects the side LN. Let us denote the midpoint of LN by P, the point where LN is tangent to the circle by Q, the intersection of the perpendicular to the side LN through vertex M by R, and the intersection of the line MP and the line perpendicular to the side LN at the point Q by S. Denote also $LM = a$, $MN = b$, $LN = c$, $MR = h_c$, $\angle LNM = \alpha$, and the radius of the circle by r. Without loss of generality, we can assume that $a > b$. Then $|PQ| = \frac{a-b}{2}$.



By the cosine law,

$$|RN| = b \cdot \cos\alpha = \frac{b^2 + c^2 - a^2}{2c}.$$

So,

$$|PR| = |PN| - |RN| = \frac{c}{2} - b \cdot \cos\alpha = \frac{c}{2} - \frac{b^2 + c^2 - a^2}{2c} = \frac{a^2 - b^2}{2c}.$$

By similarity of $\triangle MPR$ and $\triangle SPQ$, we have

$$\frac{|MR|}{|SQ|} = \frac{|PR|}{|PQ|} = \frac{\frac{a^2-b^2}{2c}}{\frac{a-b}{2}} = \frac{a+b}{c}.$$

We will now prove that $\frac{a+b}{c}$ is constant. Note first that $c \cdot h_c = (a+b+c) \cdot r$, because both sides of this equation are equal to twice the area of $\triangle LMN$. We obtain $\frac{a+b}{c} = \frac{h_c - r}{r}$, which remains constant as M moves on d. Consequently, $|QS|$ is constant, and therefore the point S is fixed.

Exercise. Show that as M moves on d, the length of the line segment AB is minimal when the points Q, S, M are collinear.

4. Given are two fixed points P and Q, and a fixed line d such that $d \perp PQ$. Denote the intersection of d and the line PQ by O. Let the sides of a right angle at the vertex P intersect the line d at points A and B. Prove that as the right angle rotates about the vertex P, the perpendicular from the vertex A to the line BQ passes through a fixed point of the plane. (Problem 145 in [1])

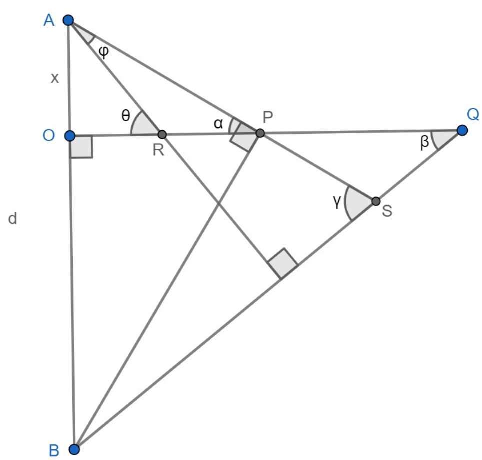
Solution. Let the perpendicular from the vertex A to the line BQ intersect PQ at R. Let us show that the point R is fixed. Suppose also that the line AP intersects BQ at S. Let us denote $|OP| = a$, $|OQ| = b$, and $|OA| = x$. First, note that since $|OB| \cdot |OA| = |OP|^2$, $|OB| = \frac{a^2}{x}$. Let us also denote $\angle APO = \alpha$, $\angle BQO = \beta$, $\angle ASB = \gamma$, $\angle RAS = \phi$, and $\angle ARO = \theta$. Then

$$\tan\gamma = \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta} = \frac{\frac{x}{a} + \frac{a^2}{bx}}{1 - \frac{a}{b}} = \frac{bx^2 + a^3}{ax(b-a)},$$

$$\tan\phi = \frac{1}{\tan\gamma} = \frac{ax(b-a)}{bx^2 + a^3},$$

$$\tan\theta = \tan(\alpha+\phi) = \frac{\tan\alpha + \tan\phi}{1 - \tan\alpha \cdot \tan\phi} = \frac{\frac{x}{a} + \frac{ax(b-a)}{bx^2+a^3}}{1 - \frac{x^2(b-a)}{bx^2+a^3}} = \frac{bx}{a^2}.$$

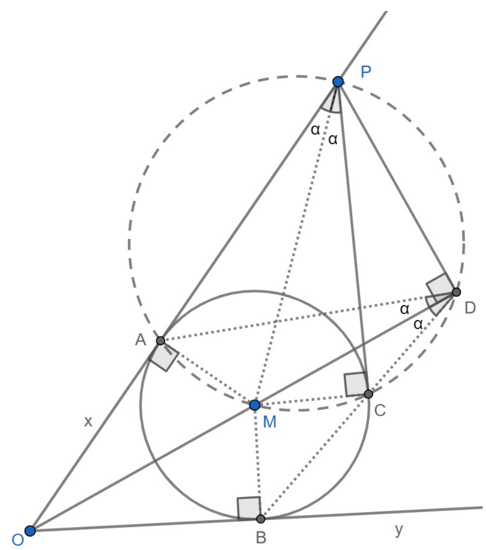
So, $|OR| = x \cdot \cot\theta = \frac{a^2}{b}$ is constant, and therefore the point R is fixed.



