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## Crux Mathematicorum

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## Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek,
Shawn Godin

## MathemAttic

No. 44

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by June 15, 2023.

MA216. The circle $x^{2}+y^{2}-2 a x=1$ cuts the $x$-axis at $P$ and $Q$. If $O$ is the origin, show that $O P \cdot O Q$ is a constant.

MA217. Prove that the alphanumeric, in which different letters represent different digits, does not have a solution.

$$
\begin{array}{r}
\text { T W E N T Y } \\
+\mathrm{T} \text { W E N T Y } \\
\hline \mathrm{C} \text { R I C K E T }
\end{array}
$$

MA218. Consider a circle with centre $C$. There are $n$ distinct points marked on the circumference of the circle so that no two points are on opposite ends of the same diameter. What is the smallest $n$ so that, no matter where the points are marked, it is possible to label two of the points $A$ and $B$ so that at least three of the points lie on the smaller $\operatorname{arc}$ from $A$ to $B$.

MA219. Solve the equation $|x|+|x-1|+|x-2|=4$.
MA220. Proposed by Titu Zvonaru, Comăneşti, Romania.
Let $A B C$ be a triangle with $A=30^{\circ}, B=80^{\circ}$. Let $D$ be the orthogonal projection of $B$ onto the side $A C$. On the line $D B$ we consider the point $E$ such that $B E=B C$ and $B$ lies between $D$ and $E$. Prove that the triangle $A E C$ is isosceles.

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

> Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le $\mathbf{1 5}$ juin 2023.

MA216. Le cercle $x^{2}+y^{2}-2 a x=1$ rencontre l'axe des $x$ en $P$ et $Q$. Si $O$ dénote l'origine, démontrer que $O P \cdot O Q$ est constant.

MA217. Dans l'alphanumérique qui suit, des lettres différentes représentent des chiffres différents. Démontrer qu'il n'y a aucune solution.

$$
\begin{array}{r}
\text { T W E N T } \\
+ \\
\mathrm{T}
\end{array} \mathrm{~W} \text { E } \mathrm{N} \text { T } \mathrm{Y} .
$$

MA218. Il y a $n$ points distincts font partie d'un cercle, de façon à ce qu'aucuns deux d'entre eux se trouvent sur un même diamètre. Quel est le plus petit $n$ pour que, quel que soit ce placement des points, il est possible d'en choisir deux d'entre eux, les nommant $A$ et $B$, de façon à ce qu'au moins trois points se retrouvent dans le plus petit arc entre $A$ et $B$.

MA219. Résoudre l'équation $|x|+|x-1|+|x-2|=4$.
MA220. Proposé par Titu Zvonaru, Comăneşti, Romania.
Soit $A B C$ un triangle tel que $A=30^{\circ}, B=80^{\circ}$ et soit $D$ la projection orthogonale de $B$ vers le côté $A C$. Sur la ligne $D B$, soit $E$ un point tel que $B E=B C$ et tel que $B$ se trouve entre $D$ et $E$. Démontrer que le triangle $A E C$ est isocèle.


## MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2022: 48(9), p. 501-502.

MA191. The sum of the digits of a set of three consecutive two-digit integers is 42 . How many such sets of integers are there?
Originally problem \#9 from 2003-2004 game \#2 in ATOM Volume VIII: Problems for Mathematics Leagues III, by Peter I. Booth, John Grant McLoughlin, and Bruce L. R. Shawyer.

There were multiple different approaches to the problem. Unfortunately, many of the submissions were incomplete as they were missing the border case with tens digit changes. Only 8 out of 15 submissions were complete. We present the solution by Catherine Jian.

Let $x$ be the integer in the middle and let its tens digit be $a$ and its ones digit be $b$, i.e., $x=10 a+b$. We have the following casework based on the value of $b$.

Case 1: $b=0$.
$x-1$ has tens digit $a-1$ and ones digit $9 . x+1$ has tens digit $a$ and ones digit 1. The sum of digits is then

$$
(a-1)+9+a+0+a+1=3 a+9=42
$$

Solving this equation gives $a=11$, which is not a valid digit.
Case 2: $b=9$.
$x-1$ has tens digit $a$ and ones digit 8. $x+1$ has tens digit $a+1$ and ones digit 0 . The sum of digits is then

$$
a+8+a+9+a+1+0=3 a+18=42
$$

Solving this equation gives $a=8$, which corresponds to one set of three consecutive integers ( $88,89,90$ ).
Case 3: $1 \leq b \leq 8$.
$x-1$ has tens digit $a$ and ones digit $b-1 . x+1$ has tens digit $a$ and ones digit $b+1$. The sum of digits is then

$$
a+b-1+a+b+a+b+1=3(a+b)=42
$$

Solving this equation gives $a+b=14$. Since $1 \leq b \leq 8$, the possible solutions for $(a, b)$ are $(9,5),(8,6),(7,7),(6,8)$, which correspond to 4 sets of integers.

Hence, in total there are 5 such sets of integers.

MA192. Suppose that $a, b, p, q, r, s$ are positive integers for which

$$
\frac{p}{q}<\frac{a}{b}<\frac{r}{s}
$$

and $q r-p s=1$. Prove that $b \geq q+s$.
Originally problem 15/2 from ATOM Volume IV: Inequalities, by Edward J. Barbeau and Bruce L. R. Shawyer.

We received 7 correct solutions for this problem. We present 2 solutions.
Solution 1, by Prithwijit De.
Observe that $a q-b p>0, b r-a s>0$ and since they are positive integers each must be at least 1. Therefore

$$
b=b(q r-p s)=q(b r-a s)+s(a q-b p) \geq q+s
$$

Solution 2, by Henry Ricardo.
The given conditions yield

$$
\begin{aligned}
\frac{1}{q s}=\frac{r}{s}-\frac{p}{q}=\left(\frac{r}{s}-\frac{a}{b}\right)+\left(\frac{a}{b}-\frac{p}{q}\right) & =\frac{b r-a s}{b s}+\frac{a q-b p}{b q} \\
& \geq \frac{1}{b s}+\frac{1}{b q}=\frac{b q+b s}{(b s)(b q)}=\frac{q+s}{b q s}
\end{aligned}
$$

which implies $b \geq q+s$.
MA193. A forester wants to plant trees in a triangular field, which is fenced as shown (the fences go from corners to the midpoints of the opposite sides). The fields are called $N, S, E$, and $W$ as shown.


The field $W$ will take 800 trees. How many trees will the field $N$ take?
Originally problem 76 from Shaking Hands in Corner Brook and other Math Problems for senior high school students, Edited by Peter Booth, Bruce Shawyer, and John Grant McLoughlin.

We received 8 correct submissions. We present the solution by Soham Bhadra, supplemented by the editor.

Let us name vertices of the triangular field $A, B, C$ and let the midpoints of $A C$, $A B$ and $B C$ be $D, E$ and $F$ respectively. Point $O$ is the intersection of $B D$ and $C E$.

Then $B D$ and $C E$ are the medians of $\triangle A B C$ and so, $O$ is the centroid. Also $O D$ and $O E$ are the medians of $\triangle O C A$ and $\triangle O B A$ respectively.
As $O D$ is a median of $\triangle O C A$,

$$
\operatorname{Area}(\Delta O C D)=\operatorname{Area}(\Delta O D A)=\frac{1}{2} \operatorname{Area}(\Delta C O A)
$$

As $O E$ is a median of $\triangle O B A$,

$$
\operatorname{Area}(\triangle O A E)=\operatorname{Area}(\triangle O E B)=\frac{1}{2} \operatorname{Area}(\triangle B O A)
$$

As $A F$ is a median of $\triangle A B C$, $\operatorname{Area}(\triangle A B F)=\operatorname{Area}(\triangle A C F)$, and as $O F$ is a median $\triangle B O C, \operatorname{Area}(\triangle B O F)=\operatorname{Area}(\triangle C O F)$.

Therefore $\operatorname{Area}(\triangle B O A)=\operatorname{Area}(\triangle C O A)$ and so $\operatorname{Area}(\Delta O D A)=\operatorname{Area}(\Delta O A E)$.
Hence

$$
\operatorname{Area}(\triangle O C D)=\operatorname{Area}(\triangle O D A)=\operatorname{Area}(\triangle O A E)=\operatorname{Area}(\triangle O E B)
$$

Then

$$
\operatorname{Area}(O D A E)=\operatorname{Area}(\triangle O D A)+\operatorname{Area}(\triangle O A E)=2 \cdot \operatorname{Area}(\triangle O E B)=1600
$$

Therefore, the field $N$ will take 1600 trees.
MA194. Let $f_{1}(x)=\frac{x-13}{x+5}$ and $f_{n+1}(x)=f_{n}\left(f_{1}(x)\right)$. Determine the value of $f_{2022}(x)$, assuming that $x$ is in the domain of $f_{2022}(x)$.
Originally problem 6 from the NLTA Senior High Math League: Season 2009/2010, Championship Game.
We received 9 solutions for this problem. The following is the solution by Catherine Jian.

By applying the given function composition, we observe that

$$
f_{2}(x)=\frac{-2 x-13}{x+2}, \quad f_{3}(x)=\frac{-5 x-13}{x-1} \text { and } f_{4}(x)=x
$$

Therefore we have

$$
f_{5}(x)=f_{1}(x), f_{6}(x)=f_{2}(x), f_{7}(x)=f_{3}(x), f_{8}(x)=x, \ldots
$$

Since $2022=4 \cdot 505+2$, we have

$$
f_{2022}(x)=f_{2}(x)=\frac{-2 x-13}{x+2}
$$

MA195. In equilateral triangle $A B C$ of side length 2, suppose that $M$ and $N$ are the mid-points of $A B$ and $A C$, respectively. The triangle is inscribed in a circle. The line segment $M N$ is extended to meet the circle at $P$. Determine the length of the line segment $N P$.
Originally problem 1.2 from Explorations in Geometry, by Bruce Shawyer.
We received 12 submissions, all correct. We present the solution by Vasile Teodorovici.


Let us set a Cartesian coordinate plane system to be centered in the center of the circle and with the $y$ axis along the altitude from $A$. It follows that $O=(0,0)$, $A=\left(0, \frac{2 \sqrt{3}}{3}\right), C=\left(1, \frac{-\sqrt{3}}{3}\right)$ and $N=\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$. Then the $y$ coordinate of $P$ is also $\frac{\sqrt{3}}{6}$. The circumradius is $\frac{2 \sqrt{3}}{3}$ and hence the equation of the circle is $x^{2}+y^{2}=\frac{4}{3}$. From this equation, we have that the $x$-coordinate of $P$ is $\frac{3 \sqrt{5}}{6}$. Finally, this implies that

$$
N P=\frac{3 \sqrt{5}}{6}-\frac{1}{2}=\frac{\sqrt{5}-1}{2} .
$$

Editor's Comments. The problem is based on the famous E3007 of American Mathematical Monthly problem proposed by Odom and Coxeter in 1982.

## SHAWYER CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2022: 48(9), p. 522-524.

## S1. Proposed by Bill Sands.

(a) Solve the alphametic

$$
B R U C E+B R U C E+\cdots+B R U C E=S H A W Y E R,
$$

where there are exactly $94 B R U C E$ s. (As usual, different letters are replaced by different digits, and no number starts with zero.)
(b)ぇ Are there any solutions if 94 is replaced by some other positive integer?

We received 2 solutions, both of which were correct. Both solutions were computer aided and found 68 solutions for part b) including the one in a). We present the solution by Ulrich Abel and Vitaliy Kushnirevych.
(a) As can be seen in part (b), the only solution with factor 94 is

$$
94 * 84206=7915364
$$

i.e., $\mathrm{BRUCE}=84206$ and SHAWYER $=7915364$.
(b) There are exactly 68 solutions. An exhaustive table in the format "Factor * BRUCE = SHAWYER", calculated by the aid of a computer, is given below:

|  |  | 8 | $23 * 94508=2173684$ |
| :---: | :---: | :---: | :---: |
| $28 * 78406=2195368$ | $34 * 40675=1382950$ | $38 * 94018=3572684$ | $42 * 58019=2$ |
| $43 * 82794=3560142$ | $48 * 74208=3561984$ | $52 * 92031=4785612$ | $54 * 48507=2619378$ |
| $58 * 58671=3402918$ | $65 * 65831=4279015$ | $70 * 30685=2147950$ | 76 * |
| $77 * 53609=4127893$ | $77 * 68924=5307148$ | $85 * 50376=4281960$ | 85 |
| $92 * 64172=5903824$ | $92 * 92761=8534012$ | $94 * 84206=791$ | $96 * 62307=5981472$ |
| $116 * 82417=95603$ | $132 * 64932=857$ | $148 * 24108=3567984$ | $150 * 40925=6138750$ |
| $152 * 46593=7082136$ | $157 * 46198=7253086$ | $158 * 38571=6094218$ | $159 * 39541=6287019$ |
| $160 * 30576=4892160$ | $168 * 26107=4385976$ | $183 * 32874=6015942$ | $185 * 50716=9382460$ |
| $205 * 35419=7260895$ | $210 * 20761=4359810$ | $212 * 18704=3965248$ | $212 * 32051=6794812$ |
| $214 * 10975=2348650$ | $214 * 34051=7286914$ | $237 * 16478=3905286$ | $238 * 40825=9716350$ |
| $252 * 16893=4257036$ | $252 * 34512=8697024$ | $264 * 34576=9128064$ | $306 * 20975=6418350$ |
| $310 * 10596=3284760$ | $354 * 10925=3867450$ | $390 * 20465=7981350$ | $393 * 12894=5067342$ |
| $410 * 10629=4357890$ | $410 * 20476=8395160$ | $410 * 20536=8419760$ | $410 * 20619=8453790$ |
| $410 * 20716=8493560$ | $414 * 10625=4398750$ | $435 * 10928=4753680$ | $460 * 10768=4953280$ |
| $524 * 16079=8425396$ | $549 * 14536=7980264$ | $610 * 10539=6428790$ | $710 * 10539=7482690$ |
| $44 * 12053=8967432$ | $838 * 10375=869425$ | $860 * 10846=932756$ | 2 |

Editor's Comments. Titu Zvonaru provided a program in Basic used to confirm the exhaustive set of solutions.

S2. Proposed by Shawn Godin.
A pair of two-digit numbers has the following properties:

1. The sum of the four digits is 25 .
2. The sum of the two numbers is 97 .
3. The product of the four digits is 864 .
4. The product of the two numbers is 1972.

