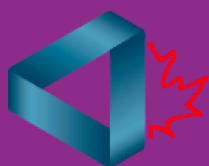


An artistic illustration of the Aurora Borealis (Northern Lights) in shades of green and blue, swirling in a dark night sky. Below the lights is a silhouette of a dense evergreen forest.

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Crux Mathematicorum with Mathematical Mayhem

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MATHEMATTIC

No. 43

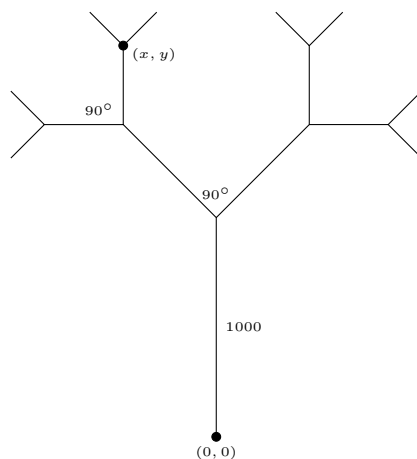
The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

*To facilitate their consideration, solutions should be received by **May 15, 2023**.*



MA211. Starting at coordinates $(0, 0)$, a line 1000 units long is drawn as indicated. This line then branches into two separate lines (which form a 90° angle, as shown). Each of these lines is 60% the length of the previous segment. The process continues. Find the (x, y) coordinates of the indicated point.



MA212. On a distant planet, railway tracks are built using one solid railway bar. A railway is built between two towns 20 km apart on a big flat section of the planet. Unfortunately the bar was made one metre too long and the constructor decided to lift it in the middle to try to make the ends fit. *Approximately* how high does he have to lift it in the middle?

MA213. A shopkeeper orders marbles made up of 19 identical packets of a larger amount and 3 identical packets of a smaller amount. A total of 224 marbles arrive loosely tossed in a container. How would you repackage the marbles properly to satisfy the shopkeeper's order? Justify your answer and show that it is unique.

MA214. *Proposed by Neculai Stanciu.*

Determine all pairs (x, y) of real numbers which satisfy

$$\sqrt{x^2 + 2x + 1} + \sqrt{x^2 - 4x + 4} + \sqrt{y^2 - 6y + 9} + \sqrt{x^2 - 2xy + y^2} = 4.$$

MA215. *Proposed by Aravind Mahadevan, Hong Kong.*

In $\triangle ABC$, $\angle B = 2\angle A$ and $\angle C = 4\angle A$. Prove that $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ where, a , b and c denote the lengths of BC , CA , and AB respectively.

.....

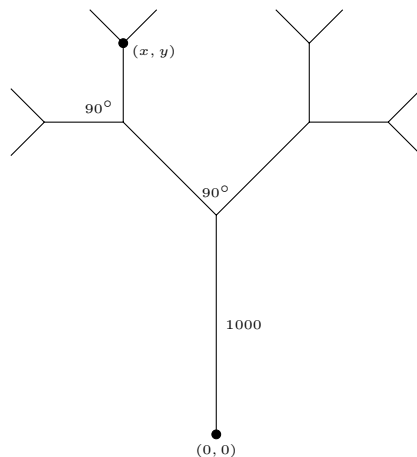
Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2023**.*



MA211. À partir du point de coordonnées $(0,0)$, on trace un segment de droite d'une longueur de 1 000 unités, comme indiqué. Ce segment de droite se scinde alors en deux segments distincts (formant un angle de 90° , tel qu'illustré). Chacun de ces segments de droites est d'une longueur correspondant à 60% de celle du segment de droite précédent. Le processus se poursuit de la même façon. Trouvez les coordonnées (x, y) du point indiqué.



MA212. La pratique sur une planète éloignée est de construire des chemins de fer en se servant d'un seul rail. Or, le chemin de fer entre deux villages à 20 km de distance a malheureusement utilisé un rail un mètre de trop long et le contremaître a décidé de corriger ceci en soulevant le rail dans son point milieu. *Environ* à quelle hauteur le rail a-t-il besoin d'être soulevé?

MA213. Un commerçant soumet une commande de billes, comprenant 19 emballages identiques à un grand nombre de billes par emballage, puis 3 emballages identiques à un plus petit nombre de billes par emballage. Or la commande lui arrive comme 224 billes dans un même sac. Déterminer comment emballer les billes de façon à répondre à la commande du commerçant. Justifier votre réponse et montrer qu'elle est la seule solution possible.

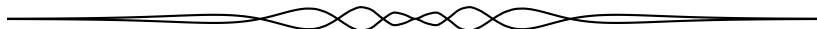
MA214. *Proposé par Neculai Stanciu, "George Emil Palade" School, Buzu, Romania.*

Déterminer tous les couples de nombres réels (x, y) tels que

$$\sqrt{x^2 + 2x + 1} + \sqrt{x^2 - 4x + 4} + \sqrt{y^2 - 6y + 9} + \sqrt{x^2 - 2xy + y^2} = 4.$$

MA215. *Proposé par Aravind Mahadevan, Hong Kong.*

Dans triangle ABC , on a $\angle B = 2\angle A$ et $\angle C = 4\angle A$. Démontrer que $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, où a , b et c dénotent les longueurs de BC , CA et AB respectivement.



MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2022: 48(8), p. 444–446.

MA186. Consider a sequence of integers $1, 3, 2, -1, \dots$, where each term is equal to the term preceding it minus the term before that. What is the sum of the first 2009 terms?

Originally question 1 from the 2009 Fifth Annual Kansas Collegiate Mathematics Competition.

We received 12 submissions, 11 of which were complete. We present a solution and a generalization.

Solution by Amy Zhai.

We are going to observe the pattern in the sequence. Notice that

$$\begin{aligned} -1 - 2 &= -3 \\ -3 - (-1) &= -2 \\ -2 - (-3) &= 1 \\ 1 - (-2) &= 3 \end{aligned}$$

So, in the sequence $\{1, 3, 2, -1, \dots\}$, six numbers $1, 3, 2, -1, -3, -2$ will be repeated. We need to determine the number of times they repeated in the first 2009 terms. When 2009 is divided by 6, we get the quotient 334 with a remainder of 5. This means that the first 2009 terms contain 334 groups of these six numbers and the first five numbers of them. Since the sum of $1, 3, 2, -1, -3, -2$ is zero, the sum of the first $334 \times 6 = 2004$ terms in the sequence is zero. The other five numbers are $1, 3, 2, -1, -3$ with a sum of 2. Therefore, the sum of the first 2009 terms is 2.

A generalization by Ivan Hadinata.

Lemma. Let $(a_n)_{n=1}^{\infty}$ be a sequence in such a way that $a_1 = a$, $a_2 = b$, and $a_{n+2} = a_{n+1} - a_n$ for every $n \in \mathbf{N}$. Then $a_1 + a_2 + \dots + a_{2009} = b - a$.

Proof. For every $n \in \mathbf{N}$, we have

$$a_{n+3} = a_{n+2} - a_{n+1} = (a_{n+1} - a_n) - a_{n+1} = -a_n$$

and then

$$a_{n+6} = -a_{n+3} = -(-a_n) = a_n, \quad \forall n \in \mathbf{N}$$

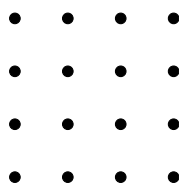
Consequently,

$$\begin{aligned} a_1 + a_2 + \dots + a_{2009} &= a_1 + a_2 + \sum_{k=1}^{2007} (a_{n+1} - a_n) \\ &= a_1 + a_2 + a_{2008} - a_1 = a_2 + a_{2008} = b - a \end{aligned}$$

since $a_2 = b$ and $a_{2008} = a_{2002} = a_{1996} = \cdots = a_{10} = a_4 = -a_1 = -a$.

By setting $a = 1$ and $b = 3$ in the lemma, we obtain the answer to the original question, which is 2.

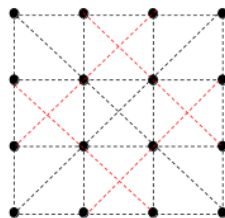
MA187. A 4×4 grid of points is uniformly distributed and a set of three points is to be randomly chosen from the grid. Each three point set has the same probability of being chosen. What is the probability that the three chosen points lie on the same straight line? Write your answer as a fraction in simplest form.



Originally question 11 from the 35th University of Alabama High School Mathematics Tournament: Team Competition (2016).

We received 3 submissions, 2 of which were correct. We present Catherine Jian's solution accompanied by Richard Hess's diagram.

In total, there are $\binom{16}{3}$ ways of choosing three points. Now we count how many ways these three points can lie on the same straight line. Connecting dots in this 4×4 grid, we can get 4 horizontal lines, 4 vertical lines and 2 main diagonal lines, each of which has 4 dots. Note that there are also 4 partial diagonal lines (one step off the main diagonal lines), each of which has 3 dots.



So in total there are $10 \times \binom{4}{3} + 4 \times \binom{3}{3}$ ways of having three points on the same line. Therefore the desired probability is

$$\frac{10 \times \binom{4}{3} + 4 \times \binom{3}{3}}{\binom{16}{3}} = \frac{44}{560} = \frac{11}{140}.$$

Editor's Comments. The incorrect solution only counted the horizontal, vertical, and main diagonal lines, for an answer of $1/14$. The problem is more complicated for larger grids, because more slopes are possible. For instance, on a 5-by-5 grid, one has to consider “steep” and “shallow” diagonals.

MA188. A farmer is selling all of her sheep, goats, and cows (she has some of each). One man offers to pay her \$100 for each sheep, \$200 for each goat, and \$400 for each cow for a total of \$4700. Another offers to pay her \$135 for each sheep, \$265 for each goat, and \$309 for each cow for a total of \$5155. How many sheep, goats, and cows does she have?

Originally question 12 from the 36th University of Alabama High School Mathematics Tournament: Team Competition (2017).

We received 10 submissions, 9 of which were correct. We present the solution by Catherine Jian.

Let the numbers of sheep, goats, and cows be s , g , and c , respectively. From the given facts, we know that $\min(s, g, c) > 0$ and can set up the following equations:

$$100s + 200g + 400c = 4700,$$

$$135s + 265g + 309c = 5155$$

Note that $135s + 265g + 309c = 5155 \implies 4c \equiv 0 \pmod{5} \implies c \equiv 0 \pmod{5}$, i.e. c has to be a multiple of 5. Letting $c = 5k$ where $k \geq 1$, the original equations can be simplified to

$$s + 2g + 20k = 47, \tag{1}$$

$$27s + 53g + 309k = 1031. \tag{2}$$

Multiplying the first equation by 27 and subtracting the second equation, we get

$$g + 231k = 238$$

which implies k can only be 1. Plugging $k = 1$, we further get $g = 7$, $c = 5$, $s = 13$. Hence there are 13 sheep, 7 goats and 5 cows.

MA189. *Proposed by Alaric Pow Ian-Jun.*

Find the range of values of the constant k such that the equation

$$(x+1)(x+3)(x+5)(x+7) = k$$

has 4 distinct solutions for x .

We received 10 submissions, of which 8 were correct. We present (with minor amendments) the solution by the Missouri State University Problem Solving Group.

Letting $x = u - 4$ and expanding, the equation becomes

$$u^4 - 10u^2 + 9 = k.$$

By the quadratic formula, the solution to this equation is

$$u = \pm \sqrt{5 \pm \sqrt{16 + k}}.$$

If the intent was for x to be real, then we must have $16 + k \geq 0$, i.e., $k \geq -16$. Since $k = -16$ only gives two values for u , it must be rejected. We must also have $5 - \sqrt{16 + k} \geq 0$ or equivalently $k \leq 9$. Since $k = 9$ only gives three values for u , it must also be rejected. Thus the range for k is $-16 < k < 9$.

If the intent was for x to be complex, then $k = -16$ and $k = 9$ are the only values that give fewer than four roots.

Editor's Comments. Some solvers only considered real solutions; others included complex solutions. Due to the ambiguity of the problem statement, both approaches were considered correct. The majority of solutions were purely algebraic, like the one above. A few solvers (S. Bhadra, H. Choi, I. Hadinata) used calculus instead, arguing in terms of the critical points of $f(x) = (x+1)(x+3)(x+5)(x+7)$. One solver, A. Mahadevan, appealed to the theory of the general quartic equation, which supplies the following criterion: the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

has 4 distinct roots (over the complex numbers) if and only if the discriminant

$$\begin{aligned} \Delta = & 256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e \\ & - 27a^2d^4 + 144ab^2ce^2 - 6ab^2d^2e - 80abc^2de \\ & + 18abcd^3 + 16ac^4e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde \\ & - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2 \end{aligned}$$

is nonzero. For us, the discriminant of

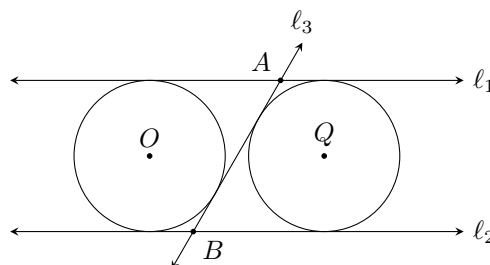
$$(x+1)(x+3)(x+5)(x+7) - k = x^4 + 16x^3 + 86x^2 + 176x + 105 - k$$

is simply

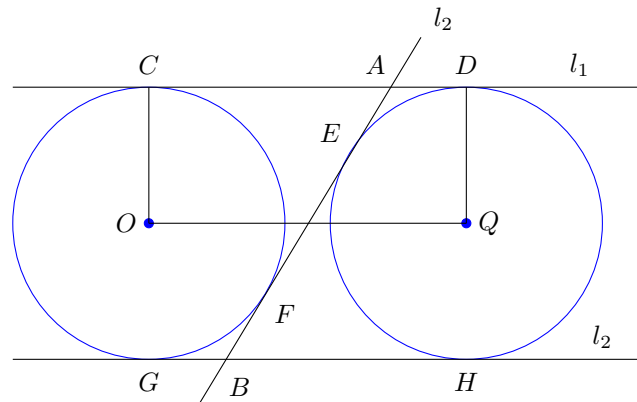
$$\Delta = -256(k-9)(k+16)^2.$$

MA190. Proposed by Jakob Denes.

Given two parallel lines ℓ_1 and ℓ_2 , the transversal ℓ_3 intersects them at points A and B respectively. Two circles with centres O and Q lie between the parallel lines on the left and on the right sides of the transversal such that the circles are tangent to all three lines. Show that $OQ = AB$.



