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MathemAttic

No. 42

The problems in this section are intended for students at the secondary school level.

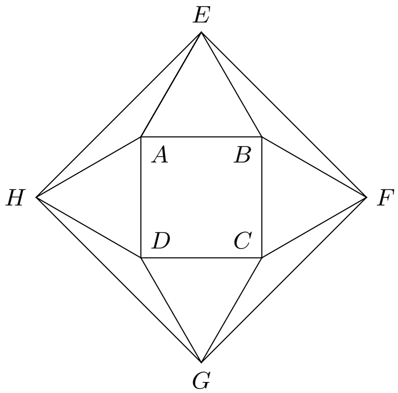
Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 30, 2023**.

MA206. Place algebraic operations $+$; $-$; $\div$; $\times$ between the numbers 1 to 9, in that order, so that the total equals 100. You may also freely use brackets before or after any of the digits in the expression and numbers may be placed together, such as 123 and 67. Two examples are given below:

$$123+45-67+8-9=100 \quad \text{and} \quad 1+((2+3)\times 4\times 5)-((6-7)\times(8-9))=100.$$

MA207. Suppose that the points E, F, G, H lie in the plane of the square $ABCD$ such that AEB, BFC, CGD, and DHA are equilateral triangles. If the area of $EFGH$ is 25, then find the area of $ABCD$.



MA208. Solve the following equation for $0 \le x < 2\pi$:

$$2^{3\cos x+3} - 2^{2\cos x+2} - 2^{\cos x+1} + 1 = 0.$$

MA209. *Proposed by Aravind Mahadevan.*

In ΔABC, D is on BC. If $\angle ADC = \theta$, prove that

$$BC\cot\theta = DC\cot B - BD\cot C.$$

MA210. *Proposed by Neculai Stanciu.*

Determine all triplets (x, y, z) of real numbers which satisfy:

$$2xy - (z + x - 1)^2 = 2xy - (x + y - 1)^2 = 2zx - (y + z - 1)^2 = 1.$$

. . . .

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

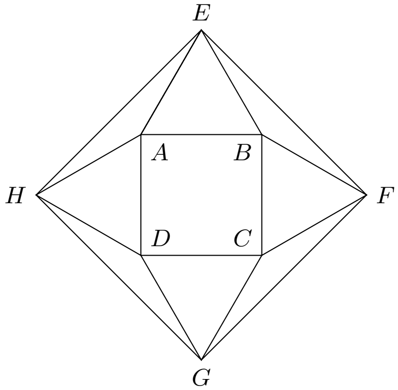
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 avril 2023**.

MA206. Placer des symboles d'opérations algébriques $+$; $-$; $\div$; $\times$ entre les chiffres 1 à 9, dans cet ordre, de façon à ce que le résultat donne 100; pour ce faire, il est permis d'utiliser des parenthèses avant ou après tout chiffre, et de regrouper les chiffres, comme 123 ou 67. Deux exemples suivent:

$$123 + 45 - 67 + 8 - 9 = 100 \quad \text{et} \quad 1 + ((2 + 3) \times 4 \times 5) - ((6 - 7) \times (8 - 9)) = 100.$$

MA207. Supposons que les points E, F, G et H se trouvent dans le plan engendré par le carré $ABCD$ de telle sorte que AEB, BFC, CGD et DHA sont des triangles équilatéraux. Sachant que l'aire de $EFGH$ est 25, trouvez l'aire de $ABCD$.



MA208. Résoudre l'équation suivante, où $0 \leq x < 2\pi$:

$$2^{3\cos x+3} - 2^{2\cos x+2} - 2^{\cos x+1} + 1 = 0.$$

MA209. *Soumis par Aravind Mahadevan.*

Dans ΔABC, le point D est situé le long du segment BC. Si $\angle ADC = \theta$, montrez que

$$BC \cot\theta = DC \cot B - BD \cot C.$$

MA210. *Proposé par Neculai Stanciu.*

Déterminer tous les triplets de nombres réels tels que

$$2xy - (z + x - 1)^2 = 2xy - (x + y - 1)^2 = 2zx - (y + z - 1)^2 = 1.$$

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2022: 48(7), p. 372–374.

MA181. In still water Aoife swims at 2 kph. She is standing on the bank of a river that is exactly 100 m wide. The river is flowing past at a speed of 1.2 kph. How long (in seconds) will it take her to swim in a straight line to the point directly across from her on the other bank of the river?

Originally question 12 from the 2018 PRISM (Problem Solving for Irish Second level Mathematicians) paper.

We received 4 submissions of which 3 were correct and complete. We present the solution by Soham Bhadra.

Aoife can swim at 2 kilometres per hour in still water. The width of the river is 100 metres. The river's speed is 1.2 kilometres per hour. So if Aoife swims in the river whose speed is 1.2 kilometres per hour in the direction indicated by the question, then the speed of Aoife during that 100 metres will be

$$\frac{2+1.2}{2} = 1.6 \text{ kph} = \frac{16 \text{ metres}}{36 \text{ seconds}}$$

Therefore the required time for Aoife to cross the river and reach the other bank is $\frac{100}{\left(\frac{16}{36}\right)} = 225$ seconds.

MA182. Antonia, Dara and Tosia are identical triplet sisters in the same class. Antonia always tells the truth, Dara always lies and Tosia sometimes lies and sometimes tells the truth. One day one of them arrives late for class. The teacher asks this late sister who she is. She answers "I am Tosia". The teacher cannot tell the girls apart so asks the other two sisters the name of the sister who was late. One of them says: "Antonia was late", and the other says: "Dara was late". Which sister was in fact late?

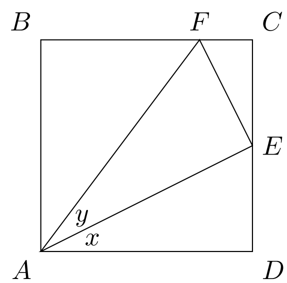
Originally question 20 from the 2015 PRISM (Problem Solving for Irish Second level Mathematicians) paper.

We received 6 submissions, all of which were correct and complete. We present the solution by Soham Bhadra, Vasiliki Lalioti, Aravind Mahadevan, and Emon Suin (done independently).

Since Antonia always tells the truth, the sister who was late could not have been Antonia as she would not have told the teacher that she is Tosia. This means that Antonia must have been one of the other two sisters whom the teacher asked.

Again, the sister who replied that Antonia was late could not have been Antonia as we have already established that Antonia was not the one who was late and Antonia always tells the truth. This means that the sister who replied to the teacher that Dara was late must have been Antonia. And since Antonia always tells the truth, we can establish the fact that it is Dara who was late.

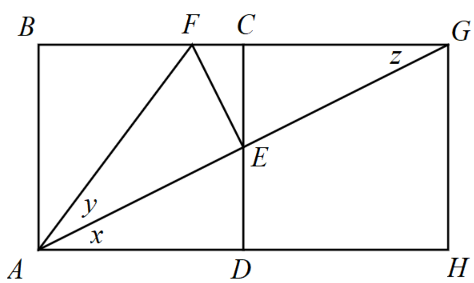
MA183. $ABCD$ is a square of sides length 4, E is the midpoint of CD, and $AE \perp EF$, as shown. If x and y are the measures of $\angle EAD$ and $\angle FAE$ respectively, prove that $x = y$.



Originally question 5 from the 17th Blundon Mathematics Contest, 2000.

We received 17 solutions for this problem. We present the solution by the Missouri State University Problem Solving Group, which uses an auxiliary square.

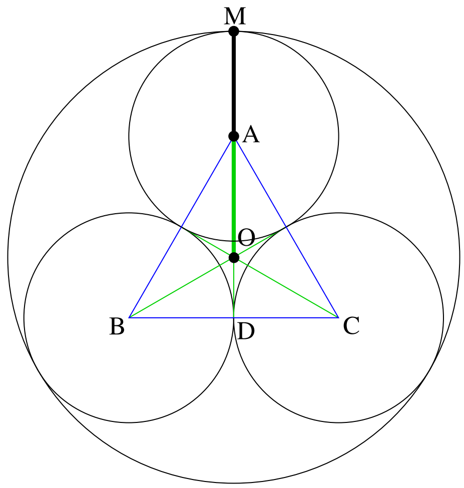
Construct square $CDHG$, extend AE to AG, and let z denote the measure of $\angle EGF$ as shown in the figure. Since $AE = EG, EF = EF$, and $\angle AEF$ and $\angle FEG$ are both right angles, $\triangle AEF \cong \triangle GEF$. Therefore $y = z$. We also have $x = z$ by alternating interior angles and the result follows.



MA184. A *unit circle* is a circle of radius 1. Two circles are said to *touch* if they have exactly one point in common. Three unit circles are drawn so that each of them touches the other two. A fourth (larger) circle is drawn around these three so that it touches each of the three unit circles. What is the radius of the large circle?

Originally question 13 from the 2018 PRISM (Problem Solving for Irish Second level Mathematicians) paper.

We received 10 *complete and correct submissions. We present the one by Denes Jakob, lightly edited.*



Let A, B, C be the centres of the three circles of radius $r = 1$. Let O be the centre of the large circle and M be the common point of the large circle and the circle with centre A. Then O,A, and M lie on the same line and the radius of the large circle is $R = OM = OA + AM = OA + 1$.

ΔABC is an equilateral triangle with side length $2r = 2$. By symmetry, the centre of the large circle, O, is also the centroid of ABC.

Let AD be the median from A to BC. Then $AD = \sqrt{3}$ and $AO = \frac{2}{3}AD = \frac{2}{\sqrt{3}}$ and we obtain

$$R = \frac{2}{\sqrt{3}} + 1.$$

MA185. Find the primes p, q, r, given that one of the numbers pqr and $p + q + r$ is 101 times the other.

Originally question 2 from the 29th Nordic Mathematical Contest, 2015.

We received 11 submissions of which 11 were correct and complete. We present the solution by Aravind Mahadevan.

Since, for any three primes p, q and r, their product is always going to be greater than their sum

$$pqr = 101(p + q + r).$$

Further, the above implies that 101 is a factor of pqr. This means that one of the primes p, q, r must be equal to 101.

Without loss of generality, let us assume that $p = 101$. Then we have:

$$\begin{aligned} 101qr &= 101(101 + q + r), \\ qr &= 101 + q + r, \\ q(r-1) &= 101 + r, \\ q &= \frac{101 + r}{r-1}, \\ q &= \frac{102 + (r-1)}{r-1}, \\ q &= 1 + \frac{102}{r-1}. \end{aligned} \tag{1}$$

We can easily see that when $r = 2$, then $q = 103$ and when $r = 103$, we have $q = 2$. No other prime values of r exist that satisfy (1) in a way that q is also prime. Thus, the values of the primes p, q and r are 2, 101 and 103.

In general, if we have three primes such that one of them is 2 and the other two are consecutive primes n and $(n + 2)$, then the product of these three primes will be equal to n times their sum. So, for example, if 101 had been replaced by 41 in the given problem, the three prime numbers in question would have been 2, 41 and 43.

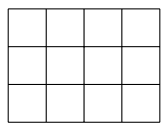
TEACHING PROBLEMS

No. 20

John Grant McLoughlin

Squares in Rectangular Grids: How many would you like?

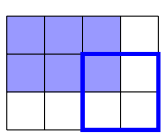
A common mathematical challenge is to count the number of squares of any size in a rectangular grid. For example, consider a 3×4 grid, as shown.

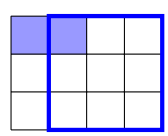


The total number of squares in this grid is 20. An observation that is helpful to note here is that it is not the area of the grid that principally makes the counting challenging to organize. The determining factor in counting the squares is the smaller of the two dimensions. In this example, the smaller number 3 restricts the count to considering only 1×1, 2×2, and 3×3 squares. Let us count these in a systematic manner that will assist in the development of this article.

Dimensions	**Number of Squares**
1×1	3×4
2×2	2×3
3×3	1×2

Let me explain the thinking behind the products on the right. The 3×4 grid consists of 3 rows and 4 columns. For our purpose we will consider the number of rows that are candidates to be the top row of a square of a given size and the number of columns to the be the left-most column of the square. For example, all three rows could be the top row of a 1×1 square. Likewise, all four columns could be the left-most column. Hence, $3 \cdot 4 = 12$ unit squares appear in the grid. Do you see that with 2×2 squares the product of $2 \cdot 3$ follows? The numbers on the product are each reduced by 1 as now the bottom row and the right-most columns are not plausible candidates (for the top row and left-most columns respectively). Continuing with this reasoning we can see that there are $1 \cdot 2$ possible squares of size 3×3. These figures illustrate the idea discussed here.

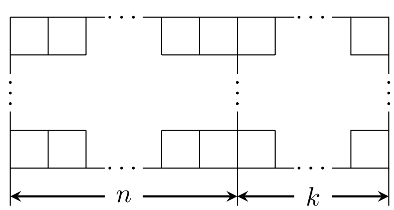




Reinforcing the earlier observation, determine how many squares appear in a 2×23 grid and also in a 5×6 grid. I will tell you that there are 70 squares in the 5×6 grid. You can check this result particularly if you want to implement the organized count as a sum of products. In contrast, the 2×23 example is much easier to consider as only 1×1 and 2×2 squares need be considered. The 68 squares are accounted for here.

Dimensions	**Number of Squares**
1×1	$2 \cdot 23$
2×2	$1 \cdot 22$

Let us shift to a general case. Without loss of generality, we will assume the dimensions are such that the number of rows is less than or equal to the number of columns. The dimensions of the grid will be $n \times (n+k)$ where k, n are non-negative integers.



The squares in this grid will range from 1×1 to $n \times n$ in dimensions. Let us consider an organized count of the total number of squares in the grid.

Dimensions	**Number of Squares**		
1×1	$n(n+k)$	$=$	$kn + n^2$
2×2	$(n-1)(n+k-1)$	$=$	$k(n-1) + (n-1)^2$
3×3	$(n-2)(n+k-2)$	$=$	$k(n-2) + (n-2)^2$
$\vdots$	$\vdots$		$\vdots$
$(n-1) \times (n-1)$	$[n-(n-2)][(n+k-(n-2)]$	$=$	$k(2) + 2^2$
$n \times n$	$[n-(n-1)][(n+k-(n-1)]$	$=$	$k(1) + 1^2$

The total number of squares is

$$\begin{aligned} k(1+2+\cdots+n) + (1^2+2^2+\cdots+n^2) &= k\left(\frac{n(n+1)}{2}\right) + n\left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= \frac{3kn(n+1) + n(n+1)(2n+1)}{6} \\ &= \frac{n(n+1)(3k+2n+1)}{6} \end{aligned}$$

Checking our 3×4 example, as in using $n = 3$ and $k = 1$, gives $\frac{3(4)(3+6+1)}{6} = 20$, as before.

The mathematics to this point has focused on counting the number of squares in a grid. While the development of the formula is likely unfamiliar to most readers, the earlier ideas are likely not. Now we move on to the twist that makes for a teaching problem.

Tell me how many squares you want

Give me a number. Then the challenge becomes to identify all possible rectangular grids that have exactly the given number of squares.

For starters, let us return to the original count of 20 squares for a 3×4 grid. Is that the only grid that will have exactly 20 squares? Reflecting momentarily it becomes apparent the answer is "no" as a 1×20 grid would also have 20 squares. Are these the only such examples? Let us check.