Determine the two numbers.
Note: This is a modification of problem M266 proposed by Bruce Shawyer [2006 : 32(7), 426] with featured solution by Geoffrey A. Kandall [2007: 33(7), 402].

We received 13 submissions, all correct. We present two solutions submitted by Ivan Hadinata and these represent common approaches taken by other solvers.

## Solution 1.

Let $x$ and $y$ be the desired pair of two-digit numbers. From the second and fourth properties given, we have $x+y=97$ and $x y=1972$. By Viéta's theorem, $x$ and $y$ are the roots of the quadratic polynomial

$$
P(a)=a^{2}-97 a+1972=(a-29)(a-68)
$$

Then $(x, y)=(29,68)$ or $(x, y)=(68,29)$. In fact, 29 and 68 satisfy all the given properties. Therefore, the two numbers are 29 and 68.

## Solution 2.

Note that $1972=2^{2} \cdot 17 \cdot 29$ and hence 1972 has only 5 two-digit positive divisors, which are $17,29,34,58,68$. In fact,

$$
1972=116 \cdot 17=68 \cdot 29=58 \cdot 34
$$

Consequently, the desired two-digit numbers are either 29 and 68 or 34 and 58 . However, 34 and 58 do not satisfy the first property. It can be readily checked that 29 and 68 satisfy all the conditions. Therefore, the two numbers are 29 and 68.

S3. If $\alpha$ and $\beta$ are the roots of $x^{2}+5 x+7=0$, find a quadratic equation with roots $\frac{1}{\alpha^{2}}$ and $\frac{1}{\beta^{2}}$.
Originally problem 40 from ATOM Volume II: Algebra - Intermediate Methods, by Bruce Shawyer.

We received 12 correct submissions. We present a standard solution approach taken on by several solvers.

The quadratic equation with roots roots $\frac{1}{\alpha^{2}}$ and $\frac{1}{\beta^{2}}$ is given by

$$
\left(\alpha^{2} x-1\right)\left(\beta^{2} x-1\right)=\alpha^{2} \beta^{2} x^{2}-\left(\alpha^{2}+\beta^{2}\right) x+1=0
$$

Therefore,

$$
x^{2}+5 x+7=x^{2}-(\alpha+\beta) x+\alpha \beta .
$$

By Viéta's formulas, we have $\alpha \beta=7$ and $\alpha+\beta=5$. Therefore,

$$
\alpha^{2} \beta^{2}=49
$$

and

$$
\alpha^{2}+\beta^{2}=(\alpha+\beta)^{2}-2 \alpha \beta=11
$$

As such, the corresponding quadratic equation is $49 x^{2}-11 x+1=0$.
$\mathbf{S 4 .}$ What is the $19^{\text {th }}$ term in the sequence $4,6,14,20, \ldots$ determined by a diagonal line (in red) in the following diagram:


We received 6 submissions, all correct. We present the solution by Missouri Problem Solving Group.

The paths below the diagonal (e.g., from 4 to 6 and from 14 to 20 ) increase by 4 steps with each iteration, so the path lengths would be $2,6,10,14, \ldots$. The paths above the diagonal (e.g. from 6 to 14 and from 20 to 32 ) also increase by 4 steps with each iteration, so those path lengths would be $8,12,16,20, \ldots$. Denote the entries on the diagonal by $a_{1}, a_{2}, a_{3}, \ldots$ The lengths of the paths between entries with odd indices are $2+8,6+12,10+16, \ldots$ or $10,18,26, \ldots$ The $k$ th term in this sequence is $8 k+2$ and

$$
a_{2 n+1}=a_{1}+\sum_{k=1}^{n}(8 k+2)=2+8 \cdot \frac{n(n+1)}{2}+2 n=4 n^{2}+6 n+4
$$

Therefore $a_{19}=4 \cdot 9^{2}+6 \cdot 9+4=382$.
A similar argument shows that $a_{2 n}=4 n^{2}+2 n$.

S5. The diagram shows the densest packing of seven circles in an equilateral triangle $A B C$.


Calculate the exact fraction of the area of $\triangle A B C$ covered by the seven circles.
Originally problem 14 from Shaking Hands in Corner Brook and other Math Problems for senior high school students, Edited by Peter Booth, Bruce Shawyer, and John Grant McLoughlin.
We received 4 submissions, all correct. We present the solution by Brian Bradie.


Let $s$ denote the length of each side of the equilateral triangle, and $r$ denote the radius of each of the seven circles. The fraction of the area of $\triangle A B C$ covered by the seven circles is then

$$
\frac{7 \pi r^{2}}{\frac{s^{2} \sqrt{3}}{4}}=\frac{28 \pi}{\sqrt{3}} \cdot \frac{r^{2}}{s^{2}}
$$

From the diagram, $A G=D C=r \sqrt{3}, \quad F G=D E=r \sqrt{3}, \quad$ and $\quad E F=2 r$, so $s=r(2+4 \sqrt{3})$, or

$$
\frac{r^{2}}{s^{2}}=\frac{1}{4(1+2 \sqrt{3})^{2}}=\frac{1}{4(13+4 \sqrt{3})} .
$$

Thus, the fraction of the area of $\triangle A B C$ covered by the circles is $\frac{7 \pi}{12+13 \sqrt{3}}$.

# TEACHING PROBLEMS 

# No. 21 <br> Nora Franzova and John Grant McLoughlin Motivating Conceptual Learning Through Mathematical Misconceptions 

This issue's Teaching Problems piece is reprinted with permission from the Proceedings of Sharing Mathematics: A tribute to Jim Totten (May 13-15, 2009), available at: unbscholar.lib.unb.ca/islandora/object/unbscholar\%3A10293/datastream

The authors Nora Franzova and John Grant McLoughlin presented on the idea of counterintuitive problems and their place in mathematical problem solving. We are grateful to Rick Brewster for granting permission to reprint the article here in the spirit of sharing mathematics.
$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$


#### Abstract

Surprising results challenge us to think in novel ways. Intuition is a wonderful thing and when the solution counters that intuition there is fertile ground for learning. Counterintuitive results offer a rich avenue for teaching conceptual ideas. The motivation for learning is enhanced through the engagement emerging from being surprised. An example of such a problem is offered here:

Suppose that a bin contains $G$ green balls and $R$ red balls. Two balls are to be selected without replacement. It is known that the probability of selecting one green and one red ball is exactly $1 / 2$. What do we know about the relationship between the number of green and red balls? It is not often that permission is given to simply try a variety of problems knowing that the instinctive responses are likely to be wrong. That's encouraged as we play with a range of elementary problems and discuss the mathematics underlying the misconceptions. The pedagogical value of integrating such examples into the teaching of mathematics will also be discussed. Of course, there may be a problem with no surprises, but in this session you may not trust your intuition on that one.


## 1 Introduction

Problem solving skills may be atop the list of what we hope our mathematics students will master. For instructors, these skills are most frequently represented by one's ability to successfully apply the four steps made famous by George Pólya. For students on the other hand, "problem solving" brings on the image of dreaded
word problems and sections of textbooks that boast titles like modeling and applications.

When we solve problems we frequently do so without consciously thinking about the four step process. The steps likely occur somewhere in the back of our mind. Nevertheless, in many instances students are taught the steps; teaching may even extend the steps and specify them for different groups of problems, or integrate the teaching of key words as a guide to apply a certain set of steps that would lead to the result. Rather than discussing the merits or flaws of common practices, we take this opportunity to articulate a place for those "intuition undermining" problems in the teaching and learning experiences associated with mathematical problem solving.

We will begin with the inclusion of a few famous "tricky" problems. Then we will reason that problems like these can play a significant role in an increasing individual's deeper understanding of the concepts at hand.

## 2 Five Problems

## Problem 1

A man drives from home to work at a speed of $50 \mathrm{~km} / \mathrm{h}$. The return trip from work to home is traveled at a speed of $30 \mathrm{~km} / \mathrm{h}$. What is the average speed for the round trip?

Many of us know this problem, and first time around we were completely taken aback by how simple, yet obviously tricky it is.

The main misconception here is a mistaken generalization caused by primarily noticing and then locking in on the key word "average" without considering the context. The concepts of the average of numbers and the average speed are quite a bit apart from each other.

Once the problem is solved we know that the average speed is

$$
\frac{2}{\frac{1}{50}+\frac{1}{30}}=37.5 \mathrm{~km} / \mathrm{h}
$$

which is not $40 \mathrm{~km} / \mathrm{h}$ as the first guess might have been. A convenient (albeit long) distance like 150 kilometres from home to work could alternatively have been used to find a total distance traveled of 300 kilometres and a total travel time of $3+5=8$ hours, thus, giving the average of $37.5 \mathrm{~km} / \mathrm{h}$.

Seeing a problem like this once, and being intrigued by it, offers insight into a concept that is unlikely to be acquired through another routine example. The conceptual understanding can be strengthened with an extension of the concept to another level. Such an extension is provided here.

Observe that the average speed ( $\mathrm{km} / \mathrm{h}$ ) must be between 30 and 50 . In fact, we know that the average would be exactly in the middle if the same amount of time had been spent at both speeds. Instead it was the distance at each speed that was the same, and hence, more time was spent at the slower speed tipping the average below the arithmetic mean of 30 and 50 (i.e. closer to 30 than 50 ). Note that the ratio of time spent at $30 \mathrm{~km} / \mathrm{h}$ to that spent at $50 \mathrm{~km} / \mathrm{h}$ must be $5: 3$ since the distances traveled were the same. Now we can apply a linear model to obtain the average. Place 30 and 50 at the endpoints of a segment and then divide the segment internally in a ratio of $3: 5$ to demonstrate the shift of the average toward the 30 , as shown in Figure 1. The segment will be split at 37.5 .


Figure 1: Partitioning the line segment $[30,50]$ into the ratio $3: 5$.

## Problem 2

An old car has to travel a two mile route. The first mile is uphill and the car (being an old one) can do a maximum of 15 mph on this uphill part. The second mile is downhill and the car can go faster there, of course. How fast does the car have to travel on this second mile to achieve an average speed of 30 mph for the trip?
(Note: It is mentioned in Stewart's College Algebra text that even Einstein once considered this problem interesting. See Mathematical Intelligencer, Spring 1990, page 41.)

The misconception in this problem is not based on the wording; it is more in the curious build up of the problem.

Notice in the solution, we require a value of $v$ to satisfy the equation

$$
\frac{2}{\frac{1}{15}+\frac{1}{v}}=30
$$

where $v$ represents the speed of the car on the downhill part. This gives $2=2+\frac{30}{v}$, and no such value of $v$ exists.

Using a simpler (less technical) approach we can logically deduce it would have to take the car zero time on the downhill part to end up with the desired average velocity.

We can see that the numbers were chosen in a very specific way. This choice was the trick. Is this then really a misconception or is this just a trick pulled from the magician's hat? There is a misconception in the sense that after reading the problem most people believe that it of course can be done. The numbers look
reasonable enough for all of us to believe we can drive so fast to achieve such an average speed. Maybe again, the word "average" is misleading. Also, please notice that if the question asked to achieve an average speed of 20 mph it would be possible and one with the knowledge about average speed computation from Problem 1 could readily solve this problem.

## Problem 3

Here we will shift our attention from average speeds to another area of mathematics that offers an abundance of intuition baffling problems, namely, probability. One such problem is stated here in its original form.

Suppose I have two six-sided cubes. Each cube has a square on three of its faces, a triangle on two of its faces and a circle on one face. If the cubes are rolled 1000 times, what combination of shapes will occur most often? (Kantowski, 1986, p. 431)

An adaptation (Problem 3) was shared in the presentation.
A standard die has a square on three faces, a triangle on two faces, and a circle on one face. Two of these dice are rolled. What is the most likely combination?

The quick answer "two squares" is entirely logical. Squares are most common and hence, two squares seem to be an obvious choice. In fact, this response is so likely that mathematicians themselves will usually exclaim this answer with confidence. How wonderful? Everyone is motivated to learn by the surprise that this is not the most likely combination. In fact, it is not even that close in that you are $1 / 3$ (as in over $33 \%$ ) more likely to get a square and a triangle. Try it. You can list all 36 possible ordered outcomes. There are only $3 \times 3=9$ that give two squares. Note, that there are 12 or $(3 \times 2)+(3 \times 2)$ that produce a square and a triangle. The surprise makes sense when one imagines two dice landing in succession. If you want two squares, there are 3 out of 6 faces that will eliminate you when the first die lands. In contrast, only the one face with the circle makes it impossible to obtain a square and a triangle.

## Problem 4

Two boxes are filled with donuts. One box contains 5 chocolate donuts and the other 5 basic glazed donuts. You want a chocolate one. You are allowed to look into the boxes, shake the boxes, and transfer any number of donuts from one to the other. After that you will be blindfolded, and the boxes will be moved around. Then you will be asked to choose randomly one box and then select a donut from it.

Even though we are still faced with only 5 chocolate donuts in the total of 10 donuts, our chances can be improved to about $72 \%$ by moving 4 of the chocolate
donuts into the other box. This leaves one box with 1 chocolate donut and one with 9 donuts including 4 of which are chocolate. That makes the probability of choosing a chocolate donut

$$
\frac{1}{2}+\frac{1}{2} \cdot \frac{4}{9}=\frac{13}{18} \approx 0.72
$$

Diagnosing the misconception may be a little more difficult here. It is related to oversimplifying the situation and some form of false assumption based upon believing that choosing a box and choosing a donut are similar. In fact, it would be good to have students explain their own errant processes to shed further light on the misconception. Such discussion would enrich the learning. Also, it is worth extending the problem to exaggerate the improvements (increased percentages) obtained by implementing the strategy with larger quantities such as 50 chocolate and 50 glazed donuts. (Also see Kending (2008).)

## Problem 5

Suppose that a bin contains $G$ green balls and $R$ red balls. Two balls are to be selected without replacement. It is known that the probability of selecting one green and one red ball is exactly $1 / 2$. What do we know about the relationship between the number of green and red balls?

