# A Taste $\mathrm{O}_{\mathrm{F}} \mathrm{Mathematics}^{\text {a }}$ 



# Aime-T_On les Mathématiques 

Volume / Tome IX<br>THE CAUT PROBLEMS

Edward Barbeau
University of Toronto

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## 1 Preface

This book contains over sixty problems that were originally published in the CAUT Bulletin, published by the Canadian Association of University Teachers. The Canadian Mathematical Society is grateful to the Canadian Association of University Teachers for granting permission for these problems to be included in this ATOM volume. The reader might ask how mathematical problems wound up in a publication whose articles generally deal with matters of university governance, academic policies and conditions of employment of the faculty and librarians of Canada's colleges and universities.

The executive director of CAUT is Dr. James Turk, who used to be a professor of sociology at the University of Toronto. During his time in Toronto, he noted the mathematics problems that I contributed to the back page of the University of Toronto Alumni Magazine, right next to the cryptic crossword. When he went to Ottawa, he wondered if the pages of the Bulletin could occasionally be enlived with a mathematical problem and asked me to supply some.

I was pleased to comply. In my view, there are many mathematical problems that are accessible to the lay person and require only a basic knowledge of arithmetic and algebra as well as some capacity for reasoning and analysis. In short, the demands are no more severe that those of bridge, cryptic crosswords, board games such as checkers or reversi and the numerous puzzles sold in the market and published in newspapers. A good problem is a work of art, as much as a well-executed song, poem or painting, and deserves a place in the popular culture. Most of these problems are not original; the
ideas come from a number of sources, some of them school texts. New problems appear on the scence regularly, and a couple of them are included here. I hope that readers enjoy trying their hand at them.

While these problems are intended for a general audience, I strongly encourage their use with school children for a number of reasons. In my experience, many children enjoy the sort of challenge that these problems provide. One of the purposes of education is to induct young people into the broader culture, and "heritage" problems in mathematics have their place in this culture. The investigation and analysis that some of these problems require reflect the authentic character of mathematics more faithfully that much of the purely technical material occupying much of the school syllabus. Other problems provide a natural application for some of the skills that pupils are expected to master at school. Finally, these are problems that pupils can take home and share with their families.

After the book was put together, the editor indicated that there was still a little room for other material. Over the years, I have learned a few mathematical card tricks which I have demonstrated to students. Since they involve only mathematical principles, they suited to anyone like myself who lacks the manual dexterity for sleight of hand. While they were not published in the CAUT Bulletin, those who have found pleasure in the problems may enjoy trying them out.

Pleasant solving!
Ed Barbeau
Department of Mathematics
University of Toronto

## 2 About the Author

Edward Barbeau is Professor Emeritus of Mathematics at the University of Toronto.

For many years, he has been involved with school education, training of contest candidates and conveying mathematics to the general public. He has chaired the committee for the Canadian Mathematical Olympiad, operated a problems correspondence program for secondary students, and accompanied the Canadian team at five International Mathematical Olympiads.

## 3 "Homework!" Problems

## 1. "This Triangle is Not Square"

Each of the numbers $1,2,3,4,5,6,7,8,9$ is to be written in exactly one of the circles in such a way that (a) the sums of the four numbers on each side of the triangle are equal; and (b) the sums of the squares of the four numbers on each side of the triangle are equal. Can you achieve this?

It may help to know that the sum of the three numbers at the vertices of the triangle must be divisible by 3 , and that the sum of the squares of these numbers must also be divisible by 3 .


## 2. "Drop the Triangle"

An equilateral triangle sit atop a square as in the diagram. All sides have length 1. A circle passes through points $P, Q$ and $R$. What is the radius of the circle?


## 3. "Forgetful Iphy!"

At noon, Iphigenia set off on a bike hike from her home in Saskatoon, maintaining a leisurely pace of 20 km per hour on the pleasantly level terrain. Later, her mother noticed that she forgot her dinner, and sent Electra off on her bike to meet her; Electra maintained a steady pace of 30 km per hour. But then the sky darkened and the storm clouds gathered. So, exactly a half hour after Electra left, Orestes was sent off to meet the others with rain gear. Orestes rode at a steady pace of 40 km per hour. All three followed the same route. As it happened, the three siblings met at exactly the same time. What time was that?

## 4. "Bikes that Pass at Noon."

Ophelia lives in Edmonton and Desdemona lives in Calgary. One day both young ladies set out on their bikes at exactly the same time, heading for the city of the other in opposite directions along the same route. They each rode at a constant speed, although Ophelia was the faster. They passed at noon. Ophelia got to Calgary at four in the afternoon while Desdemona did not get to Edmonton until nine in the evening. At what time did the ladies start out?

## 5. "The Hymns at St. Trinians."

The sexton of St Trinians was posting the hymns on the hymn board for the Sunday service. This consisted of sliding the individual digits for the hymn numbers into three slots to form the hymn numbers. He noted that there were three hymns, that he required one of each of the digits $1,2,3,4,5,6,7,8,9$, and that the hymn numbers were in the ratio $1: 3: 5$. What were the hymns for that Sunday?

## 6. "Numbering on the Octahedral Die."

An ordinary die is a cube whose faces are numbered from one to six. But dice can have other shapes. An octahedral die is the shape of a regular octahedron (formed by gluing two square-based pyramids together) which has eight triangular faces, with four of them meeting at each of its six vertices. Show how the faces can be numbered from one to eight in such a way that the sum of the numbers of the four faces joining at each vertex is always the same. It might help to begin by deciding what that sum should be.

## 7. "A Slight Difference."

The starred positions are to be filled by the digits from 1 to 8 inclusive to give two four-digit numbers so that the top number exceeds the lower and the difference between the numbers is as small as possible.


What are the two numbers?

## 8. "Coverage."

A page is placed on top of an identical page as indicated below, so that the top left corner of the top page falls on the top right comer of the bottom page, and the bottom left comer of the top page falls on the left side of the bottom page.


Explain why the top page covers more than half the area of the bottom page. (The hard part of this problem is to explain why the right edge of the top page does indeed cover the bottom right corner of the bottom page.)

## 9. "Knights \& Knaves"

A traveller to a strange island discovers that it is inhabited by knights who can make only true statements and knaves who can make only false statements. One day the traveller encountered three inhabitants, whom we will call $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, and asked, "How many knights are there among you three?" A made an answer, which the traveller missed, but which was understood by the other two. When $\mathbf{B}$ was asked what $\mathbf{A}$ said, $\mathbf{B}$ responded, "A said that there is one knight among us." "Don't believe $\mathbf{B}$," shot in C, "he is lying." What are B and C?

## 10. "A Number Game."

Cordelia and Kent play the following game. Cordelia goes first and they take alternate turns. Each selects a number from 1 to 6 inclusive that has not already been selected; the game ends in six moves. At the end of each move, the player making the move takes the sum of all the numbers selected by either player up to that point and claims all of its positive divisors. When the game is over, the score of each player is the highest number $k$ for which the player has claimed all the consecutive numbers $1,2,3, \ldots, k$ from 1 to $k$ : inclusive. The winner is that player with the highest score; if both have the same score, neither wins and the game is a tie. For example, suppose the six moves are as follows: $C: 2$; $K: 4 ; C: 1 ; K: 3 ; C: 5 ; K: 6$. The respective claims by $C$ are 1,$2 ; 1,7 ; 1,3,5,15$; and by $K$ are $1,2,3,6 ; 1,2,5,10$ and $1,3,7,21 . C$ and $K$ have the same score, 3 , and the game is a tie. The example really does not demonstrate very good play. Is there any way that Cordelia can be prevented from winning?

## 11. "Finding The Correct Time."

Stephen's clock has stopped, and he has no way of getting the correct time at his home. So he winds the clock up to make it go, and sets it at random. He then walks to the house of his colleague, Stockwell. Stockwell has a functioning clock that reads the correct time. Eventually, Stephen leaves and walks home following the way he came. Upon arrival at his house, he sets his clock to the correct time. How did he do this?

## 12. "The Swinging Lad."

Lothario has two girl friends, one in Scarborough and the other in Etobicoke, and he is equally fond of both. When he wants to see one of them, he goes to the Yonge subway station on the Bloor-Danforth line, takes the first train that comes along, and visits the appropriate girl. The trains run at equal 10 -minute intervals in each direction. However, as time goes on, he discovers that he is seeing the Scarborough girl friend about $80 \%$ of the time. Why is this?

## 13. "Keeping a Good Distance."

Place seven points on a page in such a way that among any three of them, you can find two that are three centimetres apart.

## 14. "Misinformation."

On the table, there are three boxes. One contains two black marbles, one contains two white marbles, and one contains one white and one black marble. The boxes are labelled according to their contents: BB, WW, and BW. However, the cleaning lady comes along and switches the labels so that all the boxes are incorrectly labelled. You are allowed to take one marble at a time out of any box and replace it, without looking inside, and by this process of sampling, determine the contents of all three boxes. What is the smallest number of drawings needed to do this?

## 15. "The Great Escape."

Adonis is on a railway bridge joining $\mathbf{A}$ to $\mathbf{B}, \frac{3}{8}$ of the way across from $\mathbf{A}$. He hears a train approaching $\mathbf{A}$; it is travelling 60 kilometres per hour. If he runs towards $\mathbf{A}$, he will arrive there exactly when the train does. If he runs towards $\mathbf{B}$, the train will overtake him at $\mathbf{B}$. How fast can he run?

## 16. "Ferryboats."

Two ferryboats start at the same instant from opposite sides of a river, travelling across the river on routes perpendicular to the shores. Each travels at constant speed, but one is faster than the other. They pass at a point 720 metres from the nearest shore. Both boats remain in their slips for 10 minutes before starting back. On the return trips, they meet 400 metres from the other shore. How wide is the river?