Exercise. Show that $(a+b)\cdot\cos\gamma \le b-a$.

5. Denote by x and y the sides of an angle with vertex O and a designated point P on x. A circle is tangent to x at A and to y at B; let the second tangent from P touch that circle at C. Prove that the line BC passes through a fixed point of the plane as A moves along x while B moves along y and P remains fixed. (Problem 39 in [1])

Solution. Let the center of the circle be M. Draw a perpendicular PD to the line OM. Then the points A, P, D, C, M are concyclic. Therefore, $\angle APM = \angle MPC = \angle CDM = \angle MDA$. Denote this angle by α. If we can show that $\angle BDM = \alpha$, then we can say that the points B, C, D are collinear. But this follows directly from the fact that $\triangle OAD = \triangle OBD$. Since the point D is fixed and the line BC passes through D, the proof is complete.



Exercise. Show that if the point M changes along the segment OD, then the area of $\triangle BMD$ is maximal when M is the midpoint of OD.

Problems to solve

Do you want to try some problems yourselves?

Submit your solutions at `https://publications.cms.math.ca/cruxbox/` before the 15th of July 2023. Feel free to also send us your opinion about the featured book or your recommendations for future issues of this column.

S6. Prove that if the three vertices A, A', A'' of a changing triangle are moving along three fixed and concurrent lines u, u', u'', respectively, and its two sides $A'A''$ and $A''A$ are rotating about two fixed points O and O', respectively, then the third side AA' is also rotating about a fixed point O'', which is on the line OO'. ([1], Problem 51).

S7. Prove that the lines joining a point on a circle with the endpoints of a chord of the circle divide the diameter perpendicular to the chord harmonically. ([1], Problem 32).

S8. Given are a fixed circle and two fixed points A and B on it. On the same circle two arbitrary points C and D are chosen. Let $M = AC \cap BD$ and $N = AD \cap BC$. Prove that as the points C and D change, the line MN passes through a fixed point of the plane. ([1], Problem 162).

S9. Let the tangent lines of the circumcircle of $\triangle ABC$ at the points B and C intersect at point D. Through D draw the line that is parallel to the tangent line of the circle at A. This line intersects lines AB and AC at points E and F, respectively. Show that D is the midpoint of the line segment EF. (Problem 159 in [1]).

S10. Given are a triangle ABC and its incircle touching the side BC at the point D. From a point A_1 on the side BC, a second tangent A_1A_2 of the incircle is drawn and the tangency point A_2 is connected with A by a line. The line AA_2 intersects the side BC at D_1. Prove that $\frac{DB}{DC} \cdot \frac{D_1B}{D_1C} = \left(\frac{A_1B}{A_1C}\right)^2$. ([1], Problem 35).

Remark. Some of the problems have fast and easy proofs using projective geometry. Such solutions are also welcome, but unlike [1], we will focus more on elementary ones, similar to those given for Problems 1–5 above.

Acknowledgment

Many thanks to the reviewer who suggested many corrections and improvements to this paper.

References

[1] M.P. Chernyaev, *Problems in synthetic geometry*, Rostov University, Rostov-on-Don, 1961 (in Russian).

[2] V.G. Kopp, M.P. Chernyaev, *Problems in synthetic geometry (review)*, Uspekhi Mat. Nauk, 10:4(66) (1955), 222–225 `http://mi.mathnet.ru/umn8038` (in Russian).

[3] H.S.M. Coxeter, Samuel L. Greitzer, *Geometry revisited, Series: New mathematical library 19*, Mathematical Assoc. of America, 2008.

[4] `https://www.mathnet.ru/eng/person29912`

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 30, 2023**.

4841. *Proposed by Aravind Mahadevan.*

In triangle ABC, let a, b, c denote the lengths of the sides opposite angles A, B and C, respectively. If $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are the lengths of the sides opposite angles A', B' and C', respectively, in triangle $A'B'C'$, prove that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \cos A'\cos B'\cos C'.$$

4842. *Proposed by Mihaela Berindeanu.*

In non-equilateral $\triangle ABC$, let I be the incenter, G the centroid, $IG \parallel BC$ and S be a point chosen so that $BSCG$ is a parallelogram. The parallel drawn through S to BC cuts AI in D. Show that $A,\ C,\ D,\ B$ are concyclic points.