We received 9 solutions, all correct. The following is the solution by Mingshen Zong.



In the diagram above, let C, D, G, H be the points of tangency of circles O and Q with lines l_1 and l_2 . Denote E and F the points of tangency of circles O and Q with line l_3 .

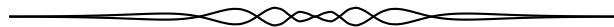
Connect OC and QD . Then both OC and QD are perpendicular to l_1 , therefore making OC parallel to QD . Since radius $OC = DQ$, $CDQO$ is a rectangle and $OQ = CD$. Similarly, we can get $OQ = GH$.

It follows from the two tangent theorem that $AC = AF$ and $AD = AE$. Since $AF = EF + AE$, we get

$$CD = AC + AD = EF + AE + AD = EF + 2AE.$$

Similarly, we can get $GH = EF + 2BF$. Then the fact that $CD = GH$ implies that $EF + 2AE = EF + 2BF$ and so $AE = BF$.

Since $AB = EF + AE + BF$ and $AE = BF$, we have $AB = EF + 2AE = CD$. Finally, the fact $CD = OQ$ proves $OQ = AB$.



PROBLEM SOLVING VIGNETTES

No. 26

Shawn Godin

Three Cute Contest Problems

I am a big fan of math contests. As a student I always looked forward to that one day a year I would get to write the *Junior Math Contest* from the University of Waterloo (I wrote the *Descartes Contest* in my last year of high school). When I was a classroom teacher I often incorporated problems from math contests – either in their original form or modified – into my class activities, homework assignments and tests. The students in my math club wrote many different contests, so they became a big part of our meetings. I have also had the good fortune to work on teams creating contests at the local level, as well as with the Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo, and with the Canadian Mathematical Society (CMS).

On Friday February 17, I gave a talk *Mining Math Contests for Problems* at the professional development day for my former board. It was nice to touch base with so many old friends and colleagues that I haven't seen for a couple of years while I have been living the dream. In the process of preparing for the talk, I did all Canadian mathematics contests that I could find online. I earmarked a long list of problems that piqued my interest for one reason or another.

This column highlights three of those problems. On their own, I don't think any of them have enough meat for a full column. However, each one has something worth looking at. It might be a clever trap left to catch those who are not being careful. It may be an alternate solution that simplifies things or gives us some insight. It may be the answer is surprising. I hope you enjoy these problems.

.....

The first problem is question 12 from Part I of the 2022–23 *Alberta High School Mathematics Competition*. The *Alberta High School Mathematics Competition* is hosted by the University of Alberta and is written in two parts. In November, students write a 16 question multiple choice contest. Then in February, a selection of students are invited to write a 5 question, full solution contest. More information and past contests can be found on their website:

What is the number of integers m , $1 \leq m \leq 300$ for which m^m is a perfect cube?

- (A) 100 (B) 101 (C) 103 (D) 104 (E) 106

Careful thinking is needed for this problem. You may reason, if $m = 3k$ then

$$m^m = m^{3k} = (m^k)^3$$

which is a perfect cube. Thus, since $1 \leq m \leq 300$ we must have $1 \leq 3k \leq 300$ or $1 \leq k \leq 100$, since m and k must be integers. So we may be tempted to pick (A).

However, if we think about it a bit more, we would see that if $m = k^3$, then

$$m^m = (k^3)^m = k^{3m} = (k^m)^3$$

which is a perfect cube. Thus, since $1 \leq m \leq 300$ we must have $1 \leq k^3 \leq 300$ or $1 \leq k \leq 6$, since m and k must be integers. Thus there are 6 more cases and we may be tempted to pick (E). However, if we think a bit more we realize that $k = 3^3 = 27 = 3 \times 9$ and $k = 6^3 = 216 = 3 \times 72$ are counted in both groups, so that the answer we are after is $100 + 6 - 2 = 104$, answer (D).

Counting questions should always be handled with care. It is often as easy to miss cases as it is to count other cases multiple times. Problems regularly can be solved in multiple ways using elementary techniques. So next time you see a counting problem resist the impulse to rush through it because you “know how to do it”.

.....

Continuing, we will look at problem B4 from the 2022 *Canadian Open Mathematics Challenge*. The *Canadian Open Mathematics Challenge* is written each November and is hosted by the CMS. The contest consists of three sections with four problems each. In sections *A* and *B*, students are rewarded full marks for a correct answer. The work for incorrect answers is checked for possible partial credit. Questions in part *C* are full solution. More information about the contest and past contests can be found on the website:

Determine all integers a for which $\frac{a}{1011 - a}$ is an even integer.

Let's attack this one by setting

$$\frac{a}{1011 - a} = 2k$$

for some integer k . Then

$$\begin{aligned} a &= 2022k - 2ka \\ a(2k + 1) &= 2022k \end{aligned}$$

and so $(2k + 1) \mid 2022k$, since all numbers are integers. Since $2k + 1$ is odd and k and $2k + 1$ are relatively prime (why?), we must have $2k + 1 \mid 1011$. As the divisors of 1011 are $\pm 1, \pm 3, \pm 337, \pm 1011$ and $a = \frac{2022k}{2k+1}$ we get

$2k + 1$	k	a	$\frac{a}{1011-a}$
-1	-1	2022	-2
1	0	0	0
-3	-2	1348	-4
3	1	647	2
-337	-169	1014	-338
337	168	1008	336
-1011	-506	1012	-1012
1011	505	1010	1010

Let's take another look at this problem through a slightly different lens. If we let $d = 1011 - a$, then $a = 1011 - d$ and the expression in the problem becomes

$$\frac{a}{1011 - a} = \frac{1011 - d}{d} = \frac{1011}{d} - 1.$$

Since 1011 is odd, then the only way $\frac{1011}{d} - 1$ can be an integer is if $d \mid 1011$, but then d must be odd and $\frac{1011}{d} - 1$ *must* be even! Thus all the divisors, d , of 1011 generate solutions and the sought for values of a are just $1011 - d$ for all possible integer divisors of 1011.

The technique makes the solution a bit more straightforward. Not so much to make much of a difference in solving the problem. However, the alternate point of view allows us to see far more:

- Each integer divisor, d , of 1011 generates a result.
- The corresponding values of a are just $1011 - d$.
- The result $\frac{a}{1011-a} = \frac{1011}{d} - 1 = d' - 1$, where d' is also an integer divisor of 1011, with $dd' = 1011$. That is all the results are one less than an integer divisor of 1011.

From this we should be able to see that we can immediately generalize this method to any case where 1011 is replaced by an *odd* integer. If we replace 1011 with an *even* integer we can still come up with an easy general solution if we build something in to rid ourselves of the *evenness*. I will leave the further exploration of this problem to interested readers.

.....

Lastly, let us look at problem 6(a) from the 2022 *Euclid Contest*. The *Euclid Contest* is a 10 question, full solution contest hosted by the CEMC at the University of Waterloo. You can check out all the CEMC contests at their website:

A function f has the property that

$$f\left(\frac{2x+1}{x}\right) = x + 6$$

for all real values of $x \neq 0$. What is the value of $f(4)$?

Functional equations are interesting because each seems to need a different method of attack, although some general techniques exist. What makes this difficult is that we are used to functions being defined *explicitly*. That is, we are usually given the expression defining our function.

Let's start by trying to dissect this function. Rewriting $\frac{2x+1}{x} = 2 + \frac{1}{x}$ we can think of how to turn this into $x + 6$. Breaking it down into steps we get:

- Subtracting 2, yields $(2 + \frac{1}{x}) - 2 = \frac{1}{x}$.
- Taking the reciprocal gives us x .
- Adding 6 leaves us with the desired $x + 6$.

So, starting with 4 we get

- $4 - 2 = 2$
- $\frac{1}{2}$
- $\frac{1}{2} + 6 = \frac{13}{2}$

There must be a nicer way! An insight is that since we want $f(4)$, we need $\frac{2x+1}{x} = 4$. We can solve for x to get

$$\begin{aligned} 2x + 1 &= 4x \\ 1 &= 2x \\ x &= \frac{1}{2} \end{aligned}$$

which corresponds with our less elegant solution. Thus the desired value of the function is $f(4) = \frac{1}{2} + 6 = \frac{13}{2}$.

That seems a bit better. However, let us try something else. Since we prefer explicit function definitions, let's try to get one. If we let $y = \frac{2x+1}{x}$ in $f(\frac{2x+1}{x}) = x + 6$, then we have $f(y) = x + 6$. If we can relate x and y we are done. Rearranging our definition of y yields

$$\begin{aligned} y &= \frac{2x+1}{x} \\ xy &= 2x+1 \\ x(y-2) &= 1 \\ x &= \frac{1}{y-2} \end{aligned}$$

and thus

$$f(y) = \frac{1}{y-2} + 6.$$

Hence, as in the previous two solutions,

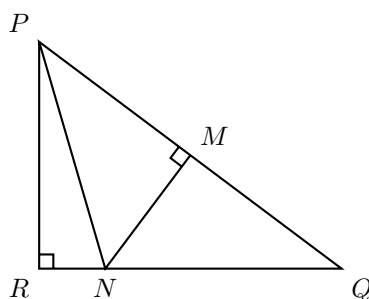
$$f(4) = \frac{1}{4-2} + 6 = \frac{13}{2}.$$

The last technique, known as a *change of variables* is useful in many situations – for instance, the previous problem – and if you continue to study mathematics you will come across it from time to time.

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I hope you enjoyed these three problems from mathematics contests. There is a vast treasure trove of contest material available online for the contest writer or problem solving enthusiast to play with. I strongly suggest you explore some of these online contest collections, especially ones that you are not familiar with. I leave the interested reader with a few more problems to play with.

1. In the diagram, $\triangle PQR$ is right-angled at R , $PR = 12$, and $QR = 16$. Also, M is the midpoint of PQ and N is the point on QR so that MN is perpendicular to PQ .



The area of $\triangle PNR$ is

- (A) 21 (B) 17.5 (C) 36 (D) 16 (E) 21.5
 (2022 *Fermat Contest*, #19)
2. What is the largest integer n with the properties that $200 < n < 250$ and that $12n$ is a perfect square?
 (2022 *Canadian Team Mathematics Contest*, Individual Problems #5)
 3. In an unnamed country, Donald and Joe are running for president. There are 3 states. Each state consists of 3 counties. Each county has 3 cities, and each city has 3 wards. Each ward has 3 electors who cast votes. To win a ward, a candidate must win $\frac{2}{3}$ of the electors; to win a city, one must win $\frac{2}{3}$ of the wards; to win a county, one must win $\frac{2}{3}$ cities; and to win a state, you have to win $\frac{2}{3}$ of the counties, and finally to win the election, you must win $\frac{2}{3}$ of the states. Abstaining from voting is not allowed.
 - (a) What is the smallest number of elector votes Donald must receive to win the election?
 What percentage of the total popular vote is this?
 - (b) What is the smallest number of total votes Joe needs to guarantee a victory?
 (2022 *W.J. Blundon Mathematics Contest*, #8)

4. Show that for any positive integer n the number

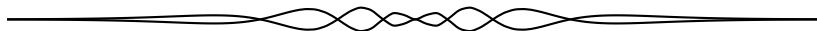
$$\underbrace{111 \dots 1}_{3^n \text{ digits}}$$

consisting of 3^n 1s, is divisible by 3^n .

(2021–22 *Alberta High School Mathematics Competition, Part II, #3*)

5. An integer container (x, y, z) is a rectangular prism with positive integer side lengths x, y, z , where $x \leq y \leq z$. A stick has $x = y = 1$; a flat has $x = 1$ and $y > 1$; and a box has $x > 1$. There are 5 integer containers with volume 30: one stick $(1, 1, 30)$, three flats $(1, 2, 15)$, $(1, 3, 10)$, $(1, 5, 6)$ and one box $(2, 3, 5)$.
- (a) How many sticks, flats and boxes are there among the integer containers with volume 36?
- (b) How many flats and boxes are there among the integer containers with volume 210?
- (c) Suppose $n = p_1^{e_1} \cdots p_k^{e_k}$ has k distinct prime factors p_1, p_2, \dots, p_k , each with integer exponent $e_1 \geq 1, e_2 \geq 1, \dots, e_k \geq 1$ and $k \geq 3$. How many boxes are there among the integer containers with volume n ? Express your answer in terms of e_1, e_2, \dots, e_k . How many boxes with volume $n = 8!$ are there?

(2022 *Canadian Open Mathematics Challenge, #C4*)



Fibonacci Sequence and Higher Order Golden Ratio

Doddy Kastanya

The Fibonacci sequence is a sequence of numbers in which each number (starting from the third number in the sequence) is formed by adding the two preceding numbers. The sequence can then be formed recursively by employing the following formula:

$$F_{n+1} = F_n + F_{n-1} \quad (1)$$

for $n \geq 1$, and $F_0 = F_1 = 1$. This is a typical textbook setup of a Fibonacci sequence. An interesting thing about the Fibonacci sequence is the fact that as the number of elements approaches infinity, the ratio between one element and the preceding one approaches φ (the lowercase Greek letter “phi”) which has a value of $\frac{1+\sqrt{5}}{2}$ or approximately 1.618. This quantity is also called the Golden Ratio.