The authors approached the problem in different ways, as summarized below with the two expressions. As this problem appeared in the abstract, it turned out that two attendees, namely, John Siggers and Gene Wirchenko, mentioned attempting the problem beforehand. As with the authors, they solved the problem in these two different ways. The first approach considers an ordering of one colour followed by the other with the balls selected separately, whereas, the second considers the selection as a whole. This latter way requires that one of each colour be selected when the two balls are pulled out simultaneously. Both expressions lead to similar mathematical results. Perhaps one of these approaches will more naturally appeal to you.

$$
\begin{align*}
& \frac{1}{2}=\frac{R}{R+G} \cdot \frac{G}{R+G-1}+\frac{G}{R+G} \cdot \frac{R}{R+G-1}  \tag{1}\\
& \frac{1}{2}=\frac{\binom{R}{1}\binom{G}{1}}{\binom{R+G}{2}} \tag{2}
\end{align*}
$$

Developing the relationship between $R$ and $G$ and following with some investigation and/or manipulation leads to the familiar triangular numbers making an appearance. That is, we find $R$ and $G$ must be consecutive triangular numbers. For example, if $R=1$, then $G=3$, but if $R=3$, we could have $G=1$ or 6 (the preceding or subsequent triangular number).

A result like this is extremely satisfying to mathematicians and math enthusiasts, but what is there to intrigue an already struggling student?

Let us take a look from that position. Seemingly simple questions put the student into uncomfortable situations; hard working students can feel that math is just an endlessly losing game, since the moment one understands how to do one problem a seemingly similar problem suddenly does not fit the mould. Adding all this up it seems that math just undermines one's intuition; thus, eroding trust in one's own ability, and bringing forth more and more frustration.
It is crucial to consider how and when we integrate counterintuitive questions into our teaching. At this point the authors go in separate ways.

One of the authors suggests that we not use "tricky" conceptual questions as our main tool, but use them only as we use "gourmet food". We have all witnessed something like the following: a small child in a very fancy restaurant asking for Kraft dinner. It makes them feel safe and at home. This metaphor suggests to us that we should only slowly introduce new "gourmet flavours" of the same old problem. Then natural appreciation can occur.

The other author is more inclined to bring out the counterintuitive questions at the outset. For example, Problem 3 may be used as an opening to provoke closer examination of the topic of probability. The conflict between the expectation and the actual result sparks a curiosity to better understand the principles.
It is not that one author's choice is right and that of the other is not. Instead we are challenged to reckon the balance of teaching, intention, beliefs, and context. While the first approach takes a settling view followed by a jolt, the latter advocates for the initial jolt as a motivation for learning and a means of unsettling any comfort. It is how the teachers manage and work with the students through the big picture of the course that will determine the merits of the path that is taken.

## 3 Conclusion

Counterintuitive results can be used as thought provoking, discussion opening tools. Allowing students to discuss their intuition with peers can facilitate group work. Desire to find one's flaw in thinking develops the reasoning process. Arguing constructively with peers will develop presentation abilities, and enhance the need for proper structure in logical arguments, thus, ideally bringing the joy of discovering and understanding.

When we approach problems we deeply rely on our intuition. One of many definitions of intuition from Wikipedia says: "Intuition - an incompletely founded concept or perception formed from associations to similar models, contexts, or scenarios, in humans frequently below the level of conscious iteration."

Each new "tricky" problem, similar to those presented here or as in sources such as [1, 3, 6, helps us to develop a better intuition. Mathematicians want this
challenge and appreciate it. As Kending (2008) suggests, mathematicians set out to find counterexamples, try to remove exceptions, look for symmetry, and rejoice over a far-reaching generalization. It is our hope that students will also benefit from us intentionally incorporating counterintuitive problems to build up and enrich their problem solving skills.

The authors welcome comments, examples, or other insights on this topic.

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# MATHEMAGICAL PUZZLES 

No. 4<br>Tyler Somer<br>Molten Gold - IV

In the previous articles in this series, we have looked at some of the geometric and arithmetic principles that govern puzzles with an apparent increase in area. Here, we now examine some commercial puzzles that use these very principles of having an extra piece "melt" into the tray. The reader is encouraged to determine the dimensions of each tray and each basic building unit.

Disclosure: I have no financial incentive with any of the puzzles presented in this article. I do own copies of these - or similar - puzzles. I am on friendly terms with the various designers, builders, and suppliers. I am motivated to have these individuals be successful, so that they can continue to design and produce interesting puzzles.

I am grateful to Puzzle Master of Saskatchewan and Bill Cutler of Illinois for their kind permission to use their stock photos and to discuss their products in this article.

Puzzle Master is a Canadian business success story, based in Saskatoon. Here are six melting-tile-style puzzles available through their web-store (puzzlemaster.ca).

From Vladimir Krasnoukov, a trio of puzzles produced by Recent Toys: Square + Triangle + , and Diamond +


In Square + , all $7+1$ pieces incorporate a variety of $45^{\circ}$ and $90^{\circ}$ angles. The smallest piece to melt into the tray is a $45^{\circ}-45^{\circ}-90^{\circ}$ triangle. Comparing this smallest piece to the other 7 , it appears that the basic triangular unit of the puzzle is smaller still. The 7 main pieces include two different sizes of $45^{\circ}-45^{\circ}-90^{\circ}$ triangles and five dissimilar quadrilaterals. Each of the five quadrilaterals can be dissected into a rectangle and either one or two $45^{\circ}-45^{\circ}-90^{\circ}$ triangles; and so, presumably, they can be further dissected to some number of building units.

In Triangle + , the tray is an equilateral triangle. Two of the pieces are also
equilateral triangles, including the smallest one to melt in. All $7+1$ pieces can be dissected to some number of $30^{\circ}-60^{\circ}-90^{\circ}$ triangles.

Diamond + has a number of similarities with Triangle + . The tray is a diamond. Two of the pieces are also diamonds, including the smallest one to melt in. All $9+1$ pieces can be dissected to some number of $30^{\circ}-60^{\circ}-90^{\circ}$ triangles.


Also from Vladimir Krasnoukov is the Fuji Puzzle, produced by Philos. The tray is a $45^{\circ}-45^{\circ}-90^{\circ}$ triangle, as is the basic building unit. The $49+1$ triangular components form $9+1$ pieces. Half the pieces are convex while the other half are concave.

From Dutch designer Niek Neuwahl, No Fit (circa 1990) is produced by Dave Janelle of Creative Crafthouse. At first glance, I would be inclined to say that Krasnoukov was inspired by this puzzle for his more recent Square + design. No Fit uses $5+1$ pieces, all with a variety of $45^{\circ}$ and $90^{\circ}$ angles, save for the smallest square to insert.
With the Impossible Square (designer unknown), produced by Winshare Puzzles, we have $16+1$ pieces in a large square tray. This one is most reminiscent of the Geometrex puzzles, mentioned in part II of this series [2022: 48(10), 594-596]; and is based on Fibonacci numbers, referenced in part I of this series [2022: 48(8), 458461]. The eight triangles, four small and four large, appear to be 2-by-5 and 3-by- 8 each, respectively. The other eight pieces are composed of glued-up rectangular bits, but those bits appear to be these various combinations: $(1,5),(2,2),(2,3)$, $(2,8)$. On this basis, the seventeenth piece is a 2 -by- 2 square.

The next four puzzles are all designed by Bill Cutler, and are available from his web-store (billcutlerpuzzles.com), Cutler has taken the basic "melting tile" idea to the extreme. He has done this in many ways: with relatively large dimensions, pieces with complex shapes, and by using parallelograms which are not rectangles.


Squeeze Me In (2018): With this puzzle, Cutler has formed a tray which starts with 90 rectangular units in an array of 9 -by- 10 . The tray is apparently filled with ten complex tiles, each composed of nine small rectangular units. The solver must add a singleton rectangle, the ninety-first unit, to the tray. Readers are invited to draw the relevant diagrams and solve the system that will determine the dimensions of the tray and the singleton, similar to the procedures of part III of this series [2023: 49(2), 73-76].
Pentominoes-MB (2016): The suffix MB is a direct reference to the "melting block" principle which inspired this design. Cutler starts with 11 of the 12 flat pentominoes - a topic for another article. A "standard" pentomino has an area of 5 square units, so the 11 pieces fill 55 square units, but it is clear these are not square components. The added singleton brings the area to 56 , but it is not obvious whether this is composed of an array of rectangles 8 -by- 7 or 7 -by8. Normally, setting up a rectangular array from 55 to 56 units, and solving the associated system, would be sufficient to determine the dimensions of the pieces which create a meaningful puzzle. Cutler then sheared the whole thing, so that the geometry is not based on a rectangle, but rather a parallelogram, with the deliberate obfuscation of deriving the 56 units. Readers are invited to determine the answers for themselves.


Ten Irregular Heptiamonds, versions 1 and 2 (2019): Each of the ten large pieces is composed of seven triangular bits, thus 70 area units in total. The singleton to add in both cases is a parallelogram composed of two triangular bits. The two variations are based on different size triangles. If we consider the trays to be based on small parallelogram components rather than triangles, there are 35 parallelograms set up in an array 7 -by- 5 to start. In the first variation, the evolution to 36 parallelograms is 6 -by- 6 , while the evolution of the second variation is 9 -by- 4 .


When he was teaching, Tyler often had mechanical puzzles in his classroom. As a freelancer, Tyler has worked with numerous inventors and co-designers to bring dozens of table-top solo-logic puzzle kits to market. He continues to design puzzles, and he spends a good deal of time in his woodshop, building his own and others' puzzle designs.

# From the bookshelf of . . . 

Doddy Kastanya

This MathemAttic feature brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to MathemAttic@cms.math.ca. Publishers are also welcome to send along books for possible review.

Coincidences, Chaos, and All that Math Jazz: Making Light of Weighty Ideas
by Edward B. Burger and Michael Starbird
ISBN 978-0-393-32931-5, 276 pages
Published by W. W. Norton \& Company, New York, 2006.

The first time I bought this book was back in the spring of 2010 when I was attending a workshop in Myrtle Beach, South Carolina. After a couple of days of heavy-duty lectures and discussions around verification and validation for nuclear system analyses, I needed a break. So, I went to a bookstore. After perusing different types of books for an hour or so, I landed my eyes on this book. Skimming through the book, I thought that this book should be enjoyable for some light readings on various math related subjects. So, I bought it - and I was not disappointed.

The book covers four major topics, each of which is expanded into three very entertaining chapters. The first topic roams through the world of uncertainties. The authors provide plentiful day-to-day examples related to coincidence (which can be viewed as things magically converging into an event), chaos (which can be viewed as things unexpectedly diverging into seemingly unconnected events), and statistics (which tries to fairly measure the uncertain). The second topic explores the world of numbers. How the secrets could remain secrets with the help of numbers: big prime numbers; how big numbers could actually get; and the search of pattern
 in numbers which includes an introduction to the famous Fibonacci sequence, are some examples covered here. The authors took an unexpected turn in the third part of the book where they present the artistic aspect of mathematics. They explore the world of fractals, discuss pleasing geometry related to the golden ratio, and touch on some ideas from topology. Finally, the book closes with a three-chapter part called Transcending Reality. The discussions in this last part of the book are more abstract since they deal with the concept of fourth dimension, infinity, and even stuff beyond infinity. The authors provide somewhat tangible examples such as dealing with checking in at a hotel while including a nice twist with the hotel having an infinite number of rooms and an
infinite number of people who would like to check in. One of the things that I like about the book is the fact that the authors have a unique way of guiding the readers through each part of the book one step at a time into an acceptance of a truly obscure concept like"infinities come in infinitely many sizes".

The main thing that I like about this book is the way the authors humorously present the materials. The concepts that could be really hard when explained with formulas become more enjoyable when presented with more practical examples. As a person who already read the book twice, I agree that the descriptions found at the back cover of the book really capture what this book is all about.
> "A book for the eternally curious, Coincidence, Chaos, and All That Math Jazz fuses a professor's understanding of the hidden mathematical skeleton of the universe with the sensibility of a stand-up comedian, making life's big questions accessible and compelling. Each chapter opens with a surprising insight from which the authors leapfrog over math and anecdote toward profound ideas about nature, art, and music. Coincidence is a book for lovers of puzzles and posers of outlandish questions, lapsed math aficionados and the formula phobic alike."

At the beginning of this article, I mentioned "the first time I bought this book". It certainly was not a mistake. Recently, I ran into this book again at a Goodwill store and bought the used copy for a buck. This time around it did not take me much time to read through the book since I already had some ideas of what would happen in the end. It was a nice refresher, nonetheless. Anyway, it was still enjoyable, and I certainly hope that you would experience the same sentiment once you are through with it.


This book is a recommendation from the bookshelf of Doddy Kastanya. Doddy is a math enthusiast working as a nuclear engineer. The love of math and physics was the reason for him to choose this field. In his spare times, among other things he likes to solve math puzzles and problems. In addition to Crux, the Project Euler has provided him with enough challenges and enjoyment in this area. Doddy and his family share their Oakville home with their four cats: Luke, Lorelai, Lincoln, and Lilian. Communications can be shared with the author via email: kastanya@yahoo.com.

# MATHEMATICS FROM THE WEB 

No. 9
This column features short reviews of mathematical items from the internet that will be of interest to high school and elementary students and teachers. You can forward your own short reviews to mathemattic@cms.math.ca.

## Nick's Mathematical Puzzles

http://www.qbyte.org/puzzles/puzzle99.html
The link will take readers to an index of problems numbered up to 160 at the time of publication. The numbered problems have names suggestive of the type of problem in some cases. A problem selection will actually take one to the set of 10 problems containing the selection. For example, a selection of Problem 52 Floor function sum or Problem 56 Partition identity would load the set of problems 51 through 60. The problems range over a wide variety of topics. All require a certain ingenuity, but usually only pre-college math. Some puzzles are original.

## The einstein tile

https://www.livescience.com/newly-discovered-einstein-tile-is-a-13
-sided-shape-that-solves-a-decades-old-math-problem
The recent discovery of a special tile called the einstein is worth learning more about. This tile can be used infinitely to tile the plane without ever repeating a pattern. This link is one of many that can shed light on the story. Teachers can use this example in conjunction with ideas around tessellations or Penrose tilings, links of which are also provided below in this issue.

## Penrose tiles

http://www.ams.org/publicoutreach/feature-column/fcarc-penrose This feature column from 2005 focuses attention on Penrose tilings. The article entitled Penrose Tiles Talk Across Miles effectively uses a blend of visuals and words to offer interested readers insight into the topic.

## Math and the Art of M.C. Escher

https://mathstat.slu.edu/escher/index.php/Math_and_the_Art_of_M._C. Escher
Anneke Bart and Bryan Clair, professors at Saint Louis University, authored a book entitled Math and the Art of M.C. Escher. The website has plenty of material suitable for the readership of MathemAttic whether one is interested in learning more about the topic or finding teaching materials. Simply delving in, it is likely that one will find paths they had not planned to be on as history of mathematics, geometry and other surprises can be learned through this site.

## OLYMPIAD CORNER

## No. 412

The problems in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by June 15, 2023.