4843. *Proposed by Vasile Cirtoaje.*

Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that

$$\sqrt{(a+3b)(a+3c)} + \sqrt{(b+3c)(b+3a)} + \sqrt{(c+3a)(c+3b)} \geq 12.$$

4844. *Proposed by Seán M. Stewart.*

Suppose n is a positive integer. Show that the value of the improper integral

$$\int_0^\infty \frac{x^{n-1}e^{-x}}{\sqrt{x}}\left(\sum_{k=0}^{n-1}\binom{2k}{k}\frac{x^{-k}}{2^{2k}(n-k-1)!}\right)dx$$

is independent of n.

4845. *Proposed by Ivan Hadinata.*

Let ABC be an acute triangle and let l be a line that is tangent to the circumcircle of ABC at point A. Suppose that X and Y are midpoints of AB and AC respectively. Finally, suppose that D, E, D_0, E_0 are orthogonal projections of the points X, Y, B, C to l respectively and $M = DY \cap EX$, $N = D_0C \cap E_0B$. Prove that the lines MN, CD, BE are concurrent if and only if $AB = AC$.

4846. *Proposed by Michel Bataille.*

The tangents to a parabola $\mathcal{P}$ at B and C intersect at A. Prove that the line through the midpoints of AB and AC is tangent to $\mathcal{P}$.

4847. *Proposed by Todor Zaharinov.*

Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer n. Find all n, such that $k \geq 4$ and

$$n = 2d_2d_3 + 3d_3d_4 + 4d_4d_2.$$

4848. *Proposed by Antonio Garcia.*

Let x, y, z all be non-negative real numbers such that $x + y + z = 3$. Prove that for positive integers m and n we have

$$\sqrt[m]{xy}\frac{\sqrt[n]{z}}{1+\sqrt[n]{z}} + \sqrt[m]{yz}\frac{\sqrt[n]{x}}{1+\sqrt[n]{x}} + \sqrt[m]{xz}\frac{\sqrt[n]{y}}{1+\sqrt[n]{y}} \leq \frac{3}{2}.$$

4849. *Proposed by Daniel Sitaru.*

Solve for real numbers:

$$\begin{cases} x^2 + y^2 + z^2 = 38 \\ \frac{xy(x^2-y^2)+yz(y^2-z^2)+zx(z^2-x^2)}{xy(x-y)+yz(y-z)+zx(z-x)} = 10 \\ x^3 + y^3 + z^3 = 160 \end{cases}$$

4850. *Proposed by George Apostolopoulos.*

Suppose that three concurrent Cevians intersect the side BC, CA, AB of a triangle ABC in points A', B', C', respectively. Let r' be the radius of a circle inscribed in $A'B'C'$; let r, R be the radii of inscribed and circumscribed circles of ABC. Prove that

$$r + r' \leq \frac{3}{4}R.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 juillet 2023**.

4841. *Soumis par Aravind Mahadevan.*

Dans le triangle ABC, notons a, b et c les longueurs des côtés opposés aux angles A, B et C, respectivement. Si les longueurs des côtés opposés aux angles A', B' et C' sont respectivement $\sqrt{a}, \sqrt{b}$ et $\sqrt{c}$, prouvez que dans le triangle $A'B'C'$ on a

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \cos A'\cos B'\cos C'.$$

4842. *Soumis par Mihaela Berindeanu.*

Dans le triangle non équilatéral ABC, désignons par I le centre du cercle inscrit et par G le centroïde. Supposons que $IG \parallel BC$ et soit S un point choisi de sorte que $BSCG$ soit un parallélogramme. La parallèle à BC passant par S coupe AI en D. Montrez que $A,\ C,\ D,\ B$ sont des points concycliques.

4843. *Soumis par Vasile Cirtoaje.*

Soient a, b et c des nombres réels non négatifs tels que $ab + bc + ca = 3$. Montrez que

$$\sqrt{(a+3b)(a+3c)} + \sqrt{(b+3c)(b+3a)} + \sqrt{(c+3a)(c+3b)} \geq 12.$$

4844. *Soumis par Seán M. Stewart.*

Supposons que n est un entier positif. Montrez que la valeur de l'intégrale impropre

$$\int_0^\infty \frac{x^{n-1}e^{-x}}{\sqrt{x}}\left(\sum_{k=0}^{n-1}\binom{2k}{k}\frac{x^{-k}}{2^{2k}(n-k-1)!}\right)dx$$

est indépendant de n.

4845. *Soumis par Ivan Hadinata.*

Soient ABC un triangle aigu et I une droite tangente au cercle circonscrit à ABC au point A. Supposons que X et Y sont respectivement les milieux de AB et AC. Enfin, supposons que D, E, D_0 et E_0 sont respectivement les projections orthogonales des points X, Y, B et C sur I et que $M = DY \cap EX$ et $N = D_0 \cap E_0B$.

Montrez que les droites MN, CD et BE sont concourantes si et seulement si $AB = AC$.

4846. *Soumis par Michel Bataille.*

Les tangentes à une parabole $\mathcal{P}$ en B et C se rencontrent en A. Montrez que la droite passant par les milieux de AB et AC est tangente à $\mathcal{P}$.