Let’s take a look at an elementary proof of this fact. The process starts by dividing the left- and right-hand sides of Eq. (1) by F_n and taking the limit as $n \rightarrow \infty$.

$$\frac{F_{n+1}}{F_n} = \frac{F_n}{F_n} + \frac{F_{n-1}}{F_n} \quad (2)$$

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{F_{n-1}}{F_n} \quad (3)$$

Since we want to show that the ratio between an element and the preceding one approaches a certain value as the number of elements approaches infinity, we can assign $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \alpha$, which also means that $\lim_{n \rightarrow \infty} \frac{F_{n-1}}{F_n} = \frac{1}{\alpha}$ where α is just an arbitrary number at this point. So, we can rewrite Eq. (3) as:

$$\alpha = 1 + \frac{1}{\alpha} \quad (4)$$

which can be rearranged into a quadratic equation:

$$\alpha^2 - \alpha - 1 = 0. \quad (5)$$

Since the elements of the Fibonacci sequence are positive, α must be the positive root of this quadratic equation, namely,

$$\alpha = \frac{1 + \sqrt{5}}{2} = \varphi \quad \square \quad (6)$$

Now, let’s perform some manipulations on the Fibonacci sequence. Begin by pairing up two consecutive elements of the sequence as in F_{2n-1}, F_{2n} for $n \geq 1$ and create another sequence by adding the components of each group. Let’s call the new sequence FS_2 (for “2-element Fibonacci Sum”) and the Fibonacci sequence F .

$$F = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \dots\}$$

$$FS_2 = \{2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, \dots\}$$

An interesting observation was made when trying to, at first, numerically determine if the ratio of two consecutive elements of FS_2 approaches a unique quantity as $n \rightarrow \infty$. It turns out that

$$\lim_{n \rightarrow \infty} \frac{FS_{2,n+1}}{FS_{2,n}} = \varphi^2$$

Worrying that this is only a fluke, I extended the exercise to create FS_3 through FS_6 :

$$FS_3 = \{4, 16, 68, 288, 1220, 5168, 21892, 92736, \dots\}$$

$$FS_4 = \{7, 47, 322, 2207, 15127, 103682, 710647, \dots\}$$

$$FS_5 = \{12, 131, 1453, 16114, 178707, 1981891, \dots\}$$

$$FS_6 = \{20, 356, 6388, 114628, 2056916, 36909860, \dots\}$$

Examining the ratios of two consecutive elements of these sequences, I observed that $\lim_{n \rightarrow \infty} \frac{FS_{3,n+1}}{FS_{3,n}} = \varphi^3$, $\lim_{n \rightarrow \infty} \frac{FS_{4,n+1}}{FS_{4,n}} = \varphi^4$, $\lim_{n \rightarrow \infty} \frac{FS_{5,n+1}}{FS_{5,n}} = \varphi^5$, and $\lim_{n \rightarrow \infty} \frac{FS_{6,n+1}}{FS_{6,n}} = \varphi^6$. So, I would like to put out a conjecture that

$$\lim_{n \rightarrow \infty} \frac{FS_{\xi,n+1}}{FS_{\xi,n}} = \varphi^\xi$$

for $\xi \geq 1$. Since I consider myself as an amateur mathematician, I am not well equipped to perform the rigorous proof for this. Instead, I would like to conclude this article by sharing a less rigorous proof for a lower value of ξ , namely $\xi = 3$. However, this method generalizes for any exponent ξ . I leave this as an exercise for the interested readers.

Let's assume that the ratio of an element to its immediate predecessor in FS_3 sequence approaches a certain value, β , as $n \rightarrow \infty$. Recalling that each component of the FS_3 sequence is formed by adding three consecutive elements of the Fibonacci sequence, we can write:

$$\beta = \lim_{n \rightarrow \infty} \frac{F_{n+2} + F_{n+1} + F_n}{F_{n-1} + F_{n-2} + F_{n-3}} \quad (7)$$

Dividing the numerator and denominator of Eq. (7) by F_{n-4} gives us

$$\beta = \frac{\lim_{n \rightarrow \infty} \frac{F_{n+2}}{F_{n-4}} + \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_{n-4}} + \lim_{n \rightarrow \infty} \frac{F_n}{F_{n-4}}}{\lim_{n \rightarrow \infty} \frac{F_{n-1}}{F_{n-4}} + \lim_{n \rightarrow \infty} \frac{F_{n-2}}{F_{n-4}} + \lim_{n \rightarrow \infty} \frac{F_{n-3}}{F_{n-4}}} \quad (8)$$

Before continuing with the proof, we need to use another property of the Fibonacci sequence that could be simply derived from $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \varphi$, which is

$\lim_{n \rightarrow \infty} \frac{F_{n+m}}{F_n} = \varphi^m$. The proof for this assertion is given below.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{F_{n+m}}{F_n} &= \lim_{n \rightarrow \infty} \left(\frac{F_{n+m}}{F_{n+m-1}} \times \frac{F_{n+m-1}}{F_{n+m-2}} \times \cdots \times \frac{F_{n+2}}{F_{n+1}} \times \frac{F_{n+1}}{F_n} \right) \\ &= \left(\lim_{n \rightarrow \infty} \frac{F_{n+m}}{F_{n+m-1}} \right) \times \left(\lim_{n \rightarrow \infty} \frac{F_{n+m-1}}{F_{n+m-2}} \right) \times \cdots \\ &\quad \times \left(\lim_{n \rightarrow \infty} \frac{F_{n+2}}{F_{n+1}} \right) \times \left(\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} \right) \end{aligned}$$

Since each of the term on the right hand of the equation equals to φ and there are m terms in total, one can write $\lim_{n \rightarrow \infty} \frac{F_{n+m}}{F_n} = \varphi^m$ \square .

Using this knowledge, Eq. (8) can be written as:

$$\beta = \frac{\varphi^6 + \varphi^5 + \varphi^4}{\varphi^3 + \varphi^2 + \varphi} \quad (9)$$

There are two possible paths for completing the proof. The first one is related to recognizing that the numerator and denominator in Eq. (9) share a common factor. Taking advantage of this fact, the conclusion of the proof will become obvious.

$$\beta = \frac{\varphi^6 + \varphi^5 + \varphi^4}{\varphi^3 + \varphi^2 + \varphi} = \frac{\varphi^4 \times (\varphi^2 + \varphi + 1)}{\varphi \times (\varphi^2 + \varphi + 1)} = \frac{\varphi^4}{\varphi} = \varphi^3 \quad \square \quad (10)$$

For the second path to complete the proof, we need to recall an important property of the Golden Ratio, which is:

$$\varphi^2 = \varphi + 1 \quad (11)$$

It should be noted that this expression follows directly from Eq. (5) and Eq. (6).

Using Eq. (11), the following expressions can be derived:

$$\varphi^3 = 2\varphi + 1 \quad (12)$$

$$\varphi^4 = 3\varphi + 2 \quad (13)$$

$$\varphi^5 = 5\varphi + 3 \quad (14)$$

$$\varphi^6 = 8\varphi + 5 \quad (15)$$

It should be noted that Eq. (12) through Eq. (15) can be generalized as

$$\varphi^n = F_{n-1}\varphi + F_{n-2}$$

for $n > 1$ (recall that the sequence starts with F_0 not F_1).

Substituting the expressions given in Eq. (11), and Eq. (12) through Eq. (15), Eq. (9) can be written as:

$$\beta = \frac{8\varphi + 5 + 5\varphi + 3 + 3\varphi + 2}{2\varphi + 1 + \varphi + 1 + \varphi} = \frac{16\varphi + 10}{4\varphi + 2} = \frac{8\varphi + 5}{2\varphi + 1} \quad (16)$$

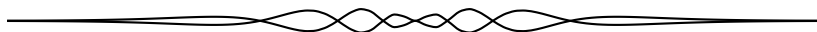
Using Eq. (15) and Eq. (12), we can complete the proof:

$$\lim_{n \rightarrow \infty} \frac{FS_{3,n+1}}{FS_{3,n}} = \frac{8\varphi + 5}{2\varphi + 1} = \frac{\varphi^6}{\varphi^3} = \varphi^3 \quad \square$$

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Doddy is a math enthusiast working as a nuclear engineer. The love of math and physics was the reason for him to choose this field. In his spare times, among other things he likes to solve math puzzles and problems. In addition to ***CruX***, the Project Euler has provided him with enough challenges and enjoyment in this area. Doddy and his family share their Oakville home with their four cats: Luke, Lorelai, Lincoln, and Lilian. Communications concerning the article can be shared with the author via email: kastanya@yahoo.com.



From the Bookshelf of ...

Trefor Bazett

This MathemAttic feature brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to MathemAttic@cms.math.ca. Publishers are also welcome to send along books for possible review.

Infinite Powers

by Steven Strogatz

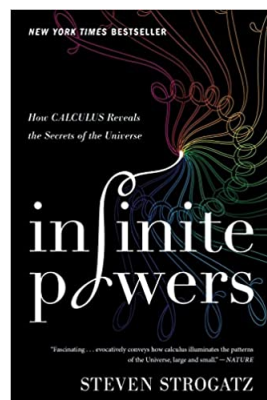
ISBN 9780358299288, hardcover, 360 pages

Published by Mariner Books, 2020.

Calculus is a field that is both tremendously powerful in its enormous range of applications while also being delightful in the puzzles it poses and answers. No wonder calculus is a required subject for so many students in STEM fields. Yet too often the joy of calculus is lost among a sea of memorized formulas and tedious computations. *Infinite Powers* is an antidote to this, a book that helps reveal both the power and beauty of calculus.

What I most appreciate about *Infinite Powers* is how wide the potential audience is and how accessibly the book is written. For a high school student who is thinking they might want to take calculus someday, but doesn't even really know what that is about, this book would be fantastic. For myself as someone who has lost track of how many times I've taught a calculus course, I still learned and grew a lot both in my understanding of calculus but perhaps more importantly as an educator and mathematics communicator. And it would be similarly excellent for anyone in between. While the book isn't a calculus textbook nor does it try to be, the primary focus of the book is on building a robust conceptual understanding of the big themes of calculus. It does an excellent job deepening and augmenting your understanding of calculus no matter what your relationship with calculus is.

At the core of the book is what Strogatz terms the Infinity Principle, the basic idea of taking a phenomenon and reimagining "it as an infinite series of simpler parts, analyze those, and then add the results back together to make sense of the original whole" (pg xvii). While a typical calculus student might see this basic idea in, for instance, the definition of the Riemann Integral, Strogatz weaves this principle throughout the book from the ancient Greeks through to contemporary problems, showing how the "Golem of Infinity" has puzzled and delighted humanity for



millennia.

One aspect I really appreciated about the book was how it authentically connected to the history and characters in the development of calculus. Today when we study calculus with all of its details so fully fleshed out we can lose track of the original motivations for why these problems were fascinating to begin with. Strogatz doesn't just go back to Newton and Leibniz, but more broadly to humanity's wrestling with the concept of infinity from Zeno to Archimedes to Galileo and Kepler. These historical figures, and the problems that motivated them, come alive on the page. I'm definitely going to sneak a few of these stories in the next time I teach calculus.

As much fun as I had with tidbits of historical development, what I found really fascinating was the many connections to applications that are relevant today. For instance, Strogatz connected the methods that Archimedes used for the quadrature of the parabola to only a few pages later a discussion of modern computer animations in movies and simulations of surgeries. With a wide range of applications in physics, engineering, technology, and biology, the question of *why* a student should study calculus is answered again and again and again. *Infinite Powers* even has a final chapter on the future of calculus that connects challenges in calculus related to things like nonlinearity and chaos to applications in the future such as artificial intelligence and modelling DNA.

As an educator, I mentioned that I learned a lot about mathematical communication by reading several of Strogatz' books, *Infinite Powers* included. Two specific lessons learned are firstly the importance of story. Telling the story of calculus together with both a cast of characters motivated by intriguing problems as well as a rich array of modern applications draws us in to calculus, building that sense of appreciation for its power and beauty as we go along. Secondly, I appreciated the focus on the key concepts and emphasizing conceptual understanding without relying solely on technical machinery. For instance, the Fundamental Theorem of Calculus is explained through a "Paint-Roller Proof" (pg 175), a metaphor that helps make the meaning of this triumph of calculus seem intuitive and clear.

Regardless of where you might be on your calculus journey, I highly recommend you check out *Infinite Powers*!

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This book is a recommendation from the bookshelf of Dr. Trefor Bazett. Trefor is an Assistant Teaching Professor at the University of Victoria and has been teaching some flavour of calculus for over a decade. Trefor is also a math YouTuber, dedicated to sharing the joys of learning mathematics with millions of students around the world.

MATHEMATICS FROM THE WEB

No. 8

This column features short reviews of mathematical items from the internet that will be of interest to high school and elementary students and teachers. You can forward your own short reviews to mathemattic@cms.math.ca.

The Lonely Runner Conjecture

http://www.openproblemgarden.org/op/lonely_runner_conjecture

Suppose k runners having distinct constant speeds start at a common point and run laps on a circular track with circumference 1. Then for any given runner, there is a time at which that runner is distance at least $\frac{1}{k}$ (along the track) away from every other runner.

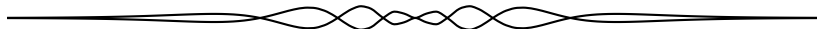
This conjecture is an example of an unsolved problem, one of hundreds from various branches of mathematics, that appears in Open Problem Garden. The discussion of this open problem mentions that the conjecture has been proven for values of k up to and including seven. An extensive bibliography accompanies the problem.

The Quest to Find Rectangles in a Square

<https://www.nytimes.com/2023/02/07/science/puzzles-rectangles-mathematics.html>

There are three ways to divide a square into three rectangles with the same proportions.

MathemAttic readers may be interested in finding the three configurations themselves. This statement was used to ask the next logical question: How can you divide a square up into four rectangles with the same proportions? The problem was posted on *Mathstodon*, a community within the social network *Mastodon*. The *New York Times* article discusses the solution techniques of several people as well as providing the results, with illustrations, for three, four, five and six rectangles. (Submitted by Rad de Peiza, Toronto Ontario)



OLYMPIAD CORNER

No. 411

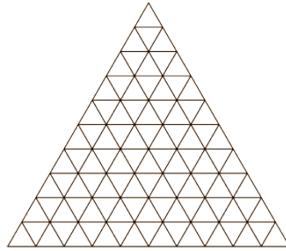
The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

*To facilitate their consideration, solutions should be received by **May 15, 2023**.*

OC621. Find all natural numbers n for which the number $n^n + 1$ is divisible by $n + 1$.

OC622. An equilateral triangle with side length n is divided into n^2 small equilateral triangles of side length 1 (as in the picture for $n = 10$). At the start, one small internal triangle (with no points in common with external sides of the large triangle) is painted in blue, and the rest are painted yellow. In one move, you can choose any of the n^2 small triangles and swap its colour and the colours of the triangles adjacent to it along its sides. Using such moves, is it possible to make the entire board one colour?



OC623. Let B and C be two points on the circumference of a circle with diameter AD such that $AB = AC$. Let P be a point on line segment BC and let M, N be points on line segments AB and AC , respectively, such that $PMAN$ is a parallelogram. Suppose PL is an angle bisector of triangle MPN with L lying on the line segment MN . If the line PD intersects MN in point Q , show that the points B, Q, L and C lie on the same circle.

OC624. A series contains 51 not necessarily different natural numbers which add up to 100. A natural number k is called *representable* if it can be represented as the sum of several consecutively written numbers in this series (perhaps one number). Prove that at least one of the two numbers k and $100 - k$ is representable, where $1 \leq k \leq 100$.