We require

$$\frac{n(n+1)(3k+2n+1)}{6} = 20.$$

Hence,

$$n(n+1)(3k+2n+1) = 120.$$

Since $n < \sqrt[3]{120} < 5$, we can check $n = 1, 2, 3, 4$.

$n = 4$	$4(5)(3k+9) > 180$	no solutions
$n = 3$	$3(4)(3k+7) = 120$	implies $k = 1$
$n = 2$	$2(3)(3k+3) = 120$	implies $k = 5$
$n = 1$	$1(2)(3k+3) = 120$	implies $k = 19$

There are three rectangular grids that provide counts of exactly 20 squares, namely, $3 \times (3+1)$, $2 \times (2+5)$, and $1 \times (1+19)$. We found a third grid with this method. That is, a 2×7 grid also has 20 squares.

With relatively small numbers, the quick observation concerning the cube root as an upper bound is helpful. However, upon sharing this article with Shawn Godin, he noted a significant improvement with respect to the need for checking values. This makes a big difference when the numbers are larger.

Shawn Godin provides the argument below to show that in fact n is less than the cube root of 3 times the number of squares, where S is the number of squares. The mathematics appears below.

In general since

$$\frac{n(n+1)(3k+2n+1)}{6} = S$$

where S is the number of squares on the grid, then

$$\begin{aligned} \frac{n(n+1)\left(2\left(n+\frac{3}{2}k+\frac{1}{2}\right)\right)}{6} &= S \\ n^3 < n(n+1)\left(n+\frac{3}{2}k+\frac{1}{2}\right) &= 3S \\ n &< \sqrt[3]{3S} \end{aligned}$$

Furthermore, many values of n can be eliminated from consideration when considering divisibility properties. For example, if n or $(n+1)$ is a multiple of 5, it becomes clear that say a resulting product which is not divisible by 5 would be impossible to obtain. Readers may wish to consider prime numbers and other divisibility properties in further reducing unnecessary values of n for consideration while working out such problems.

Concluding comments and observations

Many readers will have been introduced to counting squares within square grids. For instance, the number of squares in 1×1, 2×2, 3×3, and 4×4 grids respectively would be 1, 5, 14, and 30. These numbers correspond to 1^2, 1^2+2^2, $1^2+2^2+3^2$, and $1^2+2^2+3^2+4^2$. How does this connect to our general formula of $\frac{n(n+1)(3k+2n+1)}{6}$? In the case of a square grid, we have $k=0$ that when substituted gives the resulting sum of squares for the value of n.

The idea for my own examination of the problem grew out of seeing the opening problem in *Jim Totten's Problems of the Week* [1]. Most recently I shared the ideas discussed here in a seminar [2] with colleagues in the School of Mathematical and Statistical Sciences at University of Galway. Here is the problem as stated in [1].

> *Find the dimensions of all rectangles of size $m\times n$ which contain exactly 100 squares of all sizes, where each square has sides parallel to the edges of the given rectangle and has its corners at the grid points of the interior or boundary of the rectangle.*
>
> *For example, a 2×3 rectangle would have 2 squares of size 2 and 6 squares of size 1, for a total of 8 squares.*

The problem is left as a challenge for the reader. Indeed a 1×100 grid would work. In fact, two other grid sizes will also satisfy the requirements.

Finally I close with two additional challenges.

(i) Show that no rectangular grid other than 1×18 will have exactly 18 squares.

(ii) An 8×8 checkerboard has 204 squares. Determine if any non-square rectangular grids, aside from the 1×204 case, have exactly 204 squares.

References

[1] John Grant McLoughlin, Joseph Khoury & Bruce Shawyer (Eds.) (2013). *Jim Totten's Problems of the Week*. Singapore: World Scientific Publishing.

[2] John Grant McLoughlin. (2022). *Mathematical Logic Puzzles on a Grid*, School of Mathematical and Statistical Sciences Seminar, University of Galway, Nov. 3, 2022.

MATHEMAGICAL PUZZLES

No. 3

Tyler Somer

Molten Gold – III

The earlier articles of this series have illustrated several illusions of an increase in area, by way of rearranging pieces of a geometric dissection. In the first article, dissections of various squares and rectangles were rearranged into different shapes. The second article introduced an apparent paradox, as the shape seemed unaltered. This third investigation builds on the second.

. . . .

Consider a rectangular tray with unknown dimensions, and it contains eight identical tiles. The eight tiles appear to fill the tray, with a small bit of room – the play, once again – for the pieces to move within the tray. The mathematical magician presents a ninth tile, identical to the first eight, and declares that the tray will accommodate this ninth tile!

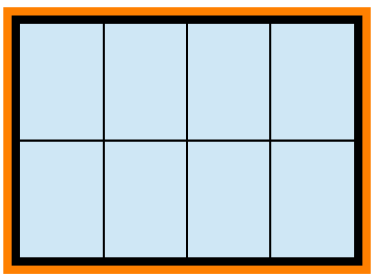


Figure 1

As with the examples presented in the earlier articles, the size of the tray and the sizes of the pieces all matter. Further, these values can be uniquely determined. To begin, realize that the pieces must be turned 90 degrees, so that they will all fit in the tray, with less play.

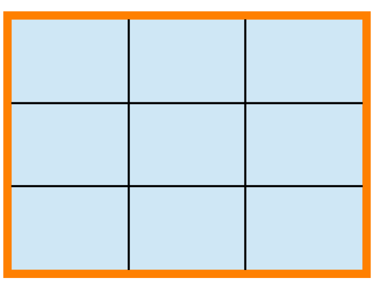


Figure 2

If the play were to be reduced to zero, nine pieces would be too tight to fit the tray. For the sake of a physical demonstration, a small amount of play is still required. For calculations, the amount of play can be ignored.

The simplest way to calculate all the components – the tiles, the tray, the play – is as follows. Assign variables x and y to the short and long dimensions, respectively, of each tile. When only eight tiles are in the tray, define the play as 1 in both the x- and y-dimensions. Thus, in Figure 1, the horizontal and vertical measurements of the tray are $(4x + 1)$ and $(2y + 1)$, respectively. In Figure 1, note that the play is distributed symmetrically around the perimeter, thus it appears to be $\frac{1}{2}$. With the play reduced, as in Figure 2, these same measurements are equal to $3y$ and $3x$, respectively. This provides a system of equations in two variables to solve:

$$\begin{aligned} 3y &= 4x + 1 \\ 3x &= 2y + 1 \end{aligned}$$

It should be a simple matter for the reader to determine and verify that $(x, y) = (5, 7)$. Each tile has an area of 35 square units. Eight tiles take up 280 square units of the tray, with plenty of play. Nine tiles, turned 90 degrees from the original, take up 315 square units, with the amount of play greatly reduced. The tray must be 21-by-15.

This simple example illustrates the principle, but a meaningful puzzle remains to be developed. With so few components, it is difficult to create a challenging puzzle. Still, a rudimentary puzzle might be composed of four double-tiles, two each of 10-by-7 and 5-by-14, as presented in Figure 3. The additional piece can fit the tray only one way (up to rotation or reflection), as presented in Figure 4. It can be said that the additional tile "melts" into the tray.

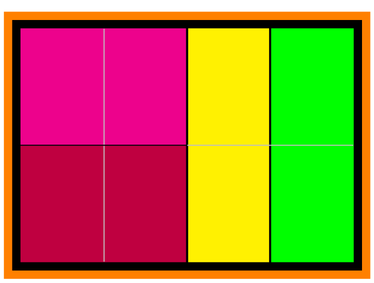


Figure 3

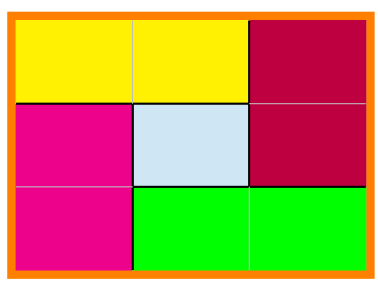


Figure 4

Also note that Figure 3 does not represent a unique solution. For example, the green and yellow pieces can be at opposite ends of the tray. Other rudimentary designs may be developed, but most will fail to be unique for one or the other tray.

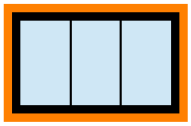


Figure 5

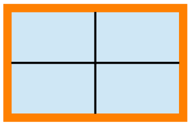


Figure 6

Consider an example with fewer pieces, as shown by Figures 5 and 6. Thee tiles, plus plenty of play, can be shifted around to accommodate a fourth tile. A similar assignment of the variables x and y leads to the system of equations:

$$
\begin{aligned}
2y &= 3x + 1 \\
2x &= y + 1
\end{aligned}
$$

and its solution $(x, y) = (3, 5)$. The tray is 10-by-6. Along with the trivial nature of this example, the reader can see that the relative amount of play is high, so much as to be labeled as sloppy.

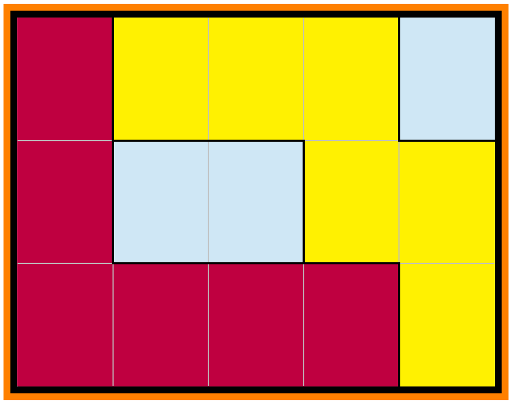


Figure 7

Moving up to 15 and 16 tiles, as in Figures 7 and 8, the appropriate system of equations is:

$$
\begin{aligned}
4y &= 5x + 1 \\
4x &= 3y + 1
\end{aligned}
$$

with the solution $(x, y) = (7, 9)$, and the tray dimensions are 36-by-28. There are now enough component parts to create super-tiles, which could lead to a meaningful puzzle. Along the x-dimension, the super-tiles could be any of $\{7, 14, 21, 28\}$ units, and similarly the y-dimension $\{9, 18, 27\}$. Most often, the super-tiles are simply larger rectangles. Occasionally, zigzag or L-shaped pieces may be used, but too many of them will reduce the challenge to a triviality, as shown.

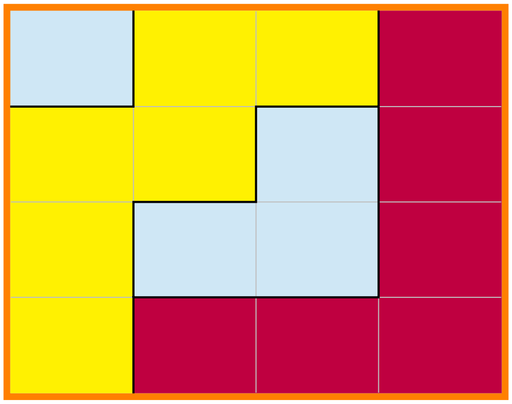


Figure 8

Readers are encouraged to investigate the many other possible designs using these $15+1$ component parts. Readers may wish to investigate larger trays with $24+1$, $35+1$, $48+1$ components, and so on. Other rectangular trays may be interesting: $27+1$, $44+1$, $55+1$, $70+2$, $90+1$, among others, for example. The designer's goal is to create a set of super-tiles that have a unique solution in both cases: n components and $n+1$ components.

The main design benefit of using larger trays is that the relative amount of play is significantly reduced. This can be quite effective when the mathematical magician presents the seemingly-filled tray to a casual observer, then produces that extra tile!

Larger pieces and trays have the drawback that they can occupy a large space. Puzzle designers might have to get creative. Woodworkers might reduce the size of the pieces, but then precision might be lost in the process. For example, it is much easier to tell apart pieces that are 15 or 16 cm long, versus those that are 15 or 16 mm. Improper cutting or sanding may render these small pieces as, say, 14.9 mm and 15.1 mm pieces, then a play of 1 mm is too much for such pieces. Laser cutting will help with this, but acrylic sheets and other specialized materials are expensive.

When he was teaching, Tyler often had mechanical puzzles in his classroom. As a freelancer, Tyler has worked with numerous inventors and co-designers to bring dozens of table-top solo-logic puzzle kits to market. He continues to design puzzles, and he spends a good deal of time in his woodshop, building his own and others' puzzle designs.

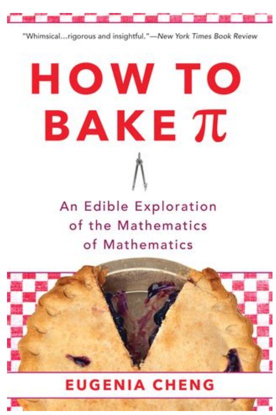
From the Bookshelf of . . .

Rebecca McKay

This MathemAttic *feature brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to* `MathemAttic@cms.math.ca`*. Publishers are also welcome to send along books for possible review.*

How to Bake π: An Edible Exploration of the Mathematics of Mathematics
by Eugenia Cheng
ISBN 978-0-465-09767-8, 288 pages
Published by Basic Books, 2015.

In 2022, my husband had an interesting idea. He proposed that various family members read the same book, book-club-style, and then discuss. After a lot of consideration, we finally decided on "How to Bake π" by Eugenia Cheng. It was a book that we had both been intending to read (for context, my husband Neil is also a mathematician) and seemed like a book that would both appeal to my mother-in-law (who is a life-long talented baker) and would be accessible to the non-mathematicians in the family. Although the books took a surprising amount of time to reach family members across Canada, we have all received our copies and most of us have finished reading it.



"How to Bake π" is an introduction to the idea of abstraction and logic and how it is used in mathematics. Cheng does this through comparisons in baking. Each chapter starts with a recipe that is then used to explain a general idea within mathematics. For example, Chapter 2 is entitled *Abstraction* and starts with a recipe for mayo or Hollandaise sauce. Cheng notes that the two recipes are quite similar. She uses this to explain how mathematicians look for situations where objects are the same with a small exception. Another example is in the chapter *Axiomatization.* This chapter starts with a recipe for Jaffa cakes which lists among the ingredients "small round flat plain cakes". Cheng uses this recipe as a springboard for discussing what constitutes a basic ingredient and what needs to be made from more basic ingredients.

The book has two parts: "Math" and "Category Theory". Part I is an engaging introduction to the basics of abstraction and logic. Part II gently guides the reader through exploring category theory. The chapter on *Relationships* starts with a recipe for porridge with the ingredients all measured in cups. Cheng notes that the important part of this recipe is not the quantity of the ingredients but rather the relationship between them and draws the comparison that category theory emphasizes the relationships between mathematical objects rather than studying the objects themselves.

Each chapter is broken up into short chunks where Cheng uses brief vignettes that are easily understood by all audiences to explain a mathematical concept. For example, in *Principles*, she uses a story of her learning to weld in high school to illustrate the importance of learning the principles of something that you are using so that you can get the best out of it and you have power when something goes wrong.

I am an applied mathematician by training and mostly teach courses aimed at engineering students. Despite having spent time with many category theorists while at Dalhousie University, I had only the vaguest idea of what category theory is ("There are arrows!" being my basic tool for discerning whether something was category theory.) Eugenia Cheng finally gave me a general idea of what category theory is and what category theorists study in a very tangible way.

What might you get out of the book? This book is written in easily digestible pieces to make math accessible. Aimed at a general audience, anyone can read this and gain an appreciation for abstract mathematical thinking. Even if you are a mathematician who is very familiar with things mathematical, the connections and presentation of the topics by the author make the book a worthwhile read. I found that reading the book gave me ideas of how to present particular concepts to undergraduate students (like the transitive property). I plan on adding other books by Eugenia Cheng to my bookshelf soon.