4847. *Soumis par Todor Zaharinov.*

Soient $1 = d_1 < d_2 < \cdots < d_k = n$ tous les diviseurs d'un entier positif n. Trouvez tous les n tels que $k \geq 4$ et

$$n = 2d_2d_3 + 3d_3d_4 + 4d_4d_2.$$

4848. *Soumis par Antonio Garcia.*

Soient x, y, z des nombres réels non négatifs tels que $x + y + z = 3$. Montrez que, pour m et n entiers positifs, on a

$$\sqrt[m]{xy}\frac{\sqrt[n]{z}}{1+\sqrt[n]{z}} + \sqrt[m]{yz}\frac{\sqrt[n]{x}}{1+\sqrt[n]{x}} + \sqrt[m]{xz}\frac{\sqrt[n]{y}}{1+\sqrt[n]{y}} \leq \frac{3}{2}.$$

4849. *Soumis par Daniel Sitaru.*

Résolvez le système d'équations suivant pour x, y et z des nombres réels.

$$\begin{cases} x^2 + y^2 + z^2 = 38; \\ \frac{xy(x^2-y^2)+yz(y^2-z^2)+zx(z^2-x^2)}{xy(x-y)+yz(y-z)+zx(z-x)} = 10; \\ x^3 + y^3 + z^3 = 160. \end{cases}$$

4850. *Soumis par George Apostolopoulos.*

Supposons que trois céviennes concourantes coupent les côtés BC, CA et AB d'un triangle ABC aux points A', B' et C', respectivement. Soit r' le rayon d'un cercle inscrit dans $A'B'C'$; soit encore r et R les rayons des cercles inscrit et circonscrit de ABC, respectivement. Montrez que

$$r + r' \leq \frac{3}{4}R.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2022: 48(10), p. 633–636.

4791. *Proposed by George Stoica.*

Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{C}^n$. Prove that if u maximizes $\prod_{i=1}^n |v_i \cdot u|$ over all unit vectors in $\mathbb{C}^n$, then for all i, $|v_i \cdot u| \geq \frac{1}{\sqrt{n}}$.

It was brought to our attention that this problem and its full solution appear in "The complex plank problem, revisited" by Oscar Ortega-Moreno, which can be found at `https://arxiv.org/pdf/2111.03961.pdf`. *We apologize for the replication of the material without proper attribution and thank our readers for pointing our the discrepancy.*

We remind our contributors that the guidelines for proposing problems can be found at `https://cms.math.ca/publications/crux/information-for-contributors/`.

4792. *Proposed by George Apostolopoulos.*

The interior bisectors of angles B and C of a triangle ABC with incenter I meet AC at D and AB at E, respectively. Suppose that $Area(BIC) = Area(AEID)$. Prove that $\angle A = 60^\circ$.

There were 16 correct solutions, most following the approach below.

The respective lengths of AE and AD are $bc/(a+b)$ and $bc/(a+c)$. Where r is the inradius, we find from $[BIC] = [AEID] = [AEI] + [ADI]$ that

$$\tfrac{1}{2}ra = \tfrac{1}{2}\left(\frac{rbc}{a+b}\right) + \tfrac{1}{2}\left(\frac{rbc}{a+c}\right),$$

whence

$$\begin{aligned} 0 &= a(a+b)(a+c) - bc(a+c) - bc(a+b) \\ &= a^3 + a^2(b+c) + abc - 2abc - bc^2 - b^2c \\ &= (a^2 - bc)(a+b+c). \end{aligned}$$

Therefore $a^2 = bc$ and

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{b^2+c^2-bc}{2bc} = \frac{1}{2} + \frac{(b-c)^2}{2bc} \geq \frac{1}{2}.$$

Hence $\angle A \leq 60^\circ$, with equality if and only $b = c$, *i.e.*, the triangle is equilateral.

Comments from the editor. One solver used analytic geometry and two solvers resorted to trigonometry, both approaches requiring considerable manipulation. Vivek Mehra also proved a similar result, that when $[BCDE] = 2[BIC]$, then $\angle A = 90°$. This is a consequence of equating

$$[ADE] = [ABC] - [BCDE] = [ABC] - 2[BIC]$$

and

$$[ADE] = bc(a+c)^{-1}(a+c)^{-1}[ABC]$$

to obtain $b^2 + c^2 = a^2$.

4793. *Proposed by Corneliu Manescu-Avram.*

Let a be an even positive integer and let p be an odd prime number such that $\gcd(a^2-1, p) = 1$. Prove that $a^{n-1} - 1$ is divisible by n, where $n = \dfrac{a^{2p}-1}{a^2-1}$.

We received 3 submissions of which 2 were correct and complete. We present the solution by the UCLan Cyprus Problem Solving Group.