## 17. "The Four-Sided Cut-Up."

Take any sheet of paper in the form of a quadrilateral (a four-sided figure); it does not have to be rectangular. Join the midpoints of adjacent sides, as in the diagram. These line segments partition the quadrilateral into four triangular "ears" and a central figure that turns out to be a parallelogram (opposite sides parallel).


Remarkably, regardless of the shape of the quadrilateral, the area of the parallelogram is equal to the sum of the areas of the four triangular ears. One way to see this is to cut off the four ears with a pair of scissors and arrange them to cover the parallelogram without overlapping. Show how this can be done.
18. "Who Gets Home First?"

Penelope and Ulysses like to have a bit of exercise each morning. They set off at exactly the same time in the morning, and run or walk the same circuit Both run at exactly the same speed, and both walk at exactly the same speed. However, they do not remain together. Penelope runs for half the time that she is out, and walks for the other half of the time. Ulysses runs for half the distance and walks for the other half of the distance. The first one returning to the house has to start the coffee. Who starts the coffee?

## 19. "Drying Out the Slurry."

The Watsyorsis Mine has just filled an open tank with $4,000 \mathrm{~kg}$ of slurry which is $99 \%$ water and $1 \%$ tailings by weight. Since it was pretty heavy to transport, the proprietors left the tank until some of the water could evaporate away. After a while, the slurry was $98 \%$ water. How much did it weigh then?

## 20. "Beating the Odds."

Three knights are told by their king that, in recognition of their services in rescuing his beautiful daughter from a fierce dragon, he will give them a chance to win a great fortune. He tells them he will seat them in a circle, and place on the head of each a hat that is either red or green. Each of them will be able to see the hats of the others, but not his own hat. Each will be asked either to guess the colour of his own hat or keep silent, but at least one of them must take a guess. If all those who speak guess correctly, then all three will receive a fortune. Otherwise, they receive nothing. Of course, the king set some rules. Beforehand, the knights are permitted to discuss what strategy they will use. However, once they are seated in a circle, there must be no further communication of any kind among them. You might think the best they can manage is to have an even chance of winning the fortune (by, for example, having a particular one take a guess). But they can in fact increase the odds in their favour. How?

## 21. "Giving a Handicap."

Two brothers. Castor and Pollux, run a 100-metre race. Castor wins the race, crossing the finish line when Pollux has run only 95 metres. Out of consideration for his brother's self-esteem, in the second race, Castor starts five meters back from the starting line (so he will run a total of 105 metres to reach the finish line), while Pollux runs the 100 metres as before. If each runs as fast as in the first race, who will win now?

## 22. "Ski Lift."

Baldur likes to ski in the Rockies. To get to the top of the mountain, he takes the cable lift The chairs are suspended at equal distances along a continuously moving cable loop that goes between the base and the peak of the mountain. Baldur noticed that as he ascended, he passed a descending seat at intervals of five seconds. It took exactly three minutes for him to reach the top. What can be said about the number of seats on the lift? How frequently can skiers be picked up at the base of the mountain? .

## 23. "The Sale."

Calpurnia agrees to go to the market with Julius to buy a new toga. These are on sale with a 25 per cent reduction in the price; however, the customer has to pay 15 per cent sales tax on the cost of the purchase. Julius finds one that he likes for 160 denarii. The sales clerk then adds the 15 per cent tax to this amount, and then substracts 25 per cent of the total amount to get the final cost "Hold on," says Julius, "you are ripping me off! You should take off the 25 per cent before you add the 15 per cent sales tax." Calpurnia is impatient with this difference of opinion and says, "Look. You are making a big deal of this. You are taking 25 per cent off and adding 15 per cent on. So why not just take 10 per cent off the price and call it a day?

Who is right?

## 24. "Points of View."

A number of identical cubical blocks are stacked up. From the front, the pile looks like:

while from the side it looks like:


What is the smallest and the largest number of blocks that could have been used?

## 25. "A Triangular Array."

Consider the following array of numbers:

| 11 |  | 15 |  |  |  | 13 |  | 8 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | 4 |  | 14 |  | 12 |  | 5 |  |
|  | 10 |  | 2 |  | 7 |  |  |  |
|  |  | 8 |  | 5 |  |  |  |  |
|  |  |  |  | 3 |  |  |  |  |

In each row but the top row, each number is the (positive) difference of the two numbers directly above it. You will observe that the numbers have been chosen from between 1 and 15, inclusive, but in this example, both the numbers 8 and 5 appear twice, and the numbers 6 and 9 do not appear at all. Is it possible to find a triangular array of numbers such as this one, with the same rule that each number in the lower row is the difference of the two immediately above it, but for which, each of the numbers from 1 to 15 appears exactly once?

As this is a challenging problem, you might want to warm up with triangular arrays with two numbers, three numbers and four numbers across the top, using respectively, the first three, the first six and the first ten positive whole numbers. A productive way to approach the problem is to ask how far away from the top row the largest numbers can be.

## 26. "The Six-Number Problem."

An amusing pastime is to start with any six different whole numbers between 1 and 25 inclusive, and use any five of them, along with the arithmetic operations,,$+- \times, \dot{\circ}$, and exponentiation (raising to a power), with suitable bracketing, to produce the sixth. For example, we can produce the number 18 from the numbers $3,5,6,9$ and 21 as follows:

$$
(9 \times(21+5))-6^{3}=18
$$

But you should also be able to get each of $3,5,6,9,18,21$ from the rest In the solution, it will be shown how each number in the set $\{2,3,5,6,7,24\}$ can be obtained from the remaining ones.

## 27. "Twice as Old."

Your age should more properly be referred to as your age last birthday, the number of complete years that you have lived. You have a given integer age for a year, and this increases by one when your birthday comes. Consider any two people, perhaps you and some friend or relative.

Assuming sufficient longevity, for how long a period is the age of the older one exactly twice the age of the younger one? Try this with several pairs and account for what happens. What is the situation in the case of twins?

## 28. "My Three Sons."

My friend came to my house. "Your sons must be growing up," he remarked.
"Yes," I said, "the product of their ages is 72. "
"That's not very informative; tell me more."
"Well, the sum of their ages is my house number."
"That is still not enough"
I replied thoughtfully, "you are right. The oldest one looks a lot like his mother."
Whereupon, he was able to deduce what the three ages (all whole numbers) were.

## 29. "Redecorating."

CAUT is having two of its offices repainted. The larger office has twice the area of the smaller. One day, a band of painters came in and they all worked on the larger office until the middle of the working day. Then they split into two equal groups. One remained in the larger office and finished it at the end of the day. The other moved to the smaller office, but were unable to finish the job by the end of the day. So, on the following day, a single painter spent the whole day completing the decorating of the smaller office. All the painters worked steadily at the same rate during the whole time. How many painters were there altogether?

## 30. "The Slow Clock."

A clock loses four minutes every hour. It was set to the correct time at 8:30 this morning. What will the actual time be when the clock shows that it is noon today?

## 31. "Area Problems."

Both the squares pictured below have area equal to one square unit. Each is divided into regions by straight lines, and the areas of these regions are indicated by upper-case letters. In the first square, each straight line joins a vertex to the midpoint of one of the opposite sides. In the second square, the sides are divided into three equal segments as indicated, and certain lines are drawn. Determine the areas of the regions.


These problems can be solved using some basic principles:
(a) the area of a triangle is half the baselength times the height;
(b) the ratio of the areas of two triangles with the same height is proportional to the ratio of the lengths of their bases;
(c) if two triangles are congruent (one can be laid on top of the other), then their areas are equal;
(d) if two triangles are similar (one is a scaled version of the other; the angles of one are equal to the corresponding angles of the other), then the ratio of their areas is equal to the ratio of the squares of their linear dimensions;
(e) a line intersecting two parallel lines makes alternating angles (like the angles in a Z ) and corresponding angles equal.
The second problem is quite challenging.

## 32. "Squares \& Cubes."

The quartets $(1,2,3,4)$, $(1,2,2,4)$ and $(2,2,4,4)$ all have the interesting property that the square of the sum of the numbers in the set is equal to the sum of the cubes of the numbers. For example

$$
(1+2+2+4)^{2}=9^{2}=81=1^{3}+2^{3}+2^{3}+4^{3}
$$

Find as many sets of six whole numbers as you can which have the same property of having the sum of their cubes equal to the square of their sum. You can repeat a number in the sextet

## 33. "Complete Coverage."

Nine farms in a square array are separated by a grid of roads, one kilometre apart, as in the diagram, A cyclist starting at the lower left comer wishes to follow a route that covers each piece of road around and between the farms at least once.


What is the minimum distance the cyclist must travel?

## 34. "Finding the Counterfeits."

The police have found a cache of six rolls of coins, each with 25 coins. They know that the coins in each of the rolls are either entirely genuine or entirely counterfeit, but they do not know how many or which of the rolls are counterfeit. The weights of each true and each counterfeit coin are known, with each counterfeit coin one gram lighter than each true coin. Show how the police can make a selection of coins from the rolls and determine with a single weighing on a graduated scale which rolls are counterfeit

## 35. "It Makes No Difference."

Suppose we take three of the first six numbers, say $(1,3,4)$ arranged in ascending order.. Arrange the remaining three numbers in descending order: $(6,5,2)$. Now pair them off and calculate the sum of the positive differences of the numbers in each pair: $(6-1)+(5-3)+(4-2)=9=3^{2}$. We can do this starting with a different choice of three numbers: $(2,3,5)$ and get the sum $(6-2)+(4-3)+(5-1)=9=3^{3}$.
Your problem is to do the same sort of thing with the first 24 positive integers. Select 12 of them and arrange them in ascending order. Now arrange the remaining 12 in descending order. Pair them off in order and calculate the sum of the positive differences of the numbers in each pair. What is the answer? Try this with different choices and try to account for what happens.

