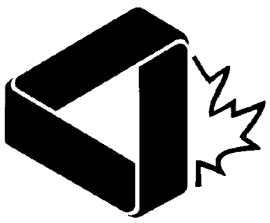
A Taste Of Mathematics



Aime–T–On les Mathématiques

Volume / Tome VII
PROBLEMS OF THE WEEK

Jim Totten

Thompson Rivers University

The ATOM series

The booklets in the series, **A Taste of Mathematics**, are published by the Canadian Mathematical Society (CMS). They are designed as enrichment materials for high school students with an interest in and aptitude for mathematics. Some booklets in the series will also cover the materials useful for mathematical competitions at national and international levels.

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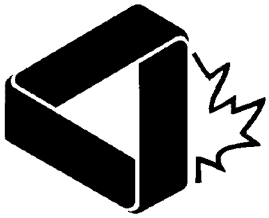

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1 Foreword

It is the hope of the Canadian Mathematical Society that this collection may find its way to high school students including those who may have the talent, ambition and mathematical expertise to represent Canada internationally. Those who find the problems too challenging at present can work their way up through other collections. For example:

1. The journal *Crux Mathematicorum with Mathematical Mayhem* (subscriptions can be obtained from the Canadian Mathematical Society, 577 King Edward, PO Box 450, Station A, Ottawa, ON, Canada K1N 6N5);
2. The book *The Canadian Mathematical Olympiad 1969–1993 L'Olympiade mathématique du Canada*, which contains the problems and solutions of the first twenty five Olympiads held in Canada (published by the Canadian Mathematical Society, 577 King Edward, PO Box 450, Station A, Ottawa, ON, Canada K1N 6N5);
3. The book *Five Hundred Mathematical Challenges*, by E.J. Barbeau, M.S. Klamkin & W.O.J. Moser (published by the Mathematical Association of America, 1529 Eighteenth Street NW, Washington, DC 20036, USA);
4. The CMS website,

   `www.cms.math.ca`

   where all of the International Mathematical Talent Search problem sets are available.

2 About the Author

Jim Totten is a faculty member at Thompson Rivers University in Kamloops, British Columbia (formerly called the University College of the Cariboo). Jim has taught post-secondary mathematics in Ontario, Nova Scotia, Saskatchewan, and British Columbia over the past 35 years. He has been an active participant in the British Columbia High School Mathematics Contest (and its precursors) since 1980, and has been a coach for Putnam competitions at three different universities since 1976. For the past 10 to 12 years he has been conducting enrichment programs from grade 1 to grade 12 in local schools. Currently, Jim is the Editor-in-Chief of the CMS problem-solving journal, Crux Mathematicorum with Mathematical Mayhem.

3 Preface

When I was a graduate student at the University of Waterloo (1968–1974), I was very fortunate to have been assigned an office near that of Ross Honsberger, a faculty member well known for his enthusiasm regarding the solving of interesting mathematics problems. As a result, when Ross found some new interesting problem and/or solution which he wanted to share with someone, he often found me as a willing listener.

The excitement that Ross showed for his "gems" was contagious. Consequently, when I first starting teaching at Saint Mary's University in 1976, I wanted to show the math students there some of these wonderful problems. The method I chose was through a "Problem of the Week". The response was better than I had ever expected! The students not only enjoyed the problems, but were sometimes found gathering at the bulletin board where the problems were posted in anticipation of getting an early start on the next one!

With this kind of positive feedback, it was only natural that when I moved to other institutions as my career evolved, the "Problem of the Week" moved with me. The problems from the first few years were all repeated when I moved to a new institution, but I have tried to avoid repetition since I arrived at my current institution in 1979. Very few of the problems in my collection are original. Some of the problems are commonly known among problem-solvers, but I believe that everyone can find something new in this collection.

The collection continues to grow. This monograph contains only problems that appeared as a "Problem of the Week" before the fall of 1986.

I would like to thank John Grant McLoughlin of the University of New Brunswick, Fredericton, and Edward J. Barbeau of the University of Toronto, for proofreading earlier drafts of this manuscript and providing very valuable feedback.

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Dedication

For showing me that not only was it acceptable to be excited and enthusiastic about mathematical problem-solving, but that it was to be strongly encouraged, I am dedicating this book to Ross Honsberger. Thanks for being there, Ross!

4 Problems

1. Find the missing digits in the following long division. (. represents a decimal point; the division has no remainder.)

$$
\begin{array}{rl}
 & X\,X\,X\,X\,.\,X\,X\,X\,X \\
X\,X\,X\,\big| & \overline{X\,X\,X\,X\,X\,X\,X} \\
 & \underline{X\,X\,X} \\
 & \quad X\,X\,X \\
 & \quad \underline{X\,X\,X} \\
 & \quad\quad X\,X\,X\,X \\
 & \quad\quad \underline{X\,X\,X\,X} \\
 & \quad\quad\quad X\,X\,.\,X\,X \\
 & \quad\quad\quad \underline{X\,X\,.\,X\,X} \\
 & \quad\quad\quad\quad X\,X\,X\,X \\
 & \quad\quad\quad\quad \underline{.\,X\,X\,X\,X} \\
 & \quad\quad\quad\quad\quad\quad 0
\end{array}
$$

2. A criminal is currently located in the centre of a circular swimming pool, and a policeman trying to apprehend him is standing at the edge of the pool. The policeman is a non-swimmer; his maximum speed is 4 times the maximum swimming speed of the criminal. Let us assume that if the criminal can get to the edge of the pool without the policeman able to reach that point before him, then the criminal can escape.

Can the criminal escape or will the policeman inevitably catch him? Why?

3. In a certain classroom, there are 5 rows with 5 seats per row arranged in a square. Each student is to change her seat by going either to the seat immediately in front or behind her, or immediately to the left or right. (Of course, not all possibilities are open to all students.) Determine whether this can be done, beginning with a full class of students.

Try to generalize to a rectangular array and find the conditions under which such a change of seats can be managed.

4. "If two of my children are selected at random, likely as not they will be of the same sex," said the Sultan to the Caliph.

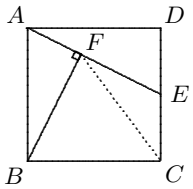
"What are the chances they will both be girls?" asked the Caliph.

"Equal to the chance that one child selected at random will be a boy," replied the Sultan.

How many boys and how many girls did he have?

5. Determine the total number of squares on an ordinary 8×8 chessboard, counting all squares of all sizes and positions with sides parallel to the sides of the chessboard.

6. Given a square $ABCD$ with E the mid-point of the side CD. (See diagram at right.) Join A to E and drop a perpendicular from B to AE at F. Prove $CF = CD$.



7. Describe all those natural numbers which can be written as the sum of two or more consecutive odd positive integers.

8. A census taker, who was very intelligent, came to a house where the three inhabitants were not at home. To get their ages, he asked the housekeeper, "What is the product of their ages?"

She replied, "252."

Then he asked, "What is the sum of their ages?"

She answered, "The same as the house number." (He could see the house number.)

The census taker then said, "Are any of them older than you?"

She replied, "No." (He apparently knew her approximate age.)

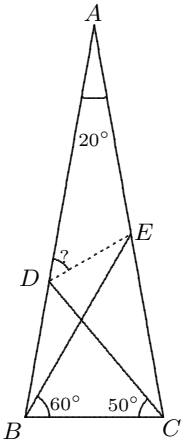
With that information, he wrote down the three ages. What are those ages?

9. The radius of the inscribed circle of a triangle is 4 and the segments into which one side is divided by the point of contact are 6 and 8. Determine the lengths of the other two sides of the triangle.

10. Two brothers sold a herd of sheep. For each sheep they received as many dollars as the number of sheep in the original herd. The money was divided as follows: the older brother took 10 dollars; then the younger brother took 10 dollars; next the older brother took another 10 dollars; and the younger brother took another 10 dollars; and so on. At the end of the division, the younger brother, whose turn it was, found that there was less than 10 dollars left for him. He took what remained. In order to even things up, the older brother gave him his penknife.

How much was the penknife worth?

11. Given an isosceles triangle ABC with $AB = AC$ and $\angle A = 20°$, let D be a point on AB with $\angle BCD = 50°$, and let E be a point on AC with $\angle CBE = 60°$. Find $\angle ADE$.



12. A convex polygon of twelve sides inscribed in a circle has (in some order) six sides of length 4 and six sides of length $6\sqrt{3}$. What is the radius of the circle?

13. Bill, Jack, and Tom went fishing together one day. Around the campfire that evening, as they elaborated the day's adventure for their wives, their tale took somewhat the following turn:

Bill: Tom caught only two fish.
Jack caught one more than Tom.
Jack and I together caught eight more than Tom.
I caught more than the others put together.

Jack: Tom caught the most.
I caught three more than Bill.
Tom's wrong when he says I didn't catch any.
Bill and Tom caught the same number.

Tom: Jack didn't catch any.
Bill is wrong when he says I only caught two.
Bill and I didn't catch the same number.
Between them, Jack and Bill caught thirteen.

The meaning of such obviously contradictory statements as these is entirely obscure until one realizes that, try as he will, no confirmed fisherman can tell the truth more than half the time, and these were no exceptions. In fact, just two of the four statements made by each man were true.

How many fish did each man catch?

14. Find all solutions of the equation $3 \cdot 2^m + 1 = n^2$, if m and n are required to be non-negative integers. Prove that you have found all of the solutions.

15. There are 25 students in a class. Among them, 17 students ride a bicycle, 13 swim, and 8 ski. No student does all three sports, and those who ride, swim, or ski have received either A or B in mathematics. However, 6 students received C, D, or F in mathematics.

How many swimmers ski?

16. The three sides and height of a triangle are four consecutive integers. What is the area of the triangle?

17. For \$480.00 a man bought a number of equally-priced cows, each cow costing an integral number of dollars. After the death of three of the cows, the man disposed of the remaining cows, selling all of them for identical integer dollar amounts, and making a profit of \$15.00 on the entire transaction. If the price the man paid for each cow was \$1.00 less than the price for which he sold each cow, find the number of cows the man originally bought.

18. The length of the perimeter of a right triangle is 60 inches and the length of the altitude perpendicular to the hypotenuse is 12 inches. Find the sides of the triangle.

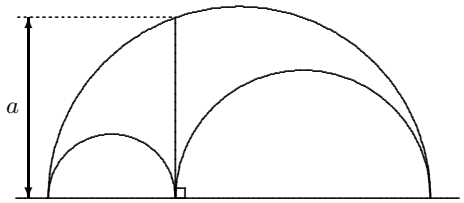
19. 1. There are five houses in a row on a particular street, each of a different colour and inhabited by men of different nationalities who have different pets, different beverages, and smoke different cigarettes.
2. The Englishman lives in the red house.
3. The Spaniard owns the dog.
4. Coffee is drunk in the green house.
5. The Ukranian drinks tea.
6. The green house is immediately to the right (<u>your</u> right) of the ivory house.
7. The Old Gold smoker owns snails.
8. Kools are smoked in the yellow house.
9. Milk is drunk in the middle house.
10. The Norwegian lives in the first house on the left.
11. The man who smokes Chesterfields lives in the house next to the man with the fox.
12. Kools are smoked in the house next to the house where the horse is kept.
13. The Lucky Strikes smoker drinks orange juice.
14. The Japanese smokes Parliaments.
15. The Norwegian lives next to the blue house.

Who drinks WATER? And who owns the ZEBRA?

20. The lengths of the sides of a triangle form an arithmetic progression with common difference d. The area of the triangle is t. Express the lengths of the sides of the triangle in terms of d and t. Solve this problem for the case $d = 1$ and $t = 6$.

[Hint: Heron's Formula for the area of a triangle, given the lengths a, b, c of its sides, is Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle.]

21. Given the diameter of a semicircle on which two small semicircles are drawn tangent externally to one another and internally to the large semicircle (as in the diagram below), and given the length a of the line segment perpendicular to the diameter at the point of external tangency, find the area between the large semicircle and the two smaller ones.



22. If $\left(r + \dfrac{1}{r}\right)^2 = 3$, find $r^3 + \dfrac{1}{r^3}$.

23. A statue 10 feet high stands on a base which is 13 feet high. A man whose eye-level is 5 feet above the (level) ground walks toward the statue, gazing at it all the while. Work out a solution using geometry alone (no calculus!) to the problem of finding the distance from the base of the statue to the point at which the man should stand in order that the statue appear tallest to him (that is, the angle between his lines of sight to the bottom and to the top of the statue is a maximum).

24. A jailer, carrying out the terms of a partial amnesty, made passes along the row of n cells as follows: in the first pass he turned every lock (that is, opening each of them); in the second pass he turned every second lock beginning at the second cell (that is, this locks, again, every second cell); in the third pass he turned every third lock beginning at the third cell. That is, in the k^{th} pass he turned every k^{th} lock beginning at the k^{th} cell. This was done for all k from 1 to n.

Which cells were finally left open?

25. It was a very hot day and the 4 couples together drank 44 bottles of pop: Ann had 2, Betty 3, Carol 4, and Dorothy 5 bottles. Mr. Brown drank just as many bottles as his wife, but each of the other men drank more than his wife: Mr. Green twice, Mr. White three times, and Mr. Smith four times as many bottles. Tell the last names of the four ladies. Show how you got your answer.

26. Let u, f, and g be functions, defined for all real numbers x, such that

$$\frac{u(x+1)+u(x-1)}{2} = f(x) \qquad \text{and} \qquad \frac{u(x+4)+u(x-4)}{2} = g(x)\,.$$

Determine $u(x)$ in terms of f and g.

27. Let $\lfloor m \rfloor$ denote the largest integer not exceeding the real number m. Show that, for all natural numbers n, the value of $\left\lfloor (2+\sqrt{3})^n \right\rfloor$ is an odd number.

28. Three numbers are in arithmetic progression, three other numbers in geometric progression. Adding the corresponding terms of these two progressions successively, we obtain

85, 76, and 84,

respectively, and adding all three terms of the arithmetic progression, we obtain 126. Find the terms in both progressions.

29. Divide the unit square into 9 equal squares by means of two pairs of parallel lines perpendicular to each other. Now remove the central square. Treat the remaining 8 squares the same way, and repeat the process.

(a) How many squares of side length $1/3^n$ remain?

(b) What is the sum of the areas of the removed squares as n becomes infinite?

30. Find all pairs a, b of positive integers satisfying the equation $2a^2 = 3b^3$.

31. An examination in three subjects, Algebra, Biology, and Chemistry, was taken by 41 students. The following table shows how many students failed in each subject, as well as in the various combinations:

subject	A	B	C	AB	AC	BC	ABC
# failed	12	5	8	2	6	3	1

(For instance, 5 students failed in Biology, among whom there were 3 who failed both Biology and Chemistry, and just 1 of these 3 failed all three subjects.)

How many students passed in all three subjects?

32. The sequence

$$2, 3, 6, 1, 8, 6, 8, 4, 8, 4, 8, 3, 2, 3, 2, 3, 2, 3, 2, 2, 4, 6, 6, 6, 6, 6, 6, 6, 4, \ldots$$

is constructed as follows:

- the first two terms are given to start things off;
- the third term is 6 because $2 \cdot 3 = 6$;
- since $3 \cdot 6 = 18$, the fourth term is 1 and the fifth is 8;
- the sixth term is $6 \cdot 1 = 6$;
- the seventh term is $1 \cdot 8 = 8$;
- the eighth and ninth terms are 4 and 8 because $8 \cdot 6 = 48$;

and so on.

(a) Show that the digit 9 never appears in the sequence.

(b) Show that the digit 7 never appears in the sequence.

(c) Show that the digit 5 never appears in the sequence.

33. Find all natural numbers, not ending in zero, which have the property that if the final digit is deleted, the integer obtained exactly divides into the original.

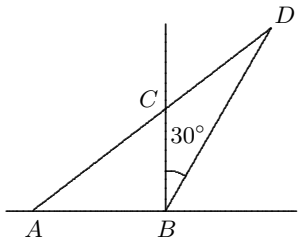
34. In a common carnival game, a player tosses a penny from a distance of about 5 feet onto a table ruled in 1-inch squares. If the penny, 3/4 inch in diameter, lands entirely within a square, the player receives a nickel, but does not get his penny back; otherwise, the player loses his penny. Assuming the penny lands on the table, and that the width of the rulings is negligible, what is the probability of a random toss winning? What should the payoff be in order that the game be played at even odds?

35. An alley 12 feet wide is blocked by two ladders placed crosswise between the buildings flanking the alley. Each ladder touches both buildings and the alley. The ladders are 13 and 37 feet long, respectively. How high is their point of "intersection" with each other?

36. If every room in a house has an even number of doors, prove that there must be an even number of entrances to the house from the outside.

37. Given the diagram to the right with $AB = CD$, $CB \perp AB$, and $\angle CBD = 30°$.

If AC has length 4 cm, what is the length of AB?



38. What is the maximum number of terms in a geometric progression with common ratio greater than 1 whose terms all come from the set of integers between 100 and 1000 inclusive?

39. Find all integers $n \geq 1$ such that the binomial coefficient $\binom{n}{k}$ is odd for all k, $0 \leq k \leq n$.

Note that the binomial coefficient $\binom{n}{k}$ can be defined as:

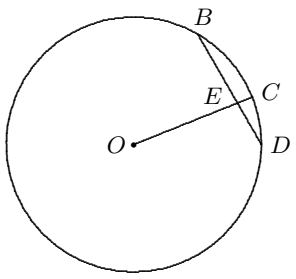
$$\binom{n}{k} \;=\; \frac{n!}{k!(n-k)!}\,.$$

40. Describe all real-valued continuous functions F satisfying the equation:

$$F(x+y) \;=\; F(x) + F(y) + 2xy\,,$$

for all real numbers x and y.

41. Let E be a point on the chord DB of a circle such that $DE = 3$ and $EB = 5$. Let O be the centre of the circle. Join OE and extend it to C (see diagram to the right). Given that $EC = 1$, find the radius of the circle.



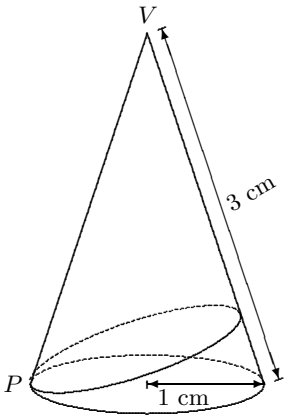
42. Semicircles are constructed outwardly on the sides of a right-triangle. A circumscribing rectangle is constructed around the whole figure so that its sides are parallel to the legs of the right-triangle. Prove that the rectangle is always a square.