By Fermat's Little Theorem, $a^{2p} \equiv a^2 \bmod p$ and since $p \nmid a^2 - 1$ it follows that

$$n = \frac{a^{2p}-1}{a^2-1} \equiv \frac{a^2-1}{a^2-1} \equiv 1 \bmod p\,;$$

that is, $n-1$ is a multiple of p.

Since a is even, n is odd and so $n-1$ is a multiple of 2.

Since p is an odd prime it is thus the case that $n-1$ is a multiple of $2p$.

Hence $a^{2p}-1 | a^{n-1}-1$ and since $n | a^{2p}-1$ we conclude that $a^{n-1}-1$ is a multiple of $n = \dfrac{a^{2p}-1}{a^2-1}$.

4794. *Proposed by Abhishek Jha.*

Let $P(x) = x^2 + bx + 1$, where b is a non-negative integer. Define $x_0 = 0$ and $x_{i+1} = P(x_i)$ for all integers $i \geq 0$. Find all polynomials $P(x)$ such that there exists a positive integer $n > 1$ which divides x_n.

We received 3 solutions, all of which were correct. We present the solution by UCLan Cyprus Problem Solving Group.

If b is even, then we have $x_1 = 1, x_2 = b+2$ and so $2|x_2$. We will show that if b is odd then $n \nmid x_n$ for every $n > 1$.

Let p be the minimal prime dividing n. Since x_i is easily seen to be odd for each i, then $p > 2$.

If $p = 3$ and $b \equiv 0 \bmod 3$, then the sequence (x_n) modulo 3 is $0, 1, 2, 2, \ldots$ so $n \nmid x_n$. If $p = 3$ and $b \equiv 1 \bmod 3$, then the sequence (x_n) modulo 3 is $0, 1, 0, 1, \ldots$

so $n \nmid x_n$. Indeed if $n|x_n$, then $3|x_n$ and so n is even. But since x_n is odd, then $n \nmid x_n$. If $p = 3$ and $b \equiv 2 \bmod 3$, then the sequence (x_n) modulo 3 is $0, 1, 1, 1, \ldots$ so $n \nmid x_n$.

So we may assume that $p \geqslant 5$. Since p is odd, there is a b' such that $2b' \equiv b \bmod p$. Then

$$P(x) \equiv (x + b')^2 + (1 - b^2) \bmod p,$$

which takes at most $\frac{p+1}{2} < p - 1$ distinct values modulo p.

So two out of $x_1, x_2, \ldots, x_{p-1}$ are congruent modulo p. Suppose $x_i \equiv x_j \bmod p$ where $i < j < p$. Then the sequence is eventually periodic with period $j - i$. If none of $x_i, x_{i+1}, \ldots, x_{j-1}$ is a multiple of p, then $p \nmid x_m$ for every $m \geqslant i$. So $p \nmid x_n$ and therefore $n \nmid x_n$.

Thus we may assume that $x_k \equiv 0 \bmod p$ for some $k < p$. We may assume that k is the minimal positive integer such that $x_k \equiv 0 \bmod p$. Then $p|x_n \iff k|n$. Since $x_1 = 1$, then $k \geqslant 2$. So $k|n$ contradicts the fact that p is the minimal prime dividing n. Thus $p \nmid x_n$ and so $n \nmid x_n$.

4795. *Proposed by Mihaela Berindeanu.*

Let (x_n) and (y_n) be two sequences of natural numbers. If (x_n) is defined by the recurrence relation $x_1 = \dfrac{1}{9}$ and $x_{n+1} = 9x_n^3$, $\forall n \geq 1$ and $y_n = 9x_n^2 + 3x_n + 1$, calculate:

$$\lim_{n\to\infty} y_1 \cdot y_2 \ldots \cdot y_n$$

There were 15 correct and 2 incomplete solutions submitted. The solvers essentially followed the same path.

By induction, it can be established that $x_n = 3^{-(3^{n-1}+1)}$ for $n \geq 1$, whereupon $\lim_{n\to\infty} x_n = 0$. Since

$$y_n = 9x_n^2 + 3x_n + 1 = \frac{1 - (3x_n)^3}{1 - 3x_n} = \frac{1 - 3x_{n+1}}{1 - 3x_n},$$

we have

$$\lim_{n\to\infty} y_1 y_2 \cdots y_n = \lim_{n\to\infty} \frac{1 - 3x_{n+1}}{1 - 3x_1} = \lim_{n\to\infty} \frac{3}{2}(1 - 3x_{n+1}) = \frac{3}{2}.$$

4796. *Proposed by Vasile Cîrtoaje and Leonard Giugiuc.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers less than 1 so that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\frac{a_1^2}{(1-a_1)^2} + \frac{a_2^2}{(1-a_2)^2} + \cdots + \frac{a_n^2}{(1-a_n)^2} \geq \left(\frac{a_1}{1-a_1} + \frac{a_2}{1-a_2} + \cdots + \frac{a_n}{1-a_n} - \frac{\sqrt{n}}{\sqrt{n}+1}\right)^2.$$

We received 7 submissions of which 6 were correct and complete. We present the solution submitted by the proposers.