## 36. "The Weight of a Full Load of Bricks."

Several bricks of various sizes are given, but none of them weighs more than one kilogram. It is known the bricks cannot be divided into two sets for which the total weight of each set is greater than one kilogram. Find the highest possible total weight of all the bricks.

## 37. "Maximizing the Area."

Below is a rectangle (not drawn to scale) that has been partitioned by straight line segments into an octagon and a hexagon. The sides of the octagon have length $1,2,3,4,5$, $6,7,8$, in some order. How should these numbers be assigned to the sides of the octagon to ensure that the area of the shaded hexagon is as large as possible?


## 38. "An Addition Problem."

Use each of the 10 digits: $0,1,2,3,4,5,6,7,8,9$, once only, to form three positive whole numbers, the largest of which is the sum of the other two. Find the largest and smallest values of these sums.

## 39. "A Tennis Match."

Agassi won the first set of nine games with a score of $6-3$ in a tennis match with Becker. There were five breaks of serve. Who served the first game? (In a set of tennis games, the competitors serve alternately. A service is broken when the server loses the game.)

## 40. "A Network of Acquaintances."

Some of the pairs of readers of the CAUT Bulletin consist of mutual acquaintances. For the rest of the pairs, the two are not acquainted. Suppose that to each reader, we assign the number of that reader's acquaintances among the others. It turns out that the total of all these numbers is even, and also that there are two readers with exactly the same number of acquaintances. Why is this so?

## 41. "Two Pairs of Whole Numbers."

Find two pairs of positive whole numbers (using each number only once) for which the sum of each pair is equal to the product of the other pair. How many examples are there?

## 42. "Going the Distance."

Goneril, Regan and Cordelia are going to visit their grandmother, who lives 60 kilometers away. To get there, they have a tandem bicycle that accommodates at most two of them and is capable of a maximum speed of 45 km per hour. Any one of the three can walk at a maximum speed of 5 km per hour. They leave at 8:50 am. Can they make it to grandmother's house by noon?

## 43. "The Longest Path."

You are given a board with nine pegs in a square $3 \times 3$ array. A string goes from one peg to. the next in straight segments to join all nine of them. An example of how this is done is given in the diagram below.


What should the configuration be to make the path as long as possible? The string should meet each peg exactly once and should not cross itself. There are eight segments. The distance between horizontally and vertically adjacent pegs, is 1 unit, the distance between diagonally adjacent pegs is about 1.41 units, and the distance between a corner peg and a peg in the middle of an opposite side is about 2.24 units.

## 44. "The Product of the Digits."

Is it possible for a positive whole number with at least two digits to be equal to the product of its digits?

## 45. "My Grandson \& I."

My grandson and I have birthdays within a week of each other. Some time ago, I was twelve times his age. Now I am only six times his age. How old is he now?

## 46. "Walking to School."

Castor walks to school in 45 minutes while it takes Pollux 30 minutes. One morning Castor set out at 8:00 a.m, while Pollux left 10 minutes later. At what time did they pass on the way to school? Assume each walked at a constant rate.
Note: It is assumed that both children start from the same address and follow the same route.

## 47. "Five Circles."

Five circles, drawn as in the diagram below, surround nine regions, five of which are contained in one of the circles and four in two of the circles. Show how the numbers $1,2, \ldots, 8,9$, can be placed in these regions so that the sum of the numbers within each of the five circles is the same.

48. "Cutting a Square into Three Parts."

Show how to cut a square into three pieces so that the pieces are the same shape, but are not all the same sizes. Then do this in the case for which two of the three pieces are the same size.

## 49. "A Not Quite Magic Square."

Place the numbers $1,2,3,4,5,6,7,8,9$ in a $3 \times 3$ square array so that
(a) the sums of the top two rows are equal,
(b) the sum of the third row is as large as possible,
(c) the column sums are equal, and
(d) the two diagonal sums are equal.

## 50. "Domestic Conviviality."

A man and his wife can drink a barrel of beer in 15 days. After drinking together for six days, the wife alone drank the remainder of the barrel in 30 days. In what time would either alone drink the whole barrel?
(It is assumed that the daily consumption of each spouse is constant, but not the same for one as the other.)

Note: This problem appeared in The High School Algebra by Robinson and Birchard, approved by the Ontario Department of Education in 1886.

## 51. "Assigning Numbers."

Assign the numbers $1,2,3,4,5,6$ to the vertices and sides of a triangle in such a way that each number occurs once and the number assigned to each side is the sum of those assigned to its end-points. In addition, can the analogous problem for a square, pentagon and hexagon be solved, where the numbers go up to 8 , 10 and 12 , respectively.

## 52. "Fair Division of Land."

A man owned a piece of land in the shape of a trapezoid; it was bounded by four straight sides. Two of these sides were parallel; one was seven furlongs long, the other seventeen furlongs. He divided it fairly between two sons by constructing a fence parallel to these two sides that partitioned the piece of land into two trapezoids of equal area. Tell me, O esteemed scholar, the length of the fence.

## 53. "A Long Wait at the Passport Office."

The passport office was full of people when I took my number. I noted that it was a perfect square. Fortunately, there was an available seat, and I saw that the number of the guy next to me was also a perfect square. Just to make conversation, I drew his attention to the coincidence. He responded, "What is even more interesting is that your number is the square of the sum of the digits of my number, and my number is the square of the sum of the digits of your number." What numbers did we have?

## 54. "Guessing the Right Bag."

A magician has a collection of tokens numbered from 1 to 20 . He distributes them among three purses, with each purse getting at least one token. A spectator is asked to select two of the purses, take one token from each, and announce the sum of the numbers on the two tokens. The magician then states correctly which bag the spectator did not select. How might the magician distribute the tokens so that the trick always works?

## 55. "Splitting a Field into Two."

A field has this shape:


Find dimensions that would allow the field to be partitioned into two subfields, each exactly the same size and shape as the other.
56. "The ubiquitous 2009."

Find a 2009 digit number which has all of these properties:
(a) it is divisible by 2009;
(b) the sum of its digits is 2009;
(c) its last four digits are 2009.

## 57. "The befuddled clerk."

I bought four items at a $7-11$ store. The clerk tried to determine their cost by multiplying their four prices and came up with the answer $\$ 7.11$. When I pointed out his error, he acknowledged that he should have added. Adding the four prices again gave the answer $\$ 7.11$. What did the four items cost?

## 58. "A day's outing."

Troilus and Cressida live near Milton, Ontario. One morning, they set out on their bicycles for Rattlesnake Point, a journey that began with a ride over level ground and terminated with an uphill pull to the top of the escarpment. They arrived at noon and picnicked there for exactly one hour before returning home by exactly the same route.

Given that they rode at $20 \mathrm{~km} / \mathrm{hr}$ on level ground, $15 \mathrm{~km} / \mathrm{hr}$ ascending the hill and $30 \mathrm{~km} / \mathrm{hr}$ descending the hill, and that the whole outing, including the picnic lasted for three hours, estimate when they left home in the morning.

## 59. "Retrieving the hat."

The Humber River flows towards Lake Ontario at the rate of $3 \mathrm{~km} / \mathrm{hr}$. Ahab is rowing upstream; his speed relative to the water is $5 \mathrm{~km} / \mathrm{hr}$. Just as he passes under the Bloor Street bridge, his hat falls out of the boat, but he does not notice that it is missing until 20 minutes later. He immediately turns around to retrieve it, again rowing $5 \mathrm{~km} / \mathrm{hr}$ relative to the water. How far below the bridge has it travelled by the time that he retrieves it?

## 60. "Labourers in the vineyard."

A vintner hired a team of men to work in his vineyard. They all worked at the same rate, and, if they all had begun and finished at the same time, they could have done the work in 6 hours.

However, the men reported to work singly at equal intervals. Once on the job, they all stayed until the work was finished. It turned out that the one who arrived first worked five times as long as the one who arrived last. How many hours did the earliest arrival work?

## 61. "Self-referencing numbers."

Determine all positive integers with no more than ten digits (when written in base 10) for which the left digit tells the number of zeros in its representation, the second digit from the left the number of ones, the third digit from the left the number of twos, and so on.

## 4 "Homework!" Answers

## 1. Answer to: "This Triangle is Not Square"

The numbers should appear in the triangle as shown below.


Of course, there are other ways to write the answer.
The numbers 2,5 and 8 must be in the vertices. The numbers 4 and 9 must be on the side between 2 and 5 ; the numbers 3 and 7 must be on the side between 2 and 8 ; and the numbers 1 and 6 must be on the side between 5 and 8 .

## 2. Answer to "Drop the Triangle"

The radius is 1 .
The most elegant way to see this is to imagine that the triangle is slid down the vertical sides of the square until its base rests on the base QR of the square.

Its apex $P$ slides 1 unit down to a point $P^{\prime}$ which is distance 1 unit from the vertices $Q$, and $R$; this point is the centre of a circle of radius 1 that contains $P, Q$ and $R$.


## 3. Answer to "Forgetful Iphy!"

Since Orestes travelled twice as fast as Iphigenia, he travelled for half the time. Since Electra travelled three-halves of the speed of Iphigenia, she travelled for two-thirds of the time that Iphigenia did. So the half-hour represents the difference between two-thirds and one-half of Iphigenia's travelling time, namely one-sixth of that time. So Iphigenia travelled for three hours until they all met at three in the afternoon.