This book is a recommendation from the bookshelf of Rebecca McKay. Rebecca lives and works in Saint John, New Brunswick and is a Teaching Professor at the University of New Brunswick. For the most part she teaches various levels of calculus and linear algebra. Growing up in Newfoundland, she attended Memorial University, then moved on to Nova Scotia for graduate school, before moving on to New Brunswick. She has been involved in the mathematics community in Atlantic Canada for more than 20 years.

MATHEMATICS FROM THE WEB

No. 7

This column features short reviews of mathematical items from the internet that will be of interest to high school and elementary students and teachers. You can forward your own short reviews to `mathemattic@cms.math.ca`*.*

Knight's Tour
`https://mathlair.allfunandgames.ca/knightstour.php`
The knights tour challenge is a familiar one to many readers. Basically the challenge is to create a path visiting every square on a square board exactly once using the moves of a knight as in a regular chess game. Those unfamiliar with the problem may wish to convince themselves quickly that such a tour is impossible on a 3×3 board. What about on a 4×4 board?

The Bridges Conference
`https://archive.bridgesmathart.org/#gsc.tab=0`
The *Bridges Conference* connects mathematics with its various artistic dimensions from fine arts to architecture to poetry and beyond. The 2023 conference will be held in Halifax July 27 - 31. Proceedings of the conferences are housed in an accessible online archive at the above link. A plethora of rich mathematical ideas can be found here.

Decathlon: the Art of Scoring Points
`https://sport.maths.org/content/decathlon-art-scoring-points-0`
This website was launched in advance of the 2012 Summer Olympics in London as part of the *Millennium Project*, which has existed since 1999. The site is a collection of articles with focus on the place of mathematics in sports. One such example relates to the scoring of decathlons. The following excerpt hints at the challenge.

> The most striking thing about the decathlon is that the tables giving the number of points awarded for different performances are rather free inventions. Someone decided them back in 1912 and they have subsequently been updated on different occasions. Clearly, working out the fairest points allocation for any running, jumping or throwing performance is crucial and defines the whole nature of the event very sensitively. Britain's Daley Thompson missed breaking the decathlon world record by one point when he won the Olympic Games 1984 but a revision of the scoring tables the following year increased his score slightly and he became the new world record holder retrospectively!

OLYMPIAD CORNER

No. 410

The problems in this section have appeared in a mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **April 30, 2023**.

OC616. Let a, b, c be integer side-lenghts of a triangle, $\gcd(a,b,c) = 1$ and all the values

$$\frac{a^2+b^2-c^2}{a+b-c}, \quad \frac{b^2+c^2-a^2}{b+c-a}, \quad \frac{c^2+a^2-b^2}{c+a-b}$$

are integers as well. Prove that

$$(a+b-c)(b+c-a)(c+a-b) \quad \text{or} \quad 2(a+b-c)(b+c-a)(c+a-b)$$

is a perfect square.

OC617. Consider a positive integer n, a circle of circumference $6n$ and $3n$ points on the circle that divide it into $3n$ small arcs so that n of these arcs have a length of 1, another n of these arcs have a length of 2, and the remaining arcs have a length of 3. Show that among the considered points there are two that are diametrically opposite.

OC618. Let $n \in \mathbb{N}, n \geq 2$. For all real numbers $a_1, a_2, \ldots, a_n$ denote $S_0 = 1$ and

$$S_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} a_{i_1} a_{i_2} \cdot \ldots \cdot a_{i_k}$$

the sum of all the products of k numbers chosen among $a_1, a_2, \ldots, a_n$, $k \in \{1, 2, \ldots, n\}$. Find the number of n-tuples $(a_1, a_2, \ldots, a_n)$ such that

$$(S_n - S_{n-2} + S_{n-4} - \ldots)^2 + (S_{n-1} - S_{n-3} + S_{n-5} - \ldots)^2 = 2^n S_n.$$

OC619. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy simultaneously the following conditions:

(a) $f(x) + f(y) \geq xy$ for all real numbers x and y;

(b) for every real number x there is a real number y such that $f(x) + f(y) = xy$.

OC620. Given a trapezoid $ABCD$ with bases AB and CD, with the circle of diameter BC tangent to the line AD, prove that the circle of diameter AD is tangent to the line BC.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 avril 2023**.

OC616. Les longueurs des côtés dun triangle sont dénotées a, b et c. Or, a, b et c sont des entiers tels que $\gcd(a, b, c) = 1$; de plus,

$$\frac{a^2+b^2-c^2}{a+b-c}, \quad \frac{b^2+c^2-a^2}{b+c-a}, \quad \frac{c^2+a^2-b^2}{c+a-b}$$

sont entiers. Démontrer que

$$(a+b-c)(b+c-a)(c+a-b) \quad \text{ou} \quad 2(a+b-c)(b+c-a)(c+a-b)$$

est un carré parfait.

OC617. Considérons un entier positif n, un cercle de circonférence $6n$ et, enfin, $3n$ points divisant le cercle en $3n$ petits arcs de sorte que n de ces arcs ont une longueur de 1, n autres de ces arcs ont une longueur de 2 et les arcs restants ont une longueur de 3. Montrez que parmi les points considérés, il y en a deux qui sont diamétralement opposés.

OC618. Soit $n \in \mathbb{N}, n \geq 2$. Étant donné des nombres réels $a_1, a_2, \ldots, a_n$, on pose $S_0 = 1$ et

$$S_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} a_{i_1} a_{i_2} \cdot \ldots \cdot a_{i_k},$$

la somme de tous les produits de k nombres choisis parmi $a_1, a_2, \ldots, a_n$, $k \in \{1, 2, \ldots, n\}$. Trouvez le nombre de n-tuples $(a_1, a_2, \ldots, a_n)$ tels que

$$(S_n - S_{n-2} + S_{n-4} - \ldots)^2 + (S_{n-1} - S_{n-3} + S_{n-5} - \ldots)^2 = 2^n S_n.$$

OC619. Trouvez toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ qui satisfont simultanément les conditions suivantes:

(a) $f(x) + f(y) \geq xy$ pour tous les nombre réels x et y;

(b) pour tout nombre réel x, il existe un nombre réel y tel que $f(x) + f(y) = xy$.

OC620. Etant donné un trapèze $ABCD$ de bases AB et CD, dont le cercle de diamètre BC est tangent à la droite AD, montrez que le cercle de diamètre AD est tangent à la droite BC.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2022: 48(7), p. 405–406.

OC591. Let $\overline{abcd}$ be a four-digit number, where $a \neq 0$ and $c \neq 0$ such that

$$\frac{\sqrt{\overline{abcd}}}{\sqrt{\overline{ab}} + \sqrt{\overline{cd}}}$$

is a rational number. Find all possible value(s) of $\overline{abcd}$.

Originally from 2022 Indonesia International Mathematics Competition Keystage 3 Individual Contest Section B (Invitational World Youth Mathematics Intercity Competition), Problem 3.

We received 8 submissions, all of which were correct and complete. We present the solution by the UCLan Cyprus Problem Solving Group.

We show that 1681 is the only solution. First, it is easy to verify that 1681 satisfies the condition. Next, we establish a general result.

Let r, s, t be positive natural numbers. Then $\frac{\sqrt{r}}{\sqrt{s}+\sqrt{t}}$ is a rational number if and only if there is a natural number k such that r/k, s/k, t/k are all perfect squares.

Indeed, if there exists a natural number k such that r/k, s/k, t/k are perfect squares, then it is trivial to establish that $\frac{\sqrt{r}}{\sqrt{s}+\sqrt{t}}$ is a rational number. Assume now that $\frac{\sqrt{r}}{\sqrt{s}+\sqrt{t}}$ is rational. Then the square

$$\left(\frac{\sqrt{r}}{\sqrt{s}+\sqrt{t}}\right)^2 = \frac{r}{s+t+2\sqrt{st}}$$

is rational, as is $\sqrt{st}$. Thus there exist a square-free natural number k and natural numbers m, n such that $s = km^2$ and $t = kn^2$. As

$$\frac{\sqrt{r}}{\sqrt{s}+\sqrt{t}} = \frac{\sqrt{r/k}}{m+n}$$

is rational, it follows that $\sqrt{r/k}$ is also rational. However, k is square-free, so r/k must be a perfect square.

Let $x = \overline{ab}$ and $y = \overline{cd}$. Then $\overline{abcd} = 100x + y$. From the general result we must have that $x = km^2$ and $y = kn^2$ for some $m, n \in \{1, 2, \ldots, 9\}$. Then $100x + y = k(100m^2 + n^2)$, hence we must have $100m^2 + n^2 = \ell^2$ for some natural number ℓ.

If $m = 1$, then $100 < 100m^2 + n^2 < 200$ so $\ell \in \{11, 12, 13, 14\}$. But then we have $\ell^2 \in \{121, 144, 169, 196\}$ and $n^2 \in \{21, 44, 69, 96\}$ which is impossible. If $m = 2$, we get $\ell \in \{21, 22\}$ which gives $\ell \in \{41, 84\}$ which is again impossible. If $m = 3$, then $\ell = 31$ which is rejected while if $m = 4$ then $\ell = 41$ and $n = 9$. Finally the cases $m = 5, 6, \ldots, 9$ are rejected because then

$$(10m + 1)^2 = 100m^2 + 20m + 1 > 100m^2 + 100.$$

Hence $(10m)^2 < \ell^2 < (10m + 1)^2$ which is impossible.

Hence the only solution is $m = 4$, $n = 9$, $\ell = 41$. Since $y = kn^2 = 81k$ is a two-digit number, we must have $k = 1$. These give 1681 as the unique solution.

OC592. For all positive integers n, k, let $f(n, 2k)$ be the number of ways an $n \times 2k$ board can be fully covered by nk dominoes of size 2×1. (For example, $f(2, 2) = 2$ and $f(3, 2) = 3$.) Find all positive integers n such that for every positive integer k, the number $f(n, 2k)$ is odd.

Originally from the 2022 European Girls' Mathematical Olympiad (EGMO), Problem 5.

We received 4 submissions, of which 3 were correct and complete. We present the solution by Oliver Geupel.

We are going to prove that n has the desired property if and only if $n + 1$ is a power of 2, that is, if $n \in \{1, 3, 7, 15, \ldots\}$.

First, suppose that n is even and consider any tiling T of a square $n \times n$ board. Reflecting T in the principal diagonal of the board yields another tiling T' which is distinct from T. Hence, the tilings come in pairs T, T'. Thus, $f(n, n)$ is even.

Next, assume that n is odd, $n = 2m + 1$. Split an $n \times 2k$ board into two $m \times 2k$ boards separated by a $1 \times 2k$ strip S. First, consider tilings of the board with the property that S is fully covered by k dominoes. The number of those tilings is $(f(m, 2k))^2$. Next, consider a tiling T such that k dominoes do not cover S. Reflecting T in S yields another tiling T' distinct from T. Hence, those tilings come in pairs T, T'. Therefore,

$$f(2m + 1, 2k) \equiv f(m, 2k) \pmod 2.$$

We obtain for $q \in \mathbb{N}$ that

$$f\left(2^q - 1, 2k\right) \equiv f\left(2^{q-1} - 1, 2k\right) \equiv \ldots \equiv f(3, 2k) \equiv f(1, 2k) \equiv 1 \pmod 2.$$

It remains to consider the case where $n = 2m + 1$ cannot be written in the form $2^q - 1$. A straightforward induction shows that repeated applications of the rule $2m + 1 \mapsto m$ eventually, lead to an even number. If n is mapped to 2ℓ in this way, we obtain that $f(n, 2\ell)$ must be even.

Editor's Comments. UCLan Cyprus Problem Solving Group mentioned that there is an exact formula for $f(n, 2k)$:

$$f(n, 2k) = \prod_{i=1}^{\lfloor n/2 \rfloor} \prod_{j=1}^{k} \left(4\cos^2 \frac{\pi i}{n+1} + 4\cos^2 \frac{\pi j}{2k+1} \right).$$

However, it appears that the exact formula cannot be used to solve the current question. The exact formula for $f(n, 2k)$ was establish by Kasteleyn in "The statistics of dimers on a lattice, I: The number of dimer arrangements on a quadratic lattice", *Physica*, 27 (12) 1961: 1209-1225 using the fact that $f(n, 2k)$ counts the perfect matchings of a graph. The graph is a quadratic lattice with vertices, the squares of the board, and edges connecting only those squares that share a common side.

OC593. Let $\triangle ABC$ be a triangle, and let C_0, B_0 be the feet of perpendiculars from C and B onto AB and AC respectively. Let Γ be the circumcircle of $\triangle ABC$. Let E be a point on Γ such that $AE \perp BC$. Let M be the midpoint of BC and let G be the second intersection of EM and Γ. Let T be a point on Γ such that TG is parallel to BC. Prove that T, A, B_0, C_0 are concyclic.

Originally 2021 Princeton University Mathematics Contest (Pumac), Individuals Final B, Problem 3.

We received 9 correct solutions. We present 2 solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

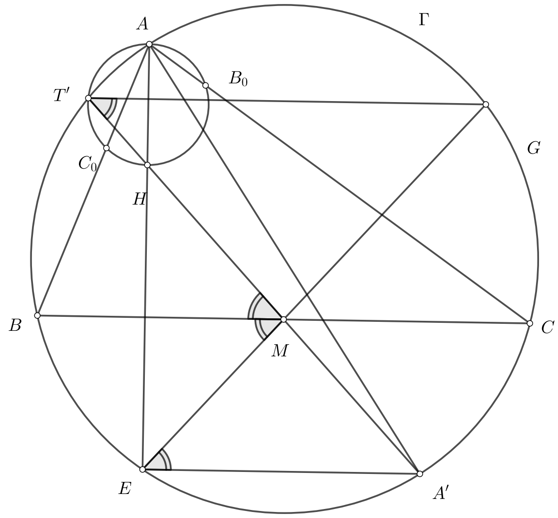
Let ω be the circumcircle of triangle AB_0C_0 and recall that the orthocenter H of ABC belongs on ω and in fact AH is a diameter of ω.

Let T' be the second point of intersection of MH with ω and let A' be the symmetric point of H with respect to M. It is well known that A' lies on Γ with AA' being a diameter of Γ. We have $\angle A'TA = \angle HT'A = 90^\circ$ and therefore T' lies on Γ as well. So it is enough to show that $T' = T$.

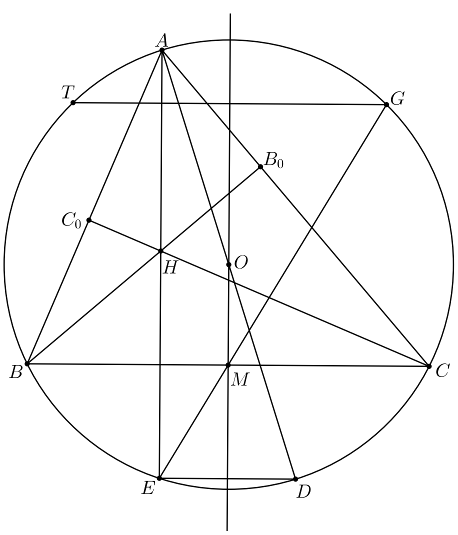
We have

$$\angle MT'G = \angle A'T'G = \angle A'EG = \angle A'EM\,.$$

We also have $AM = MH = HE$ (as E is the reflection of H on BC). So the triangle AME is isosceles and therefore $\angle HME = 2\angle A'EM = 2\angle MT'G$. Thus $\angle T'MB = \frac{1}{2}\angle HME = \angle MT'G$. This shows that $T'G$ is parallel to BM and so $T = T'$ as required.