43. Evaluate the infinite product:

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} \;=\; \frac{7}{9}\cdot\frac{26}{28}\cdot\frac{63}{65}\cdots$$

44. Ted (with his father) and Peter (with his father) went fishing. Ted hooked as many fish as his father, and Peter hooked three times as many as his father. Seventy fish were hooked in all, 80 per cent of them by the sons. Who is older, Ted or Peter?

45. A right circular cone of base radius 1 cm and slant height 3 cm is given. P is a point on the circumference of the base and the shortest path from P around the cone and back to P is drawn (see the figure to the right). What is the minimum distance from the vertex V to this path?



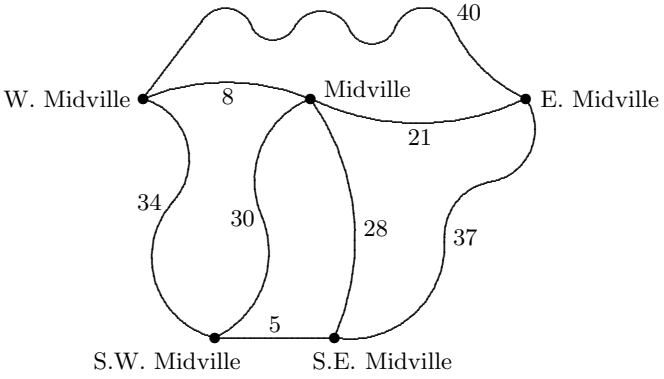
46. In a certain card game, one of the hands dealt contains:

1. Exactly thirteen cards.
2. At least one card in each suit.
3. A different number of cards in each suit.
4. A total of five hearts and diamonds.
5. A total of six hearts and spades.
6. Exactly two cards in the "trump" suit.

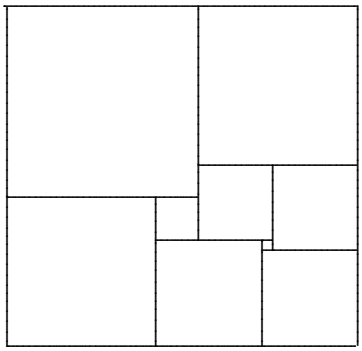
Which of the four suits—hearts, spades, diamonds, or clubs—is the "trump" suit?

47. Find all sets of three distinct natural numbers such that the sum of their reciprocals is an integer.

48. The Ramada County Department of Highways has just resurfaced the county roads, and now the yellow stripe down the middle of the road must be repainted. The truck used for this purpose is very inefficient as far as gas consumption is concerned, and thus the Department would like to have the truck travel the shortest distance possible. A road map of the county is shown below (with distances given in kilometres). The county truck is garaged in Midville, and it must return there when the job is done. How many kilometres must it travel and what route should it follow?



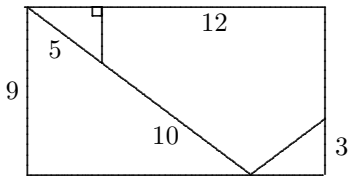
49. Depicted below is a rectangle partitioned into 9 squares. If the small square has sides of length 1, what are the lengths of the sides of the rectangle?



50. Determine the function $F(x)$ which satisfies the (functional) equation

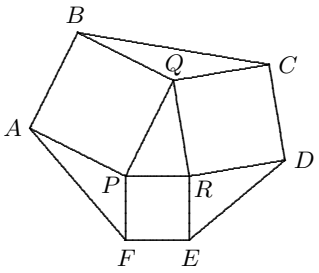
$$x^2F(x) + F(1-x) \;=\; 2x - x^4\,.$$

51. A rectangle is dissected as shown, with some of the lengths indicated. If the pieces are rearranged to form a square, what is the perimeter of the square?



52. In chess, is it possible for a knight to go from the lower left corner square of the chessboard to the upper right corner square and in the process to land exactly once on each other square?

53. Shown to the right is a triangle PQR upon whose sides have been constructed squares of areas 13, 25, and 36 square units. Find the area of the hexagon $ABCDEF$.



54. Three men possess a pile of money, their shares being $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$. Each man takes some money from the pile until nothing is left. The first man then returns $\frac{1}{2}$ of what he took, the second $\frac{1}{3}$, and the third $\frac{1}{6}$. When the total so returned is divided equally among the men it is found that each then possesses what he is entitled to. How much money was in the original pile, and how much did each man take from the pile? (Find the smallest such solution where all monetary amounts described above are positive integers).

55. The length of the sides of a rectangle are 6 and 8. If the rectangle is folded so that the two diagonally opposite vertices coincide, what will be the length of the fold?

56. Just as Liz and Irv stepped off the express train, they could hear the local pulling into the upper platform of the station. They quickly ran up the escalator, Irv taking three steps for each two that Liz took. Unfortunately, the doors of the local closed just as Irv reached the top step of the escalator.

Since they had to wait for another local to arrive, they had time to reflect about the escalator. Liz remarked that it had taken her 24 steps to reach the top; Irv noted that it had taken him 30. Assuming that Liz and Irv each climbed the escalator at constant rates, how many steps would be visible if the escalator stopped running?

57. A contractor estimated that one of his two bricklayers would take 9 hours to build a certain wall and the other 10 hours. However, he knew from experience that when they worked together, 10 fewer bricks got laid per hour. Since he was in a hurry, he put both men on the job and found it took exactly 5 hours to build the wall. How many bricks did the wall contain?

58. How many positive integers have their digits in strictly increasing order? No computer solutions, please!

59. Assume f and g are functions of $x > 0$ and that

(i) $gf' - fg' = 2(\ln g)$,

(ii) $gg' = \frac{1}{2}$, and

(iii) $g(0) = 0$.

Express f as an explicit function of x.

60. Five middle-aged couples who had all known each other for a long time were reminiscing one evening about the predictions that some of them had made many years back.

The names of the five men are Arthur, Basil, Clarence, Desmond, and Edgar; and the names of their wives, in no particular order, are Ruth, Veronica, Fanny, Polly, and Helen.

Arthur remembered that he had predicted that Clarence would marry Ruth. Basil had predicted that Edgar would marry Veronica. Clarence had been less precise about his forecast and had merely said that Arthur would marry neither Polly nor Helen. And Edgar had been firmly of the view that Basil would not marry Helen.

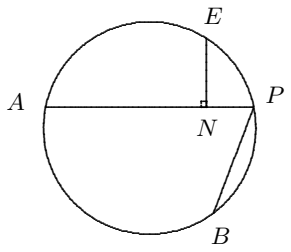
None of the five marriages that would have resulted on the basis of these predictions did, in fact, take place. As a matter of fact, all the predictions turned out to be incorrect!

Among the five women was Edgar's sister. If you knew who she was, you would be able to give all the details about who was married to whom.

(i) What five weddings would have resulted from the predictions?

(ii) Who was, in fact, married to whom, and who was Edgar's sister?

61. Given any two points A and B on the circumference of a circle, and E the mid-point of the arc AB (note that there are really two arcs that could be called AB; it does not matter which one we choose as long as the rest of the discussion is assumed to pertain only to points and arcs lying on the arc AB that we chose). Let P be any point on the arc EB and construct EN perpendicular to AP with N on the chord AP.



Prove that $AN = NP + PB$ (we are dealing only with magnitudes of line segments here).

62. I received four bills in the mail yesterday. They were from the cleaner, the dentist, the milk man, and the florist.

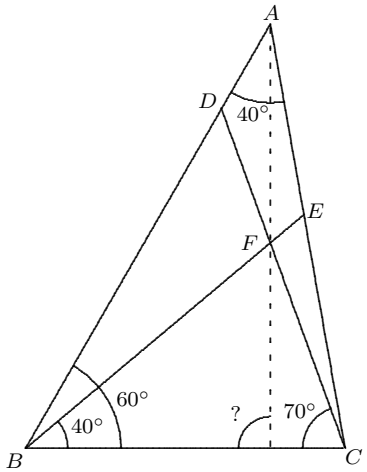
The bills from the cleaner and the dentist were both an exact amount of dollars and the cleaner's bill was half as much again more than the dentist's. I was surprised to note that the dollars in the milk man's bill were exactly the same as the cents in the florist's bill, and the dollars in the florist's bill were exactly the same as the cents in the milk man's bill.

The dentist's bill was the smallest, the milk bill the next smallest, and the total of the bills was \$55.25. What was the amount of each bill?

63. Characterize all those positive integers which can be used as the length of one side of a right-angled triangle, all of whose sides are integers.

64. Let ABC be a triangle with $\angle A = 40°$ and $\angle B = 60°$. Let D and E be points on AB and AC, respectively, such that $\angle DCB$ is $70°$ and $\angle EBC$ is $40°$. Furthermore, let F be the point of intersection of DC and EB.

Show that AF is perpendicular to BC.



65. According to Lewis Carroll in "The Hunting of the Snark" (1876):

> All Boojums are snarks.
> Every Bandersnatch is a frumious animal.
> Only animals which frequently breakfast at five o'clock tea can be snarks.
> No frumious animals breakfast at five o'clock tea.

Are any Bandersnatches Boojums?

66. List the number(s) of (all) the true statement(s) in the following list:

1. Exactly one statement in this list is false.
2. At least two statements in this list are false.
3. At most three statements in this list are false.
4. Exactly four statements in this list are false.
5. At least five statements in this list are false.
6. At most six statements in this list are false.
7. Exactly seven statements in this list are false.
8. At least eight statements in this list are false.
9. At most nine statements in this list are false.
10. Exactly ten statements in this list are false.

67. Two men are walking toward each other alongside a railway. A freight train overtakes one of them in 20 seconds and exactly 10 minutes later meets the other man coming in the opposite direction. The train passes this man in 18 seconds. How long after the train has passed the second man will the two men meet? (Constant speeds are to be assumed throughout.)

68. In the game of Yahtzee there are 5 dice. A full house consists of three numbers of one kind (sixes, for example) and two of another (fours, for example). If, on the first throw, a person gets three numbers of one kind, which is the better strategy to obtain a full house:

(i) to keep the three matched numbers and one unmatched number and try to match this unmatched other number? or

(ii) to pick up the two unmatched dice and try to throw a pair directly?

What is the probability of getting a full house with the better strategy? That is, what is the probability of obtaining a pair on the second throw or (failing that) a pair on the third throw?

69. Can a cube be decomposed into smaller cubes, no two of the same size?

70. Six colours of paint are available. Each face of a cube is to be painted a different colour. In how many different ways can this be done if two colourings are considered the same when one can be obtained from the other by rotating the cube?

71.

There were five fine ladies from Carruther
Who named their pets after each other.
From the following clues,
Can you carefully choose
The pet which belongs to Sue's mother?

Toni Taylor owns a hog;
Belle Bradkowski owns a frog;
Janet Jackson owns a crow;
The garter snake is owned by Jo;
Sue's the name they call the frog;
And "Here Jo, here Jo" brings the hog;
The name by which they call the pony
Is the name of the woman whose pet is Toni;
The final clue, which I'll now tell,
Is that Sue's mother's pet is Belle.

72. For a positive number such as 3.27, the digit 3 is referred to as the integer part of the number and .27 as the fractional part. Find a positive number which is such that its fractional part, its integer part, and the number itself are three terms in geometric progression (in that order).

73. A tireless bug lies on one end of a three-inch rubber band. Every time the bug crawls one inch toward the other end, some maniacal force stretches the band another three inches. If the bug crawls at the rate of one inch per second, and if the band doesn't break, how long will it take the bug to reach the other end?

74. The clerk picked up the dimes Jill had put on the counter. "But what stamps do you want?" he asked.

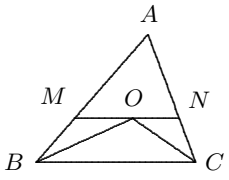
The little girl looked unhappy. "Dad wants one-cent, two-cent, three-cent, five-cent, and ten-cent," she replied. "He said to get four each of two sorts and three each of the others, but I've forgotten which."

Postal clerks have to handle tougher problems than that. "I guess he gave you the exact money for them," he told her.

"Just these dimes," she answered.

What stamps did Jill have to buy?

75. In the figure to the right, BO bisects $\angle ABC$, CO bisects $\angle ACB$, and MN is drawn through O and parallel to BC. Find the value of the perimeter of triangle AMN in terms of the lengths of AB, AC, and BC.



76. What is the integer part of

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{10000}} \, ?$$

Do not find computer solutions! This is a math problem, not a computer problem.

77. Al, Bob, and Carl live on a straight road that runs past the post office. Bob lives halfway between Al and the post office, and Carl lives halfway between Bob and the post office. One morning, with all their cars laid up, the three men find that they all need to buy supplies at the post office. By telephone they arrange for a taxi to pick up Al, then Bob, then Carl, take them to the post office, and wait briefly while they transact their business. When they reboard, the taxi drops off Carl, then Bob, and finally Al. When the three men get together a few nights later, Bob and Carl find that Al paid the entire fare of $8.40 at the end of the trip, and naturally insist on paying their shares. Assuming no complicating factors such as extra charges for extra passengers, no meter charges while waiting at the post office, and no tip, how much does each owe Al?

78. The members of a high school mathematics club met one day after school at the corner drugstore for some refreshments. Each ordered the same item, which cost more than a penny. (The drugstore only sells items that cost a whole number of pennies). The identical orders were all put on one check. If we knew the exact amount of that check, which was somewhere between two and three dollars and which was not subject to sales tax, we would be able to determine exactly how many members were present. But, believe it or not, we already have enough information.

How many members were present?

79. When Sinbad the sailor died, he left his ships to Haroun Al Rashid, the Vizier of Bagdad, and his fabled collection of diamonds to his sons and their wives. His will read like this (that is, after all the whereins and wherefores, etc., were deleted): "To Hameed, my eldest son, I leave one diamond and, to his wife, one ninth of those remaining. To Ishmael, my second son, I leave two diamonds and, to his wife, one ninth of those remaining. To Farouk, my third son, I leave three diamonds and, to his wife ...". The will continued in this manner. But when the distribution was complete, Fatima, the wife of the youngest son, complained that there were no diamonds left for her. Anyway, since she preferred pearls to diamonds, and already had enough diamonds, she didn't really care.

How many diamonds did Sinbad leave, and how many sons did he have?

80. A shipping clerk wants to package a 26–inch diameter sphere in a cubical box measuring 30 inches on each side, using eight identical small spheres in the corners of the box to prevent any movement of the large sphere.

What must the diameter of each small sphere be?

5 Solutions

1. The fact that the dividend is an integer allows us to deduce that the last four digits "brought down" are zeroes; that is, we have

$$
\begin{array}{rl}
 & X\,X\,X\,X\,.\,X\,X\,X\,X \\
X\,X\,X\,\big| & \overline{X\,X\,X\,X\,X\,X\,X} \\
 & \underline{X\,X\,X} \\
 & X\,X\,X \\
 & \underline{X\,X\,X} \\
 & X\,X\,X \\
 & \underline{X\,X\,X} \\
 & X\,X\,.\,0 \\
\text{Line } A & \underline{X\,X\,.\,X} \\
\text{Line } B & .\,X\,0\;\;0\;\;0 \\
\text{Line } C & \underline{.\,X\,0\;\;0\;\;0} \\
 & 0
\end{array}
$$

Therefore, the divisor in question evenly divides some single-digit multiple of 1000. Clearly, the units position of the divisor must be one of 0, 2, 4, 5, 6, or 8. If that digit was even (and different from 0), then the last digit in the quotient would have to be 5, implying that the divisor is a multiple of 200, in which case the division would have terminated with line A. Thus, the only choices for the units position of the divisor are 0 and 5. If the digit is 0, then the division would have again terminated with line A. We may then conclude that the divisor ends in a 5. Again, from line A and line B together, we see that line A must end in a 5. Thus, we have

$$
\begin{array}{rl}
 & X\,X\,X\,X\,.\,X\,X\,X\,X \\
X\,X\;5\;\big| & \overline{X\,X\,X\,X\,X\,X\,X} \\
 & \underline{X\,X\,X} \\
 & X\,X\,X \\
 & \underline{X\,X\,X} \\
 & X\,X\,X \\
 & \underline{X\,X\,X} \\
 & X\,X\,.\,0 \\
 & \underline{X\,X\,.\,5} \\
 & .\,5\;0\;0\;0 \\
 & \underline{.\,5\;0\;0\;0} \\
 & 0
\end{array}
$$

The divisor then is a 3-digit divisor of 5000, and when it divides 5000 it must give a single-digit quotient. This single-digit quotient can be none other than 8, which

gives a divisor of 625. Thus, we fill in as follows, since all the multiples of 625 have four or more digits except 625 itself:

$$
\begin{array}{rl}
 & 1\ 0\ 1\ 1\ .\ 0\ 0\ 0\ 8 \\
\cline{2-9}
6\ 2\ 5\ \big| & X\ X\ X\ X\ X\ X \\
 & 6\ 2\ 5 \\
\cline{2-2}
 & \quad X\ X\ X \\
 & \quad 6\ 2\ 5 \\
\cline{2-3}
 & \quad\ \ X\ X\ X \\
 & \quad\ \ 6\ 2\ 5 \\
\cline{2-4}
 & \quad\ \ \ \ X\ X\ .\ 0 \\
 & \quad\ \ \ \ 6\ 2\ .\ 5 \\
\cline{2-6}
 & \quad\ \ \ \ \ \ .\ 5\ 0\ 0\ 0 \\
 & \quad\ \ \ \ \ \ .\ 5\ 0\ 0\ 0 \\
\cline{2-9}
 & \quad\ \ \ \ \ \ \ \ \ \ 0
\end{array}
$$

The remaining digits can then be determined, as shown below:

$$
\begin{array}{rl}
 & 1\ 0\ 1\ 1\ .\ 0\ 0\ 0\ 8 \\
\cline{2-9}
6\ 2\ 5\ \big| & 6\ 3\ 1\ 9\ 3\ 8 \\
 & 6\ 2\ 5 \\
\cline{2-2}
 & \quad 6\ 9\ 3 \\
 & \quad 6\ 2\ 5 \\
\cline{2-3}
 & \quad\ \ 6\ 8\ 8 \\
 & \quad\ \ 6\ 2\ 5 \\
\cline{2-4}
 & \quad\ \ \ \ 6\ 3\ .\ 0 \\
 & \quad\ \ \ \ 6\ 2\ .\ 5 \\
\cline{2-6}
 & \quad\ \ \ \ \ \ .\ 5\ 0\ 0\ 0 \\
 & \quad\ \ \ \ \ \ .\ 5\ 0\ 0\ 0 \\
\cline{2-9}
 & \quad\ \ \ \ \ \ \ \ \ \ 0
\end{array}
$$

2. Let r be the radius of the swimming pool. The criminal will obviously try to move toward the edge diametrically opposite the policeman. If the policeman moves around the pool, the criminal should also change direction in order to ensure that the centre of the pool always lies between him and the policeman. The criminal can swim out to a circle of radius $\frac{1}{4}r$ and still ensure he can keep the centre between him and the policeman, since the circumference of his small circle is $\frac{1}{4}$ that of the whole swimming pool. He should then make for the edge nearest to him, a distance of $\frac{3}{4}r$ away. The policeman meanwhile must travel πr to reach that spot. If the speed of the swimmer is v, then the speed of the policeman is $4v$. Therefore, the time taken for the criminal to reach the edge is $\frac{3}{4}r/v$ and for the policeman is $\frac{\pi}{4}r/v$. Since $\pi > 3$, we see that the criminal reaches the edge first and thus escapes.