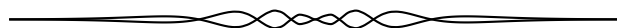
OC625. Does there exist a convex 2021-gon with vertices at points with integer coordinates and such that the lengths of all its sides are equal?

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

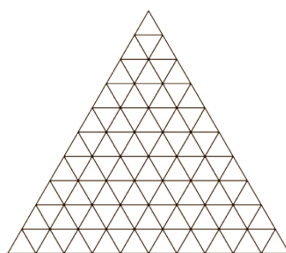
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 mai 2023.



OC621. Déterminer tous les nombres naturels n tels que $n^n + 1$ est divisible par $n + 1$.

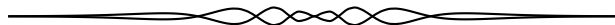
OC622. Un triangle équilatéral de côtés de longueur n est subdivisé en n^2 petits triangles équilatéraux de côtés de longueur 1, tel qu'illustré ci-bas pour le cas où $n = 10$. Au départ, un seul petit triangle est coloré bleu, ce petit triangle étant interne, donc n'ayant aucun point en commun avec le grand triangle; tous les autres petits triangles sont colorés jaune. Par la suite, on choisit un quelconque des n^2 petits triangles et on change sa couleur, ainsi que celles des triangles qui lui sont adjacents en ayant un côté en commun. Continuant ainsi, est-il possible que le tout devienne éventuellement d'une même couleur?



OC623. Deux points B et C sont placés sur la circonférence d'un cercle de diamètre AD , de façon à ce que $AB = AC$. Soit alors P un point sur le segment BC et soient M et N des points sur les segments AB et AC , respectivement, de façon à ce que $PMAN$ soit un parallélogramme. Enfin, PL bissecte le triangle MPN , où L se trouve entre M et N . Si la ligne PD rencontre MN au point Q , démontrer que les points B , Q , L et C se trouvent sur une même circle.

OC624. Une série consiste de 51 nombres naturels, pas nécessairement distincts, dont la somme est de 100. Un nombre naturel k est alors dit *représentable* s'il est la somme de nombres consécutifs de cette série (possiblement un seul nombre). Pour un quelconque nombre k , $1 \leq k \leq 100$, démontrer qu'au moins un de k et $100 - k$ est représentable.

OC625. Existe-t-il dans le plan un polygone convexe à 2021 côtés de longueurs égales, dont tous les sommets ont des coordonnées entières ?



OLYMPIAD CORNER

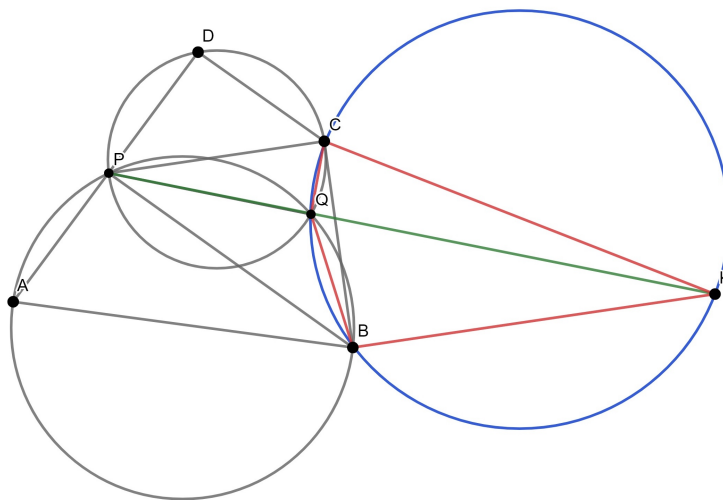
SOLUTIONS

Statements of the problems in this section originally appear in 2022: 48(8), p. 462–463.

OC596. Let $ABCD$ be a convex quadrilateral with pairwise non-parallel sides. On the side AD , choose an arbitrary point P different from A and D . The circumscribed circles of triangles ABP and CDP intersect at another point Q . Prove that the line PQ passes through a fixed point independent of the choice of point P .

Originally from the 2018 Moscow Math Olympiad, 4th Problem, Grade 9.

We received 6 correct solutions. We present the solution by Elmar Aliyev.



We know that $CDPQ$ and $ABQP$ are cyclic quadrilaterals. Hence, if $\angle BAD = \alpha$ and $\angle ADC = \beta$, then

$$\angle PQC = 180 - \beta, \angle PQB = 180 - \alpha, \angle CQB = \alpha + \beta.$$

Draw a circle through Q , C and B . The radius of this circle is fixed, because BC and $\angle CQB$ are fixed. Let the line PQ intersect the circle BCQ at a point K . So, $BKCK$ is a cyclic quadrilateral. Point K is fixed, because $\angle CQK$ and $\angle KQB$ are fixed angles. As a result, the line PQ passes through K , which is independent of the choice of point P .

OC597. For a natural number n and for a column matrix

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \in \mathcal{M}_{n,1}(\mathbb{Z}),$$

let $\delta(X)$ be the greatest common divisor of numbers x_1, x_2, \dots, x_n . Let $n \in \mathbb{N}, n \geq 2$ and let $A \in \mathcal{M}_n(\mathbb{Z})$. Prove that the following statements are equivalent:

- (a) $|\det A| = 1$;
- (b) $\delta(AX) = \delta(X)$ for all $X \in \mathcal{M}_{n,1}(\mathbb{Z})$.

Originally from the 2018 Romania Math Olympiad, 1st Problem, Grade 11, Final Round.

We received 3 solutions, all of which were correct. We present the solution by UCLan Cyprus Problem Solving Group.

Assume first that $|\det(A)| = 1$. If $\delta(X) = k$, then each x_i is a multiple of k . Letting $AX = (y_1 \ y_2 \ \dots \ y_n)^T$, then each y_i is an integer linear combination of the x_i 's and therefore also a multiple of k . Thus $\delta(AX) \geq k = \delta(X)$. On the other hand $A^{-1} = \frac{1}{\det A} \text{adj}(A) = \text{adj}(A)$ is a matrix with integer entries and determinant 1. Thus the same proof shows that $\delta(X) = \delta(A^{-1}AX) \geq \delta(AX)$. It follows that $\delta(AX) = \delta(X)$.

Assume now that $|\det(A)| \neq 1$ and pick a prime p such that $p \mid \det(A)$. Then $\det(A) \equiv 0 \pmod p$ so the columns $\mathbf{c}_1, \dots, \mathbf{c}_n$ of A are linearly dependent over \mathbb{F}_p . So there are integers x_1, \dots, x_n not all multiples of p such that each entry of $AX = x_1 \mathbf{c}_1 + \dots + x_n \mathbf{c}_n$ is a multiple of p . So for this X we have $\delta(X) \not\equiv \delta(AX) \pmod p$.

OC598. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function having the Darboux property. Prove that if f is injective on $\mathbb{R} \setminus \mathbb{Q}$, then f is continuous on \mathbb{R} .

Originally from the 2018 Romania Math Olympiad, 3rd Problem, Grade 11, Final Round.

We received 6 correct submissions. We present the solution by UCLan Cyprus Problem Solving Group.

We will first prove that f is monotonic, so assume for contradiction that this is not the case. Without loss of generality suppose that there are $x < y < z$ such that $f(y) > f(x), f(z)$. We may further assume that $f(z) \geq f(x)$. Since f has the Darboux property, it takes every value in $(f(z), f(y))$ in the interval (y, z) . Furthermore, all but countably many of them are taken by irrational numbers in (y, z) . Similarly, all but countably many values in $(f(z), f(y)) \subseteq (f(x), f(y))$ are taken by irrational numbers in (x, y) . Since $(f(z), f(y))$ is uncountable, it means that there are two distinct irrationals $w_1 \in (x, y)$ and $w_2 \in (y, z)$ such that $f(w_1) = f(w_2)$, a contradiction.

Since f is monotonic and has the Darboux property then it is well-known that it is continuous. (Monotonic functions have left and right limits at each point $a \in \mathbb{R}$ and the Darboux property guarantees that these are equal to $f(a)$.)

OC599. Given the five-element subsets A_1, A_2, \dots, A_k of the set $\{1, 2, \dots, 23\}$ such that for all $1 \leq i < j \leq k$ the set $A_i \cap A_j$ has at most three elements, prove that $k \leq 2018$.

Originally from the 2018 Poland Math Olympiad, 5th Problem, Second Round.

We received 2 correct solutions. We present the solution by Nihat Mammadli.

Assume that $k \geq 2000$. Then we have at least 2000 5-element subsets of $\{1, 2, \dots, 23\}$ such that no two have 4 common elements. If we choose any subset $\{x_1, x_2, x_3, x_4, x_5\}$ its 4-element subsets are unique and are not subsets of any other 5-element subset. The number of these 4-element subsets is $\binom{5}{4} = 5$. Since $k \geq 2000$, the number of all 4-element subsets is at least $5 \times 2000 = 10000$. But $\{1, 2, \dots, 23\}$ has $\binom{23}{4} = 8855$ 4-element subsets. So our assumption $k \geq 2000$ is wrong and $k < 2000 < 2018$.

OC600. Let k be a positive integer and let a_1, a_2, a_3, \dots be a sequence whose terms are in the set $\{0, 1, \dots, k\}$. Let

$$b_n = \sqrt[n]{a_1^n + a_2^n + \dots + a_n^n}$$

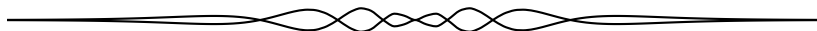
for all positive integers n . Prove that if in the sequence b_1, b_2, b_3, \dots there are infinitely many integer terms, then all the terms of the sequence are integers.

Originally from the 2018 Poland Math Olympiad, 6th Problem, Second Round.

We received 3 solutions, all of which were correct. We present the solution by Oliver Geupel.

The set $\{a_n : n \in \mathbb{N}\}$ is a subset of the finite set $\{0, 1, \dots, k\}$ and therefore finite, so it has a maximum, which we will denote by M . There is an index q such that $a_q = M$. For every index $n \geq q$, we have $M \leq b_n \leq \sqrt[n]{n} \cdot M$. Observing that $\sqrt[n]{n}$ converges to 1 for $n \rightarrow \infty$, we obtain $\lim_{n \rightarrow \infty} b_n = M$ by the squeeze theorem.

As a consequence, almost all terms of the sequence b_1, b_2, b_3, \dots lie in the open interval $(M - 1, M + 1)$. If there are infinitely many integer terms in the sequence, then it follows that infinitely many terms are equal to M . Hence, for every index $j \neq q$, there is an index $n > \max\{j, q\}$ such that $a_1^n + a_2^n + \dots + a_n^n = b_n^n = M^n$, which implies $a_j = 0$. Therefore, all terms of the sequence a_1, a_2, a_3, \dots , except for a_q , vanish. It follows $b_n = 0$ for $n < q$ and $b_n = M$ for $n \geq q$. Because M is an integer, all terms b_1, b_2, b_3, \dots are integers.



From the Lecture Notes of ...

Elyse Yeager

*In this feature of **CruX**, we share some of our favourite problems from first and second year undergraduate courses. These problems are a bit non-standard, elegant or unexpected. If you have a problem you would like to share (and it fits on one page), please send it along with its solution and a description of the course/audience it is intended for to cruX.eic@gmail.com.*



This month's column is brought to you by Elyse Yeager. Elyse is an Associate Professor of Teaching at the University of British Columbia. She is a co-author of the open-source CLP series of calculus textbooks, and organized a social-sciences-flavoured remix of the integral calculus textbook.

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Finding the demand for multiple goods, given their prices, is a satisfying and (reasonably) intuitive example of constrained optimization. It involves a thoughtful interpretation of which function is the constraint and which the objective, and it provides an opportunity to understand first partial derivatives in real-world terms.

In addition to these “softer” model-interpretation skills, the utility function used below offers students practice with using parameters (if you leave C and α as they are) and algebraic manipulation of fractional powers. The optimization itself can be solved with the method of Lagrange multipliers, or with elimination of variables.

These examples are intended for students learning constrained optimization in multivariable calculus, with an interest in business or economics. They were developed by Bruno Belevan, Parham Hamidi, Nisha Malhotra, and Elyse Yeager for UBC's commerce-focused second-semester calculus course.

Set-Up

Consider two goods. Each has its own unit price, and the utility to the consumer depends on the amount consumed of each good. There are two competing desires: to achieve a high utility (i.e. be made happy by your consumption) and to pay a low price.

The **Marshallian demand** of two goods is the consumption of each that maximizes utility, subject to a fixed budget constraint. (Think about buying the tastiest

meal possible, with only the cash you happen to have in your pocket.) Swapping the constraint and the objective function gives us **Hicksian demand**, the consumption that minimizes cost subject to a fixed utility. (For example, buying the cheapest possible meal that will still make you feel full.) I particularly like this feature of the problem: swapping the objective and constraint functions gives you a different perspective on the same model.

Another nice feature is that we can investigate our answers further with partial derivatives. The derivatives of demand with respect to price are the **price effects**: how does the price of one good affect the optimal consumption of it and the others? Does it lead to substituting one good for another, or just adjusting the quantity of the good whose price has changed?

Marshallian demand depends on a budget constraint; the partial derivative with respect to that variable describes how changes in budget lead to changes in consumption. For a **normal good**, increasing budget leads to increasing consumption. For an **inferior good**, increasing budget leads to decreasing consumption. (For example, when you have more money for lunch, you might buy juice instead of soda. An increased budget increases your consumption of juice, and decreases your consumption of soda.)

Questions and Answers

Let X and Y be two goods, with unit prices p_x and p_y , respectively. The utility of buying x units of X and y units of Y is given by the utility function, $u(x, y)$.

We will use the utility function

$$u(x, y) = Cx^\alpha y^{1-\alpha},$$

where C is a positive constant and α is a constant in the interval $(0, 1)$. This has nice properties for a utility function: $u(0, y) = u(x, 0) = 0$ gives a minimum, so the optimal consumption does involve some of each good. Its first partial derivatives are positive (“more is better”) and its second partial derivatives are negative (“diminishing returns”).

1. Find the amounts x^M of good X and y^M of good Y that maximize the utility function subject to the constraint $p_x x + p_y y = I$. That is: find the Marshallian demand functions $x^M(p_x, p_y, I)$ and $y^M(p_x, p_y, I)$. (The superscript M stands for “Marshallian,” and does not denote an exponent. I am assured that this is standard notation in economics.)