Solution 2, by Michel Bataille.



Let O and H be the circumcentre and the orthocentre of ΔABC, respectively. Let D be the point diametrically opposite to A on Γ. Let $\mathbf{R_M}, \mathbf{R_{BC}}, \mathbf{R_{OM}}$ denote the reflections across the point M, the line BC, the line OM, respectively.

Since $\overrightarrow{DA} = 2\overrightarrow{DO}$ and $\overrightarrow{HA} = 2\overrightarrow{MO}$ (a well-known result), we have $\overrightarrow{DH} = 2\overrightarrow{DM}$, that is, $\mathbf{R_M}(H) = D$. Since $\mathbf{R_{BC}}(E) = H$ (another well-known result) and $\mathbf{R_M} \circ \mathbf{R_{BC}} = \mathbf{R_{OM}}$, we have $D = \mathbf{R_{OM}}(E)$ and therefore the lines DH and EM are symmetrical with respect to OM. Thus, $\mathbf{R_{OM}}(G)$ is on DH. However, this point is also on TG (since $TG \perp OM$) and on Γ (since Γ is its own reflection in its diameter OM), hence $\mathbf{R_{OM}}(G) = T$. As a result, T, H, M, D are collinear.

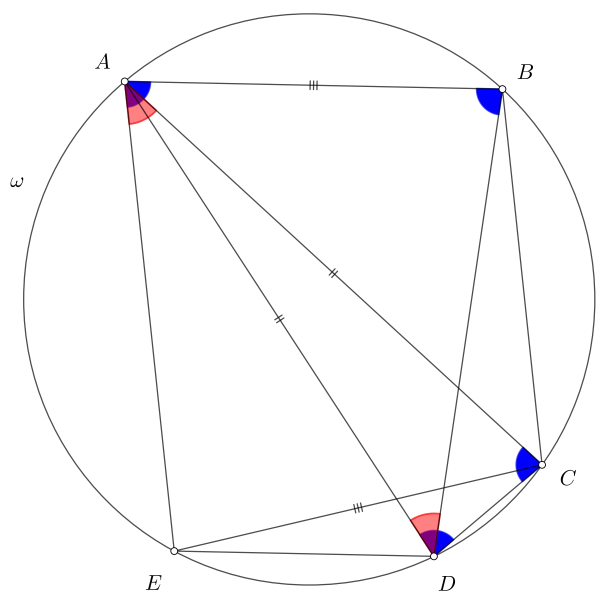
Now, AD being a diameter of Γ, we have $AT \perp TD$. Therefore, we also have $HT \perp TA$ so that T is on the circle with diameter AH. Because $HB_0 \perp B_0A$ and $HC_0 \perp C_0A$, the points B_0, C_0 are on this circle as well and we can conclude that A, T, B_0, C_0 are concyclic.

OC594. A pentagon has vertices labelled A, B, C, D, E in that order counter-clockwise, such that AB and ED are parallel and $\angle EAB = \angle ABD = \angle ACD = \angle CDA$. Furthermore, suppose that $AB = 8$, $AC = 12$, $AE = 10$. Finally, suppose that the area of triangle CDE can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers, so that b is square free, whereas a, c are relatively prime. Find $a + b + c$.

Originally 2021 Princeton University Mathematics Contest (Pumac), Geometry B, Problem 7.

We received 9 submissions of which 8 were correct and complete.

We present the solution by UCLan Cyprus Problem Solving Group.



Since $\angle ABD = \angle ACD$, then A, B, C, D are concyclic. Let ω be the circle through A, B, C, D. Since $\angle EAB = \angle ABC$ and AB is parallel to ED, then $ABDE$ is an isosceles trapezium and so E also belongs on ω.

We have $AB = 8$ and $AD = AC = 12$ (since triangle ACD is isosceles). We also have $BD = AE = 10$ (since $ABDE$ is an isosceles trapezium). Thus, letting

$\vartheta = \angle ABD$ then
$$\cos\vartheta = \frac{10^2+8^2-12^2}{2\cdot 8\cdot 10} = \frac{1}{8}\,.$$
Let $\varphi = \angle BDA$ and let R be the circumradius of ω. Then
$$12 = AD = 2R\sin\vartheta = 2R\sqrt{1-\cos^2\vartheta} = \frac{3R\sqrt{7}}{4} \implies R = \frac{16\sqrt{7}}{7}\,.$$
We now have
$$8 = AB = 2R\sin\varphi \implies \sin\varphi = \frac{\sqrt{7}}{4} \implies \cos\varphi = \frac{3}{4}$$
where we used the fact that $\varphi < \vartheta$ and so $\cos\varphi > 0$. Note that
$$\angle EAC = \angle EAB - \angle CAB = \angle CDA - \angle CDB = \angle BDA\,.$$
It follows that $EC = BA = 8$. From the isosceles trapezium $ABDE$ we have
$$DE = AB - 2(AE)\cos\vartheta = 8 - 20\cdot\frac{1}{8} = \frac{11}{2}\,.$$
Note also that $\angle DEC = \angle DAC = 180^\circ - 2\vartheta$. Thus the area of the triangle CDE is equal to
$$\frac{1}{2}(CE)(ED)\sin(\angle DEC) = \frac{1}{2}\cdot 8\cdot\frac{11}{2}\cdot 2\sin\vartheta\cos\vartheta = 44\cdot\frac{3\sqrt{7}}{8}\cdot\frac{1}{8} = \frac{33\sqrt{7}}{16}\,.$$
So $a = 33, b = 7, c = 16$ and $a+b+c = 56$.

OC595. The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by
$$a_1 = 1,\ a_{n+1} = \frac{a_n}{n} + \frac{n}{a_n}, \quad n \geq 1.$$
Prove that $\lfloor a_n^2 \rfloor = n$ for $n \geq 4$, where $\lfloor x \rfloor$ denotes the integer part of x.

Originally 1996 Bulgarian National Olympiad in Mathematics, Fourth round, Problem 4.

We received 11 submissions of which 9 were correct and complete. We present the solution by Mohammed Aassila.

We prove by induction on n that
$$\sqrt{n} + \frac{1}{(n-1)\sqrt{n}} \leq a_n < \sqrt{n+1} \qquad \forall n \geq 4.$$
Since $a_1 = 1$, then $a_2 = 2$, $a_3 = 2$. If $n = 4$, then we have $a_4 = \dfrac{13}{6} < \sqrt{5}$ and $a_4 - \sqrt{4} = \dfrac{1}{6} = \dfrac{1}{(4-1)\sqrt{4}}$.

Now, assume that the given relation is true for n. Let $f_n(x) = \dfrac{x}{n} + \dfrac{n}{x}$. Observe that f_n is strictly decreasing on $(0, n]$. Since $a_n < \sqrt{n+1}$, then

$$a_{n+1} = f_n(a_n) > f_n(\sqrt{n+1}) = \sqrt{n+1} + \frac{1}{n\sqrt{n+1}},$$

so the first inequality is proved. We have to prove the other inequality, namely $a_{n+1} < \sqrt{n+2}$. Since $a_n \geq \sqrt{n} + \dfrac{1}{(n-1)\sqrt{n}}$, then

$$a_{n+1} = f_n(a_n) \leq f_n\left(\sqrt{n} + \frac{1}{(n-1)\sqrt{n}}\right),$$

so we need to prove that

$$f_n\left(\sqrt{n} + \frac{1}{(n-1)\sqrt{n}}\right) < \sqrt{n+2},$$

which is equivalent to

$$\frac{1}{\sqrt{n}} + \sqrt{n} + \frac{1}{n(n-1)\sqrt{n}} - \frac{\sqrt{n}}{n^2-n+1} < \sqrt{n+2},$$

i.e.

$$\left(\sqrt{n} + \frac{1}{\sqrt{n}} - \sqrt{n+2}\right) + \frac{1}{n(n-1)\sqrt{n}} < \frac{\sqrt{n}}{n^2-n+1}.$$

Since $a - b = \dfrac{a^2-b^2}{a+b}$, then

$$\sqrt{n} + \frac{1}{\sqrt{n}} - \sqrt{n+2} = \frac{\left(\sqrt{n} + \frac{1}{\sqrt{n}}\right)^2 - n - 2}{\sqrt{n+2} + \sqrt{n} + \frac{1}{\sqrt{n}}} < \frac{1}{2n\sqrt{n}}.$$

Moreover, $\dfrac{1}{n(n-1)\sqrt{n}} \leq \dfrac{1}{2n\sqrt{n}}$ for all $n \geq 3$, so

$$\left(\sqrt{n} + \frac{1}{\sqrt{n}} - \sqrt{n+2}\right) + \frac{1}{n(n-1)\sqrt{n}} < \frac{1}{n\sqrt{n}} < \frac{\sqrt{n}}{n^2-n+1},$$

and the conclusion follows by the Principle of Mathematical Induction.

Chebyshev Substitutions for Binomial Integrals

Emilian Sega

One of the most popular integration techniques in calculus is the method of partial fractions. It reduces the integration of a rational function to integrating a number of simpler rational expressions.

Some integrals, although not of rational functions to begin with, can be reduced to integrals of rational functions using various and sometimes sophisticated substitutions.

In this paper we examine binomial integrals of the form $\displaystyle\int x^m(ax^n+b)^p\,dx$, where a and b are real numbers and m, n, and p are rational numbers.

We first show that if at least one of p, $\frac{m+1}{n}$, or $p+\frac{m+1}{n}$ is an integer, then a suitable substitution reduces the computation of the binomial integral to the integral of a rational function.

Let $I=\int x^m(ax^n+b)^p\,dx$ and let $x=t^{1/n}$. Then $dx=\frac{1}{n}t^{1/n-1}dt$, and consequently

$$I=\frac{1}{n}\int t^{m/n}(at+b)^p t^{1/n-1}dt=\frac{1}{n}\int t^{(m+1)/n-1}(at+b)^p dt=\frac{1}{n}\int t^q(at+b)^p dt,$$

where $q=\dfrac{m+1}{n}-1$.

Consider 3 cases.

Case I: p is an integer. Write $q=\frac{r}{s}$, where r and s are relatively prime integers and $s\geq 1$. Let $u=t^{1/s}$. Hence $t=u^s$, $dt=su^{s-1}du$ and

$$I=\frac{s}{n}\int u^{r+s-1}(au^s+b)^p du.$$

Because r, s, and p are integers, we can see that I is now the integral of a rational function.

Case II: $\frac{m+1}{n}$ is an integer. Then $q=\frac{m+1}{n}-1$ is also an integer. As p is rational, we can write $p=\frac{r}{s}$, where r and s are relatively prime integers and $s\geq 1$. Let $u=(at+b)^{1/s}$, so $t=\frac{u^s-b}{a}$, $dt=\frac{s}{a}u^{s-1}du$, and

$$I=\frac{1}{n}\cdot\frac{s}{a}\int\left(\frac{u^s-b}{a}\right)^q u^r u^{s-1}du=\frac{s}{na}\int\left(\frac{u^s-b}{a}\right)^q u^{r+s-1}du.$$

Because q, r, and s are integers, I is now the integral of a rational function.

Case III: $p + \frac{m+1}{n}$ *is an integer.* Hence $p + q = p + \frac{m+1}{n} - 1$ is an integer. Note that

$$I = \frac{1}{n}\int t^q (at+b)^p dt = \frac{1}{n}\int t^{p+q}(at+b)^p t^{-p} dt = \frac{1}{n}\int t^{p+q}(a+bt^{-1})^p dt.$$

As p is rational, we can write $p = \frac{r}{s}$, where r and s are relatively prime integers and $s \geq 1$. Let $u = (a+bt^{-1})^{1/s}$, so $t = \dfrac{b}{u^s - a}$. Then $dt = -\dfrac{bsu^{s-1}}{(u^s-a)^2}du$, so we have:

$$I = \frac{1}{n}\int \left(\frac{b}{u^s-a}\right)^{p+q} u^r \left(-\frac{bsu^{s-1}}{(u^s-a)^2}\right) du = -\frac{b^{p+q+1}s}{n}\int \frac{u^{r+s-1}}{(u^s-a)^{p+q+2}}du$$

Because $p+q$, r, and s are integers, I is now the integral of a rational function.

We now state our main result, summarizing the transformation of I into the integral of a rational function.

Theorem. Let a, b be real numbers. Let m, n, and p be rational numbers, and let $I = \displaystyle\int x^m (ax^n + b)^p\, dx$. Then the following substitutions reduce the calculation of I to the antiderivative of a rational function:

- $u = (x^n)^{1/s}$, if p is an integer and $\dfrac{m+1}{n} - 1 = \dfrac{r}{s}$, with r and s relatively prime integers and $s \geq 1$
- $u = (ax^n + b)^{1/s}$, if $\dfrac{m+1}{n}$ is an integer and $p = \dfrac{r}{s}$, with r and s relatively prime integers and $s \geq 1$
- $u = (a + bx^{-n})^{1/s}$, if $p + \dfrac{m+1}{n}$ is an integer and $p = \dfrac{r}{s}$, with r and s relatively prime integers and $s \geq 1$

Remark. The above substitutions are sometimes referred to as Chebyshev substitutions. P. L. Chebyshev proved that if none of p, $\frac{m+1}{n}$, and $p + \frac{m+1}{n}$ are integers, then I cannot be reduced to the integral of a rational function, and hence I cannot be calculated with elementary integration methods.

Example 1. Find $\displaystyle\int x^{-3/8}(x^{1/4}+1)^{-2}dx$.

Solution. Note that $m = -\frac{3}{8}$, $n = \frac{1}{4}$, $p = -2$. Since p is an integer and $\frac{m+1}{n} - 1 = \frac{3}{2}$, it follows that $r = 3$ and $s = 2$.

Following case I, we let $u = (x^{1/4})^{1/2} = x^{1/8}$. Hence $u^8 = x$, so $dx = 8u^7 du$ and

$$I = \int u^{-3}(u^2+1)^{-2}(8u^7)du = 8\int \frac{u^4}{(u^2+1)^2}du.$$

The new integral in u can be solved using partial fractions. Note that

$$\frac{u^4}{(u^2+1)^2} = \frac{(u^4+u^2)-(u^2+1)+1}{(u^2+1)^2} = \frac{u^2}{u^2+1} - \frac{1}{u^2+1} + \frac{1}{(u^2+1)^2} = 1 - \frac{2}{u^2+1} + \frac{1}{(u^2+1)^2}.$$

To find $\int \frac{1}{(u^2+1)^2} du$ we use a trigonometric substitution: let $u = \tan\theta$, so $du = \sec^2\theta d\theta$, and

$$\begin{aligned} J = \int \frac{1}{\sec^4\theta}\sec^2\theta d\theta = \int \cos^2\theta d\theta = \int \frac{1+\cos 2\theta}{2} d\theta &= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta \\ &= \frac{1}{2}\tan^{-1} u + \frac{u}{2(u^2+1)} + C \end{aligned}$$

(note that if $u = \tan\theta$, then $\sin\theta = \frac{u}{\sqrt{u^2+1}}$ and $\cos\theta = \frac{1}{u^2+1}$). Then

$$I = 8\left(u - 2\tan^{-1} u + \frac{1}{2}\tan^{-1} u + \frac{u}{2(u^2+1)}\right) = 8u - 12\tan^{-1} u + \frac{4u}{u^2+1} + C$$

Thus,

$$I = 8x^{1/8} - 12\tan^{-1}(x^{1/8}) + \frac{4x^{1/8}}{x^{1/4}+1} + C = \frac{8x^{3/8}+12x^{1/8}}{x^{1/4}+1} - 12\tan^{-1}(x^{1/8}) + C.$$

Our next Example comes from the Calculus round of the 2012 Stanford Math Tournament.