## 4. Answer to "Bikes that Pass at Noon."

Since each lady travels at constant speed, for each lady, the time taken to travel from Calgary to the meeting point bears the same ratio to the time taken to travel from Edmonton to the meeting point If " $t$ " is the time in hours of travel before noon, this means that $\frac{t}{9}=\frac{4}{t}$, so that $t=6$. So they set off at 6:00 a.m. and travelled for six hours until they met.

## 5. Answer to "The Hymns at St. Trinians."

Each hymn number must have three digits. (If the lowest number has fewer than three digits, then no more than eight digits can be used altogether. If the lowest number has at least three digits, then the other two numbers must have at least this many digits.) Since the largest number has three digits, the smallest must be less than 200 and begin with a 1 . Since the largest must be a multiple of 5 , its units digit must be a 5 . So the middle number must not begin with a 5 , and so must be less than 500 . This makes the smallest number less than 167 . The smallest number cannot end in a 7 , for this would make the middle number end in a 1 , Thus the smallest number must be one of $123,129,139$, $143,149,153,159,163$. Only one of these works and the hymn numbers were 129, 387, 645.

## 6. Answer to "Numbering on the Octahedral Die."

Consider two opposite vertices. Four of the faces surround one of the vertices, and the other four faces surround the other. The sum of the numbers on the first four faces is equal to the sum of the numbers on the second four, and all the numbers on the eight faces add up to 36 (the sum of the numbers from one to eight inclusive).
So the sum of the numbers on each set of four faces surrounding a vertex must be 18. There are eight ways of finding four numbers that add to 18: $(8,7,2,1),(8,6,3,1),(8,5,4,1),(8,5,3,2)$, $(7,6,4,1),(7,6,3,2),(7,5,4,2)$ and $(6,5,4,3)$. To describe the possibilities, pick a pair of opposite vertices (the "north" and "south" poles). In the following three tables, the top row lists the numbers around the north pole and the bottom on the adjacent faces around the south pole.

| 8 | 5 | 3 | 2 |
| :--- | :--- | :--- | :--- |
| 1 | 4 | 6 | 7 |
| 8 | 5 | 2 | 3 |
| 1 | 4 | 7 | 6 |
| 8 | 2 | 5 | 3 |
| 1 | 7 | 4 | 6 |

Note that as a matter of interest the sum of the squares of the numbers around the north pole is equal to the sum of the squares of the numbers around the south pole, and that faces that are adjacent along the equator have numbers that sum to nine.

## 7. Answer to "A Slight Difference."

If the left digit of the top number exceeds the left digit of the bottom by 2 or more, the difference between the two numbers exceeds 1,000 . So we want the top left digit to be one more than the bottom left digit. Now the last three digits of the top number should form a number as small as possible while the last three digits of the bottom number should form a number as large as possible.

$$
\begin{array}{llll}
5 & 1 & 2 & 3 \\
4 & 8 & 7 & 6
\end{array}
$$

As seen in the rows above, the two numbers are 5,123 and 4,876 .

## 8. "Answer to Coverage."

The best way to see that the right side of the top page falls on the bottom right corner of the bottom page is to look at the symmetry.


If we turn the configuration over with the bisector of angle $P Q R$ as axis of rotation, then the roles of the top and bottom pages are interchanged. Once this has been established, then the area of triangle $P Q R$ considered with base $Q R$ has half the area of the rectangle constituting the lower page with the same base and height. Since the top page covers more than this triangle, the desired conclusion follows.

## 9. Answer to "Knights \& Knaves"

Exactly one of $\mathbf{B}$ and $\mathbf{C}$ is a knight. Suppose it were B. Then A did say that there was one knight among them. A could not have said this truthfully, for then there would have been two knights and the statement would have been a lie; nor could have A said this falsely, as there would then have actually been only one knight. So the hypothesis that $\mathbf{B}$ is a knight leads to a contradiction. Let us check out whether B could be a knave. This could have happened if A had truthfully said that there were two knights among them or if A had falsely said that there were some number other than one. Thus, $\mathbf{B}$ is a knave and $\mathbf{C}$ is a knight.

## 10. Answer to "A Number Game."

Cordelia can assure a win as follows:
Cordelia begins by selecting 5. Suppose first of all that Kent selects some number other than 2. Then Cordelia on the third move can achieve a sum of 12 , and so claim all the numbers from 1 to 6 , inclusive. Kent cannot match this. Note that no sum can exceed 21, and that sums of 10 and 20 are not possible for Kent. Therefore, to claim 5, Kent must play 3 on the fourth move to achieve a sum of 15 . But then, Kent cannot achieve any of the sums $8,12,16$ and 20 , and so cannot claim 4.

Now suppose Kent selects a 2 . Then for her second move, Cordelia selects 1. So far, Cordelia can claim 1, 2, 4, 5, 8. Thus far, the numbers 5,2 and 1 have been selected. To claim 2, Kent must now select an even number to get an even sum (at the end of Kent's third move, the sum, 21, is odd). If Kent selects 4, he can now claim 1, 2, 3, 4, 6, 7, 12, but then Cordelia selects 3 to get a sum of 15 and so claim altogether $1,2,3,4,5,8,15$. Kent will never claim 5, so Cordelia wins. On the other hand if Kent selects 6 for his second move, he makes the sum 14 and can claim $1,2,7$. Cordelia's third move is to select 4 , making a sum of 18 and so claiming all of $1,2,3,4,5,6,8,9,18$. Kent cannot make a sum divisible by 4 and so Cordelia wins again.

## 11. Answer to "Finding The Correct Time."

Stephen knows from his own clock how long he was absent from home, and from Stockwell's clock how long he was at his house. The difference is the time required to walk there and back, so he adds half this difference to the time on Stockwell's clock when he left to set his own clock when he gets home.

## 12. Answer to "The Swinging Lad."

The Etobicoke train regularly arrives at the station about two minutes after the Scarborough train. If Lothario arrives at random in the 10 -minute interval between successive Scarborough trains, it is more likely that the next train will be going to Scarborough.

## 13. Answer to "Keeping a Good Distance."

The seven points should be positioned as shown below:


## 14. Answer to "Misinformation."

One drawing suffices. Take a single marble out of the box marked BW. Suppose for sake of argument, that it is black. Then, as the box was mislabelled, it must have come from the box with the two black marbles. The box with the two white marbles cannot bear the label WW, so it must be the one with the label BB. Finally, the box with the label WW must have one marble of each colour. Similar reasoning applies if the marble drawn is white.

## 15. Answer to "The Great Escape."

Adonis runs at 15 kilometres per hour. The difference between the time that Adonis needs to run the $\frac{5}{8}$ of the bridge length to $\mathbf{B}$ and the time he needs to run the $\frac{3}{8}$ of the bridge length to $\mathbf{A}$ is the time taken for the train to cross the bridge. Thus, the train crosses the bridge in the same amount of time that Adonis needs to run a quarter of its length. So the train goes at four times the speed of Adonis.

## 16. Answer to "Ferryboats."

The river is 1760 metres wide. Since the resting times of the two boats are equal, we need consider only the amount of time that each boat is in motion. Both boats are in motion the same amount of time until their first encounter, and between their encounters. Let $w$ be the width of the river. The ratio of their speeds is $(w-720): 720$ and $[w+(w-400)]$ : $(w+400)=(2 w-400):(w+400)$. Equating these ratios leads to $w^{2}-320 w=1440 w$, whence $w=1760$.

## 17. Answer to "The Four-Sided Cut-Up."

You can try this yourself with a piece of paper. After you cut off the ears, slide two opposite ears to the other ends of the inner parallelogram [(a) and (b) in the diagram], and rotate the other two ears [(c) and (d)] to face into the parallelogram. There will be an exact fit. Some of you might wish to verify mathematically that this is so.


## 18. Answer to "Who Gets Home First?"

Since Penelope runs for half the time, and we can assume that she runs faster than she can walk, she runs for more than half the distance. Thus, she is moving faster for a larger proportion of the distance than Ulysses, and so will cover the ground quicker and be back to start the coffee.

## 19. Answer to "Drying Out the Slurry."

It weighed $2,000 \mathrm{~kg}$. The weight of the tailings did not change, but after the evaporation occurred, it represented twice the total weight of the slurry at the end as it did at the beginning.
20. Answer to "Beating the Odds."

We assume the hats are assigned at random, so there are eight possibilities (each man can be given either a red or green hat). The men agree that, if any one of them sees two hats of the same colour, then he will guess that his own hat is of the other colour. Otherwise, anyone seeing a hat of each colour will keep silent. In the event (which happens with two chances in eight), that all men have the same colour hat, all will speak and make an incorrect guess. However, with six chances in eight, two men will have a hat of one colour and the third of the other colour. In this case, the first two will remain silent and the third will make a correct guess. The probability of getting the fortune is three chances out of four with this strategy.

## 21. Answer to "Giving a Handicap."

Castor and Pollux will arrive 95 metres from the starting line together. But Castor is the faster and will run the remaining five metres quicker to win the race.

## 22. Answer to "Ski Lift."

Since 3 minutes equals 180 seconds, and $\frac{180}{5}=36$, there are 36 intervals between chairs. Thus, he passes 35 chairs.
A key observation is that Baldur will pass every chair on the lift; he sees the chair immediately behind his first, and when he is almost at the top, he will see on the way down the seat immediately in front of his own. Since he sees 35 chairs on the way up, there are 36 chairs including his own on the lift. These 36 chairs make a complete circuit of the lift every six minutes ( 360 seconds), so 10 seconds elapse between seats at the platform where people get on and off.