3. Let us lay out the seating plan in the upper left corner of a chessboard, with each square representing a seat. Since the upper left corner of a chessboard is white, we see that there are 13 white and 12 black squares among the 25 squares that represent the seating plan. We are essentially being asked if each student can change her seat to one of the opposite colour. This is clearly impossible since there are different numbers of black and white squares.

In general, if we have m rows with n seats per row arranged in a rectangular array, we can lay them out on a chessboard (or an extension of one), as above. Again, if the number of squares covered by the array is odd, then there will be a different number of black and white squares, and the task cannot be completed. On the other hand if the number of squares covered is even (that is, if at least one of m and n is even), the number of white and the number of black squares is the same. We will now show that the task can actually be completed in this case. We simply pair up students who exchange their positions; thus, it suffices to find a way of pairing up adjacent squares in the array. Without any loss of generality, we may suppose that the number of rows, m, is even. Then starting in the first seat of row 1, we pair up the students in the order we find them in the first seat across all the rows. We do the same for the students in the second seats of each row, and so on. Since m is even, this results in all the students being paired with another student with whom they may exchange seats according to the rules, and the task can be completed.

4. Let b be the number of boys and let g be the number of girls among the Sultan's children. The probability of choosing two children of the same sex is equal to the probability of choosing two boys plus the probability of choosing two girls. Therefore,

$$\left(\frac{b}{b+g}\cdot\frac{b-1}{b+g-1}\right)+\left(\frac{g}{b+g}\cdot\frac{g-1}{b+g-1}\right) \;=\; \frac{1}{2}\,. \tag{1}$$

Also, the probability of choosing two girls in two picks is equal to the probability of choosing one boy in one pick. Hence,

$$\frac{g}{b+g}\cdot\frac{g-1}{b+g-1} \;=\; \frac{b}{b+g}\,. \tag{2}$$

Simplifying equation (1), we get:

$$\begin{aligned} 2b(b-1)+2g(g-1) &= (b+g)(b+g-1)\,,\\ 2b^2-2b+2g^2-2g &= b^2+2bg+g^2-b-g\,,\\ b^2+g^2-2bg &= b+g\,,\\ (b-g)^2 &= b+g\,. \end{aligned} \tag{3}$$

Simplifying equation (2) yields:

$$\begin{aligned} g(g-1) &= b(b+g-1)\,,\\ g^2-g &= b^2+bg-b\,,\\ b^2-2bg+g^2 &= b^2-2bg+g+b^2+bg-b\,,\\ (b-g)^2 &= g-b-bg+2b^2\,. \end{aligned} \tag{4}$$

From (3) and (4), we have:

$$\begin{aligned} g - b - bg + 2b^2 &= b + g\,, \\ 2b^2 &= 2b + bg\,, \\ 2b &= 2 + g\,, \\ g &= 2b - 2\,, \end{aligned} \tag{5}$$

where we have used the fact that $b \neq 0$. Putting this value of g into (3) gives us:

$$\begin{aligned} \big(b - (2b-2)\big)^2 &= b + (2b-2)\,, \\ (2-b)^2 &= 3b - 2\,, \\ 4 - 4b + b^2 &= 3b - 2\,, \\ b^2 - 7b + 6 &= 0\,, \\ (b-6)(b-1) &= 0\,. \end{aligned}$$

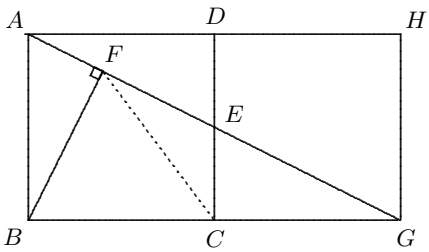
Therefore, $b = 6$ or $b = 1$. If we have $b = 1$, then equation (5) yields that $g = 0$, which means that the conversation between the Sultan and the Caliph makes no sense. On the other hand, if $b = 6$, then $g = 10$ from (5). One can check that this satisfies all the conditions. Thus, the Sultan has 6 boys and 10 girls.

5. A chessboard has squares of size $n \times n$ for $n = 1, 2, \ldots, 8$. By examining the position of the lower left corner of a square of size n on the chessboard, we see that there are k^2 possible positions to place it, where $k = 9 - n$. Thus, the total number of squares, N, is given by:

$$N \;=\; \sum_{n=1}^{8} (9-n)^2 \;=\; \sum_{n=1}^{8} n^2 \;=\; \frac{8(8+1)(2\cdot 8+1)}{6} \;=\; 204\,.$$

6. We will present here two different solutions. We note that, because $ABCD$ is a square, it is sufficient to show that $CF = BC$.

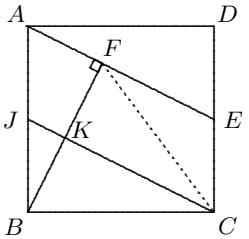
Solution 1. Adjoin to square $ABCD$ another square $DCGH$. Then AE extended passes through G (see figure to the right). Triangle BFG is right-angled. Therefore, it may be inscribed in a circle with BG as diameter. Hence, CF and BC are both radii of the same circle and must be equal.



Solution 2. Let J be the mid-point of AB. Let CJ meet BF at K (see figure to the right). Clearly, CJ and EA are parallel. Hence,

$$BK : KF \;=\; BJ : JA \;=\; 1 : 1\,.$$

That is, $BK = KF$. But JC is perpendicular to BF. Therefore, triangles FKC and BKC are congruent (SAS). Thus, $CF = BC$.



7. Suppose the integer n can be written as the sum of k ($k \geq 2$) consecutive odd positive integers beginning with the odd integer a. Then

$$n \;=\; \sum_{i=1}^{k}\big(a+2(i-1)\big) \;=\; k(a-2)+2\frac{k(k+1)}{2} \;=\; k(k+a-1)\,.$$

Now, an even value for k implies that $k+a-1$ is even (since a is odd). Hence, n is either the product of two even numbers or two odd numbers (both of which are greater than 1).

If n is the product of two even numbers, it is a multiple of 4. On the other hand, if n is any multiple of 4, say $n = 4m$ with $m \geq 1$, then n can be expressed as the sum of exactly two consecutive odd positive integers, namely

$$n \;=\; 4m \;=\; (2m-1)+(2m+1)\,.$$

If n is the product of two odd integers, it is an odd composite number. On the other hand, if n is any odd composite number, say $n = pq$, where p and q are both odd and greater than 1, then n can be expressed in the desired form as follows. Without loss of generality, we may assume that $p \leq q$. Then set $k = p$ and $a = q + 1 - p$. Then, k and a are both odd, a is positive, and thus $n = pq$ is the sum of k consecutive odd positive integers beginning with a.

Therefore, the numbers which can be written as the sum of two or more consecutive odd positive integers are all the multiples of 4 and all the odd composite numbers.

8. We first give the prime factorization of the number 252:

$$252 = 2^2 \cdot 3^2 \cdot 7\,.$$

Let us consider all possible triples (a, b, c) of integers where $a \leq b \leq c$ whose product is 252 and compute the sum $a + b + c$:

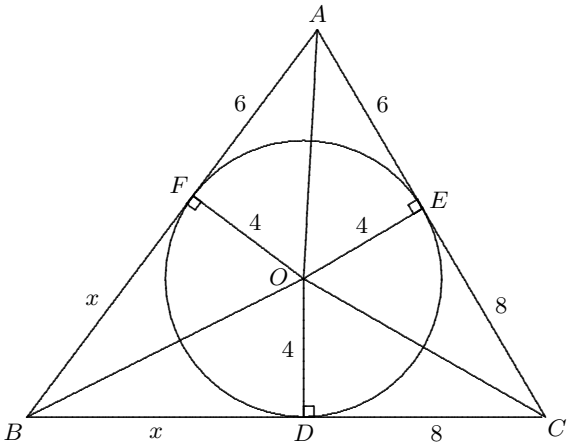
a	b	c	Sum
1	1	252	254
1	2	126	129
1	3	84	88
1	4	63	68
1	6	42	49
1	7	36	44
1	9	28	38
1	12	21	34
1	14	18	33
2	2	63	67

a	b	c	Sum
2	3	42	47
2	6	21	29
2	7	18	27
2	9	14	25
3	3	28	34
3	4	21	28
3	6	14	23
3	7	12	22
4	7	9	20
6	6	7	19

We see that the above sums are all distinct except for the triples $(1, 12, 21)$ and $(3, 3, 28)$, which both give the sum 34. Therefore, if the house number was not 34, the census taker did not need to ask another question. Since he did ask another question, we may conclude that the house number was 34 and that he had to choose

between the triples $(1, 12, 21)$ and $(3, 3, 28)$. Since the housekeeper's answer to the third question gives the census taker enough information to resolve the three ages, her age must be between 21 and 28. The answer she gives determines that the three ages are 1, 12, and 21.

9. Let the triangle be denoted by ABC. Let D, E, and F be the points of tangency of the incircle of $\triangle ABC$ with the sides BC, CA, and AB, respectively, and let O be the centre of the inscribed circle, as shown below.



Without loss of generality we may suppose that $AE = 6$ and $EC = 8$. It follows immediately that $AF = 6$ and $CD = 8$. Let $x = BD = BF$. Now the area of $\triangle AOB$ is $2(x + 6)$; that of $\triangle BOC$ is $2(x + 8)$; and that of $\triangle COA$ is $2(6 + 8) = 28$. Thus, the area of $\triangle ABC$ is the sum of these parts, namely

$$\text{Area of } \triangle ABC \;=\; 2(x + 6 + x + 8 + 6 + 8) \;=\; 4(x + 14)\,.$$

On the other hand, we know that if s is the semiperimeter of a triangle whose three sides have lengths a, b, c, then Heron's Formula states

$$\text{Area} \;=\; \sqrt{s(s-a)(s-b)(s-c)}\,.$$

For $\triangle ABC$, we have $s = (2x + 12 + 16)/2 = x + 14$, and a, b, and c are $x + 8$, 14, and $x + 6$, respectively. Therefore,

$$\text{Area of } \triangle ABC \;=\; \sqrt{(x+14)(6)(x)(8)}\,.$$

Thus, we have

$$\begin{aligned} 4(x+14) &= \sqrt{(x+14)(48x)}\,, \\ 16(x+14)^2 &= (x+14)(48x)\,, \\ x+14 &= 3x\,, \end{aligned}$$

which means that $x = 7$. Therefore, the sides of $\triangle ABC$ have lengths 13, 14, and 15. That is, the two "other" sides have lengths 13 and 15.

10. If there were x sheep in the original herd and each sheep was sold for x dollars, then the entire herd was sold for x^2 dollars. Now the two brothers took 20 dollars out of the pot each turn, until they had less than 20 dollars remaining. We are concerned with the amount remaining at this point, which is the remainder when x^2 is divided by 20.

Claim. x^2 and $(x \pm 10k)^2$ have the same remainder when divided by 20, for any integer k.

Proof: To show this, we need only show that $x^2 - (x \pm 10k)^2$ is a multiple of 20. But

$$x^2 - (x \pm 10k)^2 \;=\; x^2 - x^2 \mp 20kx - 100k^2 \;=\; 20(\mp kx - 5k^2)\,,$$

which is a multiple of 20, and the claim holds.

Thus, to determine the possible remainders of x^2 on division by 20, it is sufficient to consider $1 \le x \le 10$. By simple calculation, the remainders are 0, 1, 4, 5, 9, and 16. At the end, when it is the older brother's turn, we have that many dollars remaining. We are told that he takes 10 dollars. Consequently, the amount remaining before the older brother takes his last turn must exceed 10 dollars, and we may conclude that the amount remaining at this point was \$16. This means that the younger brother receives only \$6 on his last turn. If the older brother gives him a penknife worth \$$y$ to even things up, then from the last \$16 the older brother receives \$$(10 - y)$, and the younger brother receives \$$(6 + y)$, which must be equal. Thus, $y = 2$. That is, the penknife is worth \$2.

11. This problem is extremely challenging. Almost everyone attacks it with confidence, at some point obtaining a system of 3 linear equations in 4 unknown angle measurements, and spends much time searching for a 4^{th} independent equation.

We will display here two completely different solutions, each of which is elegant in its own right.

Solution 1. One can show that $\angle BDC = 50^\circ$; whence, $BC = BD$. Now let us use the Law of Sines in $\triangle BEC$, noting that $\angle BEC = 40^\circ$.

$$\begin{aligned}\frac{BE}{BC} \;&=\; \frac{\sin 80^\circ}{\sin 40^\circ} \;=\; \frac{2\sin 40^\circ \cos 40^\circ}{\sin 40^\circ} \;=\; 2\cos 40^\circ \\ &=\; \frac{\cos 40^\circ}{\frac{1}{2}} \;=\; \frac{\sin 50^\circ}{\sin 30^\circ}\,.\end{aligned}$$

Let $x = \angle ADE$ and $y = \angle BED$. Since $\angle ADE$ is an exterior angle to $\triangle BED$, it must be equal to the sum of the two opposite interior angles; that is, we must have $x = y + 20^\circ$. Now, applying the Law of Sines in $\triangle BED$, we get:

$$\begin{aligned}\frac{BE}{BD} \;&=\; \frac{\sin(180^\circ - x)}{\sin y} \;=\; \frac{\sin x}{\sin y} \;=\; \frac{\sin x}{\sin(x - 20^\circ)} \\ &=\; \frac{\sin\big(50^\circ + (x - 50^\circ)\big)}{\sin\big(30^\circ + (x - 50^\circ)\big)}\,.\end{aligned}$$

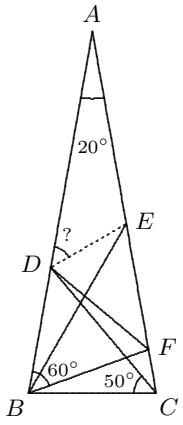
Since $BC = BD$, we have

$$\frac{\sin 50^\circ}{\sin 30^\circ} \;=\; \frac{\sin\big(50^\circ + (x - 50^\circ)\big)}{\sin\big(30^\circ + (x - 50^\circ)\big)}\,.$$

It can be shown that the only solution to this equation for $0^\circ < x < 90^\circ$ is (as you would expect) $x = 50^\circ$.

Solution 2. As in Solution 1, we will need the fact that $BC = BD$. First, we construct line BF with F on AC such that $\angle DBF = 60^\circ$. Then $\angle CBF = 20^\circ$. Hence, $\triangle CBF$ is isosceles, from which it follows that $BF = BC$. Thus, it also follows that $BF = BD$, from which we conclude that $\triangle BDF$ is isosceles. Since $\angle DBF = 60^\circ$, we see that $\triangle DBF$ is, in fact, equilateral. Therefore,

$$BD \;=\; BF \;=\; DF\,.$$



Now, $\angle EBF = 40^\circ$. Hence, $\triangle BEF$ is isosceles with $BF = FE$. Thus, $FE = DF$, and we conclude that $\triangle EFD$ is isosceles (with apex $\angle DFE = 40^\circ$). Therefore, $\angle DEF = 70^\circ$, which implies that $\angle DEB = 30^\circ$, and $\angle ADE = 50^\circ$.

12. Let r be the radius of the circle. Suppose that the sides of length 4 each subtend an angle of θ_1 at the centre of the circle, and the sides of length $6\sqrt{3}$ each subtend an angle of θ_2. Since there are six sides of each length, we know that $6(\theta_1 + \theta_2) = 360^\circ$, or $\theta_1 + \theta_2 = 60^\circ$.