OC626. Find all natural numbers $a, b, c$ such that the number $2^{a!}+2^{b!}+2^{c!}$ is a perfect cube.

OC627. Professor Srinivasa built a supercomputer, which has a screen displaying a number and two buttons. If you press the left button, the number $x$ on the screen gets replaced with the number $x^{2}-2$. If you press the right button, the number $x$ on the screen gets replaced with the number $x^{3}-3 x$. How many distinct numbers can the professor get after 10 button presses if his starting number is 2.5 ? Justify.

OC628. Given a natural number $n$, let $f(n)$ denote the sum of all natural numbers less than $n$ that are not prime. For example, $f(10)=1+4+6+8+9=28$. Find all natural numbers $n$ which satisfy $f(n)=1+\frac{n^{2}}{4}$.

OC629. You invent a new chess piece called a chevalier. Its possible moves are illustrated on the $8 \times 8$ chessboard: the black square is occupied by a chevalier and squares marked with a black dot are all the squares the chevalier can attack. How many chevaliers can you place on the chessboard so that none of them attack each other?


OC630. For natural numbers $n$, $m$, let $f_{n}(m)=1^{2 n}+2^{2 n}+\cdots+m^{2 n}$. For a given $n$, prove that there exists only a finite number of pairs of natural numbers $(a, b)$ for which $f_{n}(a)+f_{n}(b)$ is prime.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le $\mathbf{1 5}$ juin 2023.

OC626. Déterminer tous les nombres naturels $a, b, c$ tels que le nombre $2^{a!}+2^{b!}+2^{c!}$ est un cube parfait.

OC627. La professeure Saurette a construit un superordinateur doté d'un écran affichant un nombre, et muni de deux boutons. Si on appuie sur le bouton à gauche, le nombre $x$ à l'écran est remplacé par le nombre $x^{2}-2$. Si on appuie sur le bouton à droite, le nombre $x$ à l'écran est remplacé par le nombre $x^{3}-3 x$. Si le nombre à l'écran est 2.5 au départ, déterminer combien de nombres distincts peuvent apparaître après avoir appuyé 10 fois. Justifier votre réponse.

OC628. À partir d'un nombre naturel $n$, la somme des naturels plus petits que $n$ et qui ne sont pas premiers est dénotée $f(n)$. Par exemple, $f(10)=1+4+$ $6+8+9=28$. Déterminer tous les nombres naturels $n$ tels que $f(n)=1+\frac{n^{2}}{4}$.

OC629. Une nouvelle pièce aux échecs ressemble au cavalier, mais elle n'est pas capable d'attaquer les mêmes cases à partir d'où elle se trouve. Ceci est illustré ci-dessous, où la case en noir dénote celle occupée par un nouveau cavalier et les cases avec point noir sont celles attaquables par ce nouveau cavalier. Combien de nouveaux cavaliers peuvent être placés sur un échiquier $8 \times 8$, de façon à ce qu'aucun puisse attaquer aucun autre ?


OC630. Soit $f_{n}(m)=1^{2 n}+2^{2 n}+\cdots+m^{2 n}$ pour des nombres naturels $m$ et $n$. Pour $n$ donné, démontrer qu'il existe seulement un nombre fini de paires de nombres naturels $(a, b)$ telles que $f_{n}(a)+f_{n}(b)$ est premier.

# OLYMPIAD CORNER SOLUTIONS 

Statements of the problems in this section originally appear in 2022: 48(9), p. 535-536.

OC601. Let $S$ be a set containing $n^{2}+n-1$ elements, for some positive integer $n$. Suppose that the $n$-element subsets of $S$ are partitioned into two classes. Prove that there are at least $n$ pairwise disjoint $n$-element subsets in the same class.
Originally from the USAMO 2007, Problem 2.
We received 3 submissions, all of which were correct and complete. We present the solution by Oliver Geupel, Germany.

The case $n=1$ is by inspection. Next, assume that $n \geq 2$. Suppose that the set of $n$-element subsets of $S$ is partitioned into two disjoint classes $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$.

Let $\mathcal{A}$ be the class of $(n+1)$-element subsets, $T$, of $S$ with the property:
Property 1: there exists two $n$-element subsets, $T_{1}$ and $T_{2}$, of $T$ such that $T_{1} \in \mathcal{C}_{1}$ and $T_{2} \in \mathcal{C}_{2}$.
Consider a subclass $\mathcal{B}$ of $\mathcal{A}$ with the following properties:
Property 2: the elements of $\mathcal{B}$ are pairwise disjoint subsets of $S$,
Property 3: for every $T \in \mathcal{A} \backslash \mathcal{B}$ there exists an element $U \in \mathcal{B}$ such that $T \cap U \neq \varnothing$.

Let $b$ be the cardinality of $\mathcal{B}$ and observe that

$$
b \leq\left\lfloor\frac{n^{2}+n-1}{n+1}\right\rfloor=n-1
$$

Let $m=n^{2}+n-1-b(n+1)$ and

$$
V=S \backslash \bigcup_{T \in \mathcal{B}} T=\left\{v_{1}, v_{2}, \ldots, v_{m}\right\}
$$

If $b=n-1$, then $m=n$, so that $V$ is an $n$-element subset of $A$ and therefore a member of either $\mathcal{C}_{1}$ or $\mathcal{C}_{2}$, say, a member of $\mathcal{C}_{1}$. By properties 1 and 2 , the $n-1$ members of $\mathcal{B}$ yield $n-1$ more elements of $\mathcal{C}_{1}$ which are pairwise disjoint subsets of $S$, and we are done.

It remains to consider the case where $b \leq n-2$. We have $m \geq 2 n+1$. There is no loss of generality in assuming that $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\} \in \mathcal{C}_{1}$. Next, we deduce that $\left\{v_{2}, v_{3}, \ldots, v_{n+1}\right\} \in \mathcal{C}_{1}$. In fact, the hypothesis $\left\{v_{2}, v_{3}, \ldots, v_{n+1}\right\} \in \mathcal{C}_{2}$ would contradict property 3. Similarly, we obtain

$$
\left\{v_{3}, \ldots, v_{n+2}\right\}, \ldots,\left\{v_{m-n+1}, \ldots, v_{m}\right\} \in \mathcal{C}_{1}
$$

Thus we get $\left\lfloor\frac{m}{n}\right\rfloor$ disjoint $n$-element subsets of $V$ which are elements of $\mathcal{C}_{1}$. Moreover, every element of $\mathcal{B}$ yields one more element of $\mathcal{C}_{1}$. By construction, those $\left\lfloor\frac{m}{n}\right\rfloor+b$ subsets are pairwise disjoint. Finally,

$$
\left\lfloor\frac{m}{n}\right\rfloor+b=\left\lfloor n-b+1-\frac{b+1}{n}\right\rfloor+b \geq(n-b)+b=n,
$$

which completes the proof.

OC602. Let $n$ be a positive integer. Consider $2 n$ distinct lines on the plane, no two of which are parallel. Of the $2 n$ lines, $n$ are colored blue, the other $n$ are colored red. Let $B$ be the set of all points on the plane that lie on at least one blue line, and $R$ the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects $B$ in exactly $2 n-1$ points, and also intersects $R$ in exactly $2 n-1$ points.

Originally from the 2015 Asian Pacific Mathematics Olympiad, Problem 4.
We present the only solution we received, the solution by UCLan Cyprus Problem Solving Group.

Pick a blue line $\ell_{1}$ and a red line $\ell_{2}$ such that the angle formed between them is maximal. Consider the two half-lines $\ell_{1}^{\prime}$ and $\ell_{2}^{\prime}$ extended from the point of intersection, forming this maximal angle.

Any other line intersects $\ell_{1}^{\prime} \cup \ell_{2}^{\prime}$ at precisely one point. If it intersects the union of the two half-lines at two points, then it forms a larger angle with one of them. If it doesn't intersect them, then, since no two lines are parallel, it crosses the opposite two half-lines at two points, again creating a larger angle.

For every $t>0$, there is a unique circle $C_{t}$ tangent to both $\ell_{1}^{\prime}$ and $\ell_{2}^{\prime}$ at a distance $t$ from their point of intersection.

Since every line, $\ell$ intersects $\ell_{1}^{\prime} \cup \ell_{2}^{\prime}$ at precisely one point, for any large enough $t$, the circle $C_{t}$ intersects $\ell$ at two points.

Thus, choosing $t$ large enough, $C_{t}$ intersects every other line (out of the finitely many) at two points while it intersects $\ell_{1}$ and $\ell_{2}$ at one point each.

Each such circle intersects $B$ and $R$ at exactly $2 n-1$ points.

OC603. Let $A B C$ be an acute scalene triangle. Let $X$ and $Y$ be two distinct interior points of the segment $B C$ such that $\angle C A X=\angle Y A B$. Suppose that:

1) $K$ and $S$ are the feet of perpendiculars from $B$ to the lines $A X$ and $A Y$ respectively;
2) $T$ and $L$ are the feet of perpendiculars from $C$ to the lines $A X$ and $A Y$ respectively.

Prove that $K L$ and $S T$ intersect on the line $B C$.

Originally from the 2019 Balkan Mathematical Olympiad, Problem 3.
We received 7 submissions, of which 5 were correct and complete. We present the solution by Theo Koupelis, Cape Coral, Florida, USA.

For the quadrilateral $K T S L$ let point $E$ be the intersection point of the sides $K L$ and $S T$, and point $F$ be the intersection point of the diagonals $S K$ and $L T$. The quadrilaterals $A K S B$ and $A L T C$ are cyclic because $\angle B K A=\angle B S A=\angle C L A=$ $\angle C T A=90^{\circ}$. Thus, $\angle K S A=\angle K B A=90^{\circ}-\angle B A K=90^{\circ}-\angle C A L=\angle A C L=$ $\angle A T L$, and therefore $K T S L$ is cyclic.

Let point $D$ be the foot of the perpendicular from $A$ to $B C$. Then $\angle B S A=$ $\angle B D A=90^{\circ}$, and $\angle C T A=\angle C D A=90^{\circ}$, and thus point $D$ is the second intersection point of the circles $A K S B$ and $A L T C$. Therefore, for the three circles $A K S B, A L T C$, and $K T S L$ we have that the radical axes $A D, S K$, and $L T$ intersect at the same point, which is point $F$.

Let $I$ be the center of the circle $(K T S L)$, and $M, N$ be the midpoints of segments $S L, K T$, respectively. Then $I M, I N$ are the perpendicular bisectors of $S L, K T$, respectively; but $I M\|B S\| C L$, and $I N\|C T\| B K$, and thus $I$ is the midpoint of $B C$.

Finally, from Brocard's theorem for the quadrilateral $K T S L$ we have that $I$ is the orthocenter of $\triangle A F E$. Thus, $E I \perp A F$ and therefore $E I \| B I$, and thus $E$ is on the line $B C$.


OC604. Find all monic polynomials $f$ with integer coefficients satisfying the following condition: there exists a positive integer $N$ such that $p$ divides $2(f(p))!+1$ for every prime $p>N$ for which $f(p)$ is a positive integer.

## Originally from the 2016 Balkan Mathematical Olympiad, Problem 3.

We received 5 solutions, of which 4 were correct and complete. We present the solution by Missouri State University Problem Solving Group.
If $\operatorname{deg}(f)>1$, then for all sufficiently large $p, f(p)>p$. Hence, $2(f(p))!+1 \equiv$ $0+1 \not \equiv 0(\bmod p)$ and $f$ cannot have the required property. Also, $f$ cannot be constant, for then $p$ must divide the constant $2(f(p))!+1$, and there are only finitely many such $p$. A linear polynomial $f(p)=p+k$ for $k \geq 0$ does not have the required property. This is because $f(p) \geq p$ and $2(f(p))!+1 \equiv 0+1 \not \equiv 0(\bmod p)$.

Therefore, the remaining candidates are linear polynomials $f(p)=p-k$ with $k>0$. Notice that $(p-1)(p-2)=p(p-3)+2 \equiv 2(\bmod p)$.

Let $k=3$. We have

$$
\begin{aligned}
2(p-3)!+1 & \equiv(p-1)(p-2)(p-3)!+1 \\
& \equiv(p-1)!+1 \\
& \equiv-1+1(\text { by Wilson's theorem }) \\
& \equiv 0(\bmod p)
\end{aligned}
$$

Therefore, $f(p)=p-3$ satisfies the condition of the problem.
Let $k=1$. Then

$$
\begin{aligned}
2(p-1)!+1 & \equiv 2(-1)+1(\text { by Wilson's theorem }) \\
& \equiv-1(\bmod p)
\end{aligned}
$$

Therefore, $f(p)=p-1$ does not satisfy the condition of the problem.
Let $k=2$. We have

$$
\begin{aligned}
2(p-2)!+1 & \equiv(p-1)(p-2)(p-2)!+1 \\
& \equiv(p-2)(p-1)!+1 \\
& \equiv(-2)(-1)+1(\text { by Wilson's theorem }) \\
& \equiv 3(\bmod p)
\end{aligned}
$$

Only $p=3$ satisfies this condition. Therefore, $f(p)=p-2$ does not satisfy the condition of the problem.

Assume there exists $k>3$ that satisfies the condition of the problem. Then

$$
2(p-k)!+1 \equiv 0(\bmod p) .
$$

Multiply the above by $(p-3)(p-4) \cdots(p-(k-1))$ to get

$$
2(p-3)!+(p-3)(p-4) \cdots(p-(k-1)) \equiv 0(\bmod p)
$$

or

$$
-1+(-3)(-4) \cdots(-(k-1)) \equiv 0(\bmod p)
$$

The last congruence implies that infinitely many primes, $p$, divide the constant on the left-hand side of the congruence, which is impossible.
We conclude that the only polynomial satisfying the condition of the problem is $f(p)=p-3$.

OC605. Let $A B C$ be a scalene triangle with $\angle B=130^{\circ}$. Let $H$ be the foot of altitude from $B . D$ and $E$ are points on the sides $A B$ and $B C$, respectively, such that $D H=E H$ and $A D E C$ is a cyclic quadrilateral. Find $\angle D H E$.
Originally from the 2017 St Petersburg Mathematical Olympiad, Grade 9, Problem 5.

We received 6 submissions, of which 2 were correct and complete. We present the solution by Oliver Geupel, Germany.

We show that $\angle D H E=2\left(180^{\circ}-\angle B\right)=100^{\circ}$. It follows that $H$ is the circumcentre of triangle $B D E$.