Example 2. Calculate $\displaystyle\int_{2^5}^{3^5} \frac{1}{x - x^{3/5}} dx$.

Solution. Note that

$$I = \int_{2^5}^{3^5} \frac{1}{x - x^{3/5}} dx = \int_{2^5}^{3^5} \frac{1}{x(1 - x^{-2/5})} dx = \int_{2^5}^{3^5} x^{-1}(1 - x^{-2/5})^{-1} dx.$$

Then $m = -1$, $n = -\frac{2}{5}$, and $p = -1$. Since p is an integer and $\frac{m+1}{n} - 1 = -1$, it follows that $r = -1$, $s = 1$.

Following case I, we make the substitution $u = (x^{-2/5})^1$, so $u = x^{-2/5}$. Hence $x = u^{-5/2}$, $dx = -\frac{5}{2}x^{-7/2}du$. Also, when $x = 2^5$, $u = (2^5)^{-2/5} = 2^{-2} = \frac{1}{4}$, while if $x = 3^5$, $u = (3^5)^{-2/5} = 3^{-2} = \frac{1}{9}$.

Hence

$$I = \int_{1/4}^{1/9} u^{5/2}(1-u)^{-1}\left(-\frac{5}{2}u^{-7/2}\right) du = \frac{5}{2}\int_{1/4}^{1/9} \frac{1}{u(u-1)} du.$$

The last integral can now be easily solved with partial fractions:

$$I = \frac{5}{2}\int_{1/4}^{1/9}\left(\frac{1}{u-1} - \frac{1}{u}\right)du = \frac{5}{2}(\ln|u-1| - \ln|u|)\Big|_{1/4}^{1/9} = \frac{5}{2}\ln\left|1-\frac{1}{u}\right|\Big|_{1/4}^{1/9} = \frac{5}{2}\ln\frac{8}{3}.$$

Example 3. The area below the graph of $f(x) = x^3(1-x^{2/3})^{3/2}$ between $x = 0$ and $x = 1$ can be written in the form $\dfrac{k}{h}$, where k and h are relatively prime positive integers. Find $k + h$.

Solution. Note that $f(x)$ is continuous and $f(x) \geq 0$ for all x in $[0, 1]$. Thus, the area below the graph of $f(x)$ is given by

$$\text{Area} = \int_0^1 x^3(1-x^{2/3})^{3/2}dx.$$

Let $I = \displaystyle\int_0^1 x^3(1-x^{2/3})^{3/2}dx$; with our notations from above, $m = 3$, $n = \frac{2}{3}$, $p = \frac{3}{2}$, and $\frac{m+1}{n} = 6$. Since $\frac{m+1}{n}$ is an integer, following case II we set $r = 3$, $s = 2$, and let $u = (1-x^{2/3})^{1/2}$.

Then $x^{2/3} = 1 - u^2$, so $\dfrac{2}{3}x^{-1/3}dx = -2udu$, hence $dx = -3x^{1/3}udu$, and

$$I = \int_1^0 x^3u^3(-3x^{1/3}u)du = 3\int_0^1 x^{10/3}u^4du = 3\int_0^1 u^4(1-u^2)^5du$$

Using the Binomial Theorem, $(1-u^2)^5 = 1 - 5u^2 + 10u^4 - 10u^6 + 5u^8 - u^{10}$, so

$$\begin{aligned} I &= 3\int_0^1 (u^4 - 5u^6 + 10u^8 - 10u^{10} + 5u^{12} - u^{14})du \\ &= \left(\frac{3}{5}u^5 - \frac{15}{7}u^7 + \frac{10}{3}u^9 - \frac{30}{11}u^{11} + \frac{15}{13}u^{13} - \frac{1}{5}u^{15}\right)\Big|_0^1 \\ &= \frac{256}{15015}. \end{aligned}$$

Thus $k = 256$, $h = 15015$, so $k + h = 15271$.

Example 4. Find $\displaystyle\int x^{-1/2}(1-x^{-4/3})^{-5/8}dx$.

Solution. Let $I = \displaystyle\int x^{-1/2}(1-x^{-4/3})^{-5/8}dx$ and note that $m = -\frac{1}{2}$, $n = -\frac{4}{3}$, $p = -\frac{5}{8}$, $\frac{m+1}{n} = -\dfrac{3}{8}$, and $p + \frac{m+1}{n} = -1$.

Following case III, we take $r=-5$, $s=8$, and we let $u=(x^{4/3}-1)^{1/8}$. Then $u^8+1=x^{4/3}$, so $8u^7du=\dfrac{4}{3}x^{1/3}dx$, hence $dx=7u^7x^{-1/3}du$. Rewrite I as

$$\begin{aligned} I=\int x^{-1/2}(1-x^{-4/3})^{-5/8}dx &= \int x^{-1/2}(x^{-4/3}(x^{4/3}-1))^{-5/8}dx \\ &= \int x^{-1/2}x^{5/6}(x^{4/3}-1)^{-5/8}dx \end{aligned}$$

and change to u. Then we have

$$I=\int x^{1/3}u^{-5}6u^7x^{-1/3}du=\int 6u^2du=2u^3+C=2(x^{4/3}-1)^{3/8}+C.$$

Example 5. Find $\displaystyle\int\sqrt{\frac{x^{2020}}{(1-x^{2022})^3}}dx$.

Solution. Let I denote the integral in the problem and note that it can be rewritten as $I=\displaystyle\int x^{1010}(1-x^{2022})^{-3/2}dx$.

We then have $m=1010$, $n=2022$, $p=-\frac{3}{2}$, $\frac{m+1}{n}=\dfrac{1}{2}$, and $p+\frac{m+1}{n}=-1$.

Because $p+\frac{m+1}{n}$ is an integer, following case III with $r=-3$, $s=2$, we let $u=(x^{-2022}-1)^{1/2}$. Thus

$$u^2+1=x^{-2022},\quad 2udu=-2022x^{-2023}dx,$$

and hence

$$dx=-\frac{1}{1011}ux^{2023}du.$$

Also, $x^{2022}=\dfrac{1}{u^2+1}$, so $1-x^{2022}=\dfrac{u^2}{u^2+1}$.

Consequently, I can be rewritten as

$$\begin{aligned} I=\int -\frac{1}{1011}x^{1010}\frac{u^{-3}}{(u^2+1)^{-3/2}}\cdot ux^{2023}du &= -\frac{1}{1011}\int\frac{u^{-2}}{x^{3033}}\cdot x^{3033}du \\ &= -\frac{1}{1011}\int u^{-2}du=\frac{1}{1011u}+C \end{aligned}$$

Finally, since $u=(x^{-2022}-1)^{1/2}$, we have that

$$I=\frac{1}{1011\sqrt{x^{-2022}-1}}+C=\frac{x^{1011}}{1011\sqrt{1-x^{2022}}}+C.$$

The following exercises are left for the interested reader.

1. Find $\displaystyle\int \frac{x^{2021}}{\sqrt[3]{1+x^{674}}}dx.$
2. Find $\displaystyle\int x^{-3/5}(1+x^{2/5})^{-3/4}dx.$
3. Find $\displaystyle\int \frac{1}{\sqrt{x}(\sqrt[4]{x}+1)^{100}}dx.$
4. Find $\displaystyle\int \frac{x^{2009}}{1+x^{2680}}dx.$

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Reading a Math Book

Solutions to No. 1

Yagub Aliyev

The tatements of the problems in this section originally appear in 2022: 48(8), p. 481. The problems were selected from [1].

PB1. Prove that

a) if $g(x) = (f(x))^{\frac{1}{2}}$ is nonzero, then $\dfrac{g'(x)}{g(x)} = \dfrac{f'(x)}{2f(x)}$.

b) if $h(x) = \frac{f(x)}{g(x)}$ is nonzero, then $\dfrac{h'(x)}{h(x)} = \dfrac{f'(x)}{f(x)} - \dfrac{g'(x)}{g(x)}$.

Solution. (By the author in collaboration with Laman Panakhova.) One can use the limit definition for derivative to solve this problem. We will present shorter solutions using the rules of differentiation.

a) Use the chain rule $(h(f(x)))' = h'(f(x))f'(x)$ for $h(u) = \sqrt{u}$, $u = f(x)$, to write $\left(\sqrt{f(x)}\right)' = \frac{f'(x)}{2\sqrt{f(x)}}$, and then divide both sides of the resulting equality by $g(x) = (f(x))^{\frac{1}{2}}$.

b) Use the quotient rule to write $h'(x) = \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$ and then divide both sides of the resulting equality by $h(x) = \frac{f(x)}{g(x)}$.

PB2. Let $f(x) = \frac{1}{2p}x^2 + x$ and $g(x) = a - \sqrt{a^2 - 2ax - x^2}$. Find

$$\lim_{x\to 0} \frac{f(x) - g(x)}{x^2}.$$

Solution. The limit does not exist if $a \leq 0$. If $a > 0$ then one can use L'Hospital's rule twice to find the limit. It is also possible to find the limit using the method of multiplication by the conjugate. The following is the solution by Ong See Hai.

$$\begin{aligned}
&\lim_{x\to 0} \frac{\frac{x^2}{2p} + x - a + \sqrt{a^2 - 2ax - x^2}}{x^2} \\
&= \lim_{x\to 0} \frac{1}{2p} + \lim_{x\to 0} \left(\frac{x - a + \sqrt{a^2 - 2ax - x^2}}{x^2} \cdot \frac{a - x + \sqrt{a^2 - 2ax - x^2}}{a - x + \sqrt{a^2 - 2ax - x^2}} \right) \\
&= \frac{1}{2p} + \lim_{x\to 0} \frac{a^2 - 2ax - x^2 - (a - x)^2}{x^2(a - x + \sqrt{a^2 - 2ax - x^2})} = \frac{1}{2p} - \frac{1}{a}.
\end{aligned}$$

Note that if $a = 2p$, then $g(x)$ is the osculating circle of $f(x)$ at $x = 0$.

PB3. Let $f(x) = \frac{1}{x^2+1}$. Show that the equation $\frac{d^n f(x)}{dx^n} = 0$ has n real roots.

Solution. This was probably the hardest problem among these 5 questions. We will present several approaches.

Approach 1, by Ong See Hai. Use the generalisation of Rolle's theorem.

Lemma.
$$\frac{d^n f(x)}{d\,x^n} = \frac{P_n(x)}{(x^2+1)^{n+1}},$$
where $n \geq 1$ and $P_n(x)$ is a polynomial of degree n.

Proof. We proceed by induction. It is easy to derive that $\frac{df(x)}{dx} = \frac{-2x}{(x^2+1)^2}$. Hence the base case is true. Now, suppose that the claim is true for some $n > 1$, and consider $n+1$. We have to show that the claim is true for $n+1$ too.

$$\begin{aligned}
\frac{d^{n+1} f(x)}{d\,x^{n+1}} &= \frac{d}{dx}\frac{P_n(x)}{(x^2+1)^{n+1}} \\
&= \frac{(x^2+1)^{n+1}\ P_n'(x) - (n+1)(x^2+1)^n(2x)\ P_n(x)}{(x^2+1)^{2n+2}} \\
&= \frac{(x^2+1)\ P_n'(x) - (n+1)(2x)\ P_n(x)}{(x^2+1)^{n+2}} \equiv \frac{P_{n+1}(x)}{(x^2+1)^{n+2}},
\end{aligned}$$

where we set $P_{n+1}(x)$ to be equal to the numerator of the preceding expression.

It remains to show that the coefficient of x^{n+1} in $P_{n+1}(x)$ is non-zero. From the Induction Hypothesis, let the coefficient of x^n in $P_n(x)$ be $a_n \neq 0$. Then, the leading coefficient of x^{n-1} in $P_n'(x)$ is na_n. Hence, the coefficient of x^{n+1} in $P_{n+1}(x)$ is:

$$na_n - 2(n+1)a_n = a_n(n-2n-1) = -(n+2)\ a_n,$$

which is clearly non-zero.

Corollary. $\lim_{x\to\pm\infty} \frac{d^n f(x)}{dx^n} = 0$.

We are now ready to prove the main result proper.

Let $P(n)$ be the proposition that the equation $\frac{d^n f(x)}{dx^n} = 0$ has n distinct real roots. When $n = 1$, we have $\frac{d\ f(x)}{dx} = \frac{-2x}{(x^2+1)^2}$. Since the only real root is $x = 0$, the base case is true. Now, suppose that $P(n)$ is true for some $n > 1$. Let $g(x) = \frac{d^n f(x)}{dx^n} = 0$ have n distinct real roots. We wish to show that $P(n+1)$ is true too; that is, $g'(x) = \frac{d^{n+1} f(x)}{dx^{n+1}} = 0$ has $n+1$ distinct real roots.

Label the n zeroes of $g(x)$ as $x_1, x_2, ..., x_n$ in strictly increasing order. By Rolle's Theorem, there exists $c_i \in (x_i, x_{i+1})$ for each $i \in \{1, 2, .., n-1\}$, such that $g'(c_i) = 0$. Thus $g'(x)$ has at least $n-1$ distinct roots in the interval $[x_1, x_n]$.

Since x_n is the largest root, g has at least one critical point in the interval (x_n, ∞), i.e. there exists $c > x_n$ such that $g'(c) = 0$. To see why, suppose for the sake of contradiction that $g'(c) \neq 0$ for all $c > x_n$. Then, $|g'(c)| > 0$. Let $m = \inf\{|g'(c)|\} > 0$. Note that $\lim_{x\to\infty} \frac{g(x)-g(x_n)}{x-x_n} = 0$. Hence, for all $\epsilon > 0$, there exists L such that $x > L \Rightarrow |\frac{g(x)-g(x_n)}{x-x_n}| < \epsilon$. Now, by the Mean Value Theorem, there exists $c' \in (x_n, x)$ such that $g'(c') = \frac{g(x)-g(x_n)}{x-x_n}$. We have $|g'(c')| \geq m$, i.e. for all values of $x > x_n$, $|\frac{g(x)-g(x_n)}{x-x_n}| \geq m$ too. But choosing $\epsilon < m$ results in an absurdity. By a similar argument, g has at least one turning point in the interval $(-\infty, x_1)$, i.e. there exists $c < x_1$ such that $g'(c) = 0$.

Hence, $g'(x)$ has at least $(n-1)+2 = n+1$ distinct real roots. But since the roots of $g'(x)$ are precisely the roots of the polynomial $P_{n+1}(x)$ which has degree $n+1$, by the Fundamental Theorem of Algebra, $g'(x) = \frac{d^{n+1}f(x)}{dx^{n+1}}$ must have exactly $n+1$ distinct real roots.

