Official Solutions for CMO 2023

P1. Let a and b be non-negative integers. Consider a sequence $s_1, s_2, s_3, \ldots$ such that $s_1 = a$, $s_2 = b$, and $s_{i+1} = |s_i - s_{i-1}|$ for $i \geq 2$. Prove that there is some i for which $s_i = 0$.

Solution 1. First, note that for any *positive* integers x, y, we have

$$|x - y| = \max(x, y) - \min(x, y) \leq \max(x, y) - 1 < \max(x, y).$$

Clearly, the sequence $(s_i)_{i\geq 1}$ consists of non-negative integers. For a contradiction, suppose that $s_i \geq 1$ for all $i \geq 1$. Then, for any positive integer k,

$$\begin{aligned} s_{2(k+1)-1} &= s_{2k+1} = |s_{2k} - s_{2k-1}| < \max(s_{2k-1}, s_{2k}) \\ s_{2(k+1)} &= s_{2k+2} = |s_{2k+1} - s_{2k}| < \max(s_{2k}, s_{2k+1}) \\ &\leq \max(s_{2k}, \max(s_{2k-1}, s_{2k})) = \max(s_{2k-1}, s_{2k}). \end{aligned}$$

Hence, if $b_k = \max(s_{2k-1}, s_{2k})$, then

$$b_{k+1} = \max(s_{(2(k+1)-1}, s_{2(k+1)}) < \max(s_{2k-1}, s_{2k}) = b_k,$$

so $(b_k)_{k\geq 1}$ is a decreasing sequence. But it means that $b_k = 0$ for some k which gives us the desired contradiction. $\square$

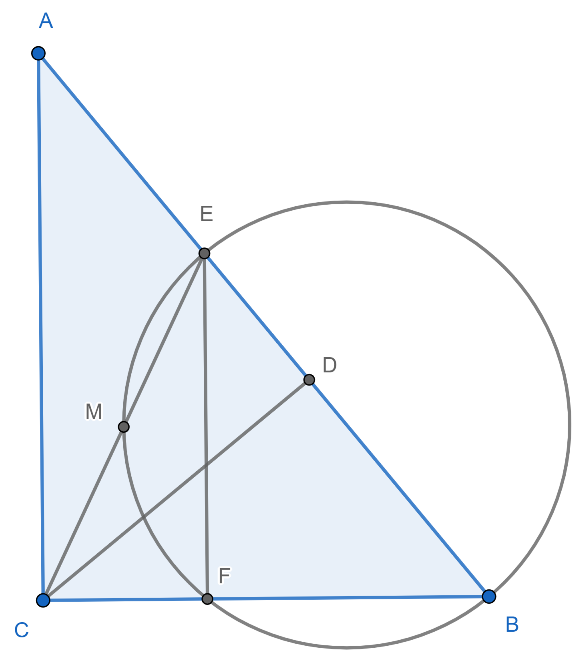
Solution 2. For non-negative integers x, y, we have $|x - y| \leq \max(x, y)$. Therefore

$$\max(s_{i+1}, s_{i+2}) = \max(s_{i+1}, |s_i - s_{i+1}|) \leq \max(s_i, s_{i+1}).$$

Thus the sequence $(\max(s_i, s_{i+1}))$ is non-increasing. Since it is bounded below by 0, it is eventually constant, that is, there exist C and N such that $\max(s_i, s_{i+1}) = C$ for all $i \geq N$. We can assume that $s_N = C$ (if not, replace N by $N + 1$). If $s_{N+1} = C$, then $s_{N+2} = |C - C| = 0$, as desired. If $s_{N+1} = 0$, then we are clearly done. Finally, if $0 < s_{N+1} < C$, then $s_{N+2} = C - s_{N+1} < C$, so $\max(s_{N+1}, s_{N+2}) < C$, which gives us the desired contradiction. $\square$

P2. An acute triangle is a triangle that has all angles less than $90°$ ($90°$ is a Right Angle). Let ABC be a right-angled triangle with $\angle ACB = 90°$. Let CD be the altitude from C to AB, and let E be the intersection of the angle bisector of $\angle ACD$ with AD. Let EF be the altitude from E to BC. Prove that the circumcircle of BEF passes through the midpoint of CE.