Since the quadrilateral $A D E C$ is cyclic, it holds

$$
\angle D E B=180^{\circ}-\angle C E D=180^{\circ}-\left(180^{\circ}-\angle A\right)=\angle A
$$

Similarly $\angle B D E=\angle C$. Let $\varphi=\angle E D H=\angle H E D$. By the law of sines in the triangles $B H D$ and $B E H$, we have

$$
\frac{\sin (\angle A+\varphi)}{\sin \left(90^{\circ}-\angle C\right)}=\frac{\sin \angle H E B}{\sin \angle E B H}=\frac{B H}{E H}=\frac{B H}{D H}=\frac{\sin \angle B D H}{\sin \angle H B D}=\frac{\sin (\angle C+\varphi)}{\sin \left(90^{\circ}-\angle A\right)}
$$

Hence

$$
\begin{aligned}
& \sin (2 \angle A+\varphi)-\sin (2 \angle C+\varphi) \\
& =(\sin (2 \angle A+\varphi)+\sin \varphi)-(\sin (2 \angle C+\varphi)+\sin \varphi) \\
& =2 \sin (\angle A+\varphi) \sin \left(90^{\circ}-\angle A\right)-2 \sin (\angle C+\varphi) \sin \left(90^{\circ}-\angle C\right)=0 .
\end{aligned}
$$



Observing that $\angle A, \angle C$, and $\varphi$ are acute angles, we see that $0^{\circ}<2 \angle A+\varphi<270^{\circ}$ and $0^{\circ}<2 \angle C+\varphi<270^{\circ}$. Also, $\angle A \neq \angle C$ by hypothesis. Therefore, the equation

$$
\sin (2 \angle A+\varphi)=\sin (2 \angle C+\varphi)
$$

can hold only if

$$
(2 \angle A+\varphi)+(2 \angle C+\varphi)=180^{\circ}
$$

that is,

$$
\varphi=90^{\circ}-(\angle A+\angle C)=\angle B-90^{\circ}
$$

Consequently,

$$
\angle D H E=180^{\circ}-2 \varphi=2\left(180^{\circ}-\angle B\right),
$$

which completes the proof.
Editor's Comments. $A B C$ being scalene is crucial for identifying the measure of $\angle D H E$. As Richard Hess pointed out, if $A B C$ is isosceles, then the measure of $\angle D H E$ is not unique. $D$ and $E$ can be selected at the intersection of any line parallel to the base $A C$. Then $D H=E H$ and $A D E C$ is an isosceles trapezoid and a cyclic quadrilateral. As the parallel line is arbitrary, the measure of $\angle D H E$ can be any value from $\left(0^{\circ}, 180^{\circ}\right)$.

# Hales-Jewett theorem through examples and exercises: Part I 

Veselin Jungić

"Last year I went fishing with Salvador Dali. He was using a dotted line. He caught every other fish."

Steven Wright, an American comedian, actor and writer

## 1 Introduction

The Hales-Jewett theorem is another of the landmarks in the development of Ramsey theory. The theorem was inspired by van der Waerden's theorem [3, 4] and a generalization of Tic-Tac-Toe, a well-known children's game. Alfred Hales and Robert Jewett published their paper Regularity and positional games [2] in 1963. Many years later Ronald Graham, Bruce Rothschild, and Joel Spencer in their book Ramsey Theory explained: "The Hales-Jewett theorem strips van der Waerden's theorem of its unessential elements and revels the heart of Ramsey theory" [1].

For the purpose of introducing an informal version of the Hales-Jewett theorem we remind the reader about the game of Tic-Tac-Toe $\left.{ }^{1}\right]$

Two players are taking turns claiming the spaces in a $3 \times 3$ grid with the goal to claim a row, a column, or a diagonal.


Tic-Tac-Toe
It's a draw!
Same but different

Hales-Jewett Theorem (Informal): In large enough dimensions, the game of Tic-Tac-Toe cannot end in a draw.

It turned out that the publication of [2] was just the beginning of the Hales-Jewett theorem's long and happy mathematical life.

Alfred Washington Hales and Robert Israel Jewett, 1937-2022, are American mathematicians. Both of them had long and distinguished academic careers, Hales at the University of California Los Angeles and Jewett at the University of Western

[^0]Washington.


When they submitted the Regularity and position games paper in 1961, Hales was 23 years old and Jewett was 24. Both were doctoral students at the time. Hales was working under the supervision of Robert P. Dilworth at the California Institute of Technology (Caltech) and Jewett was supervised by Karl Stromberg at the University of Oregon. The pair knew each other from their time as undergraduate students at Caltech.

In 1971, Hales and Jewett, together with Ronald Graham, Klaus Leeb, and Bruce Rothschild, were the first recipients of the George Pólya Prize.

To state the Hales-Jewett we need to introduce the notion of a combinatorial line. In what follows we will justify the line part in the name of these intriguing objects, but also provide the evidence that a combinatorial line is something very different than an Euclidean line.

## 2 Alphabets, Words, and Roots

In the rest of this note, for a natural number $n$ we will denote the set $\{1,2, \ldots, n\}$ by $[1, n]$.

Definition 1. For $m \in \mathbb{N}$, any set $A$ such that $|A|=m$ is called an alphabet on $m$ symbols.

Example 1. Let $A=\{a, 1, \Delta\}$. Then $A$ is an alphabet on $|A|=3$ symbols.

Definition 2. Let $A$ be an alphabet on $m$ symbols. For $n \in \mathbb{N}$, any function $w:[1, n] \rightarrow A$ is called $a$ word of length $n$ on the alphabet $A$. If $w(i)=a_{i}$, $i \in[1, n]$, then we write $w=a_{1} a_{2} \cdots a_{n}$. The set of all words of length $n$ on the alphabet $A$ is denoted by $A^{n}$. We say that $A^{n}$ is the $n$-dimensional cube on the alphabet $A$.

Example 2. Let $A=\{a, 1, \triangle\}$ be an alphabet on three symbols. Then we have that $w=a 1 a 1 a 1$ is a word of length 6 on the alphabet $A$. Here $w:[1,6] \rightarrow A$ is defined as $w(1)=w(3)=w(5)=a$ and $w(2)=w(4)=w(6)=1$. Also,
$A^{2}=\{a a, a 1, a \triangle, 1 a, 11,1 \triangle, \triangle a, \triangle 1, \triangle \triangle\}$.
Exercise 1. Let $A$ be an alphabet on $m$ symbols. How many words of length $n$ on the alphabet $A$ are there?

Definition 3. Let $A$ be an alphabet (on $m$ symbols) and let $*$ be a symbol such that $* \notin A$ (the symbol $*$ is commonly called $a$ wildcard or an active coordinate). We consider the alphabet $A_{*}=A \cup\{*\}$. Any word on the alphabet $A_{*}$, i.e., any element of $\left(A_{*}\right)^{n}=A_{*}^{n}$, for some $n \in \mathbb{N}$, that contains the symbol $*$ is called a root.

Example 3. Let $A=\{a, 1, \triangle\}$ be an alphabet on three symbols. Then we have $A_{*}=A \cup\{*\}=\{a, 1, \triangle, *\}$. By definition, $1 * \triangle \in A_{*}^{3}$ and $a * a * \in A_{*}^{4}$ are roots and the word $11 \triangle \in A_{*}^{3}$ is not.

For a root $\tau \in A_{*}^{n}$ and a symbol $a \in A$ we define the word $\tau_{a} \in A^{n}$ in the following way. For $i \in[1, n]$

$$
\tau_{a}(i)=\left\{\begin{array}{rll}
\tau(i) & \text { if } & \tau(i) \neq * \\
a & \text { if } & \tau(i)=*
\end{array}\right.
$$

Example 4. Let $A=\{a, b, c\}$ and let $\tau=* b c b \in A_{*}^{4}$ be a root. Then, $\tau_{a}=a b c b, \tau_{b}=b b c b$, and $\tau_{c}=c b c b$.

Example 5. Let $A=[1,4]$ and let $\tau=* 13 * 4 * \in A_{*}^{6}$ be a root. Then $\tau_{2}=213242$.

## 3 Combinatorial Lines

Definition 4. Let $A$ be an alphabet, let $n \in \mathbb{N}$, and let $\tau \in A_{*}^{n}$ be a root. A combinatorial line in $A^{n}$ rooted in $\tau$ is the set of words $L_{\tau}=\left\{\tau_{a}: a \in A\right\}$.
We observe that, for any root $\tau \in A_{*}^{n}$, the combinatorial line $L_{\tau}$ is a subset of the cube $A^{n}$.

Example 6. Let $A=[1,3]$ and $n=2$. Find all combinatorial lines in $A^{2}$.
Observe that the set of all roots in $A_{*}^{2}$ is given by

$$
\{\tau=* 1, \sigma=* 2, \theta=* 3, \rho=1 *, \chi=2 *, \phi=3 *, \mu=* *\}
$$

It follows that all combinatorial lines in $A^{2}$ are given by: $L_{\tau}=\{11,21,31\}$, $L_{\sigma}=\{12,22,32\}, L_{\theta}=\{13,23,33\}, L_{\rho}=\{11,12,13\}, L_{\chi}=\{21,22,23\}$, $L_{\phi}=\{31,32,33\}$, and $L_{\mu}=\{11,22,33\}$.
Here is another view of all combinatorial lines in $A^{2}$.

| $L_{\tau}$ | $L_{\sigma}$ | $L_{\theta}$ | $L_{\rho}$ | $L_{\chi}$ | $L_{\phi}$ | $L_{\mu}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 2 | 1 | 3 | 1 | 1

For yet another view of all combinatorial lines in $A^{2}=[1,3]^{2}$, we first observe the one-to-one correspondence between the 2-dimensional cube on alphabet $[1,3]$ and the set of points $P=\{(x, y): x, y \in[1,3]\}$ in the $x y$-plane.

By this correspondence, each combinatorial line corresponds to a line segment in the $x y$-plane that contains three points from the set $P$.

Observe that the south-east diagonal of the square determined by the set $P$ does not correspond to any of the combinatorial lines in $[1,3]^{2}$.


Finally, we observe that each of the combinatorial lines in $[1,3]^{2}$ corresponds to a winning position in the Tic-Tac-Toe game.


Tic-Tac-Toe: $\times$ wins!


Tic-Tac-Toe: - wins!

Again, observe that there is a winning position, the south-east diagonal, that does not correspond to any of the combinatorial lines in $[1,3]^{2}$.

Exercise 2. Let $A=\{a, b, c, d\}$. List all combinatorial lines in $A^{2}$.
Example 7. Determine combinatorial lines in $[1,3]^{3}$ rooted in $\tau=* 23, \sigma=$ $* * 3$, and $\theta=* * * \in[1,3]_{*}^{3}$.

By definition:

| $L_{\tau}$ | $L_{\sigma}$ | $L_{\theta}$ |
| :---: | :---: | :---: |
| 123 | 113 | 111 |
| 223 | 223 | 222 |
| 323 | 333 | 333 |



We observe that the points in $\mathbb{R}^{3}$ that correspond to the elements of the combinatorial line $L_{\tau}$ lie on the Euclidean line $\ell_{\tau}$ with the parametric equations $x=t, y=2, z=3, t \in \mathbb{R}$. Similarly, the combinatorial line $L_{\sigma}$ corresponds to a set of points on the line $\ell_{\sigma}$ with the parametric equations $x=t, y=t, z=3, t \in \mathbb{R}$, and $L_{\theta}$ corresponds to a set of points on the line $\ell_{\theta}$ with the parametric equations $x=t, y=t, z=t, t \in \mathbb{R}$.

In general, for $m \in \mathbb{N}$, the combinatorial line determined by a root $\mu=a_{1} a_{2} a_{3} \in$ $[1, m]_{*}^{3}$ corresponds to a set of points on the Euclidean line $\ell_{\mu}$ with the parametric equations $x=b_{1}+\alpha_{1} \cdot t, y=b_{2}+\alpha_{2} \cdot t, z=b_{3}+\alpha_{3} \cdot t, t \in \mathbb{R}$, where $b_{i}=0$ and $\alpha_{i}=1$ if $a_{i}=*$, and $b_{i}=a_{i}$ and $\alpha_{i}=0$ if $a_{i} \in[1, m]$. This set of points is obtained for the values $t \in[1, m]$.

The use of the term line in describing the set of words in $A^{3}$ determined by a root is justified by the fact that there is a one-to-one correspondence between this set and a set of collinear points in $\mathbb{R}^{3}$. The reader should keep in mind that a word $w \in A^{3}$ is a function having the set $[1,3]$ as its domain and that the set $A$, its codomain can be basically any finite set. Hence, as a mathematical object, a finite set of words that we call a combinatorial line is much different than a Euclidean line in $\mathbb{R}^{3}$.

Exercise 3. In the figure below you see four Euclidean lines passing through points in a $4 \times 4 \times 4$ cube. If a line corresponds to a combinatorial line in $[1,4]^{3}$, list all of the elements of the combinatorial line and determine the root associated with it. If a line does not correspond to a combinatorial line, briefly explain why.


Exercise 4. Let $r \in \mathbb{N}$ and let $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(r)} \in[1, m]_{*}^{n}$ be $r$ roots. We say that the corresponding combinatorial lines are focussed at $f \in[1, m]^{n}$ if $\tau_{m}^{(1)}=\tau_{m}^{(2)}=\cdots=\tau_{m}^{(r)}=f$. Let $A=[1,5]$. Give an example of five focused combinatorial lines in $A^{5}$.

Exercise 5. In the May 1947 issue of The American Mathematical Monthly, A. L. Rubinoff from the University of Toronto proposed the following problem. Suppose that a noughts and crosses (i.e., a Tic-Tac-Toe game) are played on an $n$ dimensional cube of side $m$. Show that there are exactly $\frac{(m+2)^{n}-m^{n}}{2}$ rows, columns, diagonals ... on which a win may be scored [6].

Exercise 6. Let $m, n \in \mathbb{N}$ and let $|A|=m$, i.e., let $A$ be an alphabet on $m$ symbols. Prove that the number of combinatorial lines in $A^{n}$ is $(m+1)^{n}-m^{n}$.