Now we consider the triangles formed by two radii and a side of the polygon. We will look at a single triangle for each of the different side lengths and bisect the central angle. Then, we clearly have

$$\sin\frac{\theta_1}{2} \;=\; \frac{2}{r} \qquad \text{and} \qquad \sin\frac{\theta_2}{2} \;=\; \frac{3\sqrt{3}}{r}\,.$$

Since $\cos^2 A + \sin^2 A = 1$ for all angles A, we may also conclude that

$$\cos^2\frac{\theta_1}{2} \;=\; 1 - \left(\frac{2}{r}\right)^2 \;=\; 1 - \frac{4}{r^2} \tag{1}$$

$$\text{and} \qquad \cos^2\frac{\theta_2}{2} \;=\; 1 - \left(\frac{3\sqrt{3}}{r}\right)^2 \;=\; 1 - \frac{27}{r^2}\,. \tag{2}$$

We also know that

$$\begin{aligned} \frac{1}{2} \;&=\; \sin 30^\circ \;=\; \sin\left(\frac{\theta_1}{2} + \frac{\theta_2}{2}\right) \\ &=\; \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,, \\ \therefore \qquad \frac{1}{2} \;&=\; \frac{2}{r}\cos\frac{\theta_2}{2} + \cos\frac{\theta_1}{2}\,\frac{3\sqrt{3}}{r}\,; \end{aligned} \tag{3}$$

and

$$\begin{aligned} \frac{\sqrt{3}}{2} &= \cos 30^\circ = \cos\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}\right) \\ &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,, \\ \therefore \quad \frac{\sqrt{3}}{2} &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \frac{2}{r}\cdot\frac{3\sqrt{3}}{r}\,; \end{aligned}$$

that is,

$$\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} = \sqrt{3}\left(\frac{1}{2}+\frac{6}{r^2}\right) = \frac{\sqrt{3}}{2r^2}(r^2+12)\,. \tag{4}$$

Squaring both sides of (3), and using (1), (2), and (4), we have:

$$\begin{aligned} \frac{1}{4} &= \frac{4}{r^2}\cos^2\frac{\theta_2}{2} + \frac{12\sqrt{3}}{r^2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \frac{27}{r^2}\cos^2\frac{\theta_1}{2} \\ &= \frac{4}{r^2}\left(1-\frac{27}{r^2}\right) + \frac{12\sqrt{3}}{r^2}\cdot\frac{\sqrt{3}}{2r^2}(r^2+12) + \frac{27}{r^2}\left(1-\frac{4}{r^2}\right) \\ \therefore \quad \frac{1}{4} &= \frac{4}{r^2} - \frac{108}{r^4} + \frac{18}{r^2} + \frac{216}{r^4} + \frac{27}{r^2} - \frac{108}{r^4} = \frac{49}{r^2}\,. \end{aligned}$$

Therefore, $r^2 = 4\cdot 49$, which means that $r = 2\cdot 7 = 14$ (since $r > 0$).

13. Let us agree to number the above statements from 1 to 12. Among statements 1 through 4, exactly two are correct; similarly for statements 5 through 8 and statements 9 through 12. Now let us make the following observations about these statements:

(a) Statements 1 and 10 are not both true, nor both false.
(b) Statements 7 and 9 are not both true, nor both false.
(c) Statements 8 and 11 are not both true, nor both false.
(d) Statements 2 and 5 are not both true.
(e) Statements 2 and 9 are not both true.
(f) Statements 4 and 5 are not both true.
(g) Statements 4 and 6 are not both true.
(h) Statements 4 and 8 are not both true.
(i) Statements 6 and 9 are not both true.

Furthermore, if statement 4 were true, then statements 5, 6, and 8 would all have to be false (from (f), (g), and (h) above), which is impossible, since two of the statements from 5 through 8 must be true. Therefore, statement 4 is false. Now if statement 5 were true, then statement 2 is false by (d), which implies that statements 1 and 3 must both be true (since two of Bill's statements must be true). Then Tom caught the most fish (statement 5), which was 2 fish (statement 1); but this is inconsistent with statement 3. Therefore, statement 5 must be false.

Now suppose that statement 2 is false. As above, statements 1 and 3 must be true. Hence, Tom caught 2 fish, and Jack and Bill together caught 10 fish. This implies that statements 10 and 12 are false, which means that statements 9 and 11 are true. By (i), (b), and (c) above, we see that statements 6, 7, and 8 are all

false, which is impossible. Therefore, statement 2 is true. From (e), statement 9 is false, and from (b) statement 7 is true.

Now, if statement 3 is false, then statement 1 is true, which would imply that Tom caught 2 fish and Jack caught 3. By (a) statement 10 would be false; whence, statements 11 and 12 would be true. Then by (c) we have statement 8 false, implying that statement 6 is true. But statement 6 true implies that Bill did not catch any fish, contradicting statement 12. Therefore, statement 3 is also true; whence Bill caught 7 fish (statements 2 and 3).

We also have statement 1 false, since statements 2 and 3 are true, and then we have statement 10 true by (a).

Next, suppose that statement 8 is true. Then statement 6 is false, as is statement 11 (by (c)), which means that statement 12 is true. Statements 3 and 12 together imply that Tom caught 5 fish, which is the number that Bill caught (statement 8), which contradicts the fact that Bill caught 7 fish. This inconsistency means that statement 8 must be false, which means that statements 6 and 11 are both true, and statement 12 is false. From statement 6 we see that Jack caught 10, and from statement 2 we see that Tom caught 9. In conclusion:

Bill caught 7, Jack caught 10, and Tom caught 9.

14. We first rewrite the given equation as:

$$3 \cdot 2^m \;=\; n^2 - 1 \;=\; (n-1)(n+1)\,.$$

Notice that $n-1$ and $n+1$ have the same parity, since they differ by 2.

Case (i): $n-1$ and $n+1$ are both odd.

This clearly implies that $m = 0$. Therefore, $n^2 - 1 = 3$, or $n = \pm 2$. Since we are only interested in non-negative integer solutions, the only possibility in case (i) is $(m, n) = (0, 2)$.

Case (ii): $n-1$ and $n+1$ are both even.

Let $n - 1 = 2k$. Then $n + 1 = 2(k + 1)$. Therefore,

$$3 \cdot 2^m \;=\; 2k \cdot 2(k+1) \;=\; 4k(k+1)\,, \qquad \text{or} \qquad 3 \cdot 2^{m-2} \;=\; k(k+1)\,.$$

Clearly, one of the two integers k and $k+1$ is even and the other is odd. Hence, the product $k(k+1)$ is even, which implies that $m \geq 3$. Since k and $k+1$ can have no common factor larger than 1, we conclude that one of k, $k+1$ must equal 3.

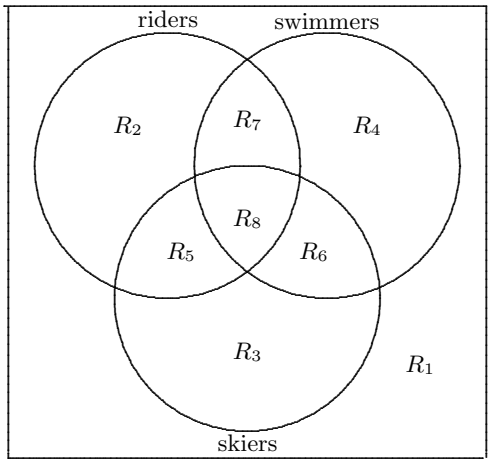
If $k = 3$, then $k + 1 = 4$. This implies that $n = 7$ and $m = 4$. Then we have $(m, n) = (4, 7)$.

If $k + 1 = 3$, then $k = 2$. This implies that $n = 5$ and $m = 3$. Then we have $(m, n) = (3, 5)$.

Therefore, there are precisely three solutions in non-negative integers to the equation $3 \cdot 2^m + 1 = n^2$, namely $(m, n) = (0, 2)$, $(m, n) = (3, 5)$, or $(m, n) = (4, 7)$.

15. We will use what mathematicians call a Venn Diagram to solve this problem. We let the class of students be represented pictorially by a rectangle, and use

circles to represent the three activities of riding bicycles, swimming, and skiing (see diagram below). This divides the interior of the rectangle into 8 parts, each representing a combination of the three activities.



From the original statements concerning grades we know that, of the 25 students in the class, there are at least 6 who lie in region 1 of the diagram. We are also told that no student does all three sports, which indicates that region 8 contains no students. Thus, regions 2, 3, 4, 5, 6, and 7 contain at most 19 students in total. We need to determine the number of students in region 6. Let R_2, R_3, R_4, R_5, R_6, and R_7 be the number of students in the regions 2, 3, 4, 5, 6, and 7, respectively. Then we have

$$\begin{aligned} R_2 + R_5 + R_7 &= 17\,, \\ R_4 + R_6 + R_7 &= 13\,, \\ R_3 + R_5 + R_6 &= 8\,. \end{aligned}$$

Summing, we have

$$R_2 + R_3 + R_4 + 2(R_5 + R_6 + R_7) \;=\; 38\,.$$

On the other hand, we also have

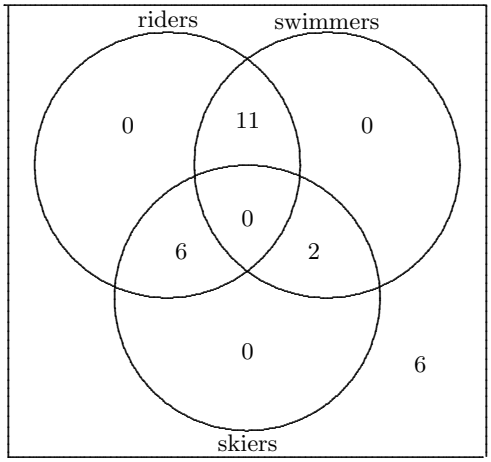
$$R_2 + R_3 + R_4 + R_5 + R_6 + R_7 \;\leq\; 19\,.$$

Subtracting the latter from the former, we get

$$R_5 + R_6 + R_7 \;\geq\; 19\,,$$

which means that regions 2, 3, and 4 contain no students. Therefore, all the students who do sports lie regions 5, 6, and 7. Since there are 17 students in regions 5 and 7, that leaves 2 students in region 6. (We can similarly determine that the number of students in region 5 is 6 and the number in region 7 is 11.)

That is, there are 2 swimmers who ski. In fact, the final numbers in all regions is given below.



16. Let the four integers be $a-1$, a, $a+1$, and $a+2$. Note that there must be at least 2 sides in any triangle which are at least as large as the largest altitude (height). Therefore, the height from the problem statement is either a or $a-1$.

Case 1. The height is a.

This means that the base must be $a-1$. Hence, the area $A = \frac{1}{2}a(a-1)$. Let s be the semiperimeter of the triangle; that is,

$$s \;=\; \tfrac{1}{2}\big((a-1)+(a+1)+(a+2)\big) \;=\; \tfrac{1}{2}(3a+2)\,.$$

From Heron's Formula, we know that

$$\begin{aligned} A^2 &= s\big(s-(a-1)\big)\big(s-(a+1)\big)\big(s-(a+2)\big) \\ &= \tfrac{1}{2}(3a+2)\cdot\tfrac{1}{2}(a+4)\cdot\tfrac{1}{2}a\cdot\tfrac{1}{2}(a-2)\,. \end{aligned}$$

Therefore, $4a^2(a-1)^2 = 16A^2 = (3a+2)(a+4)a(a-2)$, which simplifies to

$$a^3 - 16a^2 + 24a + 16 \;=\; 0\,,$$

since $a \neq 0$. We are looking for an integer solution for a. Therefore, 2 divides evenly into a^3, which means that a is even. But this further implies that 16 divides evenly into a^3, which means that 4 divides evenly into a. However, that leaves $a^3 - 16a^2 + 24a$ as a multiple of 32, which is impossible, since 32 does not evenly divide the constant term 16 of the polynomial.

Case 2. The height is $a-1$.

Again, we define A as the area and s as the semiperimeter of the triangle. In this case, we see that $s = \frac{3}{2}(a+1)$. Using Heron's Formula again, we get

$$\begin{aligned} A^2 &= s\big(s-a\big)\big(s-(a+1)\big)\big(s-(a+2)\big) \\ &= \tfrac{3}{2}(a+1)\cdot\tfrac{1}{2}(a+3)\cdot\tfrac{1}{2}(a+1)\cdot\tfrac{1}{2}(a-1) \\ &= \tfrac{3}{16}(a+1)^2(a+3)(a-1)\,. \qquad (1) \end{aligned}$$

Suppose that the base is a. Then $A^2 = \frac{1}{4}a^2(a-1)^2$. Combining this with (1), and dividing by $a-1$ (since $a \neq 1$), yields $a^3 - 19a^2 - 21a - 9 = 0$. Therefore, a divides evenly into 9, which means that $a = \pm 1$, $a = \pm 3$, or $a = \pm 9$, none of which is an actual solution of the polynomial. Suppose instead that the base is $a+2$. Then $A^2 = \frac{1}{4}(a-1)^2(a+2)^2$. Combining this with (1), and dividing by $a-1$, yields $a^3 - 3a^2 - 21a - 25 = 0$. Therefore, a divides 25 evenly, which means that $a = \pm 1$, $a = \pm 5$, or $a = \pm 25$, none of which is an actual solution of the polynomial. This leaves us with $a+1$ as the base. Then $A^2 = \frac{1}{4}(a+1)^2(a-1)^2$. Combining this with (1), and dividing by $(a-1)(a+1)^2$, yields $4(a-1) = 3(a+3)$, which has solution $a = 13$. This means the three sides are 13, 14, and 15, the altitude is 12, and the area is

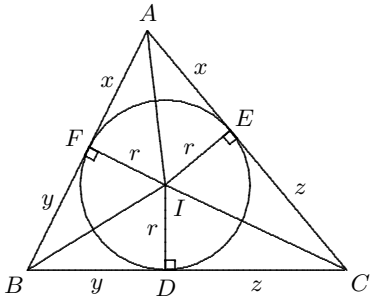
$$A \;=\; \tfrac{1}{2}(a+1)(a-1) \;=\; \tfrac{1}{2}(a^2-1) \;=\; 84\,.$$

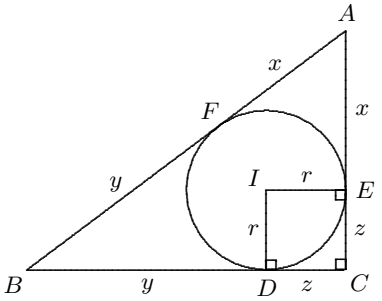
17. Let x be the number of cows purchased, and let p be the price in dollars paid for each cow. Then $px = 480$, or $x = 480/p$. If q is the price for which he sold each cow, we have $(x-3)q = 495$. Since $q = p+1$, we have

$$\begin{aligned}
\left(\frac{480}{p} - 3\right)(p+1) &= 495\,,\\
(480 - 3p)(p+1) &= 495p\,,\\
480 + 477p - 3p^2 &= 495p\,,\\
480 - 18p - 3p^2 &= 0\,,\\
p^2 + 6p - 160 &= 0\,,\\
(p+16)(p-10) &= 0\,.
\end{aligned}$$

Therefore, $p = -16$ or $p = 10$. Since p cannot be negative, we conclude that $p = 10$ and $x = 480/10 = 48$. Thus, the man bought 48 cows originally.

18. Let a, b, c be the lengths of the three sides with c the length of the hypotenuse. Let s be the semiperimeter of the triangle; that is, $s = \frac{1}{2}(a+b+c)$. If r is the radius of the inscribed circle, we have $r = s - c$. [To see this, we first observe that, if x, y, and z are the lengths of the tangents from the vertices to the points of contact with the incircle as shown in the diagram below left, then we have $s = x + y + z$. Since $x + y = c$, we have $z = s - c$. Similarly, $x = s - a$ and $y = s - b$.





If we have a right angle at C, then clearly $r = z$ as seen in the diagram above right. It follows that $r = s - c$.]

In the given problem, we have $s = 30$. Thus, $r = 30 - c$. Not only is the area of the triangle equal to $\frac{1}{2}ch$, where h is the altitude to the hypotenuse, but it can be computed as sr [to see this, simply add up the areas of the six smaller triangles in the diagram above left]. Hence, $30(30 - c) = 6c$, which simplifies to $c = 25$. Then $a + b = 2s - c = 60 - 25 = 35$ and $ab = ch = 12 \cdot 25 = 300$. This means that a and b are the roots of the quadratic equation

$$0 \;=\; x^2 - 35x + 300 \;=\; (x - 15)(x - 20)\,.$$

Therefore, the sides of the triangle are 15, 20, and 25.

19. From 15 and 10 we see that the blue house is the second from the left. From 6 we see that the first house on the left (the Norwegian's) cannot be green or ivory (nor, of course, blue). Since (by 2) the Englishman lives in the red house, the Norwegian must live in the yellow house, and smokes Kools (by 8). The horse is kept in the blue house (by 12). By 9, we thus have:

1.	2.	3.	4.	5.
Norwegian				
yellow	blue			
	horse			
Kools				
		milk		

By 6 and 2 we see that the Englishman lives in house #3 or #5.

Case 1. The Englishman lives in house #5, which must be red (by 2).

Then (by 6) house #3 is ivory and house #4 is green. This implies (by 4 and 5) that coffee is drunk in house #4, and that the Ukranian drinks tea in house #2:

1.	2.	3.	4.	5.
Norwegian	Ukranian			Englishman
yellow	blue	ivory	green	red
	horse			
Kools				
	tea	milk	coffee	

By 13, the Lucky Strikes smoker lives in house #5 and drinks orange juice, which implies that the Norwegian must drink WATER. By 7 and 14, we see that the Ukranian smokes Chesterfields. Thus, we have

1.	2.	3.	4.	5.
Norwegian	Ukranian			Englishman
yellow	blue	ivory	green	red
	horse			
Kools	Chesterfields			Lucky Strikes
WATER	tea	milk	coffee	orange juice

By 7, the person who owns snails lives in house #3 or house #4, and by 3 the dog-owner (that is, the Spaniard) also lives in house #3 or house #4. By 11 the fox-owner must be in house #1 or house #3, but the person in house #3 owns either snails or a dog (from above). Consequently, the fox-owner lives in house #1, and the Englishman must own the ZEBRA. This yields:

1.	2.	3.	4.	5.
Norwegian	Ukranian			Englishman
yellow	blue	ivory	green	red
fox	horse			ZEBRA
Kools	Chesterfields			Lucky Strikes
WATER	tea	milk	coffee	orange juice

But now 3, 7, and 14 cannot be satisfied simultaneously. Consequently, Case 1 is not possible.

Case 2. The Englishman lives in house #3, which must be red (by 2).
Then by 6 house #4 is ivory and house #5 is green. Then 4 gives us:

1.	2.	3.	4.	5.
Norwegian		Englishman		
yellow	blue	red	ivory	green
	horse			
Kools				
		milk		coffee

By examining 5 and 13, we see that tea and orange juice are drunk in houses #2 and #4 (not necessarily respectively). We may then conclude that the Norwegian drinks WATER. By considering 13 and 14, we notice that the Japanese does not drink orange juice, and by 5 he does not drink tea. Therefore, the Japanese (who smokes Parliaments) drinks coffee and must live in house #5. Thus, we have

1.	2.	3.	4.	5.
Norwegian		Englishman		Japanese
yellow	blue	red	ivory	green
	horse			
Kools				Parliaments
WATER		milk		coffee

Since the Spaniard owns the dog, he lives in house #4, which implies that the Ukranian (who drinks tea) lives in house #2. Therefore, the Spaniard drinks orange juice and by 13, he must smoke Lucky Strikes. Thus, the Englishman smokes Old Golds and owns the snails (by 7), yielding the following:

1.	2.	3.	4.	5.
Norwegian	Ukranian	Englishman	Spaniard	Japanese
yellow	blue	red	ivory	green
	horse	snails	dog	
Kools		Old Golds	Lucky Strikes	Parliaments
WATER	tea	milk	orange juice	coffee

Clearly, the Ukranian smokes Chesterfields, and the Norwegian owns the fox by 11, implying that the Japanese owns the ZEBRA. The final solution is:

1.	2.	3.	4.	5.
Norwegian	Ukranian	Englishman	Spaniard	Japanese
yellow	blue	red	ivory	green
fox	horse	snails	dog	ZEBRA
Kools	Chesterfields	Old Golds	Lucky Strikes	Parliaments
WATER	tea	milk	orange juice	coffee

Thus, the Norwegian drinks WATER, and the Japanese owns the ZEBRA.