Answer. (We can solve this both with elimination of variables and with the method of Lagrange multipliers.)

$$x^M(p_x, p_y, I) = \frac{\alpha}{p_x} I \quad \text{and} \quad y^M(p_x, p_y, I) = \frac{1-\alpha}{p_y} I$$

2. When the budget I increases, does the Marshallian demand of X resp. Y increase or decrease?

Answer. As the budget increases, the demand of both goods also increases since

$$\frac{\partial}{\partial I}x^M = \frac{\alpha}{p_x} > 0 \quad \text{and} \quad \frac{\partial}{\partial I}y^M = \frac{1-\alpha}{p_y} > 0.$$

That is, both X and Y are **normal goods**.

3. Find the amounts x^H of good X and y^H of good Y that minimize the cost $p_x x + p_y y$ subject to the constraint $u(x, y) = U$ for some constant U . That is: find the Hicksian demand functions $x^H(p_x, p_y, U)$ and $y^H(p_x, p_y, U)$.

Answer.

$$x^H(p_x, p_y, U) = \frac{U}{C} \left(\frac{p_y \alpha}{p_x (1-\alpha)} \right)^{1-\alpha}, \quad y^H(p_x, p_y, U) = \frac{U}{C} \left(\frac{p_x (1-\alpha)}{p_y \alpha} \right)^{\alpha}$$

4. What happens to the demand of goods X and Y when the price of one good changes? Does the answer change between Marshallian and Hicksian demand?

Answer. To think about the effects of price on demand, we find the signs of our first partial derivatives with respect to p_x and p_y .

$$\begin{array}{cccc} \frac{\partial}{\partial p_x}x^M < 0 & \frac{\partial}{\partial p_y}x^M = 0 & \frac{\partial}{\partial p_x}y^M = 0 & \frac{\partial}{\partial p_y}y^M < 0 \\ \frac{\partial}{\partial p_x}x^H < 0 & \frac{\partial}{\partial p_y}x^H > 0 & \frac{\partial}{\partial p_x}y^H > 0 & \frac{\partial}{\partial p_y}y^H < 0 \end{array}$$

If our goal is to spend as little money as possible (Hicksian demand), then when the price of a good drops, we will demand more of that good and less of the other. If our goal is to be as happy as possible within our set budget (Marshallian demand), then a price drop of one good will cause us to demand more of that good, but it won't change our demand for the other.

Classroom Experience

Last Spring semester (January - April 2022) was the first time I included Marshallian and Hicksian demand in my course. Coming up with scenarios to differentiate the two was quite fun for me and (I think) my students. (For example, if you want to put together the most fabulous outfit that you can afford with the gift card you got for your birthday, is that Marshallian or Hicksian demand?) Drilling into the models in real-world terms with price effects is also quite satisfying, because the abstract functions that come out of long computations suddenly have readily understandable implications.

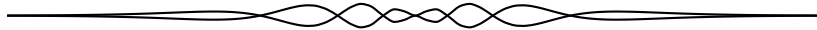
Many students exhibited difficulty working with fractional powers, even when I substituted a number for α . Keeping the proper-noun vocabulary straight (i.e. which one is Marshallian and which one is Hicksian) was predictably difficult, but

the vocabulary wasn't a priority in my learning goals, so I usually coupled the names with an explicit description of what was being optimized subject to what.

There is some friction between the language used in economics and mathematics. I mentioned that the superscripts M and H look like exponents, but are actually just short for Marshallian resp. Hicksian. To further confuse things, the variable I in Marshallian demand is short for "budget." In economics it's apparently often called "income," with the assumption that the entire income is spent on goods X and Y .

References

[1] Belevan, Hamidi, Malhotra, Yeager, *Optimal, Integral Likely*, chapter 2.6, pp. 87–94, 2021.



FOCUS ON...

No. 55

Michel Bataille

Summing an infinite series (II)

Introduction

In this second number on the infinite series, we continue to consider various methods of summation. We will present problems whose solutions resort to tools from calculus (integrals, differential equations), often combined with familiar results about power series.

With differential equations

Suppose that the series to be evaluated naturally writes as $S = \sum a_n x_0^n$, the value at x_0 of the power series $f(x) = \sum a_n x^n$. An explicit expression of $f(x)$ will obviously provide more than wanted! In the two examples that follow, such an expression is determined *via* a differential equation. First, we offer a variant of solution to problem **3920** [2014 : 76 ; 2015 : 88]:

Evaluate

$$\sum_{n=0}^{\infty} \frac{16n^2 + 20n + 7}{(4n + 2)!}.$$

Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}$ (the series converges for all real numbers x by the ratio test). Differentiating this power series term by term three times leads to

$$f'(x) = \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!}, \quad f''(x) = \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!}, \quad f'''(x) = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!}$$

so that $f(x) + f'(x) + f''(x) + f'''(x) = e^x$. Thus, f is the maximal solution of the differential equation $y''' + y'' + y' + y = e^x$ satisfying the initial conditions $y(0) = 1, y'(0) = y''(0) = 0$. The classical method leads to the general solution

$$x \mapsto \frac{e^x}{4} + Ae^{-x} + Be^{ix} + Ce^{-ix}$$

(where A, B, C are complex constants). The initial conditions then give $A = B = C = \frac{1}{4}$ and we conclude that

$$f(x) = \frac{1}{4}(e^x + e^{-x} + e^{ix} + e^{-ix}) = \frac{1}{2}(\cosh(x) + \cos(x)).$$

Now, since $16n^2 + 20n + 7 = (4n + 2)(4n + 1) + 2(4n + 2) + 1$, we see that

$$\sum_{n=0}^{\infty} \frac{16n^2 + 20n + 7}{(4n + 2)!} = f(1) + 2f'''(1) + f''(1) = \cosh(1) + \sinh(1) + \sin(1) = e + \sin(1).$$

Our second example goes back to 2001 with **2622** [2001 : 139 ; 2002 : 187]:

Find the exact value of

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{(2n+1)\binom{2n}{n}}.$$

It is readily checked that $(2n+1)\binom{2n}{n} = (n+1)\binom{2n+1}{n+1}$ so we can write the proposed series as $f(2)$ where

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\binom{2n+1}{n+1}}.$$

(With the help of the ratio test, the reader will easily verify that the radius of convergence of this power series is 4.)

Of course the factor $\frac{x^{n+1}}{n+1}$ prompts us to differentiate:

$$f'(x) = \sum_{n=0}^{\infty} \frac{x^n}{\binom{2n+1}{n+1}}$$

and after possibly some trials and errors, we obtain that for $x \in (0, 4)$,

$$(x-4)f'(x) + \frac{2f(x)}{x} = -2.$$

Therefore, f is a solution on $(0, 4)$ of the linear differential equation $x(x-4)y' + 2y = -2x$.

The general solution is classically determined as

$$y(x) = \sqrt{\frac{x}{4-x}} \left(\lambda - 2 \arcsin \left(1 - \frac{x}{2} \right) \right)$$

for some constant λ . Since $\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = 1$, the function f is given by

$$f(x) = \sqrt{\frac{x}{4-x}} \left(\pi - 2 \arcsin \left(1 - \frac{x}{2} \right) \right)$$

and the desired result is $f(2) = \pi$.

With integrals

Integrals can intervene in various ways. In our first example, we reverse the order of the preceding example, integrating instead of differentiating. The problem was proposed by *Mathematics Magazine* in 2021:

Evaluate

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{4^{2n}(2n+1)(2n+2)}.$$

We introduce the power series

$$S(x) = \sum_{n=0}^{\infty} \binom{4n}{2n} \frac{x^{2n}}{(2n+1)(2n+2)}.$$

We leave as an exercise for the reader to prove that

$$\binom{4n}{2n} \frac{1}{4^{2n}(2n+1)(2n+2)} \sim \frac{1}{4\sqrt{2\pi}} \cdot \frac{1}{n^{5/2}}$$

as $n \rightarrow \infty$ and that the series converges when $x \in (-\frac{1}{4}, \frac{1}{4}]$. From Abel's theorem, the required sum is just $S(\frac{1}{4}) = \lim_{x \rightarrow 1/4} S(x)$.

From the binomial theorem and the fact that $(-4)^n \binom{-1/2}{n} = \binom{2n}{n}$, we deduce that $\sum_{n=0}^{\infty} \binom{2n}{n} x^n = (1-4x)^{-1/2}$ for $x \in (-\frac{1}{4}, \frac{1}{4})$ and so

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{n+1}}{n+1} = \int_0^x \frac{dt}{\sqrt{1-4t}} = \frac{1 - \sqrt{1-4x}}{2}.$$

As a result, for $x \in [0, \frac{1}{4})$, we have

$$\frac{1}{2} \left(\frac{1 - \sqrt{1-4x}}{2} - \frac{1 - \sqrt{1+4x}}{2} \right) = \sum_{n=0}^{\infty} \binom{4n}{2n} \frac{x^{2n+1}}{2n+1}$$

and by integration,

$$\sum_{n=0}^{\infty} \binom{4n}{2n} \frac{x^{2n+2}}{(2n+1)(2n+2)} = \frac{1}{4} \int_0^x (\sqrt{1+4t} - \sqrt{1-4t}) dt = \frac{(1+4x)^{3/2} + (1-4x)^{3/2} - 2}{24}.$$

We readily deduce that

$$\sum_{n=0}^{\infty} \binom{4n}{2n} \frac{1}{4^{2n}(2n+1)(2n+2)} = \frac{4(\sqrt{2}-1)}{3}.$$

In our second example, problem **4687** [2021 : 450 ; 2022 : 237], an integral provides a simple expression for the partial sum of the series.

Calculate

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} - \ln 2 + \frac{1}{4n} \right).$$

As a preliminary exercise, the reader will prove that

$$\sum_{k=1}^n \frac{1}{n+k} = H_{2n} - H_n = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \ln 2 - \int_0^1 \frac{x^{2n}}{1+x} dx. \quad (1)$$

Let

$$u_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} - \ln 2 + \frac{1}{4n}.$$

From (1), it follows that $u_n = \frac{1}{4n} - \int_0^1 \frac{x^{2n}}{1+x} dx$. Therefore, if N is an integer with $N > 1$, we have

$$\sum_{n=1}^N u_n = \frac{H_N}{4} - \int_0^1 \frac{x^2(1+x^2+\cdots+x^{2N-2})}{1+x} dx = \frac{H_N}{4} - \int_0^1 \frac{x^2(1-x^{2N})}{(1-x)(1+x)^2} dx.$$

Since $\frac{x^2}{(1+x)^2(1-x)} = \frac{1}{4} \left(\frac{1}{1-x} + \frac{2}{(1+x)^2} - \frac{3}{1+x} \right)$, we obtain

$$\begin{aligned} & \int_0^1 \frac{x^2(1-x^{2N})}{(1+x)^2(1-x)} dx \\ &= \frac{1}{4} \int_0^1 (1+x+x^2+\cdots+x^{2N-1}) dx + \frac{1}{2} \int_0^1 \frac{1-x^{2N}}{(1+x)^2} dx - \frac{3}{4} \int_0^1 \frac{1-x^{2N}}{1+x} dx \\ &= \frac{H_{2N}}{4} - \frac{1}{2} \left[\frac{1}{1+x} \right]_0^1 - \frac{1}{2} \int_0^1 \frac{x^{2N}}{(1+x)^2} dx - \frac{3}{4} [\ln(1+x)]_0^1 + \frac{3}{4} \int_0^1 \frac{x^{2N}}{1+x} dx. \end{aligned}$$

Since

$$\lim_{N \rightarrow \infty} \int_0^1 \frac{x^{2N}}{(1+x)^2} dx = \lim_{N \rightarrow \infty} \int_0^1 \frac{x^{2N}}{1+x} dx = 0$$

and $\lim_{N \rightarrow \infty} (H_{2N} - H_N) = \ln 2$, it follows that

$$\sum_{n=1}^{\infty} u_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N u_n = \frac{-\ln 2}{4} - \frac{1}{4} + \frac{3 \ln 2}{4} = \frac{(\ln 4) - 1}{4}.$$

One of the sums in our next example (problem **4534** [2020 : 176 ; 2020 : 469]) can be expressed as an integral. We propose a variant of solution that takes advantage of this.

For $n \in \mathbb{N}$, evaluate

$$\frac{\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k+1)!}}{\sum_{k=0}^{\infty} \frac{1}{k!(n+k+1)}}.$$

Let $U_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k+1)!}$ and $V_n = \sum_{k=0}^{\infty} \frac{1}{k!(n+k+1)}$. The key remark is

$$V_n = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \int_0^1 x^{n+k} dx = \int_0^1 x^n \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) dx = \int_0^1 x^n \cdot e^x dx.$$

Now, an integration by parts gives

$$V_n = [x^n e^x]_0^1 - n \int_0^1 x^{n-1} e^x dx = e - n \cdot V_{n-1}$$

so that $\frac{V_n}{n!} + \frac{V_{n-1}}{(n-1)!} = \frac{e}{n!}$. It follows that

$$\frac{V_n}{n!} + (-1)^{n-1}V_0 = \sum_{k=1}^n (-1)^{n-k} \left(\frac{V_k}{k!} + \frac{V_{k-1}}{(k-1)!} \right) = e \cdot \sum_{k=1}^n \frac{(-1)^{n-k}}{k!}$$

and using $V_0 = e - 1$, a short calculation yields

$$V_n = (-1)^n \cdot en! \left(\sum_{k=0}^n \frac{(-1)^k}{k!} - \frac{1}{e} \right) = (-1)^{n+1} \cdot en! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} = en! \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k+1)!},$$

that is, $V_n = en! \cdot U_n$ and the required ratio is $\frac{1}{en!}$.