Since $P(1)$ is true and $P(n) \Rightarrow P(n+1)$ for all $n > 1$, by the Principle Of Mathematical Induction, $P(n)$ is true for all natural numbers n.

Approach 2, by the author. Use Rolle's theorem itself.

One can use substitution $x = \tan t$ for $|x| < \frac{\pi}{2}$, to make the zeros of $f(x) = \frac{1}{x^2+1}$ at $\pm\infty$ finite. Indeed, $g(t) = f(\tan t) = \frac{1}{\tan^2 t+1} = \cos^2 t$ has zeros only at the endpoints of the interval $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. By Rolle's theorem there is at least one zero of $g'(t)$ in the interval. On the other hand one can show as in Approach 1, that there is at most one zero of $g'(t)$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So, there is only one zero of $h'(t)$ in the interval. This zero is also the zero of $f'(x) = \frac{df(x)}{dx} = \frac{g'(t)}{x'(t)} = g'(t)\cos^2 t$, which also has zeros at $t = \pm\frac{\pi}{2}$. By applying Rolle's theorem again and again we can show that $f^{(n)}(x) = \frac{df^{(n-1)}(x)}{dt}\cos^2 t$ has n zeros in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and two more zeros at $t = \pm\frac{\pi}{2}$, which solves the problem.

Approach 3, by the author in collaboration with Hossaena Tedla. Use Sturm's method for the number of zeros of a polynomial.

Let us define polynomials $p_n(x)$ by

$$\left(\frac{1}{x^2+1}\right)^{(n)} = \frac{(-1)^n n! p_n(x)}{(x^2+1)^{n+1}}.$$

For any particular polynomial p_n, one can use Sturm's method [3] to check that it has exactly n real roots. For example, Sturm's sequence for $p_6(x) = 7x^6 - 35x^4 + 21x^2 - 1$ is

$$x^6 - 5x^4 + 3x^2 - \frac{1}{7},\ x^5 - \frac{10}{3}x^3 + x,\ x^4 - \frac{6}{5}x^2 + \frac{3}{35},\ x^3 - \frac{3}{7}x,\ x^2 - \frac{1}{9},\ x,\ 1.$$

We find the signs of these polynomials at $-\infty$ and $+\infty$, as $+-+-+-+$ and $+++++++$, respectively. The number of sign changes at $-\infty$ and $+\infty$ are 6 and 0, respectively. Therefore the number of real roots for for $p_6(x)$ is $6-0=6$.

n	p_n
0	1
1	$2x$
2	$3x^2 - 1$
3	$4x^3 - 4x$
4	$5x^4 - 10x^2 + 1$
5	$6x^5 - 20x^3 + 6x$
6	$7x^6 - 35x^4 + 21x^2 - 1$

For arbitrary n, one can use the fact that $p_n, p_{n-1}, \ldots, p_2, p_1, p_0$ form a generalized Sturm sequence [4]. Using

$$\begin{aligned} f^{(n)}(x) &= \left(\frac{1}{x^2+1}\right)^{(n)} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)^{(n)} \\ &= \frac{(-1)^n n!}{2i}\left(\frac{1}{(x-i)^{n+1}} - \frac{1}{(x+i)^{n+1}}\right) = \frac{(-1)^n n!\left((x+i)^{n+1} - (x-i)^{n+1}\right)}{2i\left(x^2+1\right)^{n+1}}, \end{aligned}$$

one can check that

$$p_n(x) = \binom{n+1}{1}x^n - \binom{n+1}{3}x^{n-2} + \binom{n+1}{5}x^{n-4} - \ldots,$$

and therefore, $p_{n-1}(x) = \frac{p_n'(x)}{n+1}$. Furthermore, using $\left(\frac{p_n(x)}{(x^2+1)^{n+1}}\right)' = -(n+1)\frac{p_{n+1}(x)}{(x^2+1)^{n+2}}$, we find that

$$p_{n+1} = 2xp_n - (x^2+1)p_{n-1}.$$

In particular, this implies that if $x = c$ is a root of p_n then $p_{n+1}(c)p_{n-1}(c) < 0$. Also, this implies that p_{n-1} and p_n do not have common zeros, or which is the same p_n does not have non-simple (double, triple,...) zeros. Indeed, if p_{n-1} and p_n had a common zero, then p_{n-2} would have the same zero. But $p_0 = 1$ does not have any zeros, which is a contradiction.

We checked that $p_n, p_{n-1}, \ldots, p_2, p_1, p_0$ form a generalized Sturm sequence [3,4]. Therefore we can use it to count the number of zeros of p_n. The signs of these polynomials at $-\infty$ and $+\infty$, are $+-+-\ldots$ ($n+1$ alternating signs) and $++\ldots+$ ($n+1$ times "+" signs), respectively. The number of sign changes at $-\infty$ and $+\infty$ are n and 0, respectively. Therefore, the number of real roots of $p_n(x)$ is $n-0=n$.

Note that similar treatment of the function $f(x) = e^{-x^2}$ gives Hermite polynomials $H_n(x)$. See [5] for more information about Hermite polynomials.

PB4.

a) Find the area between the parabola $f(x) = \frac{1}{2p}x^2$ and its chord AB which is perpendicular to the y-axis. Here $A(a, f(a)),\ B(b, f(b)),\ b = -a > 0,\ p > 0$.

b) Show also that this area is 2/3 of the area of the rectangle bounded by the lines AB, x-axis, $x = -a,\ x = a$.

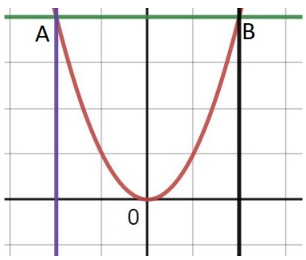
Solution. This problem is related to the quadrature of the parabola by Archimedes which in our case says that the area of a parabolic segment is $\frac{4}{3}$ the area of inscribed triangle OAB, where $O(0,0)$ is the origin. The following is the solution by Ong See Hai.

a) Simple calculation shows that the area under the parabola is

$$\int_a^b \frac{1}{2p}\, x^2\, dx = \frac{1}{2p}\left[\frac{x^3}{3}\right]_a^{-a} = \frac{1}{2p}\left[\frac{-a^3}{3} - \frac{a^3}{3}\right] = \frac{1}{2p}\left[\frac{-2a^3}{3}\right] = \frac{-a^3}{3p} = \frac{b^3}{3p}.$$

area of rectangle is $\frac{b^2}{2p} \cdot -2a = \frac{-ab^2}{p} = \frac{b^3}{p}$. Hence, desired area is given by the area of rectangle minus area under parabola, which equals to $\frac{2b^3}{3p}$ units^2.

b) This follows immediately from our work in (a): clearly, $\frac{2b^3}{3p}$ units^2 is $\frac{2}{3}$ of $\frac{b^3}{p}$ units^2.



PB5.

a) Let $f(x) = \sin\frac{1}{x}$. Does this function have a limit at $x = 0$?

Now, for parts (b) and (c) below, suppose that

$$f(x) = \begin{cases} x^n \cdot \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

b) Show that if $n = 1$, then $f(x)$ is continuous at $x = 0$ and $f'(0)$ does not exist.

c) Show that if $n = 2$, then $f(x)$ is continuous at $x = 0$, $f'(0)$ exists but $f'(x)$ is not continuous at $x = 0$.

Solution. We will solve only the first part. The remaining parts are similar to Problem 2 in [2].

a) The function $f(x) = \sin\frac{1}{x}$ does not have a limit at $x = 0$. Take, for example the sequences $x_n = \frac{1}{\frac{\pi}{2}+2\pi n}$ and $x'_n = \frac{1}{-\frac{\pi}{2}+2\pi n}$ for $n = 1, 2, \ldots$, both of which tend to zero, but $\lim_{n\to\infty} f(x_n) = 1$ and $\lim_{n\to\infty} f(x'_n) = -1$.

References

[1] Félix Lucienne, *Exposé moderne des mathématiques élémentaires*, Dunot Editeur, Paris, 1962, 1966; Russian translation: Prosveshenie, 1967; German translation: Vieweg Teubner Verlag, 1969.

[2] Yagub Aliyev, Reading a Math Book: No. 1 (Lucienne Félix: Exposé moderne des mathématiques élémentaires), *Crux Mathematicorum*, Vol. 48(8), October 2022, 477-482.

[3] A. Kurosh, *Higher Algebra*, Mir Publishers, 1980. `https://archive.org/details/kurosh-higher-algebra/page/237/mode/2up`

[4] Sturm's theorem, From Wikipedia, the free encyclopedia. `https://en.wikipedia.org/wiki/Sturm%27s_theorem#Generalization`

[5] Hermite polynomials, From Wikipedia, the free encyclopedia. `https://en.wikipedia.org/wiki/Hermite_polynomials`

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 30, 2023**.

4811. *Proposed by Nguyen Viet Hung.*

Find all positive integers n such that $\sqrt{n^3+1}+\sqrt{n+2}$ is a positive integer.

4812. *Proposed by Michel Bataille.*

Let $ABCD$ be a tetrahedron. Prove that $a = BC^2 + DA^2$, $b = CA^2 + DB^2$, $c = AB^2 + DC^2$ are the sides of a triangle. For which tetrahedra is this triangle equilateral?

4813. *Proposed by Mihai Prunescu.*

Find all plane triangles ABC such that every side is equal with the opposed angle: $BC = \angle A$, $AC = \angle B$ and $AB = \angle C$.

4814. *Proposed by Mihaela Berindeanu.*

In triangle ABC, let G be the centroid and I be the incenter. Suppose that GI is parallel to BC, AI cuts BC in E and the circumcircle in D. Show that $BD = 2ED$.

4815. *Proposed by Aravind Mahadevan.*

In triangle ABC, let a, b, c denote the lengths of the sides BC, CA and AB, respectively. If $\tan A$, $\tan B$ and $\tan C$ are in harmonic progression, prove that a^2, b^2 and c^2 are in arithmetic progression. Does the converse hold?

4816. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Let $a, b, k \geq 0$. Calculate

$$\lim_{n\to\infty} \int_0^1 x^k \sqrt{\frac{a}{x} + bn^2x^{2n}}\,\mathrm{d}x.$$

4817. *Proposed by Goran Conar.*

Let $a, b, c > 0$ be real numbers such that $abc = 1$. Prove that the following inequality holds

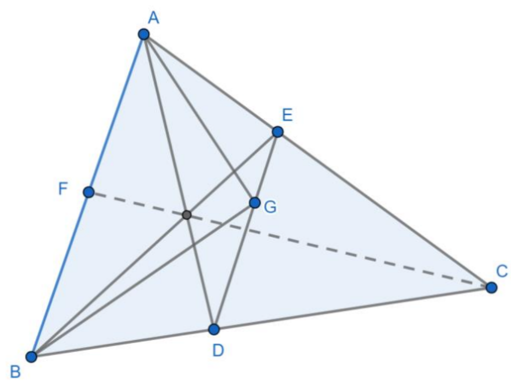
$$\frac{a^7 + a^3 + bc}{a + bc + 1} + \frac{b^7 + b^3 + ca}{b + ca + 1} + \frac{c^7 + c^3 + ab}{c + ab + 1} \geq 3\,.$$

When does equality occur?

4818. *Proposed by Yagub Aliyev.*

In triangle ABC, let $E \in AC, D \in BC, F \in AB$ such that AD, BE, CF are concurrent. Let $G \in ED$. Prove that $\left(\frac{AF}{FB}\right)^2 = \frac{DG}{GE}$ if and only if

$$\frac{1}{[ADE]^2} + \frac{1}{[BDE]^2} = \frac{1}{[AEG]^2 + [BDG]^2}$$



4819. *Proposed by Daniel Sitaru.*

Let $f : [0, 1] \to [0, 1]$ be a continuous function and $0 < a \leq b < 1$.
Prove that:

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt \geq \int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} f(t)dt\left(\int_0^{\frac{a+b}{2}} f(t)dt + \int_0^{\frac{2ab}{a+b}} f(t)dt\right)$$

4820. *Proposed by George Apostolopoulos.*

Let $ABCD$ be a square with side length a. Take interior points K, L on the sides BC and CD respectively so that the perimeter of triangle KCL equals $2a$. If the diagonal BD intersects the segments AK, AL in points N, M respectively, prove that the area of triangle AMN equals to the area of quadrilateral $KLMN$.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 avril 2023**.

4811. *Soumis par Nguyen Viet Hung.*

Trouvez tous les entiers positifs n tels que $\sqrt{n^3+1}+\sqrt{n+2}$ est un entier positif.

4812. *Soumis par Michel Bataille.*

Soit $ABCD$ un tétraèdre. Prouvez que $a = BC^2 + DA^2$, $b = CA^2 + DB^2$, $c = AB^2 + DC^2$ sont les côtés d'un triangle. Pour quels tétraèdres ce triangle est-il équilatéral?

4813. *Soumis par Mihai Prunescu.*

Trouvez tous les triangles ABC du plan pour lesquels chaque côté est égal à l'angle opposé, c'est-à-dire $BC = \angle A$, $AC = \angle B$ et $AB = \angle C$.

4814. *Soumis par Mihaela Berindeanu.*

Soit G le centre de masse du triangle ABC et soit I le centre du cercle inscrit à ABC. Supposons que GI est parallèle à BC, que AI rencontre BC en E et la circonférence en D. Montrez que $BD = 2ED$.

4815. *Soumis par Aravind Mahadevan.*

Dans le triangle ABC, on désigne par a, b et c les longueurs des côtés BC, CA et AB, respectivement. Si $\tan A$, $\tan B$ et $\tan C$ forment une progression harmonique, prouvez que a^2, b^2 et c^2 forment une progression arithmétique. La réciproque est-elle vraie?

4816. *Soumis par Ovidiu Furdui et Alina Sîntămărian.*

Étant donné $a, b, k \geq 0$. Calculez

$$\lim_{n\to\infty} \int_0^1 x^k \sqrt{\frac{a}{x} + bn^2x^{2n}}\,\mathrm{d}x.$$

4817. *Soumis par Goran Conar.*

Soient $a, b, c > 0$ des nombres réels tels que $abc = 1$. Prouvez que l'inégalité suivante est vérifiée

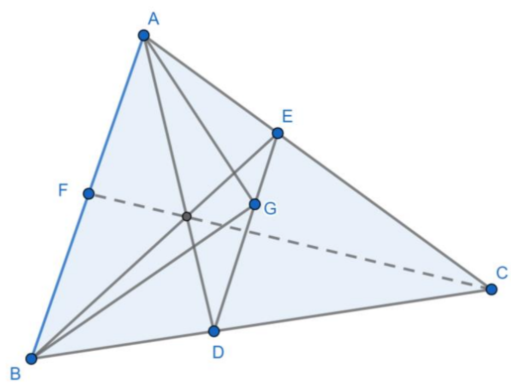
$$\frac{a^7 + a^3 + bc}{a + bc + 1} + \frac{b^7 + b^3 + ca}{b + ca + 1} + \frac{c^7 + c^3 + ab}{c + ab + 1} \geq 3\,.$$

Quand a-t-on égalité?