## 23. Answer to "The Sale."

The clerk and Julius are both correct. By the clerk's computation, adding the 15 per cent sales tax to 160 denarii yields 184 denarii, which when reduced by one quarter of its value gives 138 denarii. By Julius's computation, the 160 denarii is reduced to a net cost of 120 denarii, upon which a sales tax of 18 denarii is payable. The equality of these two approaches is encapsulated by the equation $[(160)(1.15)](0.75)=[(160)(0.75)](1.15)$, and reflects the fact that the order of multiplication does not alter the outcome. Thus, the tax is based on the actual sale price to the customer.

Calpurnia has a lot of gall in intervening. She is incorrectly applying both percentages to the same base amount. However, the clerk can exercise both shrewdness and diplomacy by accepting Calpurnia's calculation, which will yield a payable amount of 144 denarii.

## 24. Answer "Points of View."

We can imagine the blocks stacked atop a $3 \times 3$ square frame with the side view taken from the right. There must be a stack of four cubes on the centre cell, and no other cell can have a stack this high. There is at least one block in the back row of the frame, and each cell in this frame has at most one cube. The left column of the frame must have at least two blocks and the right column at least three blocks.

There can be no more than 20 blocks; the number in each cell of the frame indicates the number of cubes over that cell:

| 1 | 1 | 1 |
| :--- | :--- | :--- |
| 2 | 4 | 3 |
| 2 | 3 | 3 |

We can manage with as few as 10 blocks:

| 1 | 0 | 0 |
| :--- | :--- | :--- |
| 0 | 4 | 0 |
| 2 | 0 | 3 |

## 25. Answer to "A Triangular Array."

The top rows of the triangles can be $(3,1),(3,2),(6,2,5)$, $(6,1,4),(5,6,2),(4,6,1),(8,10,3,9),(8,10,1,6),(9,10,3,8)$, (13, 3, 15, 14, 6).

## 26. Answer to "The Six-Number Problem."

We have, for example that:

$$
\begin{aligned}
(24 \div 6)-((3+7) \div 5) & =2 \\
(2+7+24) \div(6+5) & =3 \\
((24+6) \div(3+7))+2 & =5 \\
24 \div\left(5+7-2^{3}\right) & =6 \\
(6+2+3+24) \div 5 & =7 \\
(2 \times 3)+(5+6+7) & =24
\end{aligned}
$$

## 27. Answer to "Twice as Old."

If the two people of the pair have the same birthday, then the older is twice the age of the younger for exactly one year, when the younger one attains the same age that the older was when the younger was born. In the case of twins, this happens during the first year of life when both are aged zero. Suppose that the two people have different birthdays. Then the older is twice the age of the younger for two periods totaling one year. For suppose that the older is $y$ years old when the younger is bom. During the first year of life of the younger, the older will turn $y+1$. So for the rest of their lives, their ages will differ by either $y$ or $y+1$ years depending on the time of year. Thus, one age will be double the other during the first part of the year that the younger's age is $y$ and the last part of the year that the younger's age is $y+1$. Between these two periods, for exactly one year, the age of the older is $2 y+1$.

## 28. Answer to "My Three Sons."

Probably the best way to approach this problem is the most direct way. Simply list all the possibilities for the three ages:

$$
\begin{gathered}
(1,1,72),(1,2,36),(1,3,24),(1,4,18),(1,6,12),(1,8,9),(2,2,18), \\
(2,3,12),(2,4,9),(2,6,6),(3,3,8), \text { and }(3,4,6) .
\end{gathered}
$$

My friend and I both know the house number, so he should be able to figure out the ages, unless there are two possibilities giving the same sum. All the sums are distinct except for $14=2+6+6=$ $3+3+8$. So he needs additional information to decide whether the twins are older or younger; this is provided by the last statement, so that the ages of my three sons are $3,3,8$.

## 29. Answer to "Redecorating."

In the large office, the painters did twice as much before midday as after, so they painted one-third of the large office after midday. Since there were equally many painters in both offices after midday and since the smaller office had half the area of the larger, two-thirds of the smaller office was painted on the first day. This meant that it required one painter to decorate one-third of the smaller office during a full day. So two painters could decorate two-thirds of the smaller office during a full day, while it would need four painters to decorate two-thirds of the smaller office during a half-day. So, in the afternoon of the first day, each office had four painters, and there were eight altogether.

## 30. Answer to "The Slow Clock."

Since the clock records 14 minutes of every 15 minutes of real time, the actual time elapsing over a period is $15 / 14$ of that shown by the clock. Thus, in the 3.5 hours shown by the clock, $(15 / 14) \times(7 / 2)=15 / 4$ hours actualy have elapsed. Thus, the actual time is 12:15 p.m.

## 31. Answer to "Area Problems."

We can see right away that $B+D=1 / 2$, and that $A=C+E=$ $D+E=1 / 4$, so that $C=D$. The triangles of areas $D$ and $E$ are similar with the linear dimensions of the first twice those of the second (look at the hypotenuses).
Hence $D=4 E$. It follows that $(A, B, C, D, E)=$ ( $1 / 4,3 / 10,1 / 5,1 / 5,1 / 20$ ).
For the second square, congruent triangles give us $P+T=T+$ $U+V+W=S+W=1 / 6$ and $T=W$. So $P=S=T+U+V$.
From similar triangles, we find that $T+U+V=3^{2} W=9 W=$ $9 T$. Since $1 / 6=T+U+V+W=10 W$, we deduce that $T=W=1 / 60$, and $U+V=8 T=2 / 15$.
The triangles with areas $R$ and $U$ are similar with their corresponding linear dimensions in the ratio $7: 1$. (See the diagram, in which some of the lines are drawn and the parallel corresponding sides have lengths $7 a$ and $a$.)
Hence $R=7^{2} U=49 U$, so that $1 / 3=R+V=49 U+V$. Therefore, $48 U=(49 U+V)-(U+V)=1 / 3-2 / 15=1 / 5$. Putting all of this together yields that $(P, Q, R, S, T, U, V, W)=$ (3/20, 79/240, 49/240, 3/20, 1/60, 1/240, 31/240, 1/60).


## 32. Answer to "Squares \& Cubes."

Here are some examples: $(1,2,3,4,5,6),(1,2,2,3,4,6)$, $(1,2,2,4,4,6), \quad(2,2,4,4,6,6), \quad(3,3,3,6,6,6), \quad(6,6,6,6,6,6)$. Those of you anticipating long waits in airports might wish to investigate the situation with sets of numbers of different sizes.

## 33. Answer to "Complete Coverage."

There are 24 kilometre blocks of road that must be covered. Look at the T -junctions in the middle of an edge. The cyclist must enter and leave each of these twice, thus covering at least one block for each pair of adjacent T -junctions a second time. Accordingly, the cyclist must ride at least 28 kilometres.


However, there are many ways of achieving this minimum ride; just one example is shown above.

## 34. Answer to "Finding the Counterfeits."

For any selection of coins, the number of counterfeits will be equal to the number of grams less than the true weight were the coins genuine. Thus, they should make the selection in order that the number of counterfeits determines uniquely a particular set of rolls. Accordingly, the problem is to find six positive integers (representing the number of coins taken from the six rolls) such that the sums of all of their subsets are distinct. If there were no restriction on the number of coins in the rolls, then the set $(1,2,4,8,16,32)$ would fit the bill. However, the largest number cannot exceed 25 . One possibility is ( $11,17,20,22,23,24$ ).

## 35. Answer to "It Makes No Difference."

The key observation to make is that each pair consists of one number that is no greater than 12 and one number that is at least 13. Let us see, for example, why the fourth pair of numbers cannot both be at most 12. If this were so, then it would mean that the first four numbers of the first set and the last nine (from the fourth to the twelfth) of the second set were all at most 12, which cannot occur. Similarly, one can argue that it is not possible for both numbers of any pair to be at least 13. Thus, the required sum always turns out to be:

$$
\begin{aligned}
& (24+23+22+\ldots+13)-(12+11+10+\ldots+1) \\
& \quad=(24-1)+(23-2)+(22-3)+\ldots+(13-12) \\
& =23+21+19+17+15+\ldots+1 \\
& =(23+1)+(21+3)+(19+5)+(13+11) \\
& =6 \times 24=144=12^{2} .
\end{aligned}
$$

The result generalizes to the case in which 24 is replaced by any positive even number.

## 36. Answer to "The Weight of a Full Load of Bricks."

It is possible for all the bricks to weigh a total of three kilograms; simply consider a set of three bricks each weighing a kilogram. However, the total weight cannot be more than three kilograms. Arrange the bricks in decreasing order of weight. Begin setting them aside until either all the bricks are exhausted or the total weight of those set aside just exceeds one kiIogram. Then at most two kilograms of bricks have been set aside (since the last brick cannot weigh more than one kilogram). Since we cannot have two sets weighing more than one kilogram, at most one kilogram of the bricks must remain.

## 37. Answer to "Maximizing the Area."

Let $a, b, c, d, e, f, g, h$, be the lengths of the sides, of the octagon. We must have that $b=d+f+h$ and $e+a=g+c$. It is straightforward to argue that we should have $d<h<f$ and $g<a<e$; otherwise, we can find a hexagon of larger area. Either $b$ or $c$ must equal 8 .


If $c=8, b=6$, then we must have that $(a, b, c, d, e, f, g, h)=$ $(5,6,8,1,7,3,4,2)$ for an area of 27 .
If $c=8, b=7$, then we must have that $(a, b, c, d, e, f, g, h)=$ $(5,7,8,1,6,4,3,2)$ for an area of 30 .
But, if $b=8, c=7$, then we must have $(a, b, c, d, e, f, g, f i)=$ $(4,8,7,1,6,5,3,2)$ for an area of 36 .
This problem is from a South African contest for school pupils.