Thanks to integrals, some well-oiled cogs seem in action in the solution we offer to problem 5636 proposed in 2021 in *School Science and Mathematics Association*:

Prove that

$$\sum_{n=0}^{\infty} n! \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} \right)^2 = e \sum_{n=1}^{\infty} \frac{1}{n \cdot n!}.$$

From the integral form of the remainder in Taylor's formula, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \int_0^1 \frac{(1-t)^n}{n!} \cdot e^t dt = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + e \int_0^1 \frac{u^n e^{-u}}{n!} du.$$

This remark leads to the desired result as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} n! \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} \right)^2 &= e^2 \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_0^1 u^n e^{-u} du \right)^2 \\ &= e^2 \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \int_0^1 (uv)^n e^{-(u+v)} du dv \\ &= e^2 \int_0^1 \int_0^1 \left(e^{-(u+v)} \sum_{n=0}^{\infty} \frac{(uv)^n}{n!} \right) du dv \\ &= e^2 \int_0^1 \int_0^1 (e^{-(u+v)} \cdot e^{uv}) du dv \\ &= e \int_0^1 \int_0^1 e^{(1-u)(1-v)} du dv \\ &= e \sum_{n=0}^{\infty} \left(\frac{1}{n!} \int_0^1 \int_0^1 (1-u)^n (1-v)^n du dv \right) \\ &= e \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_0^1 (1-u)^n du \right)^2 = e \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{n+1} \right)^2 \\ &= e \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+1)!} = e \sum_{n=1}^{\infty} \frac{1}{n \cdot n!}. \end{aligned}$$

To conclude, we show how to prove again Euler's result $\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \zeta(3)$ in the spirit of this number. We will make appeal to some frequently met integrals, easily calculated by induction using integration by parts: if m, k are integers such that $k \geq 0$ and $m \geq 1$, then

$$\int_0^1 x^k (\ln x)^m dx = \frac{(-1)^m m!}{(k+1)^{m+1}}.$$

We also need the following result about power series: for $x \in [0, 1)$, we have $[\ln(1-x)]^2 = 2 \sum_{n=1}^{\infty} \frac{H_n}{n+1} x^{n+1}$.

Indeed, for $x \in [0, 1)$, let $f(x) = \sum_{n=1}^{\infty} \frac{H_n}{n+1} x^{n+1}$. Then $f'(x) = \sum_{n=1}^{\infty} H_n x^n$, hence

$$(1-x)f'(x) = x + \sum_{n=2}^{\infty} (H_n - H_{n-1})x^n = x + \sum_{n=2}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

We first deduce that $f'(x) = \left(\frac{-1}{1-x}\right) \ln(1-x)$ and then conclude that since $f(0) = 0$, we have $f(x) = \frac{1}{2} [\ln(1-x)]^2$.

From this, we derive $\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} x^{n+1} = \frac{1}{2} \int_0^x \frac{(\ln(1-t))^2}{t} dt$ and the desired sum now follows from

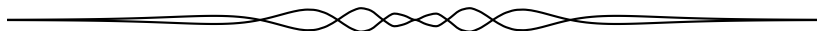
$$\begin{aligned} 2 \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} &= \int_0^1 \frac{(\ln(1-t))^2}{t} dt = \int_0^1 \frac{(\ln t)^2}{1-t} dt \\ &= \int_0^1 \left(\sum_{n=0}^{\infty} t^n (\ln t)^2 \right) dt = \sum_{n=0}^{\infty} \int_0^1 t^n (\ln t)^2 dt = \sum_{n=0}^{\infty} \frac{2}{(n+1)^3}. \end{aligned}$$

Exercises

1. (From *Bulletin de l'APMEP* in 2013) Calculate $\sum_{n=0}^{+\infty} \frac{n!}{1 \times 3 \times 5 \times \dots \times (2n+1)}$ via a differential equation satisfied by $f(x) = \sum_{n=0}^{+\infty} \frac{n!}{1 \times 3 \times 5 \times \dots \times (2n+1)} \cdot x^{2n+1}$.
2. (Problem 1195 of *The College Mathematics Journal*) Prove the following:

$$\sum_{k=1}^{\infty} \frac{H_k}{k+1} \left(\frac{\pi^2}{6} - H_{k+1,2} \right) = \frac{\pi^4}{90},$$

where $H_k = \sum_{i=1}^k \frac{1}{i}$ is the k th harmonic number and $H_{k,2} = \sum_{i=1}^k \frac{1}{i^2}$ is the k th generalized harmonic number.



PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **May 15, 2023**.



4821. *Proposed by Corneliu Manescu-Avram.*

Let k be a positive integer, $n = 2^k + 1$ and let N be the number of ordered solutions in n -tuples of positive integers to the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1.$$

Prove that $N - k$ is odd.

4822. *Proposed by Anton Mosunov.*

The n -th Chebyshev polynomial of the first kind is defined by means of the recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad \text{for } n \geq 2.$$

Prove that for all $n \geq 2$,

$$\frac{1}{3} < \int_1^{+\infty} \frac{dx}{T_n(x)^{2/n}} < \frac{1}{3} \sqrt[n]{4}.$$

4823* *Proposed by Michael Friday, modified by the editorial board.*

Given four points A, B, C, D on a circle, define the *Simson segment of A with respect to the triangle BCD* to be the smallest line segment containing the feet of all three perpendiculars dropped from A to the sides of the triangle. For any four points on a circle, prove that the Simson segments determined by each point with respect to the triangle formed by the other three all have the same length.

4824. *Proposed by George-Florin Şerban.*

Find all prime numbers p for which there are integers x and y that satisfy the conditions $11p = 8x^2 + 23$ and $p^2 = 2y^2 + 23$.

4825. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Let $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$, $n \geq 1$. Calculate

$$\sum_{n=1}^{\infty} \frac{O_n}{n(n+1)}.$$

4826. *Proposed by Paul Bracken.*

Let H_n is the n -th harmonic number $H_n = \sum_{k=1}^n 1/k$. Evaluate the following sum in closed form

$$S = \sum_{k=1}^{\infty} \frac{H_k}{k(k+1)(k+2)}.$$

4827. *Proposed by Michel Bataille.*

In the plane, two circles Γ_1 and Γ_2 , with respective centres O_1 and O_2 , intersect at A and B . Let X be a point of Γ_1 with $X \neq A, B$. The lines XA and XB intersect Γ_2 again at Y and Z , respectively. Prove that

$$YZ = \frac{AB \cdot O_1 O_2}{O_1 A}.$$

4828. *Proposed by Narendra Bhandari.*

Prove

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{\sec(x+y) \sec(x-y)}{\sec x \sec y} dx dy = \sum_{n=0}^{\infty} (-1)^n / (2n+1)^2.$$

4829. *Proposed by George Apostolopoulos.*

Let ABC be a triangle and K, L, M be interior points on the sides BC, CA, AB , respectively. Let $[XYZ]$ denote the area of a triangle XYZ .

a) Find the maximum value of the expression

$$\sqrt{\frac{[ALM]}{[ABC]}} + \sqrt{\frac{[BMK]}{[ABC]}} + \sqrt{\frac{[CKL]}{[ABC]}}.$$

b) Find the minimum value of the expression

$$\frac{[KLM]}{[ALM]} + \frac{[KLM]}{[BMK]} + \frac{[KLM]}{[CKL]}.$$

4830. *Proposed by Goran Conar.*

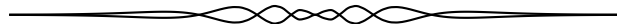
Let $a_i \in (0, \frac{1}{2})$, $i \in \{1, 2, \dots, n\}$ be real numbers such that $\sum_{i=1}^n a_i = 1$. Prove that the following inequalities hold:

$$n\sqrt{\frac{n-1}{n+1}} \leq \sum_{i=1}^n \sqrt{\frac{1-a_i}{1+a_i}} < (n+1)\sqrt{\frac{n-1}{n+1}}.$$

.....

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2023**.



4821. *Soumis par Corneliu Manescu-Avram.*

Soient k un entier positif, $n = 2^k + 1$ et N le nombre de solutions positifs ordonnées à l'équation

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = 1.$$

Démontrer que $N - k$ est impair.

4822. *Soumis par Anton Mosunov.*

Le n -ième polynôme de Tchebychev de la première sorte est défini par la récurrence

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad \text{pour } n \geq 2.$$

Démontrer que pour tout $n \geq 2$,

$$\frac{1}{3} < \int_1^{+\infty} \frac{dx}{T_n(x)^{2/n}} < \frac{1}{3} \sqrt[n]{4}.$$

4823* *Soumis par Michael Friday, modifié par le comité de rédaction.*

Pour quatre points donnés A, B, C, D sur un cercle, on définit le segment de *Simson* de A par rapport au triangle BCD comme étant le plus court segment incluant les pieds des perpendiculaires de A vers les trois côtés du triangle. Démontrer,

pour quatre points sur un cercle donné, que les segments de Simson, pour chacun d'entre eux par rapport aux trois autres, sont de même longueur.

4824. *Soumis par George-Florin Şerban.*

Déterminer tous les nombres premiers p pour lesquels il existe des entiers x et y répondant aux contraintes $11p = 8x^2 + 23$ et $p^2 = 2y^2 + 23$.

4825. *Soumis par Ovidiu Furdui et Alina Şintămărian.*

Soit $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$, $n \geq 1$. Calculer

$$\sum_{n=1}^{\infty} \frac{O_n}{n(n+1)}.$$

4826. *Soumis par Paul Bracken.*

Le n -ième nombre harmonique est défini par $H_n = \sum_{k=1}^n 1/k$. Évaluer la somme suivante, en forme close,

$$S = \sum_{k=1}^{\infty} \frac{H_k}{k(k+1)(k+2)}.$$

4827. *Soumis par Michel Bataille.*

Dans le plan, les cercles Γ_1 et Γ_2 , de centres O_1 et O_2 , se rencontrent en A et B . Soit alors X un point sur Γ_1 tel que $X \neq A, B$. Les lignes XA et XB rencontrent Γ_2 de nouveau en Y et Z respectivement. Démontrer que

$$YZ = \frac{AB \cdot O_1 O_2}{O_1 A}.$$

4828. *Soumis par Narendra Bhandari.*

Démontrer que

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{\sec(x+y) \sec(x-y)}{\sec x \sec y} dx dy = \sum_{n=0}^{\infty} (-1)^n / (2n+1)^2.$$

4829. *Soumis par George Apostolopoulos.*

Soit ABC un triangle et soient K, L, M des points intérieurs des côtés BC, CA, AB respectivement. La surface du triangle XYZ est dénotée $[XYZ]$.

a) Déterminer la valeur maximale de l'expression

$$\sqrt{\frac{[ALM]}{[ABC]}} + \sqrt{\frac{[BMK]}{[ABC]}} + \sqrt{\frac{[CKL]}{[ABC]}}.$$

b) Déterminer la valeur minimale de l'expression

$$\frac{[KLM]}{[ALM]} + \frac{[KLM]}{[BMK]} + \frac{[KLM]}{[CKL]}.$$

4830. *Soumis par Goran Conar.*

Soient $a_i \in (0, \frac{1}{2})$, $i \in \{1, 2, \dots, n\}$ des nombres reels tels que $\sum_{i=1}^n a_i = 1$.
Démontrer que les inégalités suivantes tiennent

$$n\sqrt{\frac{n-1}{n+1}} \leq \sum_{i=1}^n \sqrt{\frac{1-a_i}{1+a_i}} < (n+1)\sqrt{\frac{n-1}{n+1}}.$$



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2022: 48(8), p. 483–486.

4771. Proposed by Michel Bataille.

Let I be an open interval containing 0 and 1 and let $f : I \rightarrow \mathbb{R}$ be a differentiable, strictly increasing, convex function. If $f'(1) < 2f(1)$, prove that there exist positive real numbers a, b such that

$$\int_0^1 (f(x))^{2n+1} dx \sim a \cdot \frac{b^n}{n} \quad \text{as } n \rightarrow \infty$$

and express a and b as a function of $f(1)$ and $f'(1)$.

We received 7 solutions, all of which were correct. We present the solution by Theo Koupelis.

The function $f(x)$ is strictly increasing and convex, and thus $f'(x) > 0$, $f''(x) > 0$, and $f(1) > f'(1)/2 > 0$. Also, it has at most one zero in $[0, 1]$; if $f(x_0) = 0$, with $x_0 \in [0, 1]$, then $f(x) \leq 0$ in $[0, x_0]$, and $f(x) > 0$ in $(x_0, 1]$. We have

$$I := \lim_{n \rightarrow \infty} \int_0^1 (f(x))^{2n+1} dx = \lim_{n \rightarrow \infty} \int_0^{x_0} (f(x))^{2n+1} dx + \lim_{n \rightarrow \infty} \int_{x_0}^1 (f(x))^{2n+1} dx.$$

Expanding $f(x)$ around x_0 and 1 we get

$$\begin{aligned} f(x) &= f(x_0) - (x_0 - x) \frac{f'(x_0)}{1!} + (x_0 - x)^2 \frac{f''(x_0)}{2!} - \dots \\ &= -(x_0 - x) \cdot f'(x_0) \left[1 - (x_0 - x) \cdot \frac{f''(x_0)}{2!f'(x_0)} + \dots \right], \end{aligned}$$

and

$$f(x) = f(1) \left[1 - (1 - x) \cdot \frac{f'(1)}{f(1)} \left(1 - \frac{1 - x}{2!} \cdot \frac{f''(1)}{f'(1)} + \dots \right) \right],$$

and thus

$$\begin{aligned} I &\sim (-1)^{2n+1} \lim_{n \rightarrow \infty} \int_0^{x_0} e^{(2n+1) \ln[(x_0-x) \cdot f'(x_0)]} dx + \lim_{n \rightarrow \infty} \int_{x_0}^1 e^{(2n+1) \ln[f(1)(1-(1-x) \cdot \frac{f'(1)}{f(1)})]} dx \\ &\sim - \lim_{n \rightarrow \infty} [f'(x_0)]^{2n+1} \int_0^{x_0} (x_0 - x)^{2n+1} dx + \lim_{n \rightarrow \infty} [f(1)]^{2n+1} \int_{x_0}^1 e^{-(2n+1) \frac{f'(1)}{f(1)} (1-x)} dx \\ &\sim - \lim_{n \rightarrow \infty} [f'(x_0)]^{2n+1} \int_{x_0}^0 y^{2n+1} (-dy) + \lim_{n \rightarrow \infty} [f(1)]^{2n+1} \int_{1-x_0}^0 e^{-(2n+1) \frac{f'(1)}{f(1)} y} (-dy) \\ &\sim - \lim_{n \rightarrow \infty} [f'(x_0)]^{2n+1} \cdot \frac{x_0^{2n+2}}{2n+2} + \lim_{n \rightarrow \infty} [f(1)]^{2n+1} \cdot \frac{f(1)}{(2n+1)f'(1)} \left[1 - e^{-(2n+1) \frac{f'(1)}{f(1)} (1-x_0)} \right]. \end{aligned}$$

But $x_0 \in [0, 1)$ and thus

$$I \sim \lim_{n \rightarrow \infty} \frac{[f(1)]^{2n+2}}{(2n+1)f'(1)} \sim a \cdot \frac{b^n}{n},$$

where $a = \frac{(f(1))^2}{2f'(1)}$ and $b = (f(1))^2$. If $f(x)$ has no root in $[0, 1)$, the above analysis still holds true by setting $x_0 = 0$.