20. Let a, b, c be the lengths of the three sides. Then $a = b - d$ and $c = b + d$. The semiperimeter is $s = \frac{1}{2}(a+b+c) = \frac{3}{2}b$. From Heron's Formula, the square of the area of the triangle is

$$\begin{aligned} t^2 &= s(s-a)(s-b)(s-c) \;=\; s\big(s-(b-d)\big)(s-b)\big(s-(b+d)\big) \\ &= \frac{3b}{2}\left(\frac{b}{2}+d\right)\left(\frac{b}{2}\right)\left(\frac{b}{2}-d\right), \\ 16t^2 &= 3b^2(b+2d)(b-2d) \;=\; 3b^2(b^2-4d^2), \\ 0 &= 3b^4 - 12b^2d^2 - 16t^2. \end{aligned}$$

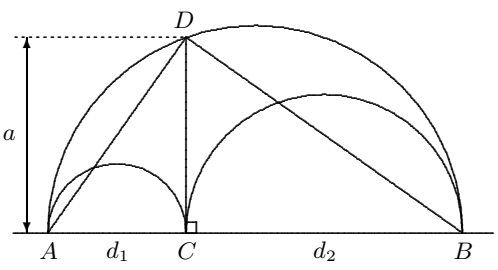
By the quadratic formula, we have

$$\begin{aligned} b^2 &= \frac{12d^2 + \sqrt{144d^4 + 192t^2}}{6} \qquad \text{since } b^2 > 0 \\ &= 2d^2 + \tfrac{2}{3}\sqrt{9d^4 + 12t^2} \\ b &= \sqrt{2d^2 + \tfrac{2}{3}\sqrt{9d^4 + 12t^2}} \qquad \text{since } b > 0. \end{aligned}$$

When $d = 1$ and $t = 6$, we get $b = 4$, which implies that $a = b - d = 3$ and $c = b + d = 5$.

21. Let the diameter of the original semicircle have endpoints A and B. Let CD be the line perpendicular to AB with C lying on AB and D on the circumference of the original semicircle. Then $CD = a$. Let $AC = d_1$ and $CB = d_2$. Then triangles ACD and DCB are similar. Therefore,

$$\begin{aligned} \frac{AC}{CD} &= \frac{DC}{CB}, \\ \text{or} \qquad \frac{d_1}{a} &= \frac{a}{d_2}. \end{aligned}$$



That is, $d_1 d_2 = a^2$. The area of the original semicircle is $\frac{1}{2}\pi\big((d_1+d_2)/2\big)^2$, and the areas of the two small semicircles are $\frac{1}{2}\pi(d_1/2)^2$ and $\frac{1}{2}\pi(d_2/2)^2$. Therefore,

the difference we seek is

$$\begin{aligned}\Delta &= \frac{\pi}{2}\left(\frac{d_1+d_2}{2}\right)^2 - \frac{\pi}{2}\left(\frac{d_1}{2}\right)^2 - \frac{\pi}{2}\left(\frac{d_2}{2}\right)^2 \\ &= \frac{\pi}{8}\big((d_1+d_2)^2 - d_1^2 - d_2^2\big) \\ &= \frac{\pi}{8}(2d_1d_2) \;=\; \frac{\pi d_1 d_2}{4} \;=\; \frac{\pi a^2}{4}\,.\end{aligned}$$

That is, the area in question is $\pi a^2/4$.

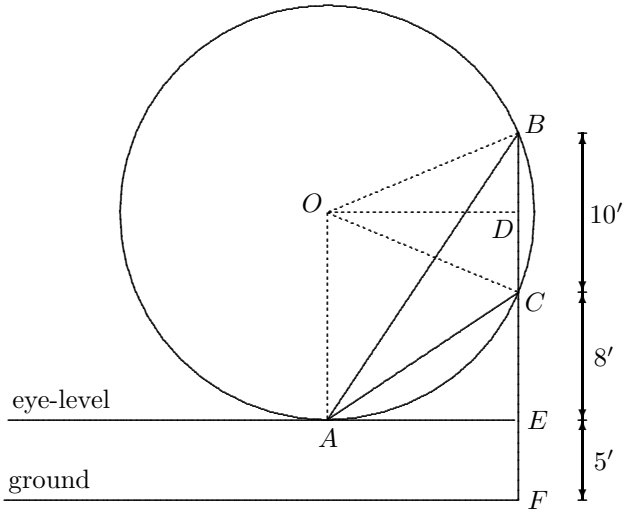
22. Since $\left(r+\frac{1}{r}\right)^2 = 3$ and

$$\left(r+\frac{1}{r}\right)^3 \;=\; r^3+3r+\frac{3}{r}+\frac{1}{r^3} \;=\; r^3+\frac{1}{r^3}+3\left(r+\frac{1}{r}\right)\,,$$

we have

$$r^3+\frac{1}{r^3} \;=\; \left(r+\frac{1}{r}\right)^3 - 3\left(r+\frac{1}{r}\right) \;=\; 3\sqrt{3}-3\sqrt{3} \;=\; 0\,.$$

23. Consider the diagram below where the statue is represented by BC and the base is represented by CF. Thus, the distance CE represents that part of the base which is above the man's eye-level, namely 8 feet.



The circle in the above diagram is the unique circle which has BC as a chord and is tangent to the line drawn at eye-level parallel to the (level) ground.

Claim. The point of tangency, A, gives the maximum angle subtended at eye-level by the points B and C.

This is easy to see if one recalls that the angle subtended at the circumference by a fixed chord is constant on each side of the chord. On the other hand, if a point outside the circle is joined to B and C, we get an angle that is smaller than $\angle BAC$. Thus, the largest angle subtended at eye-level will occur at A.

Now we wish to find the distance AE. But $AE = OD$, a radius of the circle. In $\triangle OCD$ we see that $OC^2 = OD^2 + DC^2$, by the Theorem of Pythagoras. However, $OC = OA = DC + CE = 5 + 8 = 13$ and $DC = 5$. Therefore, $OD = 12$. Hence, the distance we seek is 12 feet.

24. Notice that the k^{th} lock will be turned once in every pass whose number evenly divides k. For example, the 12^{th} lock is turned in passes numbered 1, 2, 3, 4, 6, and 12. Since all locks are locked to begin with, we need to determine all integers which have an odd number of divisors. These will correspond to the locks which remain open at the end of the ritual.

Now, if d divides k evenly, then so does k/d. If we call k/d the codivisor associated with d, then each divisor d of k can be paired with its associated codivisor. This implies that the number of divisors of k will be even unless some divisor d of k is equal to its associated codivisor; that is,

$$d \;=\; \frac{k}{d}\,.$$

This implies that $k = d^2$. Thus, k must be a perfect square. It is just as easy to verify that if k is a perfect square then it has an odd number of divisors.

Therefore, the only cells that remain open after the ritual are those whose numbers are perfect squares.

25. Since the wives drank 14 bottles in total, their husbands must have drunk 30 bottles in total. The smallest number of bottles the husbands could have drunk was for the higher factors to be matched with the smaller number of bottles drunk by each wife. This match-up gives

$$5 \cdot 1 + 4 \cdot 2 + 3 \cdot 3 + 2 \cdot 4 = 30$$

bottles of pop, exactly the correct number! Since every other match-up would have more than 30 bottles consumed, this is the correct match-up. That is, Ann Smith, Betty White, Carol Green, and Dorothy Brown are the names of the four women.

26. Note that

$$f(x+1) \;=\; \frac{u(x+2)+u(x)}{2} \qquad \text{and} \qquad f(x-1) \;=\; \frac{u(x)+u(x-2)}{2}\,.$$

Thus, we have

$$u(x) \;=\; f(x+1) + f(x-1) - \left(\frac{u(x+2)+u(x-2)}{2}\right)\,. \tag{1}$$

Using a similar argument to the above, we also note that

$$f(x+3) \;=\; \frac{u(x+4)+u(x+2)}{2} \qquad \text{and} \qquad f(x-3) \;=\; \frac{u(x-2)+u(x-4)}{2}\,.$$

Therefore,

$$f(x+3)+f(x-3) \;=\; \frac{u(x+2)+u(x-2)}{2}+g(x)\,. \tag{2}$$

Putting (1) and (2) together yields

$$u(x) \;=\; f(x+1)+f(x-1)-f(x+3)-f(x-3)+g(x)\,.$$

27. Notice that for $n \geq 1$ we have $(2+\sqrt{3})^n > 1$. Now consider $(2-\sqrt{3})^n$. Clearly, for $n \geq 1$ we have $(2-\sqrt{3})^n < 1$. From the Binomial Theorem we note that $(2+\sqrt{3})^n = a+b\sqrt{3}$ for some integers a and b. If we apply the Binomial Theorem to $(2-\sqrt{3})^n$, we can easily show that $(2-\sqrt{3})^n = a-b\sqrt{3}$ for the same integers a and b. Consequently,

$$(2+\sqrt{3})^n+(2-\sqrt{3})^n \;=\; a+b\sqrt{3}+a-b\sqrt{3} \;=\; 2a\,.$$

Therefore, $(2+\sqrt{3})^n = 2a-(2-\sqrt{3})^n$; that is, $(2+\sqrt{3})^n$ is an even number less a positive real number smaller than 1. Thus, $\lfloor (2+\sqrt{3})^n \rfloor$ must be odd.

28. Since the sum of the three terms in the arithmetic progression is 126, we know that the middle term is $126/3 = 42$. Thus, the terms in the arithmetic progression can be represented as $42-d$, 42, and $42+d$. Hence the terms of the geometric progression can be represented as $43+d$, 34, and $42-d$. A necessary condition for these three numbers to be a geometric progression is

$$(43+d)(42-d) \;=\; 34^2\,,$$

which is successively equivalent to

$$\begin{aligned} 1806-d+d^2 &= 1156\,,\\ d^2-d-650 &= 0\,,\\ (d+26)(d-25) &= 0\,, \end{aligned}$$

from which we have $d=-26$ or $d=25$. The resulting two progressions are:

(i) Arithmetic: 68, 42, 16; geometric: 17, 34, 68, or

(ii) Arithmetic: 17, 42, 67; geometric: 68, 34, 17.

29. (a) After one such division, we have $\frac{8}{9}$ of a unit remaining in the form of 8 squares of side length $\frac{1}{3}$. After a second division, we have $\frac{8}{9}\cdot\frac{8}{9}$ of a unit remaining in the form of 64 squares of side length $\frac{1}{9}$. In this manner, we see that after n divisions, we have 8^n squares of side length $\frac{1}{3^n}$ remaining.

(b) After n divisions, the area of the removed squares is $1 - \frac{8^n}{9^n}$. As n tends to infinity, this expression approaches the value 1. That is, as n goes to infinity, the sum of the areas of the removed squares is 1.

30. The left side must clearly be evenly divisible by 3. Thus, 3 divides a^2 evenly, which is only possible if 3 divides a evenly, since 3 is a prime number. Let $a = 3a_1$. Then the equation becomes

$$18a_1^2 \;=\; 3b^3\,,$$

which simplifies to

$$6a_1^2 \;=\; b^3\,.$$

Therefore, 6 evenly divides b^3, which implies that 6 evenly divides b, since it is the product of the distinct primes 2 and 3. Let $b = 6b_1$. Then our equation becomes

$$6a_1^2 \;=\; 6^3b_1^3\,,$$

which simplifies to

$$a_1^2 \;=\; 36b_1^3\,.$$

Now 36 evenly divides a_1^2, implying that 6 evenly divides a_1. Let $a_1 = 6a_2$. Our equation now transforms into

$$36a_2^2 \;=\; 36b_1^3\,,$$

which simplifies to

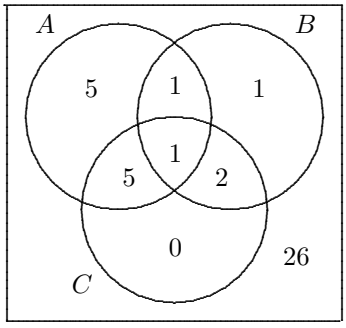
$$a_2^2 \;=\; b_1^3\,.$$

The only solution of this is that a_2 is a perfect cube c^3 and that b_1 is a perfect square c^2. Therefore, $a = 3a_1 = 18a_2 = 18c^3$ and $b = 6b_1 = 6c^2$. The general solution for positive integers a and b is

$$a \;=\; 18c^3 \qquad \text{and} \qquad b \;=\; 6c^2$$

for all positive integers c.

31. We will use a Venn diagram, as we did for the solution of problem #15. Let circle A be the set of students failing Algebra, circle B those failing Biology, and circle C those failing Chemistry. We want the number of students taking the exams (the whole rectangle), but not lying in any of the circles.



Working from right to left in the given table (which is the same as working from the centre of the Venn diagram to the outer sections), we fill in values in the diagram as above. We then conclude that 26 students pass all three subjects.

32. Since we are only ever using multiplication of single digit numbers in the manufacture of this sequence, the largest number we would ever have to "split" is 81, a 2–digit number.

Next, we observe that we never have two odd digits in a row in the sequence. For if we did have two consecutive odd digits, one of them at least would have had to come from the units position of the product of two consecutive digits appearing earlier in the sequence. This would imply that we would have had two consecutive odd digits at an earlier point in the sequence. This reasoning could be continued forever, which would imply that we have no beginning to the sequence, which is nonsense. Consequently, we cannot have two odd digits in a row.

Any digit in the sequence (after the second) arises from a product of two digits appearing earlier in the sequence. A particular digit in the sequence may be either the units digit or the tens digit of the product. Since the three digits we must show do not appear are all odd, we need only show that they cannot arise in the tens position of a product. For if they appeared in the units position, we would be forced to conclude that the product came from two consecutive odd digits, which we have seen is impossible.

(a) 9 cannot appear in the tens position since the largest product that can be formed by two digits is 81.

(b) 7 cannot appear in the tens position, since, by (a), the largest product is $8 \times 8 = 64$.

(c) 5 cannot appear in the tens position, since the only products of two digits having a 5 in the tens position are $54 = 6 \times 9$ and $56 = 7 \times 8$. One of these requires a 9 to appear in the sequence, the other a 7, both of which have been shown to be impossible by (a) and (b) above.

33. Let n be a natural number satisfying the given property. Let a be the final digit of n, and let m be the number obtained from n when the digit a is removed. Then

$$n \;=\; 10m + a\,, \qquad 1 \le a \le 9\,.$$

Furthermore, $n = mk$ for some integer k, since m divides n evenly. Thus,

$$\begin{aligned} mk &= 10m + a \\ m(k-10) &= a\,. \end{aligned}$$

This implies that m divides a evenly. Since a is a single digit and $a \neq 0$, we see that m must be a single digit also. It is now an easy task to enumerate them since n has two digits, the first of which evenly divides into the second:

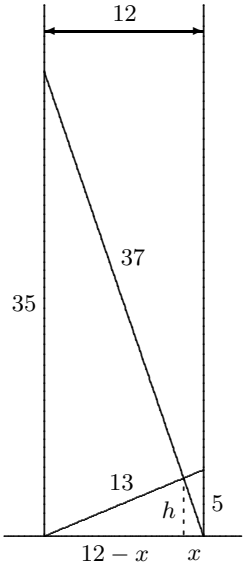
$$\begin{aligned} n \in \{&11,\, 12,\, 13,\, 14,\, 15,\, 16,\, 17,\, 18,\, 19,\, 22,\, 24,\, 26,\, 28,\\ &33,\, 36,\, 39,\, 44,\, 48,\, 55,\, 66,\, 77,\, 88,\, 99\} \end{aligned}$$

34. For a penny to land totally within the square, the centre of the penny must land within a square of side length $\frac{1}{4}$ inch centred in the ruled square. Thus, the probability of winning is the ratio of the areas of the two squares:

$$\frac{\text{area for a win}}{\text{total area}} \;=\; \frac{\left(\frac{1}{4}\right)^2}{1^2} \;=\; \frac{1}{16}\,.$$

To make the odds even, the player should get 16 cents back when he wins.

35. By using the Theorem of Pythagoras, we see that the 13 foot ladder meets the one building at a height of 5 feet above the alley, while the 37 foot ladder meets the other building at a height of 35 feet above the alley. Let h be the height above the alley of the point of intersection, and let x be the distance of this point from the building against which the 13 foot ladder leans (as in the diagram). By similar triangles, we have



$$\frac{h}{12-x} \;=\; \frac{5}{12} \qquad \text{and} \qquad \frac{h}{x} \;=\; \frac{35}{12}\,.$$

That is, $12h = 60 - 5x$ and $12h = 35x$. This yields

$$\begin{aligned} 35x &= 60 - 5x\,, \\ x &= \tfrac{3}{2}\,, \\ h &= \tfrac{35}{12}\left(\tfrac{3}{2}\right) \;=\; 4.375\,. \end{aligned}$$

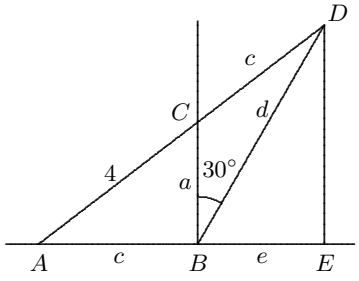
The height of the point of intersection is 4.375 feet above the alley.