## 38. Answer "to An Addition Problem."

The smallest sum is $1,026=589+437$, followed by $1,089=$ $765+324=657+432$. The largest sum is $6,021=5,987+34$, closely followed by $6,012=5,978+34$.

## 39. Answer to "A Tennis Match."

Agassi served the first of the nine games. The first player to serve served five and the second four. If the first player won $n$ of the five games he served, then he was broken $5-n$ times, and so the second player was broken $n$ times. Thus the first player won $n$ games on his service and $n$ games on the second player's service, for a total of $2 n$ games. Thus Agassi, with an even number of wins, served first.

Ross Honsberger, of the University of Waterloo, told me this problem. He discovered it in The Inquisitive Problem Solver by Paul Vaderlind, Richard Guy \& Loren Larson (Mathematical Association of America, 2002, \#64). The MAA has published several fine volumes of problems by Honsberger himself; consult the web site www.maa.org. - Ed Barbeau.

## 40. Answer to "A Network of Acquaintances."

In totalling up the number of acquaintances of all the readers, each pair of acquaintances gets counted twice, once for each of the pair. Hence, the sum will be twice the number of pairs of acquaintances and so be even. Suppose that the CAUT Bulletin is read by 50,000 people. Then the number of acquaintances of each one of them will be some number between 0 and 49, 999 inclusive. However, it is not possible for there to be a reader with no acquaintances among the other readers and, at the same time, a reader who is acquainted with everybody. This means that the 50,000 readers will each be assigned one of 49,999 numbers, so that some number must be assigned more than once.

## 41. Answer to "Two Pairs of Whole Numbers."

The only possibility is the brace of pairs $(1,5)$ and $(2,3)$. If each pair has the same sum and product, then both numbers of each pair are equal to 2 , but this possibility is excluded. Thus, for one of the pairs, the product of the numbers must exceed the sum and for the second, the sum must exceed the product. Thus, one of the numbers of one of the pairs must be equal to 1 . (If both numbers of the pair exceed 1, then the sum is no greater than twice the larger number, which in turn is no greater than the product.) For this pair, the sum exceeds the product by 1 ; for the other pair, the product exceeds the sum by 1 ; this can happen only if the numbers are 2 and 3 .

## 42. Answer to "Going the Distance."

Yes. The plan is that Goneril and Regan set out on the bike, while Cordelia starts walking. When Goneril and Regan reach a suitable point before grandmother's house, Regan dismounts and continues the journey on foot while Goneril returns to pick up Cordelia. All three sisters are to arrive at grandmother's house at the same time.
Since the tandem covers nine times the distance of a walker in any period of time, the tandem goes five units ahead and four units back, while Cordelia walks one unit. At the other end, Regan is set down one unit from her destination, while the tandem goes back four units and then ahead five units to arrive at the destination. The total length of the journey is thus six units, so that each unit is 10 km .
Goneril and Regan ride for 50 km on the tandem, taking $10 / 9$ hours to do so; Regan dismounts and takes two hours to walk the remaining 10 km . Goneril returns a distance 40 km to pick up Cordelia, and the two ride 50 km to their destination. Goneril rides a total of 140 km in $28 / 9=31 / 9$ hours; Cordelia and Regan each walk 10 km in two hours and ride 50 km in $10 / 9=11 / 9$ hours for a total travel time of $31 / 9$ hours. This is less than three hours and ten minutes ( $31 / 6$ hours), so they will see grandmother by noon.

## 43. Answer to "The Longest Path.".

Here are two long paths. There can be at most two segments of length of 2.24 units. The length of the first path is about 12.11 units and of the second about 10.06 units.


This problem was used at the 2004 KappAbel final competition for pupils in Grades 8 and 9 of Denmark, Finland, Iceland, Norway and Sweden.

## 44. Answer to "The Product of the Digits."

No. Consider, for example, the number 3,487 . The product of the digits is less than $3 \times 10 \times 10 \times 10=3,000$, which in turn is less than 3,487 . A similar argument obtains for any other number.

## 45. Answer to "My Grandson \& I."

He is eleven years old. Let $n$ be the lad's present age and $d$ the number of years between now and the earlier comparison. Then $12(n-d)=6 n-d$, so that $6 n=11 d$. Since $6 n$ is a multiple of $11, n$ must be as well. The only possibility from the context is that $n=11$.

## 46. Answer to" Walking to School."

Castor and Pollux passed each other at 8:30 a.m. Pollux arrived at the school five minutes before Castor. The latter part of the journey (after the passing) takes place in half the time of the former part of the journey. To see this, run the event backwards.

## 47. Answer to "Five Circles."

The numbers used in the five circles consist of those from 1 to 9 inclusive along with the four that are used twice. If the numbers in each circle add up to 11 , then the four numbers used twice must add up to 10 , and so be $1,2,3,4$ in some order. Trial of this possibility leads to success: the numbers appearing only in the top circles are $9,6,8$ from left to right; the numbers appearing in two of the circles are $2,4,1,3$ from left to right and the numbers appearing only in the bottom circles are 5 and 7. Are other solutions possible? The sum of the four duplicated numbers are, respectively, 15, 20, 25 and 30 when the numbers in each circle add up to $12,13,14$ and 15 .


## 48. Answer to "Cutting a Square into Three Parts."

Here are three possibilities. In the last diagram, none of the pieces are the same size.


| 12 | 9 |
| :---: | :---: |
| 14 |  |
|  |  |
| 6 |  |
| 7 |  |
| 3 |  |

## 49. Answer to "A Not Quite Magic Square."

There are essentially three possibilities: $(2,4,5 / 7,3,1 / 6,8,9)$, $(2,5,4 / 7,1,3 / 6,9,8)$, or $(5,2,4 / 1,7,3 / 9,6,8)$.
The sum of the third row cannot exceed $24(7+8+9)$, but must be odd, since the sum of all nine numbers is 45 and the sums of the first two rows are equal. The only possibility for the third row are numbers 6,8 , and 9 . Each column must add to 15 , so 7 , and hence 2 , must be in the same column as 6 . Since 2 has now been used, 5 cannot be in the same column as 8 . Hence, the columns in some order are $2,7,6 / 3,4,8 / 1,5,9$. Since each of the top two rows add up to 11 , they must be $2,4,5$ and $7,3,1$.

## 50. Answer to "Domestic Conviviality."

The wife can drink the beer in the barrel in 50 days, while it takes the husband only $\frac{150}{7}$ days. Let $w$ be the daily consumption of the wife and $h$ the daily consumption of the husband, measured as a fraction of a barrel. Then, $1=15(w+h)=6(w+h)+30 w$, from which we have $9(w+h)=30 w$, or $3 h=7 w$. Since $15 h=35 w$, we have $50 w=1$, so that $w=\frac{1}{50}$ and $h=\frac{7}{150}$, from which the result follows.

## 51. Answer to "Assigning Numbers."

While the problem can be solved just by trial and error, it helps to note that the sum of the numbers assigned to the sides is twice the sum of those assigned to the vertices. Thus, the sum of all the numbers must be divisible by 3 when the problem is solvable. This rules out the pentagon case. The sum of the numbers assigned to the vertices of the triangle, square and hexagon must be, respectively, 7, 12 and 26. Possible assignments for the vertices in order of the three figures are $\{1,2,4\},\{1,3,2,6\}$ and $\{1,3,8,2,7,5\}$. Are there others?

## 52. Answer to "Fair Division of Land."

The length of the fence is thirteen furlongs. Let $x$ be the length of the fence. Call the field whose parallel side have lengths 17 and $x$, the upper field, and the other field, the lower field. For each field, the distance between the parallel sides is its width. Recall that the area of a trapezoid is the product of its width and the average of the parallel sides. Since the upper and lower fields have the same area, their widths are in the ratio $(x+7):(17+x)$. We calculate the ratio of these widths in another way. We can think of the trapezoid as being comprised of a parallelogram and a triangle of base $17-7=10$, with the fence intersecting the triangle having a segment of length $x-7$. The ratio of the widths of the upper and lower fields is then $(17-x):(x-7)$. Equating these two ratios yields $(x+7):(17+x)=(17-x):(x-7)$, from which we obtain that $x=13$.

## 53. Answer to "A Long Wait at the Passport Office."

We had the numbers $169=(2+5+6)^{2}$ and $256=(1+6+9)^{2}$. One way to search for the answer is to start with any square number and create a sequence in which each term is the sum of the digits of the previous number. If you encounter a pair of numbers, each of which follows the number in the sequence, you have the answer. For example, starting with $4=2^{2}$, we get

$$
\begin{gathered}
4 \longrightarrow 16 \longrightarrow(1+6)^{2}=49 \longrightarrow(4+9)^{2}=169 \\
\longrightarrow(1+6+9)^{2}=256 \longrightarrow \cdots .
\end{gathered}
$$

Beginning with other squares does not lead to any further examples. If the number has at least four digits, then the number of digits of subsequent terms will eventually be less until we get down to two or three digits; then the sequence will eventually resolve itself to 1,81 or the pair $(169,256)$.

## 54. Answer to "Guessing the Right Bag."

There are at least two ways to distribute the tokens. First, he puts token 1 in the first bag, token 20 in the second, and the remaining tokens in the third. If the sum of the tokens selected is between 3 and 20 , then the second bag was not selected; if the sum is 21 , then the third bag was not selected; if the sum is between 22 and 39, then the first bag was not selected. Second, he might put tokens $1,4,7,10,16$ and 19 in the first bag, tokens 2 , $5,8,11,14,17$ and 20 in the second and tokens $3,6,9,12,15$ and 18 in the third. Is there a third possibility for the distribution?