## 4 Hints and solutions.

## Exercise $1 m^{n}$.

Exercise 2 The set of all roots in $A_{*}^{2}$ is given by $\{* a, * b, * c, * d, a *, b *, c *, d *, * *\}$. It follows that all combinatorial lines $A^{2}$ are

| a $a$ <br> b $a$ <br> c $a$ <br> d $a$ | a $b$ <br> b $b$ <br> c $b$ <br> d $b$ | a $c$ <br> b $c$ <br> c $c$ <br> d $c$ | a $d$ <br> b $d$ <br> c $d$ <br> d $d$ | $\begin{aligned} & a \mathrm{a} \\ & a \mathrm{~b} \\ & a \mathrm{c} \\ & a \mathrm{~d} \end{aligned}$ | $\begin{aligned} & b \mathrm{a} \\ & b \mathrm{~b} \\ & b \mathrm{c} \\ & b \mathrm{~d} \end{aligned}$ | $\begin{aligned} & c \mathrm{a} \\ & c \mathrm{~b} \\ & c \mathrm{c} \\ & c \mathrm{~d} \end{aligned}$ | $d$ a <br> $d \mathrm{~b}$ <br> $d \mathbf{c}$ <br> $d \mathrm{~d}$ | $\begin{aligned} & \mathrm{a} \text { a } \\ & \mathrm{b} \text { b } \\ & \mathrm{c} \mathrm{c} \\ & \mathrm{~d} \text { d } \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |

Exercise 3. Observe that $L_{2}=\left\{\begin{array}{llll}124,2 & 2 & 3 & 2 \\ 2 & 4 & 41\} \text {. Since the first and }\end{array}\right.$ third letter in each word change to different letters at different times, this set of words cannot be obtained from a root. It follows that $L_{2}$ is not a combinatorial line. The remaining three Euclidean lines correspond to combinatorial lines:

| Root | Combinatorial line | Euclidean Line |
| :---: | :---: | :---: |
| $\tau=* 32$ | $L_{\tau}=\{132,232,332,432\}$ | $L_{1}$ |
| $\sigma=* * *$ | $L_{\sigma}=\{111,222,333,444\}$ | $L_{3}$ |
| $\theta=24 *$ | $L_{\theta}=\{241,242,243,244\}$ | $L_{4}$ |

Exercise 4 For example, consider the following five roots:
$\alpha=1 * * * *, \beta=15 * * *, \gamma=155 * *, \delta=1555 *, \varepsilon=1 * 5 * *$.

The corresponding combinatorial lines are given by:

| $L^{(\alpha)}$ | $L^{(\beta)}$ | $L^{(\gamma)}$ | $L^{(\delta)}$ | $L^{(\varepsilon)}$ |
| :---: | :---: | :---: | :---: | :---: |
| 11111 | 15111 | 15511 | 15551 | 11511 |
| 12222 | 15222 | 15522 | 15552 | 12522 |
| 13333 | 15333 | 15533 | 15553 | 13533 |
| 14444 | 15444 | 15544 | 15554 | 14544 |
| 15555 | 15555 | 15555 | 15555 | 15555 |

Since $L_{5}^{(\alpha)}=L_{5}^{(\beta)}=L_{5}^{(\gamma)}=L_{5}^{(\delta)}=L_{5}^{(\varepsilon)}=15555$, the combinatorial lines $L^{(\alpha)}, L^{(\beta)}, L^{\gamma)}, L^{(\delta)}$ and $L^{(\varepsilon)}$ are focused with the focus 15555.

Exercise 5. Here is a solution by Leo Moser from 1948. Consider the "cube" of side $k$ inside a cube of side $k+2$. Clearly, every win will determine exactly one pair of surface elements, while each surface element determines exactly one win. Hence the number of wins will be half of the number of surface elements, which is the result stated [5].

Exercise 6. Observe that the number of combinatorial lines equals to the number of roots in $(A \cup\{*\})^{n}$. The number of all words of length $n$ on the alphabet $A \cup\{*\}$ is $(m+1)^{n}$. Since the number of all words of length $n$ on the alphabet $A$ is $m^{n}$ it follows that
$(\#$ of combinatorial lines $)=\left(\#\right.$ of roots in $\left.(A \cup\{*\})^{n}\right)=(m+1)^{n}-m^{n}$.

## References

[1] R. L. Graham, B. Rothschild, and J. H. Spencer. Ramsey Theory. John Wiley and Sons, New York, 2nd edition, 1990.
[2] A.W. Hales and R.I. Jewett. Regularity and positional games. Transactions of the American Mathematical Society, 106(2):222-229, 1963.
[3] V. Jungić. An introduction of van der Waerden's theorem through examples and exercises, Part I. Crux Mathematicorum, 48(7): 412-420, 2022.
[4] V. Jungić. An introduction of van der Waerden's theorem through examples and exercises, Part II. Crux Mathematicorum, 48(8):468-474, 2022.
[5] L. Moser. Solution to problem E 773 [1947, 281]. American Mathematical Monthly, 55:99, 1948.
[6] A.L. Rubinoff. Problem E 773. American Mathematical Monthly, 54:281, 1947.


## PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by June 15, 2023.
4831. Proposed by Goran Conar.

Let $p(x)=x^{n}+a_{n-1} x^{n-1}+a_{n-2} x^{n-2}+\cdots+a_{1} x+1$ be a polynomial of degree $n$, $n$ even, with all roots positive. Prove that for every $b \in \mathbb{N}$ the following inequality holds

$$
\lfloor p(-b)\rfloor-(b+1)^{n} \geq 0
$$

(where $\lfloor x\rfloor$ denotes the integer part of $x$ ).
4832. Proposed by Michel Bataille.

Let $A \in \mathcal{M}_{n}(\mathbb{C})$ and $\theta \in \mathbb{R}$. If $A^{k+1}=O_{n}$ for some positive integer $k$, prove that the matrix

$$
I_{n}+2 \sum_{j=1}^{k} \cos (j \theta) A^{j}
$$

is invertible.
4833. Proposed by Mihaela Berindeanu.

Let $A B C$ be an acute triangle, with the circumcircle $\Gamma_{1}$ and the circumcenter $O$. The circumcircle of $\triangle A O B$ is $\Gamma_{2}$, which cuts $A C$ in $P$ and $B C$ in $R$. If $C O \cap A B=\{Q\}$ show that $Q P^{2}+R C^{2}=P C^{2}+Q R^{2}$.
4834. Proposed by Michael Friday, modified by the editorial board.

Given a triangle $A B C$ with sides $a=B C, b=C A, c=A B$, prove that a point $D$ lies on the arc $A C$ opposite $B$ of its circumcircle if and only if

$$
\frac{a}{a^{\prime}}+\frac{b}{b^{\prime}}=\frac{c}{c^{\prime}}
$$

where $a^{\prime}, b^{\prime}, c^{\prime}$ are the lengths of the perpendiculars dropped from $D$ to the lines $B C, C A, A B$, respectively.
4835. Proposed by George Stoica.

Prove that the four complex numbers $z_{i}, i=1, \ldots, 4$, are the consecutive vertices of a cyclic quadrilateral (or are collinear) in the complex plane if and only if the number $\frac{\left(z_{1}-z_{4}\right)\left(z_{3}-z_{2}\right)}{\left(z_{1}-z_{2}\right)\left(z_{3}-z_{4}\right)}$ is real.
4836. Proposed by Mohammad Bakkar.

Prove the following formula:

$$
\frac{\pi^{3}}{32}=\prod_{n=1,2 n+1 \notin \mathcal{P}}^{\infty} \frac{4 n(n+1)}{(2 n+1)^{2}}
$$

where $\mathcal{P}$ is the set of prime numbers.
4837. Proposed by George Apostolopoulos.

Let $A B C$ be a given triangle and let $M, N$ be the interior points on the side $B C$ such that $B M=C N$. Prove that

$$
(A B+A C) \cdot\left(\frac{1}{A M}+\frac{1}{A N}\right)>4
$$

4838. Proposed by Daniel Sitaru.

Find:

$$
\int_{0}^{1} \frac{2 x-1}{e^{2 x}+4 x} d x
$$

4839. Proposed by Byungjun Lee.
$A B C D E F G$ is a regular heptagon, and a parabola $\Gamma$ with directrix $l$ is tangent to four lines $C D, E F, G A$, and $A B$. Let $C D \cap \Gamma=P, G A \cap \Gamma=Q$, and $E F \cap l=X$. Prove the following statements:
i) The three points $Q, B$, and $E$ are collinear.
ii) The three points $P, Q$, and $X$ are collinear.


## 4840. Proposed by Phan Ngoc Chau.

Prove that the following inequality

$$
\frac{1+a \sqrt{b c}}{a+\sqrt{b c}}+\frac{1+b \sqrt{c a}}{b+\sqrt{c a}}+\frac{1+c \sqrt{a b}}{c+\sqrt{a b}} \geq 1+\frac{4}{a+b+c}
$$

holds for all non-negative real numbers such that: $a b+b c+c a=1$. When does equality occur?

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le $\mathbf{1 5}$ juin 2023.
4831. Soumis par Goran Conar.

Soit $p(x)=x^{n}+a_{n-1} x^{n-1}+a_{n-2} x^{n-2}+\cdots+a_{1} x+1$ un polynôme de degré pair $n$, tel que toutes les racines sont positives. Démontrer que pour tout $b \in \mathbb{N}$, l'inégalité suivante est vérifiée

$$
\lfloor p(-b)\rfloor-(b+1)^{n} \geq 0
$$

(où $\lfloor x\rfloor$ dénote la partie entière de $x$ ).
4832. Soumis par Michel Bataille.

Soient $A \in \mathcal{M}_{n}(\mathbb{C})$ et $\theta \in \mathbb{R}$. Si $A^{k+1}=O_{n}$ pour un certain entier positif $k$, démontrer que la matrice

$$
I_{n}+2 \sum_{j=1}^{k} \cos (j \theta) A^{j}
$$

est inversible.
4833. Soumis par Mihaela Berindeanu.

Soit $A B C$ un triangle acutangle et soit $O$ le centre de son cercle circonscrit $\Gamma_{1}$. Le cercle circonscrit de $\triangle A O B$, que l'on dénotera $\Gamma_{2}$, rencontre $A C$ en $P$ et $B C$ en $R$. Si $C O \cap A B=\{Q\}$, démontrer que $Q P^{2}+R C^{2}=P C^{2}+Q R^{2}$.
4834. Soumis par Michael Friday, modifié par le comité de rédaction.

Soit un triangle $A B C$, les longueurs de ses côtés étant $a=B C, b=C A, c=A B$. Soit alors $D$ un point sur le cercle et soient $a^{\prime}, b^{\prime}, c^{\prime}$ les longueurs des perpendiculaires de $D$ vers les lignes $B C, C A, A B$ respectivement. Démontrer que $D$ se trouve sur l'arc $A C$ opposé à $B$ si et seulement si

$$
\frac{a}{a^{\prime}}+\frac{b}{b^{\prime}}=\frac{c}{c^{\prime}} .
$$

4835. Soumis par George Stoica.

Démontrer que les quatre nombres complexes $z_{i}, i=1, \ldots, 4$, sont, dans l'ordre, les sommets d'un quadrilatère cyclique (ou sont alignés) si et seulement si le nombre $\frac{\left(z_{1}-z_{4}\right)\left(z_{3}-z_{2}\right)}{\left(z_{1}-z_{2}\right)\left(z_{3}-z_{4}\right)}$ est réel.
4836. Soumis par Mohammad Bakkar.

Démontrer la formule

$$
\frac{\pi^{3}}{32}=\prod_{n=1,2 n+1 \notin \mathcal{P}}^{\infty} \frac{4 n(n+1)}{(2 n+1)^{2}},
$$

où $\mathcal{P}$ est l'ensemble des nombres premiers.
4837. Soumis par George Apostolopoulos.

Soit $A B C$ un triangle et soient $M, N$ des points intérieurs du côté $B C$, tels que $B M=C N$. Démontrer que

$$
(A B+A C) \cdot\left(\frac{1}{A M}+\frac{1}{A N}\right)>4 .
$$

4838. Soumis par Daniel Sitaru.

Déterminer

$$
\int_{0}^{1} \frac{2 x-1}{e^{2 x}+4 x} d x
$$

## 4839. Soumis par Byungjun Lee.

Soient $A B C D E F G$ un heptagone régulier et $\Gamma$ la parabole, de directrive $l$, tangente aux quatre droites $C D, E F, G A$ et $A B$. Soient $C D \cap \Gamma=P, G A \cap \Gamma=Q$, et $E F \cap l=X$. Démontrer les affirmations suivantes.
i) Les trois points $Q, B$ et $E$ sont alignés.
ii) Les trois points $P, Q$ et $X$ sont alignés.

4840. Soumis par Phan Ngoc Chau.

Montrer que l'inégalité

$$
\frac{1+a \sqrt{b c}}{a+\sqrt{b c}}+\frac{1+b \sqrt{c a}}{b+\sqrt{c a}}+\frac{1+c \sqrt{a b}}{c+\sqrt{a b}} \geq 1+\frac{4}{a+b+c}
$$

est vérifiée pour tous nombres réels non négatifs tels que $a b+b c+c a=1$. Quand avons-nous égalité?

## SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.
Statements of the problems in this section originally appear in 2022: 48(9), p. 562-565.

## 4781. Proposed by Michel Bataille.

Let $\mathcal{C}$ denote the set of all pairs of coprime positive integers and let

$$
d(m, n)=\operatorname{gcd}\left\{2 m(m+n), n(10 m+n), n^{2}+10 m^{2}+2 m n\right\}
$$

Find $\{d(m, n) ;(m, n) \in \mathcal{C}\}$.
We received 12 submissions and 11 of them were all complete and correct. We present a solution by the majority of solvers.

Observe that

$$
(10 m+9 n) \cdot(n(10 m+n))-50 n \cdot(2 m(m+n))=9 n^{3}
$$

and

$$
(9 m+n) \cdot(2 m(m+n))-2 m \cdot(n(10 m+n))=18 m^{3}
$$

Therefore $d(m, n) \mid \operatorname{gcd}\left(9 n^{3}, 18 m^{3}\right)$. Since $m$ and $n$ are coprime, it follows that $d(m, n)$ must be a divisor of 18 . As

$$
d(1,1)=1, d(1,4)=2, d(2,1)=3, d(1,2)=6, d(4,5)=9, \text { and } d(5,4)=18
$$

the set we are looking for consists of all the positive divisors of 18 .
4782. Proposed by Eugen J. Ionascu.