4772. *Proposed by Mihaela Berindeanu.*

Find all functions $f : (0, \infty) \rightarrow (0, \infty)$ such that $f(kx + f(y)) = \frac{y}{k} \cdot f(xy + 1)$ for all $x, y \in (0, \infty)$, where $k > 0$ is a real and fixed parameter.

We received 6 submissions and they were all complete and correct. We present a solution by the majority of solvers, slightly modified by the editor.

It is easy to verify that $f(x) = \frac{k}{x}$ is a solution to the given functional equation. We show that this is the only solution.

Suppose there exist $x, y > 0$ such that $kx + f(y) = xy + 1$, then the functional equation implies that $y = k$. Therefore, if $y \neq k$, then $x = \frac{f(y)-1}{y-k}$ (so that $kx + f(y) = xy + 1$) must be non-positive. In other words, if $y > k$, then $f(y) \leq 1$; if $y < k$, then $f(y) \geq 1$.

Let $y > 1$. Set $x = \frac{y-1}{y}$ so that $xy + 1 = y$. Then the functional equation becomes

$$f\left(k - \frac{k}{y} + f(y)\right) = \frac{y}{k} f(y).$$

We claim that $f(y) = \frac{k}{y}$. Indeed, if $f(y) > \frac{k}{y}$, then $k - \frac{k}{y} + f(y) > k$ and thus we have

$$1 \geq f\left(k - \frac{k}{y} + f(y)\right) = \frac{y}{k} f(y) > 1,$$

a contradiction. Similarly, we can deduce that $f(y) < \frac{k}{y}$ is impossible.

We have shown that $f(y) = \frac{k}{y}$ whenever $y > 1$. Now let y be an arbitrary positive number. Setting $x = \frac{1}{k}$ in the functional equation so that $kx + f(y) > 1$ and $xy + 1 > 1$, we obtain that

$$\frac{k}{1 + f(y)} = f(1 + f(y)) = \frac{y}{k} f\left(\frac{y}{k} + 1\right) = \frac{y}{k} \cdot \frac{k}{\frac{y}{k} + 1} = \frac{ky}{y + k}.$$

Solving the above equation, we conclude that $f(y) = \frac{k}{y}$. This finishes the proof.

Editor's Comment. Several solvers pointed out that this problem has appeared in the Individual Competition of the 2012 Middle European Mathematical Olympiad (MEMO) for the special case $k = 1$, and the solution for the case $k = 1$ extends to all positive k naturally.

4773. *Proposed by George Stoica.*

Suppose that x_1, x_2, \dots, x_n , where $n \geq 3$, are nonnegative real numbers such that $x_1 + x_2 + \dots + x_n = 2$ and $x_1x_2 + x_2x_3 + \dots + x_{n-1}x_n = 1$. Prove that at most three of the x_i 's are non-zero, and at least one of the x_i 's equals 1.

We received 12 correct solutions. The strategy of the solution below was followed by five solvers. Three proved the result by induction.

Let $u = \sum\{x_i : 1 \leq i \leq n, i \text{ odd}\}$ and $v = \sum\{x_i : 1 \leq i \leq n, i \text{ even}\}$. Then $u + v = 2$. Since

$$4 \geq (u + v)^2 - (u - v)^2 = 4uv \geq 4 \sum_{i=1}^{n-1} x_i x_{i+1} = 4,$$

equality holds throughout, so that $u = v = 1$, $uv = \sum_{i=1}^{n-1} x_i x_{i+1}$ and $x_i x_j = 0$ when i and j have opposite parity and $|i - j| \geq 3$.

Suppose that k is the smallest index for which $x_k \neq 0$. Then $x_{k+1+2i} = 0$ for $j \leq -1$ and $j \geq 1$. It follows that $x_i = 0$ whenever i and $k + 1$ are distinct indices with the same parity (so that the sum u or v containing x_{k+1} has a single nonzero term). Thus $x_{k+1} = 1$, $x_{k+2i} = x_{k+1+(2i-1)} = 0$ for $i \geq 2$, $x_k = x$ and $x_{k+2} = 1 - x$ where $0 < x \leq 1$. The result follows.

4774. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with inradius r . Let D, E, F be chosen on the sides BC, CA, AB respectively, so that AD, BE and CF bisect the angles of ABC . Prove that

$$FE^2 + ED^2 + DF^2 \geq 9r^2.$$

All but one of the 15 submissions were correct; we feature a composite of the almost identical solutions submitted independently by Mohamed Amine Ben Ajiba, by Nandan Sai Dasireddy, and by the proposer.

Let $a = BC, b = CA, c = AB$ be the side lengths of triangle ABC , and let Δ, s, R be its area, semiperimeter, and circumradius. We will use square brackets to denote areas. Our goal is to apply Weitzenböck's inequality to triangle DEF , which tells us that

$$FE^2 + ED^2 + DF^2 \geq 4\sqrt{3} \cdot [DEF]; \quad (1)$$

see, for example, O. Bottema et al., *Geometric Inequalities*, formula (4.4), page 42. Because E and F are the feet of the bisectors of the angles at B and C , we have

$$AE = \frac{bc}{c+a} \quad \text{and} \quad AF = \frac{bc}{a+b},$$

whence

$$[AFE] = \frac{1}{2} AE \cdot AF \sin A = \frac{bc \cdot \Delta}{(c+a)(a+b)}.$$

Similarly,

$$[BDF] = \frac{ca \cdot \Delta}{(a+b)(b+c)} \quad \text{and} \quad [CED] = \frac{ab \cdot \Delta}{(b+c)(c+a)}.$$

Thus,

$$[DEF] = \Delta - ([AFE] + [BDF] + [CED]) = \frac{2abc \cdot \Delta}{(a+b)(b+c)(c+a)}.$$

The AM-GM inequality implies that

$$(a+b)(b+c)(c+a) \leq \left(\frac{(a+b) + (b+c) + (c+a)}{3} \right)^3 = \frac{64s^3}{27},$$

while, with the help of the formulas $abc = 4R\Delta$, and $\Delta = rs$, Mitrinovic's inequality (namely $3\sqrt{3}R \geq 2s$) implies that

$$[DEF] \geq \frac{27}{64s^3} \cdot 8Rr^2s^2 = \frac{27Rr^2}{8s} \geq \frac{3\sqrt{3}}{4}r^2.$$

Plugging this last inequality into (1) gives us

$$FE^2 + ED^2 + DF^2 \geq 4\sqrt{3} \cdot [DEF] \geq 9r^2,$$

as desired. Note that equality holds if and only if triangle ABC is equilateral.

Editor's comments. About half the submissions were based on formulas for DE , EF , FD as in last month's featured solution to problem 4767 (and several earlier problems) combined with considerable algebraic manipulations. Walther Janous, with the help of his computer, proved the stronger estimate,

$$FE^2 + ED^2 + DF^2 \geq \frac{27Rr^2(181R + 214r)}{128s^2}.$$

4775. Proposed by H. A. ShahAli.

Suppose that A and B are positive numbers such that $A < B$, $A + B < \pi$ and let

$$f(x) = \sin(xA) \sin(A + xB) - \sin(xB) \sin(B + xA)$$

be defined on $0 < x < 1/2$. Prove that the graph of $y = f(x)$ is never tangent to the x -axis.

We received 8 solutions, of which five were correct, two were incomplete and one was incorrect. We present 2 solutions.

Solution 1, by Walther Janous and Didier Pinchon (done independently).

Let $P = B - A$ and $Q = B + A$. Note that $0 < P < Q < \pi$. Then, using the product of sines to difference of cosines formula, we find that

$$\begin{aligned} f(x) &= \frac{1}{2} \{ [\cos(Px + A) - \cos(Qx + A)] - [\cos(Px - B) - \cos(Qx + B)] \} \\ &= \frac{1}{2} \{ [\cos(Px + A) - \cos(Px - B)] - [\cos(Qx + B) - \cos(Qx + A)] \} \\ &= \sin \frac{Q}{2} \sin(P(\frac{1}{2} - x)) - \sin \frac{P}{2} \sin(Q(\frac{1}{2} + x)). \end{aligned}$$

Differentiating twice yields

$$f''(x) = -P^2 \sin \frac{Q}{2} (\sin(P(\frac{1}{2} - x))) + Q^2 \sin \frac{P}{2} (\sin(Q(\frac{1}{2} + x))).$$

We have that $f(0) = 0$, $f(\frac{1}{2}) < 0$ and $f''(x) > 0$ on $[0, \frac{1}{2}]$. To see this, note that $x^{-2} \sin x$ with derivative $x^{-3}(x - 2 \tan x) \cos x$ is decreasing, on $(0, \frac{\pi}{2})$ so that $P^2 \sin \frac{Q}{2} < Q^2 \sin \frac{P}{2}$. Therefore, the graph of $y = f(x)$ lies below the line joining $(0, 0) = (0, f(0))$ and $(\frac{1}{2}, f(\frac{1}{2}))$, i.e. $f(x) < 2f(\frac{1}{2})x < 0$ for $0 < x \leq \frac{1}{2}$.

Thus, there are no values of $f(x)$ for which $0 < x \leq \frac{1}{2}$ and $f(x) = 0$, so its graph never crosses the x -axis.

Solution 2, by Kai-Wai Lau.

With the notation of Solution 1, the tangency of the graph of $y = f(x)$ to the x -axis is equivalent to

$$f(x) = \sin(P(\frac{1}{2} - x)) \sin \frac{Q}{2} - \sin(Q(\frac{1}{2} + x)) \sin \frac{P}{2} = 0,$$

$$f'(x) = -P \cos(P(\frac{1}{2} - x)) \sin \frac{Q}{2} - Q \cos(Q(\frac{1}{2} + x)) \sin \frac{P}{2} = 0,$$

for some $x \in [0, \frac{1}{2}]$. Considering this as a linear system with variables $\sin \frac{Q}{2}$ and $\sin \frac{P}{2}$, we find, after converting the trigonometric products to sums, that its determinant is equal to

$$h(x) = A \sin(A + 2Bx) - B \sin(B + 2Ax).$$

Since $h(\frac{1}{2}) < 0$ and

$$h'(x) = 4AB \sin(Q(x + \frac{1}{2})) \sin P(\frac{1}{2} - x) > 0$$

on $(0, \frac{1}{2})$, it follows that $h(x) < 0$ on $[0, \frac{1}{2}]$ and the linear system has only the trivial solution. However, this is inconsistent with the fact that neither $\sin \frac{Q}{2}$ nor $\sin \frac{P}{2}$ is zero.

Therefore the graph is $y = f(x)$ is never tangent to the x -axis.

4776. *Proposed by Nguyen Viet Hung.*

For each positive integer n , find

$$\left\lfloor \frac{1}{\{\sqrt{n^2 + 3n + 4}\}} \right\rfloor$$

where $[a]$ and $\{a\}$ denote integer part and fractional part of a , respectively.

We received 32 correct solutions from 31 solvers. Nine of them provided the solution below.

Since $n + \frac{3}{2} < \sqrt{n^2 + 3n + 4} < n + 2$, then $[\sqrt{n^2 + 3n + 4}] = n + 1$ and $1/2 < \{\sqrt{n^2 + 3n + 4}\} < 1$. Therefore

$$1 < \frac{1}{\{\sqrt{n^2 + 3n + 4}\}} < 2$$

and the answer is 1.

Comment from the editor. The majority of solvers started from

$$n + 1 < \sqrt{n^2 + 3n + 4} < n + 2,$$

and then showed that

$$\sqrt{n^2 + 3n + 4} - (n + 1) = (n + 3)/(\sqrt{n^2 + 3n + 4} + n + 1)$$

lay between $(n + 3)/(2n + 3)$ and $(n + 3)/(2n + 2)$.

4777. Proposed by Goran Conar, modified by the Editorial Board.

Let $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \geq 1$ such that $\sum_{i=1}^n \frac{1}{x_i} = 1$. Prove

$$\frac{n}{1/2 + n^2} < \sum_{i=1}^n \frac{1}{\frac{1}{2} + x_i^2} < \frac{2}{3}.$$

We received 17 submissions, and in most of them, it is noticed that the upper bound is reached for $n = 1$ only (and $x_1 = 1$), and the lower bound is reached when $n \geq 1$ and $x_i = n$, $i = 1, \dots, n$. Overall 11 submissions were correct and complete, and we present two solutions.

Solution 1, by Brian Bradie.

For the case $n = 1$, $x_1 = 1$ and

$$\frac{1}{1/2 + 1^2} = \sum_{i=1}^1 \frac{1}{\frac{1}{2} + x_i^2} = \frac{2}{3},$$

that is,

$$\frac{n}{1/2 + n^2} = \sum_{i=1}^n \frac{1}{\frac{1}{2} + x_i^2} = \frac{2}{3}.$$

Suppose $n \geq 2$. For each $i = 1, 2, \dots, n$, let $y_i = \frac{1}{x_i}$. Then $0 < y_1, y_2, \dots, y_n \leq 1$, $\sum_{i=1}^n y_i = 1$, and the desired inequality becomes

$$\frac{n}{1/2 + n^2} < \sum_{i=1}^n \frac{2y_i^2}{y_i^2 + 2} < \frac{2}{3}.$$

Moreover, the condition that each y_i be positive combined with the condition that the sum of the y_i is equal to 1 implies that none of the y_i can be equal to 1, so $0 < y_1, y_2, \dots, y_n < 1$. Let

$$f(y) = \frac{2y^2}{y^2 + 2},$$

and start with the inequality on the right. For $0 < y < 1$, $f(y) < \frac{2}{3}y$, as this is equivalent to

$$0 < 2y(y-1)(y-2).$$

Thus,

$$\sum_{i=1}^n \frac{2y_i^2}{y_i^2 + 2} < \sum_{i=1}^n \frac{2}{3}y_i = \frac{2}{3} \sum_{i=1}^n y_i = \frac{2}{3}.$$

For the inequality on the left, note that equation of the line tangent to the graph of f at $y = \frac{1}{n}$ is

$$t\ell(y) = \frac{1}{1/2 + n^2} + \frac{2n^3}{(1/2 + n^2)^2} \left(y - \frac{1}{n} \right),$$

and

$$f(y) - t\ell(y) = 4 \frac{(ny - 1)^2(2n^2 - 2ny - 1)}{(1 + 2n^2)^2(2 + y^2)} \geq 0$$

for $0 < y < 1$ with equality for $y = 1/n$. Therefore,

$$\begin{aligned} \sum_{i=1}^n \frac{2y_i^2}{y_i^2 + 2} &\geq \sum_{i=1}^n \left(\frac{1}{1/2 + n^2} + \frac{2n^3}{(1/2 + n^2)^2} \left(y_i - \frac{1}{n} \right) \right) \\ &= \frac{n}{1/2 + n^2} + \frac{2n^3}{(1/2 + n^2)^2} \left(\sum_{i=1}^n y_i - 1 \right) \\ &= \frac{n}{1/2 + n^2}. \end{aligned}$$

Finally,

$$\frac{n}{1/2 + n^2} \leq \sum_{i=1}^n \frac{1}{\frac{1}{2} + x_i^2} \leq \frac{2}{3}.$$

Equality holds on the right only for the case $n = 1$, but holds on the left for any n when $x_i = n$ for each $i = 1, 2, \dots, n$.