4818. *Soumis par Yagub Aliyev.*

Dans le triangle ABC, soit $E \in AC$, $D \in BC$ et $F \in AB$ tels que AD, BE et CF sont concourants. Soit $G \in ED$. Montrez que $\left(\frac{AF}{FB}\right)^2 = \frac{DG}{GE}$ si et seulement si

$$\frac{1}{[ADE]^2} + \frac{1}{[BDE]^2} = \frac{1}{[AEG]^2 + [BDG]^2}$$



4819. *Soumis par Daniel Sitaru.*

Soit $f : [0, 1] \to [0, 1]$ une fonction continue et $0 < a \leq b < 1$.
Montrez que

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt \geq \int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} f(t)dt \left(\int_0^{\frac{a+b}{2}} f(t)dt + \int_0^{\frac{2ab}{a+b}} f(t)dt \right)$$

4820. *Soumis par George Apostolopoulos.*

Soit $ABCD$ un carré de longueur de côté a. Considérons des points intérieurs K et L sur les côtés BC et CD respectivement, de sorte que le périmètre du triangle KCL soit égal à $2a$. Si la diagonale BD coupe les segments AK et AL en des points N et M respectivement, prouvez que l'aire du triangle AMN est égale à l'aire du quadrilatère $KLMN$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2022: 48(7), p. 421–424.

4761. *Proposed by Michel Bataille.*

Let ABC be a triangle neither isosceles nor right-angled, and let O be its circumcentre. Let A', B', C' be the respective reflections of A, B, C in O and let A_1, B_1, C_1 be the reflections of O in $B'C', C'A', A'B'$, respectively. Prove that the circumcircles of the triangles OAA_1, OBB_1, OCC_1 and ABC have a common point.

We received nine submissions, all of which are correct, and feature two of them.

Solution 1, by Oliver Geupel.

Consider the problem in the plane of complex numbers where the affix of any point Z is denoted by the respective lower-case letter z. Suppose that the affix of O is 0 and $a\overline{a} = b\overline{b} = c\overline{c} = 1$. We show that the circles (OAA_1), (OBB_1), (OCC_1) and (ABC) have the common point S with affix

$$s = -\frac{ab + bc + ca}{a + b + c}.$$

Note that $a_1 = b' + c' = -b - c$. Since $\angle A$ is not a right angle, the points A_1 and O are distinct. Since $AB \neq AC$, the points A, A_1, and O are not collinear. Hence, $\triangle OAA_1$ is non-degenerate. Let D denote the circumcenter of $\triangle OAA_1$. By a standard formula (see [1], p. 108), we have

$$d = \frac{aa_1\left(\overline{a} - \overline{a_1}\right)}{\overline{a}a_1 - a\overline{a_1}} = \frac{a(-b-c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}{\frac{1}{a}(-b-c) - a\left(-\frac{1}{b} - \frac{1}{c}\right)} = \frac{a(ab + bc + ca)}{bc - a^2}.$$

Thus,

$$\overline{d} = \frac{\frac{1}{a}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)}{\frac{1}{bc} - \frac{1}{a^2}} = \frac{a + b + c}{a^2 - bc}.$$

The second intersection S of the circumcircles of $\triangle ABC$ and $\triangle OAA_1$ is the reflection of A in the axis OD. We know (see [1], p. 98) that the affix of the reflection W of a point Z over a line XY is

$$w = \frac{(x - y)\overline{z} + \overline{x}y - x\overline{y}}{\overline{x} - \overline{y}}.$$

Therefore,

$$s = \frac{d}{a\overline{d}} = -\frac{ab + bc + ca}{a + b + c}.$$

Similarly, S lies on the circumcircles of $\triangle OBB_1$ and of $\triangle OCC_1$.

References

[1] Chen, E. (2016), *Euclidean Geometry in Mathematical Olympiads.* Washington, DC: Mathematical Association of America.

Solution 2, by the UCLan Cyprus Problem-Solving Group.

Assume that A, B, C are points of the unit circle and are represented by the complex numbers a, b, c. Then B', C' are represented by $-b, -c$, their midpoint by $-(b+c)/2$, and so A_1 is represented by $-(b+c)$.

Let z be the complex number representing the point of intersection of the circumcircles of triangles OAA_1 and ABC. Since $z, 0, a, -(b+c)$ represent concyclic points, then the cross ratio in any order, for example

$$\frac{z-(-(b+c))}{z-0}\cdot\frac{a-0}{a-(-(b+c))},$$

must be real; thus we have

$$\frac{z+b+c}{z}\cdot\frac{a}{a+b+c}=\frac{\overline{z+b+c}}{\overline{z}}\cdot\frac{\overline{a}}{\overline{a+b+c}}=\frac{bc+cz+zb}{bc}\cdot\frac{bc}{bc+ca+ab}\,.$$

This leads to the quadratic equation

$$z(a+b+c)(z(b+c)+bc)-a(z+b+c)(ab+bc+ca)=0\,.$$

The product of the roots of this equation is

$$-\frac{a(b+c)(ab+bc+ca)}{(a+b+c)(b+c)}\,.$$

The number is well-defined by our assumptions: Should $b+c=0$, then b and c would represent the ends of a diameter, and the triangle would be right-angled at A; if $a+b+c=0$, then the center of gravity is equal to the circumcenter, in which case the triangle would be equilateral.

Since both circles contain the point A, an obvious root of the quadratic equation is a, whence the second root must be

$$-\frac{ab+bc+ca}{a+b+c}\,.$$

This is symmetric in a, b, c, so z also lies on the circumcircles of OBB_1 and OCC_1. *Remark.* Note that we used the assumption that ΔABC is not isosceles only to eliminate the possibility of an equilateral triangle (so that $a+b+c$ is nonzero). Of course, when the triangle is not isosceles, then $0, a, -(b+c)$ are not collinear. Our proof continues to be valid in the case of a nonequilateral isosceles triangle if we replace the circumcircle of OAA_1 by the straight line passing through O, A, A_1 should they be collinear.

4762. *Proposed by Didier Pinchon and George Stoica.*

Prove that

$$\sum_{i=1}^{n} a_i \left(\prod_{1\le j\le n, j\ne i} \left(\frac{a_i + a_j}{a_i - a_j} \right) \right) = \sum_{i=1}^{n} a_i$$

for any distinct complex numbers $a_1, \ldots, a_n$.

All 7 received submissions were correct. We present two solutions.

Solution 1, by Ulrich Abel.

We prove the result by using the following well-known facts on divided differences (eg Lemma 1 in J. Schwaiger, On a characterization of polynomials by divided differences, *Aequationes Mathematicae*, **48** (1994), 317–324):

$$[a_1, \ldots, a_n]\, f(x) \quad := \quad \sum_{i=1}^{n} \frac{f(a_i)}{\prod_{1\le j\le n, j\ne i} (a_i - a_j)},$$

$$[a_1, \ldots, a_n]\, x^r \quad = \quad \begin{cases} 0 & (r = 0, \ldots, n-2), \\ 1 & (r = n-1), \\ \sum_{i=1}^{n} a_i & (r = n). \end{cases}$$

Define $f(x) = \prod_{j=1}^{n} (x + a_j)$. Then $f(a_i) = 2a_i \prod_{1\le j\le n, j\ne i} (a_i + a_j)$.

Because $f(x) = x^n + x^{n-1} \sum_{j=1}^{n} a_j + p(x)$, where p is a polynomial of degree less than or equal to $n-2$, we obtain

$$2\sum_{i=1}^{n} a_i \prod_{1\le j\le n, j\ne i} \left(\frac{a_i + a_j}{a_i - a_j} \right) = \sum_{i=1}^{n} \frac{f(a_i)}{\prod_{1\le j\le n, j\ne i} (a_i - a_j)} = [a_1, \ldots, a_n]\, f(x)$$

$$= \quad [a_1, \ldots, a_n]\, x^n + [a_1, \ldots, a_n]\, x^{n-1} \sum_{j=1}^{n} a_j + [a_1, \ldots, a_n]\, p(x)$$

$$= \quad \sum_{i=1}^{n} a_i + \sum_{j=1}^{n} a_j + 0 = 2\sum_{i=1}^{n} a_i.$$

Solution 2, by Seán M. Stewart.

Consider the rational function

$$F(x) := \prod_{j=1}^{n} \frac{x + a_j}{x - a_j}.$$

By a partial fraction decomposition, we have

$$F(x) = \prod_{j=1}^{n} \frac{x+a_j}{x-a_j} = A_0 + \sum_{i=1}^{n} \frac{A_i}{x-a_i},$$

where $A_0 = 1$ and

$$A_i = \lim_{x\to a_i} (x-a_i)F(x) = \lim_{x\to a_i} (x-a_i) \prod_{j=1}^{n} \frac{x+a_j}{x-a_j} = 2a_i \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a_i+a_j}{a_i-a_j},$$

as $a_1, \ldots, a_n$ are distinct complex numbers. So

$$\prod_{j=1}^{n} \frac{x+a_j}{x-a_j} = 1 + \sum_{i=1}^{n} \frac{2a_i}{x-a_i} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a_i+a_j}{a_i-a_j}. \tag{1}$$

We now equate the coefficients of $1/x$ in the Laurent series expansion of (1) about $x = \infty$. Setting $x = \frac{1}{z}$, this is equivalent to equating the coefficients of z in the Laurent series expansion of

$$\prod_{j=1}^{n} \frac{1+za_j}{1-za_j} = 1 + \sum_{i=1}^{n} \frac{2za_i}{1-za_i} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a_i+a_j}{a_i-a_j}. \tag{2}$$

about $z = 0$. For the left hand side of (2) we see that

$$[z] \prod_{j=1}^{n} \frac{1+za_j}{1-za_j} = [z] \prod_{j=1}^{n} (1+za_j) \sum_{n=0}^{\infty} (za_j)^n = 2\sum_{i=1}^{n} a_i.$$

Here $[z^n]$ denotes the coefficient operator extracting the coefficient of z^n in a formal power series $A(z)$. Similarly, for the right hand side of (2) we see that

$$\begin{aligned}[z] \sum_{i=1}^{n} \frac{2za_i}{1-za_i} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a_i+a_j}{a_i-a_j} &= [z] \sum_{i=1}^{n} 2za_i \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a_i+a_j}{a_i-a_j} \sum_{n=0}^{\infty} (za_i)^n \\ &= \sum_{i=1}^{n} 2a_i \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a_i+a_j}{a_i-a_j}.\end{aligned}$$

Thus on equating equal coefficients for z in (2), we find

$$\sum_{i=1}^{n} a_i \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a_i+a_j}{a_i-a_j} = \sum_{i=1}^{n} a_i,$$

as required to prove.

4763. *Proposed by William Weakley.*

Let K be a field and let S be a nonempty subset of K that is closed under subtraction.

a) For all K and S, characterize the functions $f : S \to K$ such that

$$f(x)f(y) = f(x - y) \text{ for all } x, y \in S.$$

b) As K and S vary, what finite cardinalities can the set of such functions have?

We received 5 submissions out of which 4 were complete and correct. We present the solution by the Missouri State University Problem Solving Group, lightly edited.

a) Since S is nonempty and closed under subtraction, S is an additive subgroup of K. In particular, $0 \in S$. Let $x \in S$. Then

$$f(0) = f(x - x) = f(x)^2. \tag{1}$$

Letting $x = 0$ we see that $f(0) \in \{0, 1\}$. If $f(0) = 0$, then (1) implies $f(x) = 0$ for all $x \in S$. If $f(0) = 1$, then (1) implies $f(x) \in \{-1, 1\}$ for all $x \in S$. Therefore f is either identically 0 or it is a homomorphism from the additive group S to the multiplicative group $\{\pm 1\}$. Conversely, given such a homomorphism f and $x, y \in S$, the function f satisfies

$$f(x - y) = f(x)f(y)^{-1} = f(x)f(y),$$

since the elements of $\{\pm 1\}$ are their own inverses.

b) Let $Hom(S, \{\pm 1\})$ denote the set of all homomorphisms from S to $\{\pm 1\}$. Defining

$$\begin{aligned}(f \oplus g)(s) &= f(s)g(s),\\ (r \odot f)(s) &= f(s)^r\end{aligned}$$

for $f, g \in Hom(S, \{\pm 1\})$, $r \in \mathbb{F}_2$, and $s \in S$ makes $Hom(S, \{\pm 1\})$ an $\mathbb{F}_2$-vector space. Therefore, if $Hom(S, \{\pm 1\})$ is finite, it contains 2^d elements where d is the dimension of the vector space.

To show that all nonnegative values of d are possible, first consider $d \geq 1$. Let $K = S = \mathbb{Q}[x]/p(x)$, where p is an irreducible polynomial of degree d. Then $S \cong \mathbb{Q}^d$ and $|Hom(S, \{\pm 1\})| = 2^d$, since there are two choices for each of the d basis vectors to map to.

For $d = 0$, consider $K = \mathbb{F}_2$. Then $Hom(S, \{\pm 1\}) = Hom(S, \{1\})$ and there is just one homomorphism.

Therefore, including the function that is identically zero, the possible finite values for the number of functions satisfying the conditions of the problem are $2^d + 1$ with $d \geq 0$.

4764. *Proposed by Muhammad Afifurrahman.*

Let $(a_n)_{n=1}^{\infty}$ denote an arithmetic progression with common difference d. If both a_1 and d are positive, prove that

$$\frac{a_{2p}}{a_{q+r}} + \frac{a_{2q}}{a_{r+p}} + \frac{a_{2r}}{a_{p+q}} \geq 3$$

for all $p, q, r \in \mathbb{N}$.

We received 21 submissions of which 20 were correct and complete. We present the solution by Soham Bhadra, slightly modified.

Since $a_n = a_1 + (n-1)d$ for every $n \in \mathbb{N}$, it easily follows that

$$a_n - a_m = (n-m)d \text{ for every } n, m \in \mathbb{N}.$$

The inequality we want to prove is equivalent to

$$\frac{a_{2p}}{a_{q+r}} + \frac{a_{2q}}{a_{r+p}} + \frac{a_{2r}}{a_{p+q}} - 3 \geq 0.$$

We have that

$$\begin{aligned}
&\frac{a_{2p}}{a_{q+r}} + \frac{a_{2q}}{a_{r+p}} + \frac{a_{2r}}{a_{p+q}} - 3 \\
&= \left(\frac{a_{2p}}{a_{q+r}} - 1\right) + \left(\frac{a_{2q}}{a_{r+p}} - 1\right) + \left(\frac{a_{2r}}{a_{p+q}} - 1\right) \\
&= \frac{(2p-q-r)d}{a_{q+r}} + \frac{(2q-r-p)d}{a_{r+p}} + \frac{(2r-p-q)d}{a_{p+q}} \\
&= d\left(\frac{(p-q)+(p-r)}{a_{q+r}} + \frac{(q-r)+(q-p)}{a_{r+p}} + \frac{(r-p)+(r-q)}{a_{p+q}}\right) \\
&= d\left(\left(\frac{p-q}{a_{q+r}} - \frac{p-q}{a_{r+p}}\right) + \left(\frac{q-r}{a_{r+p}} - \frac{q-r}{a_{p+q}}\right) + \left(\frac{r-p}{a_{p+q}} - \frac{r-p}{a_{q+r}}\right)\right) \\
&= d\left((p-q) \cdot \frac{(p-q)d}{a_{q+r}\, a_{r+p}} + (q-r) \cdot \frac{(q-r)d}{a_{r+p}\, a_{p+q}} + (r-p) \cdot \frac{(r-p)d}{a_{p+q}\, a_{q+r}}\right) \\
&= d^2\left(\frac{(p-q)^2}{a_{q+r}\, a_{r+p}} + \frac{(q-r)^2}{a_{r+p}\, a_{p+q}} + \frac{(r-p)^2}{a_{p+q}\, a_{q+r}}\right).
\end{aligned}$$

Since squares of real numbers are non-negative and by assumption $a_n > 0$ for all n, the expression in the last line is clearly non-negative, concluding the proof.