Let $A$ be the set of all integers $n$ such that $1 \leq n \leq 2022$ and $\operatorname{gcd}(n, 2022)=1$. For every nonnegative integer $j$, let $S(j)=\sum_{n \in A} n^{j}$. Show that

$$
S(j)(\bmod 2022) \equiv\left\{\begin{array}{llr}
0 & \text { if } & 336 \quad \chi j \\
672 & \text { if } 336 \mid j
\end{array}\right.
$$

We received 8 submissions, 7 of which were correct and complete. We present the solution by the Cyprus Group.
We have $2022=2 \cdot 3 \cdot 337$ and $|A|=2 \cdot 336=672$. Furthermore, by the Chinese Remainder Theorem,

- all elements of $A$ are congruent to $1 \bmod 2$,
- 336 elements of $A$ are congruent to $1 \bmod 3$ and the other 336 are congruent to $2 \bmod 3$,
- for each $m \in\{1,2, \ldots, 336\}$ exactly two elements of $A$ are congruent to $m \bmod 337$.
(i.e. the system $x \not \equiv 0 \bmod 2, x \equiv 1 \bmod 3, x \not \equiv 0 \bmod 337$ has $1 \cdot 1 \cdot 336$ solutions modulo 2022).

So, for any $j$ we have

$$
S(j)=\sum_{n \in A} n^{j} \equiv \sum_{n \in A} 1 \equiv 672 \equiv 0 \bmod 2
$$

and

$$
S(j)=\sum_{n \in A} n^{j} \equiv 336\left(1^{j}+2^{j}\right) \equiv 0 \bmod 3
$$

We also have

$$
S(j)=\sum_{n \in A} n^{j} \equiv 2\left(1^{j}+2^{j}+\cdots+336^{j}\right) \bmod 337
$$

If $336 \mid j$, then by Fermat's Little Theorem $n^{j} \equiv 1 \bmod 337$ for every $n \in$ $\{1,2, \ldots, 336\}$, and thus $S(j) \equiv 2 \cdot 336 \equiv 672 \bmod 337$. Since also $S(j) \equiv 0 \bmod 2$ and $S(j) \equiv 0 \bmod 3$, then $S(j) \equiv 672 \bmod 2022$.

Suppose now that $336 \nmid j$ and let $g$ be a primitive root modulo 337 . Then $g^{j} \not \equiv$ $1 \bmod 337$ and so

$$
g^{j} S(j)=g^{j}+(2 g)^{j}+\cdots+(336 g)^{j} \equiv S(j) \bmod 337
$$

where the last congruence follows since $0, g, 2 g, \ldots, 336 g$ are all incongruent modulo 337 and thus form a complete modulo system modulo 337.
Since $g^{j} \not \equiv 1 \bmod 337$ and $g^{j} S(j) \equiv S(j) \bmod 337$, it follows that $S(j) \equiv 0 \bmod$ 337. So in this case we have $S(j) \equiv 0 \bmod 2022$.

Editor's Comments. What does it mean to know a number? The Chinese Remainder Theorem says if you know an integer modulo several distinct primes, then you know it modulo their product. This principle was universally adopted by our solvers, all of whom showed (one way or another) that $S(j) \equiv 0 \bmod 2$ and 3 while $S(j) \equiv-2$ or $0 \bmod 337$ depending on whether $j$ is divisible by 336 or not. But where are these numbers coming from? If we define

$$
S_{m}(j)=\sum_{\substack{1 \leq a \leq m \\ \operatorname{gcd}(a, m)=1}} a^{j}
$$

then the key result is that

$$
S_{n m}(j) \equiv \phi(n) S_{m}(j) \bmod m
$$

whenever $\operatorname{gcd}(n, m)=1$. This is because the equation $x \equiv a \bmod m$ has exactly $\phi(n)$ solutions $x \bmod n m$. Using a computer, W. Janous determined $S_{m}(j)$ completely for all $1 \leq m \leq 96$ and asked a very interesting question: How large can the set $\left\{S_{m}(j) \bmod m: j=0,1,2, \ldots\right\}$ be?

## 4783. Proposed by Mihaela Berindeanu.

Prove that the points $A, B, C$ on the unit circle are the vertices of an equilateral triangle if and only if the corresponding complex numbers $a, b, c$ satisfy

$$
\frac{a}{2(b+c)-a}+\frac{b}{2(c+a)-b}+\frac{c}{2(a+b)-c}=-1
$$

We received 17 submissions, all of which were correct. Our featured solution comes in two steps: almost everybody had a similar approach to the first step, while the second step brought forth two approaches that are particularly simple.

Step 1. Prove that the given equation is equivalent to $a+b+c=0$.
Note first that

$$
\frac{a}{2(b+c)-a}+\frac{1}{3}=\frac{2(a+b+c)}{3(2(b+c)-a)} .
$$

Therefore,

$$
\begin{aligned}
0 & =\sum\left(\frac{a}{2(b+c)-a}+\frac{1}{3}\right) \\
& =\frac{2(a+b+c)}{3}\left(\frac{1}{2(b+c)-a}+\frac{1}{2(c+a)-b}+\frac{1}{2(a+b)-c}\right) \\
& =\frac{6(a+b+c)(a b+b c+c a)}{(2(a+b)-c)(2(b+c)-a)(2(c+a)-b)}
\end{aligned}
$$

It follows that $a+b+c=0$ or $a b+b c+c a=0$. But the latter possibility is equivalent to the former, as follows: Because $|a|=|b|=|c|=1$,

$$
a b+b c+c a=0 \quad \text { if and only if } \quad 0=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\bar{a}+\bar{b}+\bar{c}
$$

and a complex number equals zero if and only if its conjugate, namely $a+b+c$, equals zero.
Step 2. Prove that the points $A, B, C$ on the unit circle are the vertices of an equilateral triangle if and only if the corresponding complex numbers $a, b, c$ satisfy $a+b+c=0$.

Here are two easy proofs:
Solution 1, independently provided by Michel Bataille, Mohamed Amine Ben Ajiba, Aravind Mahadevan, Corneliu Manescu-Avram, Madhav R. Modak, and the proposer.
The circumcenter of triangle $A B C$ is represented by the origin, its centroid by $\frac{a+b+c}{3}$, and its orthocenter by $a+b+c$. Two of these points coincide if and only if $a+b+c=0$, if and only if all three points coincide, if and only if the triangle is equilateral.

Solution 2, independently provided by the UCLan Cyprus Problem Solving Group and by Walther Janous.

We may assume (rotating if necessary) that $a=1$. If $b=x+i y$, then we must have $c=-1-x-i y$; since $|b|=|c|$ then $-1-x= \pm x$. This gives $x=-1 / 2$ and, thus, $y= \pm \sqrt{3} / 2$. So $a, b, c$ represent the vertices of an equilateral triangle.

## 4784. Proposed by Salem Malikic.

Given is a triangle $A B C$ and point $D$ on the extension of $C A$ beyond $A$ such that $A D=A B$. Let $E$ be the intersection point of the angle bisector of angle $B A C$ and side $B C$, and $F$ the midpoint of $A E$. If $C F$ intersects side $A B$ at $G$, prove that the points $D, G$, and $E$ are collinear.
We received 23 submissions, all of which were correct, and we sample 4 of the various types of solutions.
Solution 1, a composite of the eight submissions that were based on Menelaus's theorem.

By Menelaus's Theorem applied to triangle $A B E$ with line $C G$ we have

$$
\frac{A G}{G B} \cdot \frac{B C}{E C} \cdot \frac{E F}{F A}=1
$$

Because $E F=F A$ it follows that

$$
\begin{equation*}
\frac{A G}{G B}=\frac{E C}{B C} . \tag{1}
\end{equation*}
$$

Since $A E$ is an angle bisector, then $B E / E C=A B / A C$, or because $A B=A D$ and $A$ lies between $D$ and $C$, it follows that $D C=D A+A C$ and

$$
\begin{equation*}
\frac{B C}{B E}=\frac{D C}{D A} . \tag{2}
\end{equation*}
$$

Note that $G$ and $E$ have been defined to lie in the interiors of the sides $A B$ and $B C$ of triangle $A B C$, while $D$ lies outside the triangle on the line $A C$. We can therefore apply Menelaus's theorem to triangle $A B C$ and the pointe $D, G, E$ :

$$
\frac{A G}{G B} \cdot \frac{B E}{E C} \cdot \frac{D C}{D A}=\frac{E C}{B C} \cdot \frac{B E}{E C} \cdot \frac{B C}{B E}=1 .
$$

Thus, the points $D, G, E$ are collinear.
Editor's warning. Before applying Menelaus's theorem (or, if you prefer, the "converse" of Menelaus's theorem) to a triangle and three points (one on each side, possibly extended), one must first determine that exactly one or three of the points lie outside the triangle. (Should zero or two points lie outside the triangle, then the three points are the feet of concurrent or parallel cevians, and are certainly not collinear.) Directed line segments are often used to emphasize the matter. If the reader is one of the several correspondents who failed to establish that exactly
one of the three given points was external, please slap yourself on the wrist and promise never to do it again.


Solution 2 is a composite of the nine submissions based on properties of a trapezoid.
Because $\triangle D A B$ is isosceles,

$$
\angle B D A=\frac{\pi-\angle D A B}{2}=\frac{\pi-(\pi-\angle B A C)}{2}=\frac{\angle B A C}{2}=\angle E A C .
$$

Consequently, the lines $D B$ and $A E$ are parallel, and $A D B E$ is a trapezoid. Let $K$ be the point where $C F$ intersects $B D$. The result follows immediately for those who know the theorem which says that the line that passes through the intersection of the nonparallel sides of a trapezoid (namely $C=A D \cap B E$ and the midpoints of the parallel sides (namely $F$ and $K$ ), passes also through the intersection of the two diagonals (namely $G=A B \cap E D$ ). For those who do not know the theorem, here is a simple proof: The homothety with center $C$ that takes $A$ to $D$ must take $E$ to $B$ and, therefore, the midpoint $F$ of $A E$ to the midpoint $K$ of $D B$. Denote by $G^{\prime}$ the intersection of the diagonals $A B$ and $E D$ of $A D B E$. The homothety with center $G^{\prime}$ that takes $A$ to $B$ must take $E$ to $D$ and, thus, the midpoint $F$ of $A E$ to the midpoint $K$ of $B D$. It follows that $G^{\prime}$ is on the line $F K$ which (by definition) also contains $C$. But $G$ was defined to be the point where $C F$ intersects $A B$, whence $G$ and $G^{\prime}$ must coincide, which proves that $D, G$, and $E$ are collinear.

Solution 3, by Roy Barbara (and is typical of the five solutions that used cartesian coordinates).


Let $A$ be the origin of the system of cartesian coordinates, and place $E$ on the positive $y$-axis at $(0,2)$; consequently $F=(0,1)$, as in the accompanying figure. The positive $x$ axis is chosen so that both coordinates of $B=(b, m b)$ are positive. Note that because $A E$ bisects $\angle B A C$ while $A B=A D$, the point $D$ must be the reflection of $B$ in the $x$-axis; that is $D=(b,-m b)$. The equation of

$$
A B \text { is } y=m x, \text { of } A D \text { is } y=-m x, \text { and of } B E \text { is } y=\frac{m b-2}{b} x+2 .
$$

Because $C$ is the intersection of $B E$ and $A D$, it must satisfy

$$
C=\left(\frac{-b}{m b-1}, \frac{m b}{m b-1}\right)
$$

It follows that $C F$ is the line $y=-\frac{1}{b} x+1$, which meets the line $A B$ (namely, $y=m x)$ at

$$
G=\left(\frac{b}{m b+1}, \frac{m b}{m b+1}\right)
$$

We see that the vectors

$$
\overrightarrow{E D}=(b,-(m b+2)) \quad \text { and } \quad \overrightarrow{E G}=\left(\frac{b}{m b+1}, \frac{-(m b+2)}{m b+1}\right)
$$

have proportional coordinates, whence the points $E, D, G$ are collinear.

Solution 4, by Michel Bataille. We use barycentric coordinates relative to ( $A, B, C$ ). With the familiar notation $a=B C, b=C A, c=A B$, we immediately see that $D(b+c: 0:-c)$; we know that $E=(0: b: c)$. Furthermore, we have $F=(b+c: b: c)[$ since $(b+c)(A+E)=(b+c) A+b B+c C]$ so that $G=(b+c: b: 0)$. Finally,

$$
\left|\begin{array}{ccc}
b+c & b+c & 0 \\
0 & b & b \\
-c & 0 & c
\end{array}\right|=\left|\begin{array}{ccc}
b+c & 0 & 0 \\
0 & b & b \\
-c & c & c
\end{array}\right|=0
$$

and, therefore, $D, G, E$ are collinear.
4785. Proposed by George Apostopoulos.

Let $A B C D$ be a cyclic quadrilateral with circumradius $R$ and area $F$. Prove that

$$
\frac{\sum \tan ^{2} \frac{A}{2}}{\sum \cos ^{4} \frac{A}{2}} \leq \frac{16 R^{4}}{F^{2}}
$$

where the sums are taken over all the angles of the quadrilateral.
We received 8 solutions for this problem. The following is the solution by UCLan Cyprus Problem Solving Group.

Let $h_{1}$ be the height of triangle $A B D$ with base $B D$ and $h_{2}$ the height of triangle $C B D$ with base $B D$. We have $B D=2 R \sin A$ and $h_{1}+h_{2} \leqslant 2 R$, thus

$$
F^{2} \leqslant\left(2 R^{2} \sin A\right)^{2}=16 R^{4} \sin ^{2} \frac{A}{2} \cos ^{2} \frac{A}{2}
$$

Since $C=180^{\circ}-A$ then

$$
F^{2} \tan ^{2} \frac{C}{2}=F^{2} \cot ^{2} \frac{A}{2} \leqslant 16 R^{4} \cos ^{4} \frac{A}{2}
$$

Summing up cyclically we obtain the required inequality.