Solution 2, by Oliver Geupel.

We fix the problem to show that

$$\frac{n}{\frac{1}{2} + n^2} \leq \sum_{i=1}^n \frac{1}{\frac{1}{2} + x_i^2} \leq \frac{2}{3},$$

with equality in the left relation if and only if $x_1 = x_2 = \dots = x_n = n$, whereas equality in the right part holds if and only if $n = 1$. With the substitution $y_i = 1/x_i$

we have $0 < y_i \leq 1$, $\sum_{i=1}^n y_i = 1$. Writing $f(x) = \frac{x^2}{2+x^2}$, the desired inequalities become

$$\frac{n}{2n^2+1} \leq \sum_{i=1}^n f(y_i) \leq \frac{1}{3}.$$

First, we prove the left inequality. If for some $k \in \{1, 2, \dots, n\}$ it holds $y_k^2 \geq 2/3$, then it follows that $n \geq 2$ and

$$\frac{n}{2n^2+1} \leq \frac{2}{9} < \frac{1}{4} \leq f(y_k) < \sum_{i=1}^n f(y_i),$$

and we are done. So assume $y_i^2 < 2/3$ for all i . Since the second derivative of the function $f(x)$ is $f''(x) = \frac{12}{(2+x^2)^3} (\frac{2}{3} - x^2) > 0$ for $0 < x < \sqrt{2/3}$, we see that f is strictly convex, and it follows by Jensen's inequality that

$$\frac{n}{2n^2+1} = nf\left(\frac{1}{n} \sum_{i=1}^n y_i\right) \leq \sum_{i=1}^n f(y_i).$$

Moreover, by the equality condition of Jensen's inequality, equality holds if and only if $y_1 = y_2 = \dots = y_n = 1/n$. This completes the proof of the left inequality.

It remains to prove the right inequality. For $n = 1$, equality holds by inspection. In the following suppose that $n \geq 2$. For $a, b \in (0, 1)$, we have

$$\begin{aligned} f(a+b) - f(a) - f(b) &= \frac{ab[8 - 4ab - a^3b - 2a^2b^2 - ab^3]}{(a^2+2)(b^2+2)((a+b)^2+2)} \\ &= \frac{ab[4(1-ab) + 4 - ab(a+b)^2]}{(a^2+2)(b^2+2)((a+b)^2+2)} > 0, \end{aligned}$$

which yields $f(a) + f(b) < f(a+b)$. By a straightforward induction we finally obtain

$$\sum_{i=1}^n f(y_i) < f\left(\sum_{i=1}^n y_i\right) = \frac{1}{3}.$$

The proof is complete.

4778. *Proposed by Adnan Sadik, modified by the Editorial Board.*

Let $f(x) = 2^{x^2}$. Prove that for any $n \in \mathbb{N} \setminus \{1\}$, there exists $a, b \in \mathbb{N} \setminus \{1\}$ such that

1. $\gcd(a, b) = 1$,
2. $n | \phi(a)$ and $n | \phi(b)$,
3. $a, b \leq f(n)$,

where Euler's totient function $\phi(n)$ is the number of positive integers less than n and coprime to n .

We received 6 submissions and they were all complete and correct. We present a solution by Samuel Figueredo and the proposer (independently).

We first recall a classical statement from elementary number theory: for any $a \in \mathbb{N} - \{1\}$, $n \in \mathbb{N} - \{1\}$, we have $n \mid \phi(a^n - 1)$. Indeed, the order of a modulo $a^n - 1$ is clearly n , and Euler's theorem states that

$$a^{\phi(a^n - 1)} \equiv 1 \pmod{a^n - 1}.$$

The basic property of the order immediately implies that $n \mid \phi(a^n - 1)$.

Now, back to the problem. For a fixed n , we can take $a = 2^n - 1$, $b = (2^n - 1)^n - 1 = a^n - 1$ and easily verify that they satisfy the 3 required conditions. Alternatively, we can take $a = 2^n - 1$ and $b = 2^n + 1$. One can check $n \mid \phi(2^n + 1)$ in a similar way.

Editor's Comment. The other five solutions we received more or less relied on effective versions of Dirichlet's theorem on the arithmetic progression $\{1 + kn\}$, which are overkill for this problem. In particular, these five solutions used either Zsigmondy's theorem or Bang's theorem.

We remark that the assumption $f(x) = 2^{x^2}$ can be significantly weakened using Linnik's theorem. Let a and d be coprime such that $1 \leq a \leq d - 1$; we let $P(a, d)$ be the least prime in the arithmetic progression $(a + kd)_{k=1}^\infty$. Linnik (1944) showed that there exist positive constants c and L such that $P(a, d) < cd^L$. The best-known record on the constant L is due to Xylouris (2011), where he showed that $P(a, d) < Cd^5$, where C is an effectively computable constant.

Thus, if we take $f(x) = Cx^5$ instead, the statement of the problem remains true. Indeed, given an integer $n \geq 2$, we can take $a = n^2$ and $b = P(1, n)$. Note that if we write $n = \prod p_i^{\alpha_i}$ in its prime factorization, then $n \mid \phi(a) = \prod (p_i - 1)p_i^{2\alpha_i - 1}$. Since b is a prime and $b \equiv 1 \pmod{n}$, it follows that $\gcd(a, b) = 1$ and $n \mid \phi(b)$.

4779. Proposed by Marian Ursărescu.

Let $0 < a < b$ and let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) and with $f(a) = f(b)$. Prove that there exist distinct $c_1, c_2 \in (a, b)$ such that

$$\sqrt{b}f'(c_1) + \sqrt{a}f'(c_2) = 0.$$

We received 9 submissions, of which 7 were correct and complete. We present two solutions.

Solution 1, submitted independently by Henry Ricardo and the proposer.

Applying the Mean Value Theorem (MVT) on the subinterval $[a, \sqrt{ab}] \subset [a, b]$, we see that there is a point $c_1 \in (a, \sqrt{ab})$ such that

$$\frac{f(\sqrt{ab}) - f(a)}{\sqrt{ab} - a} = f'(c_1), \quad \text{or} \quad f(\sqrt{ab}) - f(a) = \sqrt{a}(\sqrt{b} - \sqrt{a})f'(c_1). \quad (1)$$

The MVT applied to the interval $[\sqrt{ab}, b]$ yields $c_2 \in (\sqrt{ab}, b)$ such that

$$\frac{f(b) - f(\sqrt{ab})}{b - \sqrt{ab}} = f'(c_2), \quad \text{or} \quad f(b) - f(\sqrt{ab}) = \sqrt{b}(\sqrt{b} - \sqrt{a}) f'(c_2). \quad (2)$$

Adding (1) and (2) gives us

$$f(b) - f(a) = (\sqrt{b} - \sqrt{a}) (\sqrt{a} f'(c_1) + \sqrt{b} f'(c_2)) = 0,$$

which implies that $\sqrt{a} f'(c_1) + \sqrt{b} f'(c_2) = 0$ since $\sqrt{b} \neq \sqrt{a}$. Furthermore, since $(a, \sqrt{ab}) \cap (\sqrt{ab}, b) = \emptyset$, c_1 and c_2 are distinct.

Solution 2, by Didier Pinchon.

For $x \in [a, b]$, $\sqrt{ax} \in [a, b]$ and $\sqrt{bx} \in [a, b]$. So, let g be the function defined by

$$g(x) = f(\sqrt{ax}) + f(\sqrt{bx}).$$

On $[a, b]$, g is a continuous function which is differentiable on (a, b) , with

$$g'(x) = \frac{1}{2\sqrt{x}} (\sqrt{a} f'(\sqrt{ax}) + \sqrt{b} f'(\sqrt{bx})).$$

As $g(a) = f(a) + f(\sqrt{ab})$ and $g(b) = f(\sqrt{ab}) + f(b)$, $g(a) = g(b)$ results from $f(a) = f(b)$. Using Rolle's theorem for g , there exists $c \in (a, b)$ such that $g'(c) = 0$. This is equivalent to $\sqrt{a} f'(\sqrt{ac}) + \sqrt{b} f'(\sqrt{bc}) = 0$. The result is therefore proven with $c_1 = \sqrt{ac}$ and $c_2 = \sqrt{bc}$, which are distinct because $a < b$.

4780. *Proposed by Florică Anastase.*

Let $0 < a < b$, $m = \frac{a+b}{2}$ and $f: [a, b] \rightarrow \mathbb{R}$ differentiable with derivative continuous on $[a, b]$ such that $f(m) = 0$. Prove that

$$2a^3 \int_{-a}^a (f'(x))^2 dx \geq 3 \left(\int_{-a}^a f(x) dx \right)^2.$$

We received 8 submissions of which 5 were correct and complete. We present a solution to a revised problem statement by Didier Pinchon.

A preliminary remark. We begin with two reformulations of the problem.

Statement 1 : Let $a > 0$ and $f: [-a, a] \rightarrow \mathbb{R}$ be differentiable with continuous derivative on $(-a, a)$ and $f(0) = 0$. Prove that

$$2a^3 \int_{-a}^a (f'(x))^2 dx \geq 3 \left(\int_{-a}^a f(x) dx \right)^2.$$

Statement 2 : Let $0 < a < b$ and $f: [a, b] \rightarrow \mathbb{R}$ be differentiable with continuous derivative on (a, b) and $f\left(\frac{a+b}{2}\right) = 0$. Prove that

$$\frac{(b-a)^3}{4} \int_a^b (f'(x))^2 dx \geq 3 \left(\int_a^b f(x) dx \right)^2.$$

Both statements are equivalent to the following statement.

Statement 3: Let $g : [-1, 1] \rightarrow \mathbb{R}$ be differentiable with continuous derivative on $(-1, 1)$ and $g(0) = 0$. Prove that

$$2 \int_{-1}^1 (g'(t))^2 dt \geq 3 \left(\int_{-1}^1 g(t) dt \right)^2.$$

To prove that statement 1 is equivalent to statement 3, the change of variable $x = at$ is used, with $f(at) = g(t)$ and $af'(at) = g'(t)$.

To prove that statement 2 is equivalent to statement 3, the change of variable $x = \frac{a+b}{2} + \frac{b-a}{2}t$ is used, with $f(x) = g(t)$ and $\frac{b-a}{2}f'(x) = g'(t)$.

We are now ready for the solution.

For a function $F(t)$, continuous on an interval $[a, b]$, the notation $[F(t)]_a^b$ designates the increment $F(b) - F(a)$ of $F(t)$ between a and b .

Two integration by parts give, using $g(0) = 0$,

$$\int_{-1}^0 g(t) dt = [(t+1)g(t)]_{-1}^0 - \int_{-1}^0 (t+1)g'(t) dt = - \int_{-1}^0 (t+1)g'(t) dt,$$

and

$$\int_0^1 g(t) dt = [(t-1)g(t)]_0^1 - \int_0^1 (t-1)g'(t) dt = - \int_0^1 (t-1)g'(t) dt.$$

Therefore

$$\int_{-1}^1 g(t) dt = \int_{-1}^1 h(t)g'(t) dt,$$

where the function $h(t)$ on $[-1, 1]$ is defined by

$$h(t) = \begin{cases} -(t+1), & -1 \leq t \leq 0, \\ -(t-1), & 0 < t \leq 1. \end{cases}$$

From the Cauchy–Schwarz inequality for integrals, it follows that

$$\left(\int_{-1}^1 g(t) dt \right)^2 \leq \int_{-1}^1 (h(t))^2 dt \cdot \int_{-1}^1 (g'(t))^2 dt.$$

Since

$$\begin{aligned} \int_{-1}^1 (h(t))^2 dt &= \int_{-1}^0 (t+1)^2 dt + \int_0^1 (t-1)^2 dt \\ &= \left[\frac{(t+1)^3}{3} \right]_{-1}^0 + \left[\frac{(t-1)^3}{3} \right]_0^1 \\ &= \frac{2}{3}, \end{aligned}$$

we have

$$\left(\int_{-1}^1 g(t) \, dt \right)^2 \leq \frac{2}{3} \int_{-1}^1 (g'(t))^2 \, dt,$$

which proves the statement.

Editor's Comments. Raymond Mortini and Rudolf Rupp considered whether the bound given in statement 2 was the best possible. They supplied the following example which they noted is not a C^1 function. Let

$$q(x) = \begin{cases} \frac{(x-a)^2}{2} - \frac{(b-a)^2}{8} & \text{if } a \leq x \leq (a+b)/2 \\ \frac{(x-b)^2}{2} - \frac{(a-b)^2}{8} & \text{if } (a+b)/2 \leq x \leq b. \end{cases}$$

Then q is continuous on $[a, b]$, $q((a+b)/2) = 0$ and

$$\int_a^b (q'(x))^2 \, dx = \frac{12}{(b-a)^3} \left(\int_a^b q(x) \, dx \right)^2.$$

Theo Koupelis provided the following counterexample to the problem's original statement. If $f(x) = x - 2$ in $[1, 3]$, then $\int_{-1}^1 1^2 \, dx = 2$ and $\int_{-1}^1 (x-2) \, dx = -4$. Clearly $2 \cdot 1^3 \cdot 2 < 3 \cdot (-4)^2$.