4765. *Proposed by Omar Sonebi.*

Let n be a natural number and let $S(n)$ denote the sum of digits of n in the decimal notation. Is the sequence $S(2^n)$ eventually strictly increasing?

We received 9 solutions; only 3 were completely correct. We present Soham Bhadra's submission.

The sequence $S(2^n)$ is not eventually strictly increasing. Obviously $n \equiv S(n) \pmod 9$. Let us assume that $S(2^n)$ was eventually increasing. Now, let's pick some large $k \equiv 0 \pmod 6$ so $2^k \equiv 1 \pmod 9$ and $S(2^k) = 9a+1$ for some a. Then, by hypothesis,

$$\begin{aligned} S(2^{k+1}) &\geq 9a+2, & S(2^{k+2}) &\geq 9a+4,\\ S(2^{k+3}) &\geq 9a+8, & S(2^{k+4}) &\geq 9a+16,\\ S(2^{k+5}) &\geq 9a+23, & S(2^{k+6}) &\geq 9a+28, \end{aligned}$$

hence the sequence $S(2^n)$ is at least asymptotic to $9n/2$. On the other hand, $S(2^n)$ is bounded by 9 times the number of digits that 2^n has, which is at most $n/3$ for large n as $\log_2(10) > 3$. Hence $S(2^n) \lesssim 3n$. Contradiction!

Editor's Comments. This is a very old problem. Every submission followed the same strategy: produce incompatible estimates on the growth rate of $S(2^n)$.

The upper bound

$$\limsup_{n\to\infty} \frac{S(2^n)}{n} \leq 9\log_{10}(2) = 2.709\ldots$$

comes from standard properties of "digimetric" functions.

Meanwhile, assuming that $S(2^n)$ is eventually strictly increasing, the lower bound

$$\liminf_{n\to\infty} \frac{S(2^n)}{n} \geq \frac{27}{6} = 4.5$$

comes from the fact that $S(2^n) \equiv 2^n$ has period 6 modulo 9, with residues $1, 2, 4, 8, 7, 5, 1, \ldots$. Empirical evidence suggests that $S(2^n)$ *does* grow linearly—this is an open problem!

4766. *Proposed by Le Hoang Long and Ngo Thai Binh, modified by the Editorial Board.*

Given a regular (2^n-1)-gon $A_0 \ldots A_{2^n-2}$, with sides of length $a_1 = A_0A_1$ and diagonals $a_k = A_0A_k$, prove that

$$\frac{1}{a_1} = \sum_{k=1}^{n-1} \frac{1}{a_{2^k}}\ .$$

We received 9 solutions. We present the solution by C. R. Pranesachar, slightly edited.

We may assume without loss of generality that the regular (2^n-1)-gon is inscribed in the unit circle; denote by O the center of the circle. Fix $k \in \{1, \ldots, 2^n-2\}$. Let $\theta = \frac{\pi}{2^n-1}$ so that $\angle A_0OA_k = 2k\theta$. Using the formula for the length of a chord we get

$$a_k = 2\sin(k\theta) \text{ for all } 1 \leq k \leq 2^n-2.$$

Hence the equality we want to prove is equivalent to

$$\frac{1}{\sin(\theta)} = \sum_{k=1}^{n-1} \frac{1}{\sin(2^k\theta)}. \tag{1}$$

Using the trigonometric identity $\csc(2x) = \cot(x) - \cot(2x)$ gives us

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{1}{\sin(2^k\theta)} &= \sum_{k=1}^{n-1} \left[\cot\left(2^{k-1}\theta\right) - \cot\left(2^k\theta\right)\right] \\
&= \cot(\theta) - \cot(2^{n-1}\theta) \\
&= \frac{\cos(\theta)\sin(2^{n-1}\theta) - \sin(\theta)\cos(2^{n-1}\theta)}{\sin(\theta)\sin(2^{n-1}\theta)} \\
&= \frac{\sin(2^{n-1}\theta - \theta)}{\sin(\theta)\sin(2^{n-1}\theta)} \\
&= \frac{1}{\sin(\theta)},
\end{aligned}$$

where in the last line we used the fact that for $\theta = \frac{\pi}{2^n-1}$ we have

$$(2^{n-1}\theta - \theta) + 2^{n-1}\theta = (2^n - 1)\theta = \pi,$$

and so $\sin(2^{n-1}\theta - \theta) = \sin(2^{n-1}\theta)$. This completes the proof of (1) and hence also of the desired equality.

4767. *Proposed by George Apostolopoulos.*

Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E and F be chosen on sides BC, CA and AB so that AD, BE and CF bisect the angles of ABC. Prove that

$$\frac{DE}{AB} + \frac{EF}{BC} + \frac{FD}{CA} \leq \frac{3}{4}\left(1 + \frac{R}{2r}\right).$$

There were 10 correct solutions among the 13 submissions that we received — the other three were flawed. We feature two solutions: the first is a note by Michel Bataille that describes how the result follows quickly from previous Crux *problems, and the second, almost entirely also by Bataille, to provide readers the details of a typical solution.*

Solution 1, by Michel Bataille.

Let $a = BC, b = CA, c = AB$, as usual. In **2502** [2000 : 45 ; 2001 : 53], it was proved that

$$DE \leq \frac{2c+a+b}{8}, \quad EF \leq \frac{2a+b+c}{8}, \quad FD \leq \frac{2b+c+a}{8}.$$

It readily follows that the inequality holds if

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \leq \frac{3R}{2}.$$

We are done since this is one of the inequalities of **3087** [2005 : 459,462 ; 2006 : 521].

Solution 2, by Parviz Khalili (based on a solution by Michel Bataille).

Most of the following solution appeared as a solution by Michel Bataille to problem **12182** in *The American Mathematical Monthly*, **129**(1), 2022. It is only the final estimate (after equation (1) below) that differs from Bataille's solution. Compare also the solution by Subhankar Gayan to *Crux* Problem 4609 [**47**(6) 2001:321].

Let $a = BC, b = CA, c = AB$ be the side lengths of triangle ABC. Since AD, BE, CF are the angle bisectors, we know that $AE = \frac{bc}{a+c}$ and $AF = \frac{bc}{a+b}$. Applying the Law of Cosines, we obtain

$$\begin{aligned}
EF^2 &= AE^2 + AF^2 - 2\,AE \cdot AF\,\cos A = \frac{b^2c^2}{(a+c)^2} + \frac{b^2c^2}{(a+b)^2} - \frac{bc(b^2+c^2-a^2)}{(a+b)(a+c)} \\
&= \frac{bc}{(a+b)^2(a+c)^2}\left[bc[(a+b)^2+(a+c)^2] - (b^2+c^2-a^2)(a+b)(a+c)\right] \\
&= \frac{bc[a^2(a+b)(a+c) + 2a^2bc + 2ab^2c + 2abc^2 - a^2b^2 - ab^2c - ab^3 - a^2c^2 - ac^3 - abc^2]}{(a+b)^2(a+c)^2} \\
&= \frac{abc}{(a+b)^2(a+c)^2}[a(a+b)(a+c) - a(b-c)^2 - b(b-c)^2 - c(b-c)^2] \\
&\leq \frac{a^2bc}{(a+b)(a+c)},
\end{aligned}$$

with equality if and only if $b = c$.

By the AM-GM inequality (applied three times),

$$EF \leq \frac{a\sqrt{bc}}{\sqrt{(a+b)(a+c)}} \leq \frac{a\sqrt{bc}}{\sqrt{2\sqrt{ab}\cdot 2\sqrt{ac}}} = \frac{\sqrt{a}\,\sqrt[4]{b}\sqrt[4]{c}}{2} \leq \frac{2a+b+c}{8}.$$

Similar inequalities hold for FD and DE; therefore,

$$\frac{EF}{a} + \frac{FD}{b} + \frac{DE}{c} \leq \frac{3}{4} + \frac{1}{8}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right), \qquad (1)$$

with equality if and only if $a = b = c$. With R the circumradius, r the inradius, and $s = \frac{a+b+c}{2}$, we have $ab + bc + ca = s^2 + 4Rr + r^2$ and $abc = 4Rrs$. Applying

Gerretsen's inequality ($s^2 \le 4R^2 + 4Rr + 3r^2$) we get

$$\begin{aligned} \frac{b+c}{a} + \frac{a+c}{b} + \frac{a+b}{c} &= \frac{2s(ab+bc+ca) - 3abc}{abc} \\ &= \frac{2s(s^2+4Rr+r^2) - 12sRr}{4sRr} \\ &= \frac{s^2 - 2Rr + r^2}{2Rr} \\ &\le \frac{4R^2+2Rr+4r^2}{2Rr} = 1 + 2\left(\frac{R}{r} + \frac{r}{R}\right). \end{aligned}$$

Substitute in (1) and apply Euler's inequality ($2r \le R$) to get

$$\frac{EF}{a} + \frac{FD}{b} + \frac{DE}{c} \le \frac{7}{8} + \frac{1}{4}\left(\frac{R}{r} + \frac{r}{R}\right) \le \frac{7}{8} + \frac{1}{4}\left(\frac{R}{r} + \frac{r}{2r}\right) \le \frac{3}{4}\left(1 + \frac{R}{2r}\right).$$

This completes the solution.

Editor's comments. The *Monthly* problem, mentioned in the second solution, called for an upper bound of $\frac{3R}{4r}$, which is greater than the bound here, namely $\frac{3}{4}(1+\frac{R}{2r})$. Janous observed that such estimates are misleading: Since the quantity $\frac{R}{r}$ can be arbitrarily large (for example, when an isosceles triangle has a small apex angle), these upper bounds are unbounded, but $\frac{EF}{a}$ and the other two summands are easily seen to be less than 1 (as in the estimate for EF^2 in solution 2 above). Indeed, Janous proved that their sum is less than 2; specifically,

$$\frac{EF}{a} + \frac{FD}{b} + \frac{DE}{c} \le \frac{47R+2r}{31R+2r}.$$

It remains an open question whether, except for the equilateral triangle, the sum is always less than $\frac{3}{2}$. We challenge the readers to send us a proof or counterexample.

4768. *Proposed by Mihaela Berindeanu.*

Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, such that $f(2x) \cdot f\left(\frac{y}{2}\right) \le f(xy) + 2x + \frac{y}{2}$ holds for all real numbers x, y.

We received 13 solutions, all correct. We present the solution by Michel Bataille.

We consider the equivalent problem of finding all $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x)f(y) \le f(xy) + x + y, \tag{1}$$

for all x, y. It is easy to see that the function $x \mapsto x + 1$ is a solution. We show that there is no other solution.

Let f be a solution. With $x = y = 1$, (1) gives $(f(1))^2 \le f(1) + 2$, hence $f(1) \le 2$.

With $x = y = -1$, (1) gives $(f(-1))^2 \le f(1) - 2$, hence $(f(-1))^2 \le 0$ and therefore $f(-1) = 0$.

From $f(-x)f(-1) \leq f(x) - x - 1$, we then deduce that for all real x,

$$f(x) \geq x + 1. \tag{2}$$

With $x = y = 0$, (1) yields $(f(0))^2 \leq f(0)$, hence $f(0) \leq 1$. Since $f(0) \geq 1$ (from (2) with $x = 0$), we must have $f(0) = 1$ and therefore

$$f(x) = f(x)f(0) \leq f(0) + x + 0 = 1 + x$$

for all x. With (2) we conclude that $f(x) = x + 1$ for all x and the proof is complete.

4769. *Proposed by Nguyen Tien Lam.*

Let a, b, c, d be positive integers such that $a > b > c > d$ and

$$a^2 - ac + c^2 = b^2 - bd + d^2.$$

Prove that $ab - cd$ is not prime.

We received 15 *submissions, out of which* 13 *were correct and complete. We present the solution by Prithwijit De, lightly edited.*

Observe that if $a^-ac + c^2 = b^2 - bd + d^2$ we have

$$\begin{aligned}(ac - bd)(a^2 - ac + c^2) &= ac(b^2 - bd + d^2) - bd(a^2 - ac + c^2) \\ &= (bc - ad)(ab - cd).\end{aligned} \tag{1}$$

Using the condition $a > b > c > d$ we see that $ab - cd > 0$, $ac - bd > 0$, and $a^2 - ac + c^2 > ac - ac + c^2 > 0$. Therefore $bc - ad > 0$. Further

$$ab - cd > ac - bd > bc - ad > 0. \tag{2}$$

In particular $ac - bd > 1$.

Suppose $ab - cd$ is prime. Then the inequality (2) implies that $ac - bd$ is relatively prime to $ab - cd$ and from (1) it follows that $ac - bd$ must therefore divide $bc - ad$. But this is absurd since $ac - bd > bc - ad > 0$. Hence $ab - cd$ is not prime.

4770. *Proposed by Boris Čolaković.*

Prove that for all acute triangles with angles A, B, C, the following inequality holds:

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \geq \frac{5}{4}.$$

We received 14 submissions, of which 12 were correct and complete. We present here a solution which is a composite of almost identical solutions submitted (independently) by Šefket Arslanagić, Mohamed Amine Ben Ajiba, Daniel Văcaru, and the proposer.

The Popoviciu's inequality (see C. Niculescu & L.-E. Persson, *Convex functions and their applications: A contemporary approach*, CMS Books in Mathematics, Springer, 2006, page 12, Theorem 1.1.8.) is the following: let f be a real continuous function on a real interval I. Then f is convex if and only if

$$\frac{f(x)+f(y)+f(z)}{3}+f\left(\frac{x+y+z}{3}\right)\geq\frac{2}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]$$

for all $x,y,z\in I$. If f is stricly convex on I, then the equality holds if and only if $x=y=z$.

Because the angles A,B,C of an acute triangle belong to the interval $(0,\pi/2)$, this inequality is written for the strictly concave function $\cos(x)$ on this interval:

$$\begin{aligned}&\frac{\cos A+\cos B+\cos C}{3}+\cos\left(\frac{A+B+C}{3}\right)\\&\leq\frac{2}{3}\left[\cos\left(\frac{A+B}{2}\right)+\cos\left(\frac{B+C}{2}\right)+\cos\left(\frac{C+A}{2}\right)\right].\end{aligned}$$

As $A+B+C=\pi$, this is equivalent to

$$\frac{\cos A+\cos B+\cos C}{3}+\frac{1}{2}\leq\frac{2}{3}\left[\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right], \tag{1}$$

because, for example,

$$\cos\left(\frac{A+B}{2}\right)=\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)=\sin\frac{C}{2}.$$

The equality holds if and only if $A=B=C$, that is for an equilateral triangle.

Using elementary trigonometric transformations of products into sums, it is verified that for any three angles A,B,C satisfying $A+B+C=\pi$,

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{4}\left[\cos A+\cos B+\cos C-1\right].$$

Therefore

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{2}$$

From (1) and (2), we get:

$$\begin{aligned}\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\quad&\geq\quad\frac{\cos A+\cos B+\cos C}{2}+\frac{3}{4}\\&\geq\quad\frac{5}{4}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},\end{aligned}$$

which is the requested result. The equality is verified if and only if $A=B=C$, i.e. the triangle is equilateral.