Using the substitution

$$x_i = \frac{a_i}{1-a_i}, \qquad i = 1, 2, \ldots, n\ ,$$

we need to prove that when

$$\frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} + \cdots + \frac{x_n}{x_n+1} = 1$$

we have

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq \left(x_1 + x_2 + \cdots + x_n - \frac{\sqrt{n}}{\sqrt{n}+1} \right)^2 .$$

Let $S = x_1 + x_2 + \cdots + x_n$. By the Cauchy-Schwarz inequality, we have

$$\left[\sum_{i=1}^{n} x_i(x_i+1) \right] \left(\sum_{i=1}^{n} \frac{x_i}{x_i+1} \right) \geq \left(\sum_{i=1}^{n} x_i \right)^2 ,$$

and so, $\sum_{i=1}^{n} x_i^2 + S \geq S^2$. Thus, it is sufficient to show that

$$S^2 - S \geq \left(S - \frac{\sqrt{n}}{\sqrt{n}+1} \right)^2 ,$$

which is equivalent to $S \geq \dfrac{n}{n-1}$. To prove this, we rewrite the hypothesis in the form

$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \cdots + \frac{1}{x_n+1} = n-1.$$

By the Cauchy-Schwarz inequality, we have

$$\left[\sum_{i=1}^{n} (x_i+1) \right] \left(\sum_{i=1}^{n} \frac{1}{x_i+1} \right) \geq n^2,$$

which is equivalent to $(S+n)(n-1) \geq n^2$, and so, $S \geq \frac{n}{n-1}$.

Equality occurs when $a_1 = a_2 = \cdots = a_n = \dfrac{1}{n}$.

4797. *Proposed by Goran Conar.*

Let $x, y, z > 0$ be real numbers. Prove the following inequality:

$$\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} \geq x + y + z\,.$$

We received 41 submissions, 37 of which are correct. In only 13 solutions, it is noticed that $x = y = z$ is a necessary and sufficient condition to obtain an equality. We present 2 different solutions proposed by a majority of solvers.

Solution 1. From the usual arithmetic and geometric means comparison, we have

$$\frac{1}{2}\left(\frac{yz}{x}+\frac{zx}{y}\right)\geq\sqrt{\frac{yz}{x}\cdot\frac{zx}{y}}=z,$$

with equality if and only if $\dfrac{yz}{x}=\dfrac{zx}{y}$, i.e. $x=y$.

Adding the inequalities obtained by cyclic permutations of x, y, z gives the result, with equality if and only if $x=y=z$.

Solution 2. Expanding the inequality

$$x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2\geq 0,$$

and dividing by 2, gives

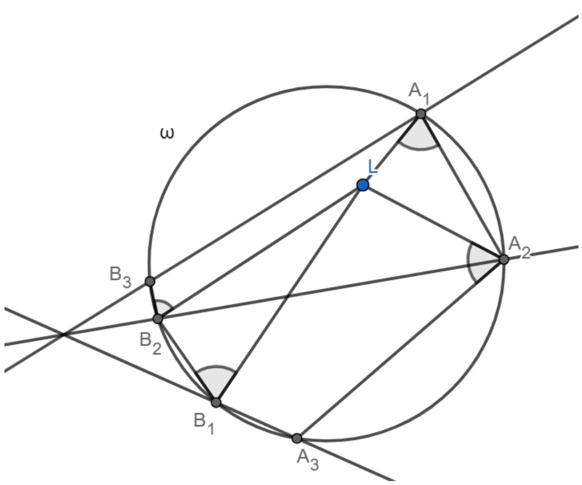
$$x^2y^2+y^2z^2+z^2x^2\geq xyz(x+y+z),$$

The result is therefore obtained after a division by xyz. From the initial inequality, it is evident that this is an equality if and only if $x=y=z$.

Editor's Comments. We have forgotten to point out that this problem statement has been modified by the Editorial Board, for which we ask our readers to accept our apologies. Indeed, the original problem submitted by Goran Conar seemed too sophisticated to us. (The current problem first appeared in Problem 7 of 2007 Irish Math Olympiad).