36. Let k be the number of doors between two rooms of the house, and let n be the number of entrances to the house from the outside. Let us now consider each room of the house and count the number of doors through which one could exit the room. If we do this for every room in the house, our count will be even since every room has an even number of doors. Say our count is $2N$. Furthermore, each door between two rooms of the house will have been counted twice and each door to the outside will have been counted once. That is,

$$2N \;=\; 2k + n\,.$$

This implies that n is even. Therefore, we have an even number of doors to the outside.

37. Let $a = BC$ and $c = AB$. Drop a perpendicular from D to the line AB, meeting it at E. Let $e = BE$ and $d = BD$. Since CB is parallel to DE, we have



$$\frac{c}{e} \;=\; \frac{4}{c}\,;$$

that is, $c^2 = 4e$. Since CB is parallel to DE, we also have $\angle BDE = 30°$, which implies that $d = 2e$ and

$$DE \;=\; d\sin 60° \;=\; 2e\frac{\sqrt{3}}{2} \;=\; e\sqrt{3}\,.$$

From the similar triangles ABC and AED, we have

$$\begin{aligned}
\frac{e\sqrt{3}}{a} &= \frac{c+e}{c} = 1+\frac{e}{c}\,,\\
\frac{4e\sqrt{3}}{a} &= 4+\frac{4e}{c} = 4+c \quad (\text{since } 4e=c^2),\\
c^2\sqrt{3} &= (4+c)a\,,\\
3c^4 &= (4+c)^2a^2\,,\\
3c^4 &= (16+8c+c^2)(16-c^2)\,,\\
3c^4 &= 256+128c-8c^3-c^4\,,\\
4c^4+8c^3-128c-256 &= 0\,,\\
4c^3(c+2)-128(c+2) &= 0\,,\\
(c+2)(4c^3-128) &= 0\,.
\end{aligned}$$

Therefore, $c=-2$ or $c=\sqrt[3]{32}$. Since $c=-2$ is impossible, we have

$$AB = c = \sqrt[3]{32} = \sqrt[3]{\frac{64}{2}} = \frac{4}{\sqrt[3]{2}}\,.$$

38. Note first that the progression 100, 200, 400, 800 has 4 terms. Let us now try to create a progression which has more than 4 terms. Let a be the first number in the progression. Let b be the last. Without loss of generality, we may assume that $a<b$. Let r be the common ratio. Then the progression is of the form:

$$a\,,\quad ar\,,\quad ar^2\,,\quad ar^3\,,\quad \ldots\,,\quad ar^{n-1}\,(=b)\,,$$

where n is the number of terms (which we wish to find). In order that all of these terms are integers, it is clear that r must be rational, say $r=p/q$, where p and q have no common factors. Note that r must be less than 2 if we are to get a progression longer than 4 elements. Also notice that for all the terms to be integers, q must evenly divide a a total of $n-1$ times. That is,

$$a = kq^{n-1} \qquad \text{and} \qquad b = kp^{n-1}\,,$$

where $100\le a<b\le 1000$, and k is an integer. These restrictions on a and b also establish that

$$r^{n-1} \le 10\,.$$

This inequality implies that if we are to get a long progression, we require a value of r which is close to 1.

Thus, if we are to get at least 5 terms, then a must be divisible by at least the fourth power of q. Let us first suppose that $q=2$. Then $p=3$. The smallest 3–digit multiple of 16 is 112, which gives the 5–term sequence 112, 168, 252, 378, 567. The next multiple of 16 is 128 gives the 6-term sequence 128, 192, 288, 432, 648, 972. It is clear that, if we were to start with a larger first term, even the

sixth term would exceed 3 digits. Thus, to find a progression with at least 7 terms, we need to consider $q \geq 3$. We also see that a must be divisible by at least the sixth power of q and b must be divisible by at least the sixth power of p. That is, $a \geq 3^6 = 729$ and $b \geq 4^6 = 4096$, which has has more than 3 digits.

Thus, the maximum number of terms in a geometric progression with common ratio greater than 1 and all of whose terms are integers between 100 and 1000 is 6.

39. We shall prove below that $\binom{n}{k}$ is odd for all k, $0 \leq k \leq n$, for exactly those integers n which are one less than a power of 2; that is, for $n = 2^i - 1$ for some positive integer i. The solution presented below is not the only way to proceed, but it is one of the easiest to write out. It assumes at least some knowledge of binary numbers.

First note that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot \cdots \cdot (n-k+1)}{k \cdot (k-1) \cdot \cdots \cdot 1}. \tag{1}$$

In this last fraction, numerator and denominator each have k factors. We are interested in the number of times that the prime 2 appears as a factor in the numerator and denominator.

Now suppose that $n = 2^i - 1$ for some positive integer i. Consider the integers k and $n - k + 1 = 2^i - k$ for $1 \leq k \leq n$. If 2^a evenly divides one of them for some positive integer a, it must also evenly divide the other. This implies that the factors in the numerator and denominator of expression (1) can be paired as $(n, 1)$, $(n-1, 2)$, $\ldots$, $(n-k+1, k)$ so that both members of the pair are evenly divisible by the same power of 2. Therefore, in this case, all occurrences of the prime 2 can be cancelled in numerator and denominator, leaving us with $\binom{n}{k}$ odd, as desired.

We now need to show that no other numbers will work. Let us now express n in binary notation. Note that a number of the form $2^i - 1$ has the binary representation $111\ldots1$ with i occurrences of the binary digit 1. Suppose n is not of this form. Then there is at least one binary digit 0 embedded in the binary representation of n. Let us suppose that the binary representation of n has the form $a0111\ldots1$, where a is some string of binary 0s and 1s, and there are b binary 1s at the end. We will now show that $\binom{n}{k}$ is even when $k = 2^b$.

The numbers n, $n-1$, $\ldots$, $n-k+1 = n-2^b+1$ will all begin (in binary) with the string $a0$. Also the last b binary digits will contain all possible combinations of 0s and 1s. On the other hand, the numbers k, $k-1$, $\ldots$, 1 will contain all possible combinations of 0s and 1s for the last b binary digits (assuming sufficiently many leading zeroes). By pairing up the factors with the same last b binary digits we have the pairs $(n, k-1)$, $(n-1, k-2)$, $\ldots$, $(n-k+2, 1)$, and the final pair $(n-k+1, k)$. Except for the last pair mentioned, both numbers of the pair are divisible by the same powers of 2, and thus those powers of 2 could be cancelled in the numerator and denominator of expression (1).

For the final pair above, however, we can observe that k is divisible by 2^b (and no higher power of 2), but $n-k+1$ ends in at least $b+1$ zeroes (in binary) and is thus divisible by at least 2^{b+1}. Therefore, there is at least one more factor of 2 in the numerator than in the denominator. Hence, $\binom{n}{k}$ must be even.

40. The reader can verify that any function of the form $F(x) = x^2 + kx$ (where k is any constant) satisfies the functional equation. We will show that there are no others.

First we will examine $G(x) = F(x) - x^2$. Then

$$G(x+y) \;=\; F(x+y) - (x+y)^2 \;= F(x) + F(y) - x^2 - y^2 \;= G(x) + G(y)\,.$$

Such a function is often called an additive function. If we set $x = y = 0$, we see that $G(0) = 0$. Also, using the fact that G is additive, it is easy to see by induction that $G(nx) = nG(x)$ for any positive integer n. Note as well that

$$G(-x) \;=\; G\big(x + (-2x)\big) \;=\; G(x) + G(-2x) \;=\; G(x) + 2G(-x)\,,$$

from which we conclude that

$$G(-x) \;=\; -G(x)\,.$$

This allows us to conclude that $G(nx) = nG(x)$ for all integers n.

Next, we wish to show that $G(ax) = aG(x)$ for all rational numbers a. Let $a = p/q$, where p and q are integers with q different from 0. Then

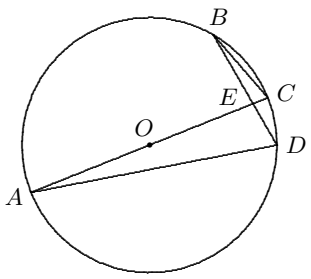
$$q\big(G(ax)\big) \;=\; G\big(q(ax)\big) \;=\; G(px) \;=\; pG(x)\,;$$

whence,

$$G(ax) \;=\; (p/q)G(x) \;=\; aG(x)\,.$$

Since F is continuous, G is also continuous. Thus, $G(rx) = rG(x)$ for all real numbers r, since any real number can be considered as the limit of a sequence of rational numbers. Let $G(1) = k$. Then $G(x) = xG(1) = kx$. Therefore, $F(x) = x^2 + kx$.

41. Extend the line OC to a diameter AOC. Then triangles BEC and AED are similar, since $\angle BCA$ and $\angle BDA$ are both subtended by the chord BA, and $\angle CBD$ and $\angle CAD$ are both subtended by the chord CD. Consequently,

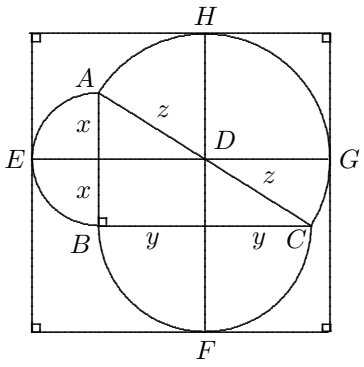


$$\begin{aligned} \frac{EB}{CE} &= \frac{EA}{DE}\,, \\ \frac{5}{1} &= \frac{EA}{3}\,, \\ \text{or} \qquad EA &= 15\,. \end{aligned}$$

Therefore, AOC, the diameter, is 16; whence, the radius is 8.

42. Let ABC be the right-triangle in question with the right angle at B. Let the lengths of AB, BC, and AC be $2x$, $2y$, and $2z$, respectively. Let D be the mid-point of AC. Then D is the centre of the semicircle of radius z mounted on AC.

Join D to the points of tangency between the outer rectangle and the semicircles, as in the diagram. Now the lines DE and DF pass through the centres of the smaller semicircles, since D is the mid-point of AC. Thus, the lines DE and DG are parts of the same line EG, since both are perpendicular to parallel sides of the rectangle. Similarly, DF and DH are parts of the same line FH. It remains only to show that EG and FH have the same length. However, from the diagram, it is clear that both have length $x + y + z$.



43. Factor both the numerator and denominator first:

$$\begin{aligned} n^3 - 1 &= (n-1)(n^2+n+1)\,; \\ n^3 + 1 &= (n+1)(n^2-n+1)\,. \end{aligned}$$

Note that

$$n^2 - n + 1 \;=\; (n-1)^2 + (n-1) + 1 \;=\; k^2 + k + 1\,,$$

where $k = n - 1$. That is, $n^2 - n + 1$ and $n^2 + n + 1$ run through the same set of integers as n ranges over all the integers. Now

$$\begin{aligned} \prod_{n=2}^{k} \frac{n^3-1}{n^3+1} &= \prod_{n=2}^{k} \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)} \\ &= \frac{1\cdot 7}{3\cdot 3}\cdot\frac{2\cdot 13}{4\cdot 7}\cdot\frac{3\cdot 21}{5\cdot 13}\cdots\frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)} \end{aligned}$$

The second factor in each numerator cancels with a factor in the denominator immediately to its right. The first factor in each numerator cancels with a factor in the denominator two terms to its left. After all cancellations have been made, the product simplifies to

$$\prod_{n=2}^{k} \frac{n^3-1}{n^3+1} \;=\; \frac{2(k^2+k+1)}{3k(k+1)} \;=\; \frac{2}{3}\cdot\frac{k^2+k+1}{k^2+k} \;=\; \frac{2}{3}\left(1+\frac{1}{k^2+k}\right).$$

Now

$$\begin{aligned} \prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} &= \lim_{k\to\infty} \prod_{n=2}^{k} \frac{n^3-1}{n^3+1} = \lim_{k\to\infty} \frac{2}{3}\left(1+\frac{1}{k^2+k}\right) \\ &= \frac{2}{3}\lim_{k\to\infty}\left(1+\frac{1}{k^2+k}\right) \;=\; \frac{2}{3}\,. \end{aligned}$$

44. The first question that immediately comes to mind with this problem is "How many fishermen were there?". One can correctly argue that in order to answer the question asked we must have only three men: namely Ted and Peter (who are father and son, but not necessarily in that order) and the father of the older one. In fact, there is enough information to actually show mathematically that there can only be three fishermen.

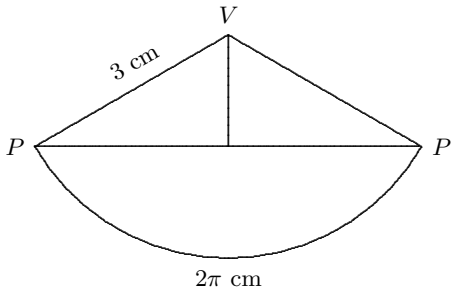
Let us suppose instead that there are four distinct fishermen. Let the number of fish caught by Peter's father be x and the number of fish caught by Ted's father be y. Then Peter caught $3x$ fish and Ted caught y fish. This would yield the following system:

$$\begin{aligned} x + y &= 14 \qquad \text{(fish caught by fathers)}\,, \\ 3x + y &= 56 \qquad \text{(fish caught by sons)}\,, \end{aligned}$$

which has the solution $x = 21$ and $y = -7$, clearly impossible. Thus, we conclude that there are only three fishermen involved. That is, either Peter is Ted's father or vice versa. This means that 56 fish (that is, 80% of 70 fish) were hooked by the sons, Peter and Ted.

Assume first that Peter is Ted's father. Then Ted and Peter both catch the same number of fish, which must be 28. But this means that Peter catches only twice as much as his father (who caught $14 = 70 - 56$ fish), instead of three times as much (as stated in the problem). Thus, we conclude that Ted is Peter's father. Then, Peter caught 42 fish and Ted 14 fish, which would be the same as the number caught by his father (the remaining person). In answer to the question asked, we conclude that Ted must be older than Peter.

45. Since this problem essentially deals with a surface and not with a solid or volume, we will first flatten out the surface by cutting along the line joining P and V. When this is flattened out, it becomes a circular sector with centre V with the point P being represented on the circumference at both ends of the sector (see the figure below). The radius of the sector is 3 cm and the arc length is 2π cm (since the arc length corresponds to the circumference of the base of the cone). Since the ratio of arc length to radius is the measurement (in radians) of the angle subtended at the centre, we see that the central angle for this sector is $\frac{2}{3}\pi$ radians, or 120°.



If we now consider the original problem, it amounts to asking what the minimum distance is between the two representations of the point P. Clearly, this distance is measured along the chord joining the two end-points of the sector (since the sector is less than a semi-circle). We are not concerned, however, with the length of the chord, but rather with the shortest distance between it and the centre V. Obviously, we want to measure this distance along the radius which bisects the sector. Each of the two congruent triangles arising from this construction is right-angled; whence, the distance we are after is $3\cos(60^\circ)$ cm; that is, 1.5 cm.

46. We provide two different solutions for this problem.

Solution I. The first three statements describe the distribution of cards in the hand. By starting with the distribution 1–2–3–7 (where this means one suit has only 1 card in it, a second suit has 2 cards, a third suit has 3 cards, and the fourth suit has 7 cards, without any regard to which suit is hearts, which is clubs, etc.) and examining the different possible distributions systematically, one discovers that there are only three distributions that satisfy all of the first three statements, namely 1–2–3–7, 1–2–4–6, and 1–3–4–5. By considering statement 6 we can immediately rule out 1–3–4–5, since no suit has exactly 2 cards in it. In the case 1–2–3–7, we can also see that no two suits have a total of 6 cards in them, and thus statement 5 could not possibly be satisfied, leaving us with only one possible distribution, namely 1–2–4–6. By using statements 4 and 5 together, we conclude that there are 6 clubs, 4 hearts, 2 spades and 1 diamond; whence, SPADES are trump.

Solution II. Let c, d, h, and s represent the number of clubs, diamonds, hearts, and spades, respectively, in the hand. Then statements 4 and 5 yield

$$\begin{aligned} & h+d &=& \ 5\,, \\ \text{and} \qquad & h+s &=& \ 6\,; \end{aligned}$$

whence, $2h+s+d=11$. Since we know that $c+s+d+h=13$, we may conclude that $c=h+2$. Since every suit is represented in the hand, we also see that h is at most 4 (statement 4, since $d \geq 1$) and that c is at least 3 (since $h \geq 1$).

Since there is a different number of cards in every suit, and there are 13 cards in total, there are only 3 possibilities for the distribution:

$$1\text{–}2\text{–}3\text{–}7\,, \qquad 1\text{–}2\text{–}4\text{–}6\,, \qquad \text{and} \qquad 1\text{–}3\text{–}4\text{–}5\,.$$

The number of cards in the trump suit is 2; this eliminates the third of the above possibilities. If the first possibility is the proper distribution, then (since $c=h+2$) we have $c=3$ and $h=1$, which requires that the number of diamonds, d, is 4, which is impossible. Thus, the distribution is 1–2–4–6 from which we must have

$$c \ = \ 6\,, \qquad h \ = \ 4\,, \qquad s \ = \ 2\,, \qquad d \ = \ 1\,.$$

This implies that SPADES are trump.

47. Let the three natural numbers be a, b, c. Without any loss of generality, we may assume that $1 \leq a < b < c$. We are given that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \;=\; n\,,$$

where n is an integer. Because the numbers are distinct the smallest possible values are $a = 1$, $b = 2$, and $c = 3$. Thus,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \;\leq\; \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \;=\; \frac{11}{6} \;<\; 2\,.$$

Consequently, the only acceptable value for n is 1. This implies that $a \neq 1$. Thus, $2 \leq a < b < c$. We also note that

$$1 \;=\; \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \;<\; \frac{1}{a} + \frac{1}{a} + \frac{1}{a} \;=\; \frac{3}{a}\,.$$

This means that $a < 3$. The only possible value of a is then $a = 2$. Our given equation then becomes

$$\begin{aligned} \frac{1}{2} + \frac{1}{b} + \frac{1}{c} \;&=\; 1 \qquad \text{with } 2 < b < c \\ \text{or} \qquad \frac{1}{b} + \frac{1}{c} \;&=\; \frac{1}{2} \qquad \text{with } 2 < b < c. \end{aligned}$$

But

$$\frac{1}{2} \;=\; \frac{1}{b} + \frac{1}{c} \;<\; \frac{1}{b} + \frac{1}{b} \;=\; \frac{2}{b}\,.$$

Therefore, $b < 4$, which implies that $b = 3$. We now have

$$\frac{1}{3} + \frac{1}{c} \;=\; \frac{1}{2}\,,$$

from which we see that $c = 6$. Therefore, the only solution for $a < b < c$ is $(a, b, c) = (2, 3, 6)$.