Solution. We provide two solutions.



Solution 1:

Let $\angle CBA = x$. Then $\angle ACD = 90° - \angle CAD = x$, so $\angle ACE = x/2$.

Therefore $\angle BCE = 90° - x/2$, and

$$\angle CEB = 180° - \angle BCE - \angle CBE = 180° - (90° - x/2) - x = 90° - x/2 = \angle BCE,$$

whence $|BC| = |BE|$.

Let the midpoint of EC be M, and as $|BE| = |BC|$, we have $\angle BME = 90°$.

Since $\angle BFE = 90°$, this implies that $BFME$ is cyclic, which proves the problem.

Solution 2:

Let $|AB| = c$, $|BC| = a$, and $|CA| = b$. Since ABC and CBD are similar (right-angled) triangles, we have

$$\frac{|CD|}{b} = \frac{|DB|}{a} = \frac{a}{c},$$

hence $|CD| = ab/c$ and $|DB| = a^2/c$. Thus $|AD| = c - |DB| = b^2/c$. As $|CE|$ is the angle bisector, let $x = |ED|$, and then

$$\frac{x}{b^2/c - x} = \frac{|DE|}{|EA|} = \frac{|CD|}{|CA|} = \frac{ab/c}{b} = \frac{a}{c}.$$

This gives $x = ab^2/c^2 - (a/c)x$, so

$$x = \frac{ab^2}{c(a+c)} = \frac{a(c^2-a^2)}{c(a+c)} = \frac{a(c-a)}{c}.$$

Therefore

$$|BE| = |BD| + |DE| = \frac{a(c-a)}{c} + \frac{a^2}{c} = \frac{ac}{c} = a = |BC|.$$

Let the midpoint of EC be M, and as $|BE| = |BC|$, we have $\angle BME = 90^\circ$.

Since $\angle BFE = 90^\circ$, this implies that $BFME$ is cyclic, which proves the problem. $\square$

P3. William is thinking of an integer between 1 and 50, inclusive. Victor can choose a positive integer m and ask William: "does m divide your number?", to which William must answer truthfully. Victor continues asking these questions until he determines William's number. What is the minimum number of questions that Victor needs to guarantee this?

Solution. The minimum number is 15 questions.

First, we show that 14 or fewer questions is not enough to guarantee success. Suppose Victor asks at most 14 questions, and William responds with "no" to each question unless $m = 1$. Note that these responses are consistent with the secret number being 1. But since there are 15 primes less than 50, some prime p was never chosen as m. That means the responses are also consistent with the secret number being p. Therefore, Victor cannot determine the number for sure because 1 and p are both possible options.

Now we show that Victor can always determine the number with 15 questions. Let N be William's secret number. First, Victor asks 4 questions, with $m = 2, 3, 5, 7$. We then case on William's responses.

Case 1. *William answers "no" to all four questions.*

N can only be divisible by primes that are 11 or larger. This means N cannot have multiple prime factors (otherwise $N \geq 11^2 > 50$), so either $N = 1$ or N is one of the 11 remaining primes less than 50. Victor can then ask 11 questions with $m = 11, 13, 17, \ldots, 47$, one for each of the remaining primes, to determine the value of N.

Case 2. *William answers "yes" to $m = 2$, and "no" to $m = 3, 5, 7$.*

There are only 11 possible values of N that match these answers (2, 4, 8, 16, 22, 26, 32, 34, 38, 44, and 46). Victor can use his remaining 11 questions on each of these possibilities.