## 55. Answer to "Splitting a Field into Two."

A possible way of partitioning the field is shown.


Starting at the point $P$ and proceeding in the direction of the arrows, we can trace along the corresponding segments of the two subfields. The measurements indicated must satisfy

$$
a+d=b+c \quad \text { and } \quad a+b=c .
$$

We can partition the field when

$$
(a, b, c, d, e)=(2,4,6,3,1) .
$$

## 56. Answer to "The ubiquitous 2009."

There are many solutions to this problem.
One strategy is to start with $x$ copies of $4018=2 \times 2009$ whose digital sum is 13 , insert a certain number of zeros, and finish with $y$ copies of 2009 whose digital sum is 11 . This requires that $13 x+11 y=2009$ and $4(x+y)<2009$.

For example, $(x, y)=(152,3)$ works. Thus, we construct a solution by juxtaposing 152 copies of 4018, following this by $1389=2009-4 \times 155$ zeros and concluding with 3 copies of 2009.

## 57. Answer to "The befuddled clerk."

It is convenient to work in cents, so as not to be confounded by the decimal point. If the four prices in cents are $a, b, c, d$, we have to solve $a+b+c+d=711$ and $a b c d=711,000,000=2^{6} \times 3^{2} \times 5^{6} \times 79$.

Exactly one price is a multiple of 79 ; at most three prices are even; at most three prices are divisible by 5 . Using these facts and some trial and error, we arrive at the four prices $\$ 1.20, \$ 1.25$, $\$ 1.50, \$ 3.16$.

## 58. Answer to "A day's outing."

It took them 3 minutes to cover each kilometre on level ground, 4 minutes going uphill and 2 minutes going downhill. Therefore, the time taken to travel each kilometre on the route in both directions is always 6 minutes.

Since they actually rode for two hours, or 120 minutes, the route must be 20 kilometres long. If the entire route were level, it would take them 60 minutes to get to the Point; if they had to travel uphill all the way there, it would have taken them 80 minutes to arrive. Therefore, they must have left between $10: 40 \mathrm{am}$ and 11:00 am.

## 59. Answer to "Retrieving the hat."

Relative to the water, he is pulling away from the hat as fast as he subsequently returns to it, so that he takes 20 minutes to retrieve the hat. By this time, it has travelled one kilometer towards Lake Ontario from the bridge.

## 60. Answer to "Labourers in the vineyard."

The answer is 10 hours.
It is clear from the context that there are at least two men. However, you do not need to know exactly how many men there are to answer the question. The key observation is that the sum of the time span worked by the first and last men is equal to the sum of the time span worked by the second and second last man, and so on, so that the average amount of time worked by the first and last men is the average working time for all of the men. Since we know this average to be 6 hours, the sum of the number of hours of the first and last men is 12 . Thus, the first man worked 10 hours and the last man 2 hours. If there were two men, their hours of work would be 10 and 2 ; if three men, 10, 6 and 2 ; if four men, $10,7+\frac{1}{3}, 4+\frac{2}{3}$, and 2 ; and, if five men, $10,8,6,4$, and 2.

## 61. Answer to "Self-referencing numbers."

The numbers are 1210, 2020, 21200, 3211000, 42101000, 521001000 and 6210001000.

We begin with some observations. The sum of the digits in the number is equal to the number of digits in the number. The left digit cannot be zero; if the digit 1 occurs exactly once, it must be in the second position from the left; the two left digits are either 20 or else sum to at least 3 (they cannot be 10 or 11). The number must have at least four digits. These facts can be used to narrow down the search.

A second approach is to decide on the number of digits, and act recursively, beginning with any number of that many digits whose digital sum is equal to the number of digits and then construct a new number whose left digit is the number of zeros in the original number, and so on. This will not necessarily lead to a result the first time.

The third approach is more analytical. Suppose the number is $\left(a_{0} a_{1} \cdots a_{n}\right)_{10}$ where $a_{0}$ is the number of zeros, $a_{1}$ the number of ones, and so on until $a_{n}$ is the number of occurrences of the digit $n$. Then counting the sum of digits in two different ways, we have

$$
a_{0}+a_{1}+a_{2}+\cdots+a_{n}=1 \times a_{1}+2 \times a_{2}+\cdots+n \times a_{n}
$$

from which

$$
a_{0}=a_{2}+2 a_{3}+\cdots+(n-1) a_{n} .
$$

Suppose that $a_{0}=r>0$. Then $a_{r} \geq 1$. Suppose that $a_{r}$ exceeds 1 , then the sum on the right is at least $2(r-1)$. Since $2(r-1)$ cannot exceed $a_{0}=r$, we must have $r \leq 2$. If $r=1$, then $a_{0}=a_{2}=1$ and $a_{k}=0$ for $k \geq 3$. If $r=2$, then, since $a_{3}$ cannot be positive, $a_{0}=a_{2}=2$ and $a_{k}=0$ for $k \geq 3$.

Suppose that $a_{r}=1$. Since $a_{0}=r$ and $(r-1) a_{r}=r-1$, we must have $a_{2}=1$. In all cases, it is straightforward to fill in the admissible values of $a_{1}$.

## 5 Mathematics in a deck of cards

As a reward to the intrepid reader who has tried the problems, I would like to present a number of mathematical card tricks that have come my way over the years. None of them require sleight of hand or deception; they all depend on the application of simple mathematical principles, without losing their mystery for the uninitiated. The "magician" performs the trick to the amazement of the "subject".

## 1. Three questions for 27 options.

The magician deals 27 cards into three 9 -card columns and asks the subject to secretly select one of the cards, but tell him which column contains it. Once the magician has this information, he gathers up the three columns, one on top of the next, and then deals the cards across into three 9 -card columns. He then ascertains from the subject which column contains the selected card and again deals the cards across into three 9 -card columns. Upon being told a third time which column contains the selected card, he is able to identify it.

The trick is based on dealing out the cards so that the first answer narrows the selected card down to one of nine cards, the second answer to one of three cards and the third answer down to a unique card. This trick is fairly well known, sometimes in the form of dealing only 7 cards to a column. Often it is set up, so that the named column is gathered up in the middle, so that the selected card turns out to be in the very middle of the deck.

## 2. The flipover.

Select the ten hearts from ace to ten, inclusive, and arrange them in increasing order in a fan. The magician presents the fan, cards face down, to the subject and asks her to pull out two adjacent cards, turn them over and reinsert them face up into the spot whence they were taken. Thus, if $4 \circlearrowleft$ and $5 \circlearrowleft$ were removed, the 5 will be where the 4 was, face up, and vice versa. He asks the subject to continue performing several times the following: cut the deck and put one end before the other, and pull out two adjacent cards, turn them over and restore them in place (either card chosen can be face down or face up). Then the magician does something sight unseen by either person and
then shows the fan; all the even cards are facing one way and the odd cards the other. What has the magician done, and why does it work?

The key to this is that parity of the cards in the fan alternate, and the actions, in a more general sense, preserve the alternation. Since cutting the deck is like moving the cards around in a ring, we will assume the cards start face down in a ring, ignore the cut, and just focus on the turnover. In each position in the ring, the cards are in one of two states $E U-O D$ (even-up, odd-down) or $E D-O U$ (even-down, odd-up). These states alternate with position, and continue to alternate with each flip. Turning over a single card and restoring it into the same position reverses the state of that card.

As a hint, it may be pointed out that whatever the magician did at the end should work if no operations at all were carried out.

## 3. Still complete in the halves.

Two packs of 13 cards, one consisting of the 13 spades in order from ace to king and the other consisting of the 13 hearts in reverse order from king to ace are placed face down on the table and subjected to a rough riffle shuffle. This means that they are incorporated into a single pile, with cards incorporated in bunches alternately from the two packs. (For a perfect riffle, the cards are mixed one alternately from each pack.)

The top thirteen cards are taken from the united pack. It turns out that each of the ranks from ace to king appears exactly once among them. The same is true for the pile left behind. Why is that?

Note that in the incorporated pack, the hearts and spades remain in the same order; they are just interspersed. Suppose, for example, that the top thirteen cards contain no six of spades. Then at most five spades made it into the top thirteen, the ace through five. Therefore, at least eight hearts must be there, the king through six. Thus, the six of hearts must be present.

## 4. Picking the correct pair.

The magician deals onto the table ten pairs of cards, and asks the subject to select one of the pairs silently. The magician then gathers the pairs up and deals them into four rows of five cards each. Upon being told which rows contain the two cards of the chosen pair, the magician can identify them.

This is easy to explain, as it simply depends on producing a one-one correspondence between the ten pairs and the number of ways of picking two rows out of four, with the possibility of a row being selected twice. The magician picks the cards up keeping the pairs together, and then carefully deals each pair into two particular rows. For example, the ten pairs can be dealt into rows $(1,1),(1,2),(1,3)$, $(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)$.

A less transparent way of dealing into rows is possible. Keep in mind the four words ATLAS, BIBLE, GOOSE and THIGH. The words have ten different letters, each occurring exactly twice. Each letter appears in a different pair of the words, and each pair of words has exactly one letter in common (with each word having one letter appearing twice). Cued by these words, you can deal the pairs accordingly.

## 5. Go to the top!

A pack of the thirteen spades is thoroughly shuffled and the cards are laid out from left to right on the table. We adopt the usual convention that $A=1, J=11, Q=12$ and $K=13$. If the leftmost card is $k$, then the $k$ th card from the left is taken from its position and placed in the first position at the left. The order of the remaining cards is left undisturbed. This move is repeated. It is found that, regardless of the original order of the cards, eventually the ace is brought to the left and the process stops. Why is this?