The reader is encouraged to investigate what happens when we remove the word "distinct" from the problem statement.

48. Notice first that, except for Midville, each town has 3 roads leading away from it (an odd number). Thus, at least one road leading away from each "perimeter" town must be travelled twice; for a town with an odd number of roads must be either the beginning or the end of a "tour" if each road is travelled once only. If we choose to travel an "interior" road twice (that is, a road leading to Midville), then we must choose two of them, in order to keep the parity of travelled roads even at Midville also. Therefore, we must compare the interior distances with the perimeter distances:

– W. Midville to E. Midville: $8 + 21 < 40$; shorter by interior roads.

– S.W. Midville to W. Midville: $34 < 30 + 8$; shorter by perimeter road.

– S.E. Midville to S.W. Midville: $5 < 28 + 30$; shorter by perimeter road.

– E. Midville to S.E. Midville: $37 < 21 + 28$; shorter by perimeter road.

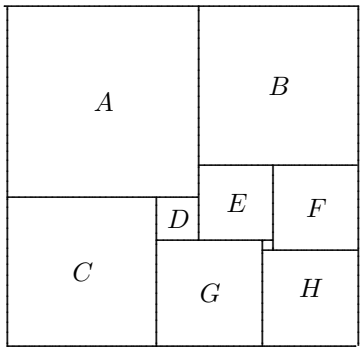
Thus, the shortest routes between adjacent perimeter towns are 5, 34, 37, and $8+21$. These are the candidates for the roads to be traveled twice. There are now two alternatives (in order to keep the parity of roads travelled at each perimeter town even): 34 and 37, or 5 and $8+21=29$. Clearly, the latter pair is shorter. Therefore, the roads which are traveled twice are those with the distances 5, 8, and 21.

There are several routes which will work. One such is the following:

$$28,\ 5,\ 30,\ 8,\ 40,\ 21,\ 8,\ 34,\ 5,\ 37,\ 21.$$

The total distance for any such route is 237 km.

49. Let the squares be labelled as in the diagram to the right. If we let x be the length of the side of square E, then squares F, H, G, and B have sides of length $x+1$, $x+2$, $x+3$, and $2x+1$, respectively. Now by considering the line separating squares E and G, we conclude that square D has sides of length 4. We can now conclude that squares C and A have sides of length $x+7$ and $x+11$, respectively. By considering the line separating squares A and B, we have successively



$$\begin{aligned}(x+11)+4 &= (2x+1)+x\,,\\ x+15 &= 3x+1\,,\\ x &= 7\,.\end{aligned}$$

This completely solves the rectangle, which must have dimensions 33 across and 32 down.

50. Given that the equation

$$x^2F(x)+F(1-x) \;=\; 2x-x^4 \tag{1}$$

holds for all real values x, it must also hold if we replace x by $1-x$, yielding

$$(1-x)^2F(1-x)+F(x) \;=\; 2(1-x)-(1-x)^4\,. \tag{2}$$

Solving (1) for $F(1-x)$ gives

$$F(1-x) \;=\; 2x-x^4-x^2F(x)\,. \tag{3}$$

By substituting (3) into (2) and simplifying, we get

$$(1-x^2+2x^3-x^4)F(x) \;=\; 1-2x^2+2x^3-2x^5+x^6\,.$$

If we solve this by polynomial division, we have $F(x)=1-x^2$.

51. To determine the perimeter of the square, it is sufficient to know its area. Clearly, the area of the square will be exactly the same as the area of the rectangle. Thus, our task is simply to find the area of the rectangle. The height is given to be 9 units. It only remains to find the width.

The small triangle with hypotenuse 5 is similar to the large triangle with hypotenuse 15. Thus, its height is in the same proportion to the height of the large triangle as its hypotenuse is. Therefore, the height of this small triangle is 3. The Theorem of Pythagoras then tells us that the base of the small triangle is 4. Consequently, the width of the rectangle is 16.

We then conclude that the area of the rectangle is 144 square units. Since the square must have the same area, the side of the square must be 12 units; whence, the perimeter is 48 units.

Note that this assumes that a square can, in fact, be generated from the given pieces, although we have not actually shown how to generate the square. We leave that as a further puzzle to the interested reader to show that the pieces can actually be rearranged to form a square!

52. First note that on any move of a knight in chess, the colour of the square it leaves differs from the colour of the square on which it lands. Starting at any point on the board, the knight must make exactly 63 moves to land on every square exactly once. This means that the colour it ends up on will differ from the colour it starts out on, since 63 is an odd number. Since the opposite corners of a chessboard have the same colour, we have shown that the stated problem is impossible.

53. There are two separate problems here. The first is to compute the area of triangle PQR. Once we have done that, we will show that the area of each of the four triangles is the same! Thus, the area of the hexagon is the sum of the areas of the 3 squares plus 4 times the area of triangle PQR; that is, $74 + 4[PQR]$.

First of all, we find the area of PQR. The sides clearly have lengths of $\sqrt{13}$, 5, and 6. We will use Heron's Formula for the area of a triangle which states that the area A of a triangle is given by

$$A \;=\; \sqrt{s(s-a)(s-b)(s-c)}\,,$$

where a, b, c are the sides of the triangle, and s is the semi-perimeter (that is, $s = (a+b+c)/2$). By using this formula, we find that the area of PQR is 9 square units. (There is another method to find the area which uses less machinery, but requires a subtle observation. We note that $\triangle PQR$ can be obtained by adjoining the two right triangles $(3, 4, 5)$ and $(2, 3, \sqrt{13})$ along the legs of side length 3. The area of $\triangle PQR$ is thus $\frac{1}{2} \cdot 2 \cdot 3 + \frac{1}{2} \cdot 3 \cdot 4 = 9$.)

Secondly, we show that all four triangles have the same area. We will actually only show that APF and PQR have the same area, but the same argument will show that BQC and DRE also have the same area as PQR. Since two squares have P as a common vertex, the angles QPR and APF must sum to 180° (that is, a straight line!). Consequently, if we rotate triangle APF about the point P until points A and Q coincide, then points F, P, R will all lie on a straight line. Thus,

if we consider FP (rotated) and PR as the respective "bases" of these triangles, we must conclude that the triangles have equal altitudes. However, they also have equal bases, since $FP = PR$. Therefore, they must have equal areas.

In conclusion, the total area of the hexagon is $74 + 4(9) = 110$ square units.

54. Let x, y, z be the amount of money taken originally from the pile by the first man, the second man, and the third man, respectively. Then the amount of money in the original pile is clearly $x + y + z$. Let this total amount be designated by T. After returning the amounts stipulated above, the three men have left

$$\tfrac{1}{2}x\,, \qquad \tfrac{2}{3}y\,, \qquad \text{and} \qquad \tfrac{5}{6}z\,,$$

respectively. The amount of money to be divided must then be:

$$\tfrac{1}{2}x + \tfrac{1}{3}y + \tfrac{1}{6}z\,.$$

Let this amount be designated by P. Then if $P/3$ is added to each of the amounts the three men have remaining, they end up with their rightful shares: $T/2$, $T/3$, and $T/6$, respectively. Thus, we must have

$$\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{3}P &= \tfrac{1}{2}T\,, \\ \tfrac{2}{3}y + \tfrac{1}{3}P &= \tfrac{1}{3}T\,, \\ \tfrac{5}{6}z + \tfrac{1}{3}P &= \tfrac{1}{6}T\,. \end{aligned}$$

To clear the fractions let us multiply these equations by 18 yielding:

$$9x + 6P \;=\; 9T\,, \qquad 12y + 6P \;=\; 6T\,, \qquad 15z + 6P \;=\; 3T\,.$$

Replacing P by $x/2 + y/3 + z/6$ and replacing T by $x + y + z$ gives:

$$\begin{aligned} 9x + (3x + 2y + z) &= 9(x + y + z)\,, \\ 12y + (3x + 2y + z) &= 6(x + y + z)\,, \\ 15z + (3x + 2y + z) &= 3(x + y + z)\,. \end{aligned}$$

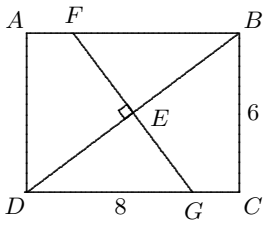
This system simplifies to

$$\begin{aligned} 3x - 7y - 8z &= 0\,, \\ -3x + 8y - 5z &= 0\,, \\ -y + 13z &= 0\,, \end{aligned}$$

which has solution $x = 33z$ and $y = 13z$. This gives a total of $47z$ for the amount of money in the original pile. It only remains to choose the minimum value of z so that all amounts will be integers.

Clearly, z must be divisible evenly by 6 (since the third man must be able to return $\frac{1}{6}$ of his original take). It is easy to see that if we select $z = 6$ all values in the problem are integers. Thus, the smallest size of the original pile is $6 \times 47 = 282$.

55. In the diagram to the right the line FG is the fold which takes the point B to the point D. Clearly, FG is the perpendicular bisector of BD. By the Theorem of Pythagoras, line BD has length 10, whence DE has length 5. We can also deduce that the triangles BDC and GDE are similar, since they each contain $\angle BDC$ and a right-angle. Thus,



$$\frac{DE}{DC} \;=\; \frac{GE}{BC}\,, \qquad \text{or} \qquad \frac{5}{8} \;=\; \frac{GE}{6}\,.$$

From this we get $GE = 15/4$. Since the fold is FG, and GE is half of this fold, we conclude that the length of the fold is $15/2$ units.

56. Let s be the number of steps visible if the escalator stopped running. Let v be the rate at which the steps disappear at the top of the escalator. Let r be Irv's rate of ascent, and let t be the time he took to clear the escalator. Let R and T have similar meanings for Liz. (We will assume that all three rates above are measured in number of steps per unit of time.) Clearly,

$$rt \;=\; 30 \qquad \text{and} \qquad RT \;=\; 24\,,$$

from which we can conclude that

$$r \;=\; \frac{30}{t} \qquad \text{and} \qquad R \;=\; \frac{24}{T}\,.$$

But $r = \frac{3}{2}R$. Thus,

$$\begin{aligned} \frac{30}{t} \;&=\; \frac{36}{T}\,, \\ \text{or} \qquad 5T \;&=\; 6t\,. \end{aligned}$$

We may also observe that each of them climbed s steps minus the number of steps which disappeared while they were climbing. Consequently, we have

$$30 \;=\; s - vt \qquad \text{and} \qquad 24 \;=\; s - vT\,,$$

or $vt = s - 30$ and $vT = s - 24$. Therefore,

$$6(s-30) \;=\; 6vt \;=\; 5vT \;=\; 5(s-24)\,,$$

which implies that

$$\begin{aligned} 6s - 180 \;&=\; 5s - 120 \\ \text{or} \qquad s \;&=\; 60\,. \end{aligned}$$

There would be 60 steps visible if the escalator stopped running.

57. Let x be the number of bricks in the wall. Then the first bricklayer lays $\frac{1}{9}x$ bricks per hour since it would take him 9 hours to build the wall himself. The second bricklayer would lay $\frac{1}{10}x$ bricks per hour by similar reasoning. The number of bricks laid per hour when they both work together would be

$$\frac{x}{9} + \frac{x}{10} - 10\,.$$

But since they took 5 hours to build the wall, they must have laid $\frac{1}{5}x$ bricks per hour. Setting these two expressions equal to each other and solving for x yields a value for x of 900 bricks.

58. Since each of the digits 1 through 9 will either be present or absent in such a number and we must have at least one such digit present (we cannot include the number 0), there are $2^9 - 1 = 511$ ways to select the set of digits of the number. Since we then only need to arrange them in increasing order, we have a total of 511 numbers.

There is also an interesting, albeit longer, approach to this problem: Notice first that all single digit integers satisfy the condition trivially. Next notice that the number of such integers having n digits with the leading digit k is the same as the number of such integers having $n-1$ digits with the leading digit greater than k. This allows us to establish the following table for the number of such integers having n digits with the leading digit k:

	n									Row
k	1	2	3	4	5	6	7	8	9	Sum
1	1	8	28	56	70	56	28	8	1	256
2	1	7	21	35	35	21	7	1	0	128
3	1	6	15	29	15	6	1	0	0	64
4	1	5	10	10	5	1	0	0	0	32
5	1	4	6	4	1	0	0	0	0	16
6	1	3	3	1	0	0	0	0	0	8
7	1	2	1	0	0	0	0	0	0	4
8	1	1	0	0	0	0	0	0	0	2
9	1	0	0	0	0	0	0	0	0	1

This gives us a grand total of 511, which is the value we seek.

For those of you familiar with Pascal's triangle you will notice that it is present (inverted) in the above table. You may also notice (if you had not already done so) that the sum of the entries in the n^{th} row of Pascal's triangle is 2^n.

59. From (ii) we see that $2gg' = 1$, which is the same as saying that the derivative of g^2 is 1. Thus, we can conclude that $\big(g(x)\big)^2$ must be $x + C$. By using condition (iii), we can conclude that $C = 0$. That is,

$$\big(g(x)\big)^2 \;=\; x\,.$$

Therefore, $g(x) = \sqrt{x}$.

When one considers condition (i), the left side of the equation appears to be similar to the derivative of the quotient f/g. By dividing both sides of the equation by g^2, the left side becomes exactly the derivative of the quotient f/g. Therefore,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \;=\; \frac{2\ln g(x)}{\big(g(x)\big)^2} \;=\; \frac{\ln x}{x}\,.$$

Thus, we see that

$$\frac{f(x)}{g(x)} \;=\; \int \frac{\ln x}{x}\,dx \;=\; \frac{(\ln x)^2}{2} + K$$

(where K is a constant of integration), from which we conclude that

$$f(x) \;=\; \tfrac{1}{2}x^{\frac{1}{2}}(\ln x)^2 + Kx^{\frac{1}{2}} \;=\; \tfrac{1}{2}\sqrt{x}\big((\ln x)^2 + 2K\big)\,.$$

60. (i) To determine the marriages based on the predictions is straightforward. They are:

1. Clarence & Ruth
2. Edgar & Veronica
3. Arthur & Fanny (since Polly and Helen are eliminated)
4. Basil & Polly
5. Desmond & Helen

(ii) Since all the predictions were false, we can immediately deduce the following marriages:

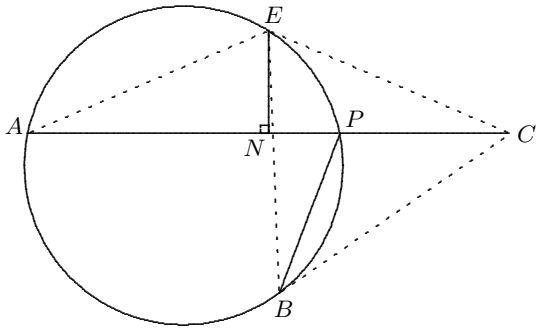
Basil & Helen
Arthur & Polly (since Helen is already paired with Basil)

Furthermore, since Clarence cannot be married to Ruth, he must be married to Fanny or Veronica. Similarly, Edgar must be married to Fanny or Ruth. We are told that knowing who Edgar's sister is would enable us to solve the problem. The only way this fact could possibly help is if Edgar's sister was either Fanny or Ruth, since he would clearly not marry his sister. If Edgar's sister were Fanny (that is, if Edgar were married to Ruth), we still have a choice in matching Clarence and Desmond with Fanny and Veronica; in fact, either match is consistent with the other clues. However, we are told that knowing Edgar's sister would determine all the marriages. Therefore, Edgar's sister must be Ruth, and Edgar is married to Fanny. This implies that Clarence is married to Veronica and Desmond is married to Ruth.

In summary, Edgar's sister must be Ruth and the marriages are:

1. Basil & Helen
2. Arthur & Polly
3. Edgar & Fanny
4. Clarence & Veronica
5. Desmond & Ruth

61. The solution given below is due to John Ciriani. Extend AP to the point C where $EC = EA$ (see diagram below). This will yield $\angle EAN = \angle ECN$. But $\angle EAN = \angle EBP$, since they are both subtended at the circumference by the chord EP; whence, $\angle ECN = \angle EBP$.



Now we are given that $EA = EB$, and we have constructed $EC = EA$. Thus, we may conclude that $EC = EB$. This further implies that $\angle EBC = \angle ECB$; whence, $\angle PBC = \angle PCB$, yielding $PB = PC$.

Finally, $AN = NC$ (since $EC = EA$). Thus,

$$AN \;=\; NC \;=\; NP + PC \;=\; NP + PB\,,$$

and we are finished.

62. Let c be the amount of the cleaner's bill, d the amount of the dentist's bill, m the amount of the milk man's bill, and f the amount of the florist's bill. Then c and d are integers and $c = \frac{3}{2}d$. Let a be the dollars in the milk man's bill and b be the dollars in the florist's bill. Then

$$m \;=\; a + \frac{b}{100} \qquad \text{and} \qquad f \;=\; b + \frac{a}{100}\,.$$

Thus, $m + f = a + b + \frac{1}{100}(a + b)$, from which we see that $a + b = 25$. Since $m + f + c + d = 55.25$, we have $c + d = 30$; whence, $d = 12$ and $c = 18$. Now $m > d = 12$ and $f > d = 12$. Thus, $a \geq 12$ and $b = 25 - a \geq 12$. Consequently, either $a = 12$ and $b = 13$, or vice versa. But $f > m$ implies that $b > a$. Therefore,

$$d \;=\; \$12.00\,, \qquad m \;=\; \$12.13\,, \qquad f \;=\; \$13.12\,, \qquad c \;=\; \$18.00\,.$$

63. Let us say that a positive integer is *admissible* if it can be used as one of the sides of a right-angled triangle, all of whose sides are integers. We will show that all positive integers except 1 and 2 are admissible. To show that n is admissible, it is sufficient to show that n^2 can be written as the difference of two squares (by the Theorem of Pythagoras).