Case 3. *William answers "yes" to $m = 3$, and "no" to $m = 2, 5, 7$.*

There are 5 possible values of N (3, 9, 27, 33, and 39). Similar to Case 2, Victor can ask about these 5 numbers to determine the value of N.

Case 4. *William answers "yes" to multiple questions, or one "yes" to $m = 5$ or $m = 7$.*

Let k be the product of all m's that received a "yes" response. Since N is divisible by each of these m's, N must be divisible by k. Since $k \geq 5$, there are at most 10 multiples of k between 1 and 50. Victor can ask about each of these multiples of k with his remaining questions. $\square$

P4. There are 20 students in a high school class, and each student has exactly three close friends in the class. Five of the students have bought tickets to an upcoming concert. If any student sees that at least two of their close friends have bought tickets, then they will buy a ticket too.

Is it possible that the entire class buys tickets to the concert?

(Assume that friendship is mutual; if student A is close friends with student B, then B is close friends with A.)

Solution 1. It is impossible for the whole class to buy tickets to the concert.

If two students A and B are close friends, and A has bought a ticket to the concert while B has not, then A is *enticing* B. We call this pair (A, B) an *enticement.*

In order for a student to change their mind and buy a ticket, they first be enticed by at least 2 of their 3 close friends. That means they can only entice at most 1 other friend. Therefore, the total number of enticements among the students decreases by 1 whenever a student changes their mind to buy a ticket.

Initially, the maximum number of enticements is 15 (each of the initial 5 students with tickets has 3 friends to entice). Assume, for the sake of contradiction, that the entire class ends up buying tickets. After the first 14 people buy tickets, the number of enticements is at most $15 - 14 = 1$. This is not enough to convince the last person to buy a ticket, since they need 2 enticements.

Therefore, it is impossible that the entire class buys tickets. □

Solution 2. We shall use the term *friendship* to denote an unordered pair of students who are close friends. Since each of the 20 students is part of exactly 3 friendships, there are exactly 30 friendships in the class. (We could also represent friendships as edges in an undirected graph whose vertices are the 20 students.)

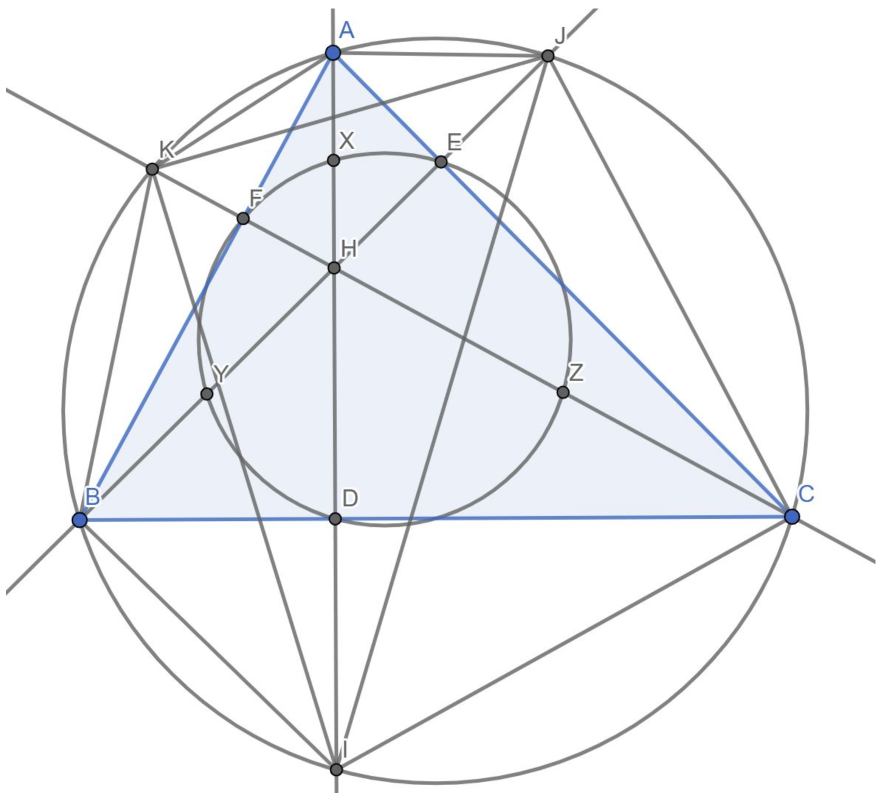
We say that a friendship is *used* if one of the students in that friendship buys a ticket after the original five buyers, and the other student already has a ticket at that time. Each time a ticket is purchased after the original five purchases, at least two friendships are used. Observe that no friendship gets used twice.