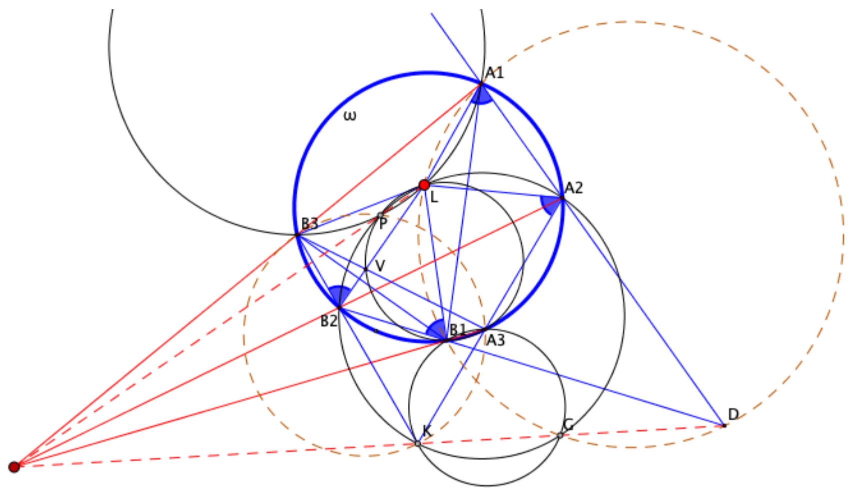
4798. *Proposed by Jason Fang.*

A point L is randomly selected inside circle ω, 6 points $A_1, A_2, A_3, B_1, B_2, B_3$ (As and Bs are in clockwise order) lie on ω such that $\angle LA_1A_2=\angle LA_2A_3=\angle LB_1B_2=\angle LB_2B_3$. Prove that A_1B_3, A_2B_2, A_3B_1 are concurrent or parallel.



We received 4 solutions, all complete and correct, and will feature the solution by Theo Koupelis.

We shall use directed angles to construct a family of five circles that have the same radical center (that is, there exists a point, possibly at infinity, common to the radical axes of pairs of these circles taken two at a time). Let the circumcircle of $\triangle A_1LB_1$ intersect the line A_1A_2 for the second time at point D. From the circle (A_1LB_1D) we get $\angle LB_1B_2 = \angle LA_1A_2 = \angle LA_1D = \angle LB_1D$ and, thus, points D, B_1, B_2 are collinear.



Let the circle (A_1LB_1D) intersect the circumcircle of $\triangle B_2LA_2$ for the second time at point G, and let the line DG intersect the circle (B_2LA_2G) for the second time at point K. For the circle (LA_1DG) we have

$$\angle LB_2B_3 = \angle LA_1A_2 = \angle LA_1D = \angle LGD = \angle LGK = \angle LB_2K,$$

and, thus, points K, B_2, B_3 are collinear. From the circle (B_2LA_2GK) we get

$$\angle LA_2K = \angle LB_2K = \angle LB_2B_3 = \angle LA_2A_3,$$

and thus the points A_2, A_3, K are colinear. But then

$$\angle B_1A_3K = \angle B_1A_3A_2 = \angle B_1A_1A_2 = \angle B_1A_1D = \angle B_1GD = \angle B_1GK,$$

and, thus, points K, A_3, B_1, G are concyclic.

Let V be the point of intersection of B_3A_3 and LB_2. Then

$$\begin{aligned}\angle LVA_3 = \angle LVB_3 &= \angle B_2VB_3 = \angle VB_2B_3 + \angle B_2B_3V\\ &= \angle LB_2B_3 + \angle B_2B_3A_3 = \angle LB_1B_2 + \angle B_2B_1A_3 = \angle LB_1A_3\end{aligned}$$

and, thus, points V, L, B_1, A_3 are concyclic.

Let the circle (VLB_1A_3) intersect the circle (B_2LA_2GK) for the second time at point P. Then $\angle B_3KP = \angle B_2KP = \angle B_2LP = \angle VLP = \angle VA_3P = \angle B_3A_3P$ and, thus, the points B_3, P, A_3, K are concyclic. We now have

$$\begin{aligned}\angle B_3PL &= \angle B_3PA_3 + \angle A_3PL = \angle B_3KA_3 + \angle A_3B_1L \\ &= \angle B_3KA_3 + \angle A_3B_1B_2 + \angle B_2B_1L = \angle B_3KA_3 + \angle A_3B_3B_2 + \angle A_2A_1L \\ &= \angle B_3KA_3 + \angle A_3B_3K + \angle A_2A_1L = \angle B_3A_3K + \angle A_2A_1L \\ &= \angle B_3A_3A_2 + \angle A_2A_1L = \angle B_3A_1A_2 + \angle A_2A_1L = \angle B_3A_1L\end{aligned}$$

and, thus, points B_3, P, L, A_1 are concyclic.

The circles (KA_3B_1G), (B_2LA_2GKP), and ω intersect by pairs, whence their radical axes

$$GK, B_1A_3, A_2B_2$$

intersect at the radical center, which could be at infinity. Similarly, the circles (VLB_1A_3P), (B_2LA_2GKP), and ω intersect by pairs, whence their radical axes

$$LP, A_2B_2, B_1A_3$$

intersect at that same radical center. Finally, the circles ω, (B_3PLA_1), and (B_2LA_2GKP) intersect by pairs, whence their radical axes

$$A_2B_2, LP, A_1B_3$$

must intersect in that same radical center. Therefore, the lines GK, B_1A_3, A_2B_2, LP, A_1B_3 intersect at a common point, which could be at infinity. In particular, the lines A_1B_3, A_2B_2, A_3B_1 are concurrent or parallel.