First let us consider an odd positive integer, say $2k + 1$. Note that

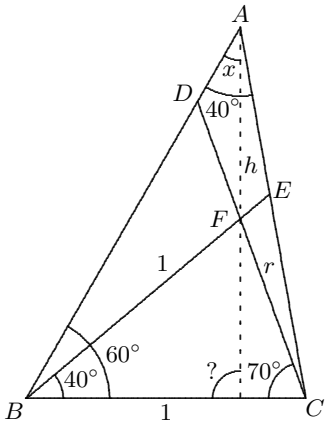
$$(k + 1)^2 \;=\; k^2 + (2k + 1)\,.$$

This shows that any odd number can be written as the difference of two squares. In order for the squares to be non-zero, we require k to be at least 1. Thus, any odd number larger than 1 is admissible.

Once n is admissible, any integer multiple of n is also admissible, since we could clearly scale the triangle which worked for n to one that works for the particular multiple of n. Thus, any integer other than a power of 2 is admissible. However, by considering a 3–4–5 triangle, we see that 4 is admissible and, thus, so are all larger powers of 2.

It only remains to show that 1 and 2 are not admissible. The only way that their squares could be written as the difference of two squares is when the smaller value is 0, which does not correspond to the side of a triangle. It only remains to show that neither 1 nor 2 could be placed on the hypotenuse. The small number of possibilities here can be readily exhausted by the reader.

64. Note that $\angle BFC$ is 70°; whence, $\triangle BFC$ is isosceles with $BF = BC$. Let us choose a unit of measurement so that $BF = BC = 1$. Let us also denote the lengths CF and AF by r and h, respectively, and let $\angle BAF = x$ (measured in degrees).



By applying the Law of Sines to $\triangle ABF$ and $\triangle ACF$, we obtain (respectively)

$$h = \frac{\sin 20^\circ}{\sin x} \qquad \text{and} \qquad h = \frac{r \sin 10^\circ}{\sin(40^\circ - x)}\,.$$

From these two results we may conclude that

$$\frac{\sin(40^\circ - x)}{\sin x} = \frac{r \sin 10^\circ}{\sin 20^\circ}\,. \tag{1}$$

By applying the Law of Sines to $\triangle BCF$, we can see that $r = \dfrac{\sin 40^\circ}{\sin 70^\circ}$, which together with (1) yields

$$\frac{\sin(40^\circ - x)}{\sin x} = \frac{(\sin 40^\circ)(\sin 10^\circ)}{(\sin 20^\circ)(\sin 70^\circ)} = \frac{2(\cos 20^\circ)(\sin 10^\circ)}{\sin 70^\circ}\,,$$

since $\sin 40^\circ = 2(\sin 20^\circ)(\cos 20^\circ)$. The right side further simplifies to

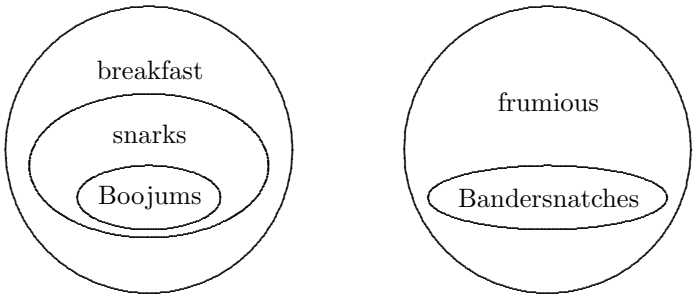
$$2\sin 10^\circ \;=\; \frac{\sin 10^\circ}{\sin 30^\circ}\,,$$

since $\sin 70^\circ = \cos 20^\circ$ and $\sin 30^\circ = \frac{1}{2}$. Thus, we have

$$\frac{\sin 10^\circ}{\sin 30^\circ} \;=\; \frac{\sin(40^\circ - x)}{\sin x} \;=\; \frac{\sin 40^\circ \cos x - \cos 40^\circ \sin x}{\sin x} \;=\; \sin 40^\circ \cot x - \cos 40^\circ\,.$$

Clearly, $x = 30^\circ$ is a solution. Since $\sin 40^\circ \cot x - \cos 40^\circ$ is a strictly decreasing function, we conclude that $x = 30^\circ$ is the only solution.

65. The easiest way to solve this problem is to employ what are called Venn Diagrams. Represent the different sets of objects by ellipses and use the above statements to determine properties such as set inclusion. Let the sets be described by the keywords: Boojums, snarks, Bandersnatches, frumious (for frumious animals), breakfast (for those who breakfast at five o'clock tea). Statement 1 states that Boojums is a subset of snarks. Statement 3 states that snarks is a subset of breakfast. Statement 2 states that Bandersnatches is a subset of frumious. Statement 4 states that the sets frumious and breakfast have no common elements.



Thus, no Bandersnatches are Boojums.

66. We first note that, among the four statements 1, 4, 7, and 10, at most one can be true, and consequently there are at least three statements which are false. From this observation we can conclude that statement 1 is false and statement 2 is true, which further implies that statement 10 is false and statement 9 is true. If statement 3 were true, then statements 4 and 7 would both be false, which would give four false statements, which would contradict statement 3; this absurd condition can only be avoided if statement 3 is false. On the other hand, if statement 8 is true, then we also conclude that it must be false; whence, statement 8 must be false.

Since we must still have either statement 4 or 7 false, we conclude that statement 5 is true and statement 4 is false. Now statement 7 becomes self-contradictory; whence, statement 7 is false and statement 6 is true.

The list of true statements is 2, 5, 6, and 9.

67. This is a rather tricky problem, as it appears at first glance that there is not enough information to solve it.

Let L be the length of the train, A the distance covered by the first man in 20 seconds, and B the distance covered by the second man in 18 seconds. Thus, the train travels a distance $L + A$ in 20 seconds and a distance $L - B$ in 18 seconds. Therefore, in 10 minutes (that is, 600 seconds) the train travels a distance D, where

$$\frac{(L+A)}{20} \;=\; \frac{D}{600} \;=\; \frac{(L-B)}{18}\,,$$

since each expression measures the speed of the train, which is constant. Equating these expressions in pairs, we find

$$D \;=\; 30(L+A) \quad \text{and} \quad 9(L+A) \;=\; 10(L-B)\,,$$

which can be rewritten as $L = 9A + 10B$.

In 10 minutes the first man travels $30A$. When we begin the 10 minute interval, the first man is aligned with the back of the train; that is, he is a distance L from the front of the train. Thus, the distance separating the two men at the moment the train reaches the second man is

$$D + L - 30A \;=\; 30(L+A) + L - 30A \;=\; 31L \;=\; 31(9A + 10B)\,.$$

Now if the first man takes 20 seconds to walk a distance A and the second man takes 18 seconds to walk a distance B, then they reduce the distance between themselves by $9A + 10B$ in 180 seconds. Therefore, it will take them

$$\frac{31(9A+10B)180}{9A+10B} \;=\; 5580$$

seconds to cover the distance between them. From this must be deducted the 18 seconds needed for the train to pass the second man. So the required time is 5562 seconds, or 1 hour 32 minutes and 42 seconds.

68. Without any loss of generality, we can assume that we have 1-1-1-2-3 on the five dice after one roll.

The first strategy says that we take one of the unmatched dice and reroll it. Say we select the die with 2; we then try to roll a 3 in the next two rolls (halting if we don't require both rolls). Our chance of success on the first roll is clearly $\frac{1}{6}$. If we fail on the first roll (which will happen $\frac{5}{6}$ of the time), we have another $\frac{1}{6}$ chance of rolling a 3 on the last roll. Thus, our chances of success on the second roll are $\frac{5}{6} \cdot \frac{1}{6}$. This gives a total probability of rolling a full house using the first strategy of $\frac{1}{6} + \frac{5}{36}$, which is $\frac{11}{36}$.

The second strategy requires that we take both singleton dice and reroll them. Success is indicated by a match of these two dice, but they should be different from 1 (the value of the other three dice). We thus have 5 chances out of 36 possibilities for a probability of $\frac{5}{36}$. We are also successful if we fail on the first throw (with a probability of $\frac{31}{36}$) and succeed on the next throw, which has

probability $\frac{31}{36} \cdot \frac{5}{36}$, or $\frac{155}{1296}$. Thus, the overall probability of success using the second strategy is $\frac{5}{36} + \frac{155}{1296}$, which is $\frac{335}{1296}$.

Clearly, the probability of the first strategy is the larger; hence, the first strategy is the preferred strategy under the given assumptions. That is, it is better to roll one die than two dice, and with this strategy the probability of success is $\frac{11}{36} = \frac{396}{1296}$.

69. This three-dimensional problem can be most easily solved by first examining a two-dimensional problem, namely what happens on one face of the large cube we start with.

Let us suppose that a cube can be decomposed into smaller cubes, no two of the same size. Then consider the bottom face of the large cube. The above decomposition will in fact subdivide that face into a number of smaller squares, no two of the same size. There must be a unique smallest square on that face coming from a cube A. The cubes on the bottom face which border A must all be larger, which implies that the space lying above the smallest square on the bottom face is surrounded by taller "walls" formed by the adjoining cubes. By a similar argument, there must be a unique smallest square lying on the upper face of the cube A, which comes from a cube B. This argument can be repeated ad infinitum. Since a decomposition implies a finite number of smaller cubes, we must answer NO to the question posed.

70. Since colour #1 must appear on one face we may assume that it will appear on the top face. For the bottom face we now have a choice of 5 possible colours. Let colour #2 be the colour we choose for the bottom face. Let colour #3 be any one of the remaining 4 colours. Clearly we can rotate the cube so that colour #3 will be on the front face. For the rear face we now have a choice of 3 possible colours, and after choosing that colour there are 2 ways to distribute the remaining 2 colours. This gives us a total of $5 \times 3 \times 2 = 30$ different ways to paint the cube.

71. The limerick at the top states implicitly that no owner named her pet after herself. By analyzing the first 6 lines of rhyming couplets, we can conclude:

1. Toni Taylor owns a hog named Jo.
2. Belle Bradkowski owns a frog named Sue.
3. Janet Jackson owns a crow.
4. Jo owns a garter snake.
5. Sue owns a pony (by process of elimination).

What about the pony's name? It cannot be Jo or Sue, since those names are already assigned to different pets. It cannot be Toni or Belle, since neither of their pets' names is Toni. Thus, the pony is named Janet. Therefore, Janet Jackson's pet's name is Toni. Then the last pet, namely Jo's garter snake, is named Belle. The final clue then allows us to identify Sue's mother as Jo. Her pet is the garter snake.

72. Let a be the fractional part and b be the integer part of such a number. If r is the common ratio of the geometric progression, then we have

$$ar \;=\; b \qquad \text{and} \qquad br \;=\; a+b\,.$$

From the first equation we see that $r = b/a$ and dividing the second equation by a yields

$$\begin{aligned} \frac{rb}{a} \;&=\; \frac{a+b}{a} \;=\; 1+\frac{b}{a} \\ \text{or} \qquad r^2 \;&=\; 1+r \end{aligned}$$

Solving for r using the quadratic formula, we get $r = \frac{1}{2}(1+\sqrt{5})$ since it is clear that $r > 0$.

To find the progression, we continue. Since $a < 1$, we see that $0 \le b < r < 2$. Since $b = 0$ would imply that a was also 0, we are forced to conclude that $b = 1$ and $a = 1/r$. Thus, the three numbers in the geometric progression are

$$\frac{\sqrt{5}-1}{2}\,, \qquad 1\,, \qquad \text{and} \qquad \frac{\sqrt{5}+1}{2}\,.$$

73. The key idea behind this problem is that the rubber band stretches 3 inches; it does not simply become three inches longer. Therefore, part of the 3–inch increase will be behind the bug.

Let x_n be the number of inches covered by the bug at the end of the n^{th} second just before the band is stretched, and let y_n be the fraction of the band covered. The band is $3n$ inches in length at this point. Then $x_1 = 1$ and

$$x_n \;=\; x_{n-1}\left(\frac{3n}{3(n-1)}\right)+1 \;=\; x_{n-1}\left(\frac{n}{n-1}\right)+1\,,$$

for $n \ge 2$. We also see that, for all $n \ge 1$,

$$y_n \;=\; \frac{x_n}{3n} \;=\; \frac{x_{n-1}}{3(n-1)}+\frac{1}{3n} \;=\; y_{n-1}+\frac{1}{3n}\,.$$

This implies that

$$y_n \;=\; \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\,.$$

We want to know the smallest value of n such that the above expression is greater than 1. By straightforward calculation we can determine n to be 11. That is, on the 11^{th} crawl the bug goes over the end of the rubber band.

One interesting observation that can be made is: no matter how long the band is, or how much it stretches, the bug will still be able to crawl to the end if it lives long enough and the band does not break. The reason for this is that the expression in parentheses above can be made as large as we wish simply by choosing the value of n large enough.

74. Let v, w, x, y, z be the number of 1–cent, 2–cent, 3–cent, 5–cent, and 10–cent stamps, respectively. Clearly, $v + w + x + y + z = 17$. Since she only had dimes, we can conclude that

$$v + 2w + 3x + 5y \;=\; 10k\,, \quad \text{for some integer } k. \tag{1}$$

It should also be clear that each of v, w, x, y, z are equal to either 3 or 4, and precisely 2 of them are equal to 4 (an even number). From (1) we can see that among v, x, y exactly one is even, which implies that w is 3 or 4, and z is the opposite value, which we can write as $z = 7 - w$. Thus, our first relationship above can now be rewritten as $v + x + y = 10$. Upon subtracting from (1), we obtain

$$\begin{aligned} 2w + 2x + 4y &= 10k - 10 \;=\; 10(k-1)\,, \\ \text{or} \qquad w + x + 2y &= 5(k-1)\,. \end{aligned}$$

From this we must conclude that $y = 4$; whence, by above arguments $v = x = 3$. Substituting in (1), we obtain $2w = 10k - 32$, from which we have $w = 4$ and $z = 3$.

Thus, she must buy 3 each of the 1–cent, 3–cent, and 10–cent stamps, and 4 each of the 2–cent and 5–cent stamps for a total of 70 cents.

75. Since MN and BC are parallel, we can first observe that $\angle MOB = \angle OBC$ and $\angle NOC = \angle OCB$. However, we are also told that $\angle MBO = \angle OBC$ and $\angle NCO = \angle OCB$. These statements imply that triangles MOB and NOC are both isosceles. Consequently, $MB = MO$ and $NC = NO$; whence, we see that the perimeter of triangle AMN is equal to $AB + AC$.

76. First denote the given sum by S. Now consider the function $f(x) = 1/\sqrt{x}$ on the interval from 1 to 10000. Let us further denote by A the area under the curve $y = f(x)$ on the interval from 1 to 10000. By breaking up this interval into subintervals from i to $i + 1$ (for i ranging from 1 to 9999) and by mounting on each of these subintervals a rectangle whose height is as small as possible but still large enough to cover all the area under $f(x)$ on that subinterval, we can observe that $A < S$.

By considering rectangles on each subinterval whose height is as large as possible, but still small enough to be completely covered by the area under $f(x)$ on that subinterval, we can observe that $S - 1 < A$. By recognizing that A can be calculated using integration on $f(x)$, we can compute the value of A to be 198. But then the two inequalities above imply that $198 < S < 199$; whence, the integer part of S must be 198.

77. We will assume that the only fair way to share the cost is for each person to share equally with the others for that part of the ride which was taken in common. Thus, if we let the distance that Carl traveled be a unit distance, then the distances traveled by Bob and Al are 2 and 4 units, respectively. Then Carl traveled 1/4 of the distance and shared that cost with both Al and Bob. Thus, Carl's share is 1/12 of the total, which is \$0.70. Now Bob must also pay \$0.70 for the leg that he shared with Carl and Al, but he must split the cost of one more fourth of the total distance with Al alone, which is a further 1/8 of the cost or \$1.05 for a total

share of \$1.75. Thus, Carl owes Al \$0.70, and Bob owes Al \$1.75. (This leaves poor Al holding the bag to pay the rest, or \$5.95.)

Most people who attempt this problem assume (erroneously) that each of the participants should pay as if he had the cab to himself for his entire trip.

78. Let x be the number of student members present, and let y be the cost per member in pennies. Then xy lies between 200 and 300. If we knew the value of the total purchase, we are told that we would be able to determine exactly the value of x. The only way this is possible is if $x = y$ and this value is a prime number. The only prime number whose square lies between 200 and 300 is 17. Thus, there were 17 members present (and they each spent 17 cents).

79. This problem is easier to analyze when one looks at it backwards rather than forwards. Consequently let us assume that the diamonds have already been distributed among all the heirs. Let us now have the heirs replace the diamonds they received (in reverse order).

Let n be the number of sons. After k of the sons and their wives have replaced their diamonds, we see that the number of diamonds replaced in total is given by

$$\frac{(9^k - 8^k)(n-8)}{8^{k-1}} + 8k\,.$$

This can be verified for small values of k, and can be formally proved by mathematical induction, which we leave as an exercise for the reader. This means the total number of diamonds originally is the value when $k = n$. That is, the total number of diamonds is

$$\frac{(9^n - 8^n)(n-8)}{8^{n-1}} + 8n\,.$$

It should be clear that since the number of diamonds is an integer, we must have $n - 8$ an integer multiple of 8^{n-1}. This is only possible for $n = 1$ and $n = 8$. But $n = 1$ is impossible, since Sinbad has at least 3 sons, namely Hameed, Ishmael, and Farouk. Thus, Sinbad has 8 sons and a total of 64 diamonds.

80. Suppose that the cube is situated with one corner at the origin of a 3–dimensional coordinate system such that the entire cube is in the first octant (that is, all three coordinates for each point on or in the cube are non-negative). If we let the radius of each of the small spheres be r, then the centre of the sphere situated closest to the origin has coordinates (r, r, r). The distance of this point from the origin is $d = r\sqrt{3}$. On the other hand, the large sphere has its centre located at $(15, 15, 15)$; whence, its distance from the origin is $D = 15\sqrt{3}$. But the distance separating the centres of the two spheres (which are touching) is the sum of the radii; that is, $r + 13$. Therefore, we have

$$\begin{aligned} D &= d + r + 13\,, \\ \text{or} \qquad 15\sqrt{3} &= r\sqrt{3} + r + 13\,, \end{aligned}$$

from which it follows that $r = 29 - 14\sqrt{3}$. Thus, the diameter is $58 - 28\sqrt{3}$.

ATOM

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