This probably needs to be performed a few times until you begin to see the dynamic. Basically, the ace either stays in its original position, or gets shoved to the right, until it is suddenly brought to the leftmost position. If the ace starts out in the $n$th position, then one of the left $n-1$ cards must have rank $n$ or bigger. One can see that one such card eventually gets "hit", whereupon the ace either comes to the first position or moves one position to the right. This is a nice example for discussion of induction.

## 6. Which card comes last?

The magician takes 16 cards from the deck and places them upsidedown in a stack on the table. The subject is asked to remove from the top fewer than half of them, leaving a stack of between 9 and 15 cards. The magician then picks this up and shows the subject (but not himself) the cards in the stack. If the subject removed $k$ cards, the subject is asked to remember the value of the $k$ th card from the bottom.

The magician then takes up the stack, cards upsidedown, and deals the cards alternately to the bottom of the stack and face-up onto the table until only one card remains in the stack. This card turns out to be the one identified earlier by the subject.

This is a manifestation of a Josephus situation; a group of people are arranged in a circle, and each $r$ th person is eliminated until only one remains. Here, $r=2$. In the present situation, suppose that $n$ individuals numbered from 1 to $n$ are in a ring, and we start with individual 1 and eliminate every second one as we count around. If $f(n)$ is the last individual to remain, we can see that $f\left(2^{m}\right)=1$ for every non-negative integer $m$ and that $f(n+1)=f(n)+2$ when $n+1$ is not a power of 2 . Then $f(16-k)=17-2 k=(16-k)-(k-1)$ (for $1 \leq k \leq 7$ ), so that the final card is the $k$ th card from the end of the $16-k$ cards.

## 7. A little hidden algebra.

The magician takes 26 cards from a regular deck and places it face down on the table. He then turns over the cards one by one to show the subject that the deck is randomly mixed, and then restores the 26 cards to the original position; call this the stock. Handing the remaining 26 cards to the subject, he instructs the subject to place a card face up on the table. We will use the equivalence $A=1$, $J=Q=K=10$. If the card turned up is $k$, the subject then places on top of it sufficiently many cards face up to count up to ten. The ranks of the additional cards are immaterial, the subject counting $k$, $k+1, \cdots, 10$ until she reaches 10 . Then the subject starts a new pile by placing one of the remaining cards on the table, and performing the same operation. This is repeated as long as there are sufficiently many cards and there are at least three piles. (In the rare case that there are not enough cards to form three piles, the subject can "borrow" from the top of the stock.)

The subject then turns three of the piles over and puts the rest of the cards face down on top of the 26 -card stock left by the magician. The subject is then to turn over the top card on each of the three piles, add them and count down that many cards in the stock (the 26 -cards augmented by the leftovers). While the subject is doing this, the magician predicts what the terminal card will be.

For example, suppose the subject turns over a 4 ; then she will place on top of it face up six more cards, counting as she goes $5-6-7-8-9-10$. If the three piles chosen are built on, say, 4, 3 and 8 , then the three piles built up on them will have, respectively, 7,8 and 3 cards. Eight cards will be returned to the stock, which will now have a total of 34 cards. When the subject turns over the three piles and reveals the top cards, these will, of course, be 4,3 and 8 , and the subject will count down 15 cards into the stock. This will go through the eight returned to the stock and end up with the seventh from the top of the original stock of 26 .

Remarkably, no matter what cards are turned face up, the count will go down to the seventh card from the top of the 26 -card stock, and it is this card that the magician must memorize. I usually convince students that it works in the following way. Suppose that the three cards turned up are all tens. Then twenty three cards are returned to the stock, and we have to count down 30 cards to the seventh from the top of the original stock. For every reduction of one in the sum of the three cards, there is one more card in the three piles and one fewer returned to the stock. At the same time, there is one fewer card to count down, so we will always wind up in the same place.

I am indebted to Peter Taylor of Queen's University for showing me this nice trick.

## 8. A quick reversion to order.

Begin with a new deck of cards in which the suits appear in order, ranked in order. A remarkable fact is that eight perfect inside riffle shuffles (where the top and bottom cards of the deck remain in position) will restore the deck to its original order. If, like me, you cannot perform a perfect riffle shuffle, you can deal them to obtain the inverse effect of a riffle and still get a striking effect. Suppose that the cards are numbered from 0 to 51, inclusive, and are originally in this order from top to bottom. Deal the cards face up alternately into
left and right piles, 0 to the left, 1 to the right, and so on. Pick up the piles, putting the right pile on the left one, turn the incorporated deck upside down and repeat. Now 0 goes to the left, 2 to the right, 4 to the left, 6 to the right and so on. Repeat the process.

Each time the process is repeated and the deck incorporated, the value of the card in any given position gets multiplied by 2 modulo 51 . Since $2^{8} \equiv 1(\bmod 51)$, eight repetitions will bring the cards back to the original order. However, when the cards are dealt face up, students can see how the order changes from one deal to the next and some interesting things occur. Try it!

While one generally cannot go into the number theory involved for most school students, the investigation of how long it takes this shuffle to return a deck to its original order for various numbers of cards is worthwhile.

## 9. A two-person operation.

This stunning trick is a recent one that I learned about in the Mathematical Intelligencer. In this case, the magician requires a confederate who is in on the trick. The magician picks five cards out of the deck at random, and passes them to the confederate without looking at them. The confederate returns four of them, and the magician identifies the fifth.

As you might suspect, the four cards selected by the confederate code the fifth card; the order in which the four cards are returned is important. The first task is to allow the magician to identify the suit of the card. Among any five cards, there are two cards of the same suit, so the confederate plans to retain one of them and pass the other to the magician first. He has a choice of which of the two to return.

Imagine the thirteen ranks from ace to king arranged in a circle, like the numbers on the face of a clock. If we go around the circle clockwise, one of the cards must follow the other in no more than six steps. For example, if the cards are two and ten, then the two follows the ten in five steps $(J, Q, K, A, 2)$. The confederate keeps the follower card and passes the other; in the example, he retains the two and passes the ten.

There are three remaining cards that will be passed to the magician. There are six possible order in which they may be passed, and the six orders will determine how far ahead of the first passed card the retained card is. One way of doing this is to imagine the 52 cards of the deck ordered lexicographically along suits and ranks; the clubs in order from ace to king are ranked lowest, then the diamonds, then the hearts and then the spades. Suppose that the last three passed cards are $P, Q, R$ in lexicographic order. Then if the retained card is $1,2,3,4,5,6$ steps beyond the first passed card is the confederate passes the last three cards in order: $P Q R, P R Q, Q P R, Q R P, R P Q$, $R Q P$.

For example, if the confederate holds the five cards $9 \mathbf{\$}, 2 \diamond, 5 \diamond$, $10 \diamond, 4 \boldsymbol{\downarrow}$, he might decide to retain the two of diamonds and pass the ten of diamonds first. The other three cards in lexicographic order are $9 \boldsymbol{\downarrow}, 5 \diamond$ and $4 \boldsymbol{\uparrow}$. Then, to code the two of diamonds, he passes to the magician the cards $10 \diamond, 4 \uparrow, 9 \boldsymbol{\uparrow}, 5 \diamond$.

## 10. Mutually referencing sequences.

This remarkable card trick is due to the American philosopher, Charles Peirce. Place on the table twelve hearts ranked from ace to queen face up with the ace on top. Take in hand twelve spades, also ranked, but face down with the ace on top. (The kings are not used.) Deal the spades into two piles alternately, ace to the left, two to the right, and so on, until the queen is the last card dealt to the right pile. Put the queen to one side, and replace it with the ace of hearts. Now place the left pile on top of the right, turn the combined pile upside down, and deal again into two piles; the left pile receives in order
 $7 \boldsymbol{\uparrow}, J \uparrow$. Put the last card, $J \boldsymbol{\uparrow}$, to one side beside $Q \boldsymbol{\uparrow}$, and replace it with the top heart, 20 . Once again, place the left pile on top of the right, turn the combined pile over, and deal alternately into two piles. Replace the last card dealt by the top heart on the heart pile. Continue doing this (twelve times) until all the spades are replaced by hearts. You will observe that the last card dealt is always a spade. The spades will come be put aside in the following order: $Q, J, 9,5$, $10,7, A, 2,4,8,3,6$. On the table will be the hearts in two piles; place the left on the right and then spread the combined pile face up onto the table.

The cards will appear in the order: $7,8, J, 9,4, Q, 6,10,3,5$, $2, A$. Let $A=1, J=11$ and $Q=12$, you will observe that the $m$ th spade is $n$ if and only if the $n$th heart is $m$. For example, the third spade is the 9 , while the ninth heart is the 3 .

That was Act One. Here is Act Two. Pick a small number, say 4 , and place the king of hearts on the face-up pile of hearts. Turn the pile over and deal the cards face-up into four piles consecutively. The king will appear on the first pile. Pick up this pile, place it on the second, then the combined piles onto the third and then finally everything onto the fourth. Spread the hearts face-up and discard the king; they will appear in order: $9,10, A, J, 6,2,8, Q, 5,7,4,3$. The third heart is the ace. So cut the sequence of spades to put the three in first position, without disrupting the cyclic order of spades. You will find that the heart and spade sequence is related as before; the $m$ th spade is $n$ if and only if the $n$th heart is $m$.

This process can be repeated using another small number. Suppose 5 is chosen. When you put the king at the end of the hearts and deal into five piles, you will find that the king appears on the third pile. Number the positions of the piles: I, II, III, IV, V. Then we carry out a similar procedure to combine the heart, placing pile III on pile I, the combined on pile IV, then on II and on $\mathbf{V}$ (going three positions over each time). Once again the spade sequence can be cut to produce the mutual referencing of the spade and heart sequences.

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