4799. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\lim_{n\to\infty} \int_1^2 \sqrt[n]{\lfloor x^n \rfloor}\mathrm{d}x,$$

where $\lfloor x \rfloor$ denotes the floor of $x \in \mathbb{R}$.

We received 17 solutions, all of which were correct. We present the solution by Ulrich Abel and Vitaliy Kushnirevych.

We prove that

$$I_n := \int_1^2 \sqrt[n]{\lfloor x^n \rfloor} dx \to \frac{3}{2} \qquad (n \to \infty).$$

Since $k/(k+1) \leq \lfloor t \rfloor / t \leq 1$, if $k \leq t \leq k+1$ $(k \in \mathbb{N})$, we have $t/2 \leq \lfloor t \rfloor \leq t$, for $t \geq 1$. Hence,

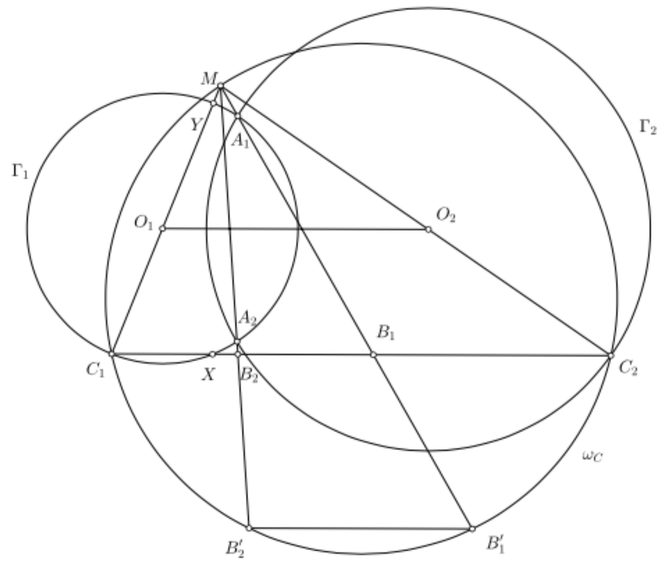
$$\sqrt[n]{\frac{1}{2}} \int_1^2 x\ dx \leq I_n \leq \int_1^2 x\ dx.$$

Since $\lim_{n\to\infty} \sqrt[n]{1/2} = 1$, our claim is a consequence of the sandwich theorem.

4800. *Proposed by Michel Bataille.*

Two circles Γ_1, Γ_2, with centres O_1, O_2 and distinct radii, intersect in A_1 and A_2. Let C_1 on Γ_1 and C_2 on Γ_2 be such that the line C_1C_2 is parallel to O_1O_2 with A_1, A_2, O_1, O_2 not on C_1C_2. Let lines C_1O_1 et C_2O_2 intersect at M and lines MA_1, MA_2 intersect C_1C_2 in B_1, B_2, respectively. Prove that the circumcircles of ΔMB_1B_2 and ΔMC_1C_2 are tangent.

We received 5 submissions, all correct, and feature the solution by the UCLan Cyprus Problem-Solving Group.



Let X be the other point of intersection of C_1C_2 with Γ_1 and Y the other point of intersection of C_1M with Γ_1. Then C_1Y is a diameter of Γ_1, so $YX \perp C_1C_2$. We also have $A_1A_2 \perp O_1O_2$, thus $A_1A_2 \perp C_1C_2$ and $A_1A_2 \parallel XY$. Because parallel transversals intercept a circle in equal arcs, it follows that

$$\angle C_2C_1A_2 = \angle XC_1A_2 = \angle A_1C_1Y = \angle A_1C_1M$$

(using directed angles). Similarly we have $\angle A_2C_2C_1 = \angle MC_2A_1$. It follows that A_2 is the isogonal conjugate of A_1 with respect to the triangle MC_1C_2. So we also have

$$\angle C_1MB_2 = \angle C_1MA_2 = \angle A_1MC_2 = \angle B_1MC_2\,.$$

This is enough to guarantee the final result. Indeed, let ω_C be the circumcircle of MC_1C_2 and let B_1' and B_2' be the other points of intersection of MB_1 and MB_2 respectively with ω_C. Then $\angle C_1MB_2' = \angle B_1'MC_2$ showing that B_1B_2 is parallel to $B_1'B_2'$. Thus the homothety with center M mapping B_1' to B_1 also maps B_2' to B_2. Thus it maps ω_C to the circumcircle ω_B of MB_1B_2. Consequently, the centres of ω_B and ω_C are collinear with M and the two circles are tangent at M.