## 4786. Proposed by Florică Anastase-Călăraşi.

In $\triangle A B C$, prove that the following relationship holds:

$$
\left(\sum_{c y c} \frac{1}{m_{a} m_{b}}\right)\left(4 s+\sum_{c y c} \frac{b c}{a}\right) \geq \frac{54}{s}
$$

We received 10 submissions, 9 of which are correct. We present here the solution by Mohamed Amine Ben Ajiba.
If $a, b, c$ are the lengths of the sides of the triangle $A B C(a=B C, b=A C, c=A B)$, and $m_{a}, m_{b}, m_{c}$ the lengths of its medians from vertices $A, B, C$, respectively, the median formulas are

$$
4 m_{a}^{2}=2 b^{2}+2 c^{2}-a^{2}, 4 m_{b}^{2}=2 c^{2}+2 a^{2}-b^{2}, 4 m_{c}^{2}=2 b^{2}+2 a^{2}-c^{2}
$$

Therefore

$$
\begin{aligned}
\left(4 m_{b} m_{c}\right)^{2} & =\left(2 c^{2}+2 a^{2}-b^{2}\right)\left(2 b^{2}+2 a^{2}-c^{2}\right) \\
& =4 a^{4}+2 a^{2}\left(b^{2}+c^{2}\right)-\left(2 b^{4}-5 b^{2} c^{2}+2 c^{4}\right) \\
& =\left(2 a^{2}+b c\right)^{2}-2\left[(b+c)^{2}-a^{2}\right](b-c)^{2} \leq\left(2 a^{2}+b c\right)^{2}
\end{aligned}
$$

the last inequality being true because $b+c>a$, and

$$
m_{b} m_{c} \leq \frac{2 a^{2}+b c}{4}
$$

The equality holds if and only if $b=c$. In a similar way, we have

$$
m_{a} m_{b} \leq \frac{2 c^{2}+a b}{4}, m_{c} m_{a} \leq \frac{2 b^{2}+a c}{4}
$$

and therefore

$$
\sum_{c y c} \frac{1}{m_{a} m_{b}} \geq \sum_{c y c} \frac{4}{2 a^{2}+b c}
$$

Using $2 s=a+b+c$, we observe that

$$
4 s+\sum_{c y c} \frac{b c}{a}=\sum_{c y c} \frac{2 a^{2}+b c}{a}
$$

and thus

$$
\begin{equation*}
2 s\left(\sum_{c y c} \frac{1}{m_{a} m_{b}}\right)\left(4 s+\sum_{c y c} \frac{b c}{a}\right) \geq\left(\sum_{c y c} a\right)\left(\sum_{c y c} \frac{4}{2 a^{2}+b c}\right)\left(\sum_{c y c} \frac{2 a^{2}+b c}{a}\right) \tag{1}
\end{equation*}
$$

By Hölder's inequality,

$$
\left(\sum_{c y c} a\right)\left(\sum_{c y c} \frac{4}{2 a^{2}+b c}\right)\left(\sum_{c y c} \frac{2 a^{2}+b c}{a}\right) \geq\left(\sum_{c y c} \sqrt[3]{a \cdot \frac{4}{2 a^{2}+b c} \cdot \frac{2 a^{2}+b c}{a}}\right)^{3}=108
$$

and that with (1) is equivalent to the expected result.
Equality holds if and only if the triangle $A B C$ is equilateral.
4787. Proposed by Toyesh Prakash Sharma.

Evaluate

$$
\int_{-\infty}^{\infty} \frac{\tan ^{2} x}{x^{2}} \cdot \frac{d x}{4+\sec ^{2} x}
$$

(Hint: use Lobachevsky Integral formula. Can you solve this using another method?)
We received 18 solutions using Lobachevsky's Integral Formula, all of which were correct. In addition, Abel, Barnette, and UCLan Problem Solving Group provided alternative solutions not involving Lobachevsky's Integral Formula. The first featured solution derives and applies Lobachevsky's Integral Formula. The second featured solution provides one of the alternative solutions.

Solution 1, by Yunyong Zhang.
Denoting the integral to be evaluated by $I$, and letting $x=y+n \pi$, we have

$$
\begin{aligned}
I & =\sum_{n=-\infty}^{\infty} \int_{n \pi}^{(n+1) \pi} \frac{\sin ^{2} x}{x^{2}\left(1+4 \cos ^{2} x\right)} d y \\
& =\sum_{n=-\infty}^{\infty} \int_{0}^{\pi} \frac{\sin ^{2}(n \pi+y)}{(n \pi+y)^{2}\left(1+4 \cos ^{2}(n \pi+y)\right)} d y \\
& =\sum_{n=-\infty}^{\infty} \int_{0}^{\pi} \frac{\sin ^{2} y}{(n \pi+y)^{2}\left(1+4 \cos ^{2} y\right)} d y .
\end{aligned}
$$

Using the known formula

$$
\pi^{2} \csc ^{2}(\pi x)=\sum_{n=-\infty}^{\infty}\left[\frac{1}{(x-n)^{2}}\right]
$$

and letting $x=-\frac{y}{\pi}$, we obtain

$$
\csc ^{2} y=\frac{1}{\pi^{2}} \sum_{n=-\infty}^{\infty}\left[\frac{1}{\left(\frac{y}{\pi}+n\right)^{2}}\right]=\sum_{n=-\infty}^{\infty} \frac{1}{(y+n \pi)^{2}} .
$$

Thus,

$$
I=\int_{0}^{\pi} \frac{1}{1+4 \cos ^{2} y} d y=\int_{0}^{\pi} \frac{\sec ^{2} y}{5+\tan ^{2} y} d y=\left.\frac{1}{\sqrt{5}} \arctan \left(\frac{\tan y}{\sqrt{5}}\right)\right|_{0} ^{\pi}=\frac{\pi}{\sqrt{5}}
$$

Solution 2, by Ulrich Abel and Vitaliy Kushnirevych.
We start from

$$
J=2 \int_{0}^{\infty} \frac{\sin ^{2} x}{x^{2}} \cdot \frac{d x}{3+2 \cos (2 x)}
$$

The function

$$
f(x)=\frac{2}{(3+2 \cos (2 x))}
$$

is an even $\pi$-periodic function that can be developed in a cosine-Fourier series:

$$
f(x)=\frac{2}{\sqrt{5}}+\sum_{k=1}^{\infty} a_{k} \cos (2 k x)
$$

with certain coefficients $a_{k}, k=1,2,3, \ldots$ This Fourier series can be integrated term by term (see, for example, Tom Apostol's textbook Mathematical Analysis, Theorem 11.16, part c)):

$$
J=\frac{2}{\sqrt{5}} \int_{0}^{\infty} \frac{\sin ^{2} x}{x^{2}} d x+\sum_{k=1}^{\infty} a_{k} \int_{0}^{\infty} \frac{\sin ^{2} x}{x^{2}} \cos (2 k x) d x
$$

It remains to show that

$$
I_{k}:=\int_{0}^{\infty} \frac{\sin ^{2} x}{x^{2}} \cos (2 k x) d x=0 \quad \text { for all } k=1,2,3, \ldots
$$

Using

$$
\begin{aligned}
2\left(\sin ^{2} x\right) \cos (2 k x) & =(1-\cos (2 x)) \cos (2 k x) \\
& =\cos (2 k x)-\frac{1}{2} \cos (2(k-1) x)-\frac{1}{2} \cos (2(k+1) x)
\end{aligned}
$$

and integrating by parts we obtain

$$
\begin{aligned}
I_{k} & =\frac{1}{2} \int_{0}^{\infty} \frac{\cos (2 k x)-\frac{1}{2} \cos (2(k-1) x)-\frac{1}{2} \cos (2(k+1) x)}{x^{2}} d x \\
& =\frac{1}{2} \int_{0}^{\infty} \frac{2 k \sin (2 k x)-(k-1) \sin (2(k-1) x)-(k+1) \sin (2(k+1) x)}{x} d x
\end{aligned}
$$

It follows from $\int_{0}^{\infty} \frac{\sin (\beta x)}{x} d x=\frac{\pi}{2}$ for all $\beta>0$ that

$$
I_{k}=\frac{1}{2}\left(2 k \cdot \frac{\pi}{2}-(k-1) \cdot \frac{\pi}{2}-(k+1) \cdot \frac{\pi}{2}\right)=0
$$

4788. Proposed by Albert Natian.

Solve

$$
\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{k+1}}\left(x^{2^{k}}+1\right)^{2}=0
$$

for real $x$ given that $|x|<1$.
We received 21 solutions, all of which were correct. We present the solution by Henry Ricardo.

We have

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{k+1}}\left(x^{2^{k}}+1\right)^{2} & =\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{k+1}}\left(x^{2^{k+1}}+2 x^{2^{k}}\right)+\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{k+1}} \\
& =\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{k+1}} x^{2^{k+1}}+\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{k}} x^{2^{k}}+\frac{1}{3} \\
& =\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{k+1}} x^{2^{k+1}}+x+\sum_{k=1}^{\infty} \frac{(-1)^{k}}{2^{k}} x^{2^{k}}+\frac{1}{3} \\
& =\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{k+1}} x^{2^{k+1}}+x-\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{k+1}} x^{2^{k+1}}+\frac{1}{3} \\
& =x+\frac{1}{3}=0
\end{aligned}
$$

so that $x=-1 / 3$ is the only real solution.
4789. Proposed by Byungjun Lee.

Side $B C$ of triangle $A B C$ is divided by interior points $D$ and $E$, so that $4 B D=$ $4 C E=B C$. Circle $\Gamma$ passing through $A$ and tangent to the segment $C D$ has center $O$, and meets segments $A B, A C$, and $A D$ again at $X, Y$, and $Z$, respectively. Suppose that $\Gamma$ and $E X$ both bisect the segment $C Z$. Prove that points $C, Y, Z$, and $O$ are concyclic.


We received 2 solutions after publication of the problem, both of them correct. The folowing is the original solution submitted by the author of the problem.


Let $W$ be the midpoint of $C Z$, then $W$ lies both on $\Gamma$ and $E X$. Let $P=A D \cap E X$, $F=B C \cap \Gamma$, and $G=B C \cap Y W$. By applying Menelaus' theorem to triangle $D E P$ and line $C W Z$, we have

$$
\frac{D C}{C E} \cdot \frac{E W}{W P} \cdot \frac{P Z}{Z D}=1 .
$$

By applying Menelaus' theorem to triangle $D E P$ and line $B X A$, we have

$$
\frac{D B}{B E} \cdot \frac{E X}{X P} \cdot \frac{P A}{A D}=1
$$

By multiplying equations (1) and (2), and given that $D B=C E, D C=B E$, $P Z \cdot P A=W P \cdot X P, E W \cdot E X=E F^{2}$ and $Z D \cdot A D=D F^{2}$, we get

$$
1=\frac{D B}{C E} \cdot \frac{D C}{B E} \cdot \frac{P Z \cdot P A}{W P \cdot X P} \cdot \frac{E W \cdot E X}{Z D \cdot A D}=\frac{E F^{2}}{D F^{2}}
$$

which gives $D F=E F=C E$.
Assume that $Z A$ and $W Y$ are not parallel, and let $Z A \cap W Y=Q$.
By applying Menelaus' theorem to triangle $D G Q$ and line $C Y A$, we have

$$
\frac{D C}{C G} \cdot \frac{G Y}{Y Q} \cdot \frac{Q A}{A D}=1
$$

By applying Menelaus' theorem to triangle $D G Q$ and line $C W Z$, we have

$$
\frac{D C}{C G} \cdot \frac{G W}{W Q} \cdot \frac{Q Z}{Z D}=1
$$

By multiplying equations (4) and (5), and using $G Y \cdot G W=G F^{2}, A D \cdot Z D=D F^{2}$, and $Q A \cdot Q Z=Y Q \cdot W Q$, we get

$$
1=\frac{D C^{2}}{C G^{2}} \cdot \frac{G Y \cdot G W}{A D \cdot Z D} \cdot \frac{Q A \cdot Q Z}{Y Q \cdot W Q}=\frac{D C^{2}}{C G^{2}} \cdot \frac{G F^{2}}{D F^{2}}
$$

So

$$
\frac{C G}{G F}=\frac{D C}{D F}=3
$$

Since

$$
D G=D F+G F=\frac{1}{2} C F+\frac{1}{4} C F=\frac{3}{4} C F=\frac{1}{2} C D
$$

$G$ is the midpoint of $C D$, so $Z A$ and $W Y$ are parallel, which contradicts to the assumption. It follows that $Z A$ and $W Y$ must be parallel, and $A Z W Y$ becomes an isosceles trapezoid. From $C A=C Z$ and $O A=O Z$, triangles $C A O$ and $C Z O$ are congruent, so $\angle O C Y=\angle O C Z$. Then $O$ lies both on the bisector of $\angle Y C Z$ and the perpendicular bisector of $Y Z$, so we can conclude that points $C, Y, Z$, and $O$ are concyclic.

## 4790. Proposed by Aravind Mahadevan.

Find $x$ and $y$ such that

$$
x \cos ^{3} y+3 x \sin ^{2} y \cos y=14 \quad \text { and } \quad x \sin ^{3} y+3 x \cos ^{2} y \sin y=13
$$

We received 29 submissions, of which 22 were correct and 7 were incomplete. We present the solution by Brian Bradie.
Adding the given equations yields

$$
x(\cos y+\sin y)^{3}=27
$$

subtracting $x \sin ^{3} y+3 x \cos ^{2} y \sin y=13$ from $x \cos ^{3} y+3 x \sin ^{2} y \cos y=14$ yields

$$
x(\cos y-\sin y)^{3}=1
$$

From here,

$$
\cos y+\sin y=\frac{3}{\sqrt[3]{x}} \quad \text { and } \quad \cos y-\sin y=\frac{1}{\sqrt[3]{x}}
$$

SO

$$
\cos y=\frac{2}{\sqrt[3]{x}} \quad \text { and } \quad \sin y=\frac{1}{\sqrt[3]{x}}
$$

Applying the fundamental trig identity,

$$
1=\cos ^{2} y+\sin ^{2} y=\frac{5}{\sqrt[3]{x^{2}}}
$$

so

$$
x= \pm 5 \sqrt{5} .
$$

For $x=5 \sqrt{5}$,

$$
\cos y=\frac{2}{\sqrt{5}} \quad \text { and } \quad \sin y=\frac{1}{\sqrt{5}}
$$

which implies

$$
y=\arcsin \frac{1}{\sqrt{5}}+2 n \pi
$$

for $x=-5 \sqrt{5}$,

$$
\cos y=-\frac{2}{\sqrt{5}} \quad \text { and } \quad \sin y=-\frac{1}{\sqrt{5}}
$$

which implies

$$
y=\arcsin \frac{1}{\sqrt{5}}+(2 n+1) \pi
$$

Thus, the solutions of the system of equations

$$
x \cos ^{3} y+3 x \sin ^{2} y \cos y=14 \quad \text { and } \quad x \sin ^{3} y+3 x \cos ^{2} y \sin y=13
$$

are

$$
x=5 \sqrt{5}, \quad y=\arcsin \frac{1}{\sqrt{5}}+2 n \pi
$$

and

$$
x=-5 \sqrt{5}, \quad y=\arcsin \frac{1}{\sqrt{5}}+(2 n+1) \pi
$$

for any integer $n$.


[^0]:    ${ }^{1}$ We will learn that the settings of the Hales-Jewett theorem exclude one of the diagonals.