If all 20 students buy tickets, then three friendships are used when the last student buys a ticket. This would imply that the number of used friendships is at least $14 \times 2 + 3 = 31$, which is more than the number of friendships. This contradiction proves that it is not possible that the entire class buys tickets. □

P5. An acute triangle is a triangle that has all angles less than 90° (90° is a Right Angle). Let ABC be an acute triangle with altitudes AD, BE, and CF meeting at H. The circle passing through points D, E, and F meets AD, BE, and CF again at X, Y, and Z respectively. Prove the following inequality:

$$\frac{AH}{DX} + \frac{BH}{EY} + \frac{CH}{FZ} \geq 3.$$

Solution. Let the circumcircle of ABC meet the altitudes AD, BE, and CF again at I, J, and K respectively.



Lemma (9-point circle)**.** *I, J, K are the reflections of H across BC, CA, AB. Moreover, D, E, F, X, Y, Z are the midpoints of HI, HJ, HK, HA, HB, HC.*

Proof. Since $ABDE$ and $ABIC$ are cyclic, we see that

$$\angle EBD = \angle EAD = \angle CAI = \angle CBI.$$

Hence the lines BI and BH are reflections across BC. Similarly, CH and CI are reflections across BC, so I is the reflection of H across BC. The analogous claims for J and K follow. A $\times 2$ dilation from H now establishes the result. $\square$

From this lemma, we get $AI = 2XD$, $BJ = 2EY$, and $CK = 2FZ$. Hence it is equivalent to showing that

$$\frac{AH}{2DX} + \frac{BH}{2EY} + \frac{CH}{2FZ} \geq \frac{3}{2},$$

which is in turn equivalent to

$$\frac{AH}{AI} + \frac{BH}{BJ} + \frac{CH}{CK} \geq \frac{3}{2}. \tag{*}$$

Let $a = JK$, $b = KI$ and $c = IJ$. Again by the lemma we find $AH = AK = AJ$, so by Ptolemy's theorem on $AKIJ$,

$$AJ \cdot KI + AK \cdot IJ = AI \cdot JK.$$

Substituting and rearranging,

$$\begin{aligned} AH \cdot b + AH \cdot c &= AI \cdot a \\ AH \cdot (b + c) &= AI \cdot a \\ \frac{AH}{AI} &= \frac{a}{b+c}. \end{aligned}$$

Similarly,

$$\frac{BH}{BJ} = \frac{b}{c+a} \qquad \text{and} \qquad \frac{CH}{CK} = \frac{c}{a+b}.$$

Plugging these back into (*), the desired inequality is now

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

This is known as Nesbitt's Inequality, which has many proofs. Below is one such proof.

Add 3 to both sides and rearrange:

$$\begin{aligned} &\left(\frac{a}{b+c} + 1\right) + \left(\frac{b}{c+a} + 1\right) + \left(\frac{c}{a+b} + 1\right) \geq \frac{3}{2} + 3 \\ \iff \quad & \frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b} \geq \frac{9}{2} \\ \iff \quad & (a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq \frac{9}{2} \\ \iff \quad & \frac{(b+c) + (c+a) + (a+b)}{3} \geq \frac{3}{\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}} \end{aligned}$$

which is true by the AM-HM inequality. $\square$

