A Taste Of Mathematics

Aime–T–On les Mathématiques

Volume / Tome XVI

RECURRENCE RELATIONS

Iliya Bluskov

University of Northern British Columbia

The ATOM series

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The Author

Iliya Bluskov is a professor of mathematics at the University of Northern British Columbia, Canada. He was born in Bulgaria and received his B.Sc. in Mathematics from the University of Plovdiv, Bulgaria, M.Sc degree in Mathematics from the University of Victoria, Canada, and Ph.D. in Mathematics from Simon Fraser University, Canada. Dr. Bluskov has broad experience in mathematical competitions and training for such competitions. His first encounter with mathematical competitions was during his school years when he started participating in math competitions, ranging from all the rounds of the Bulgarian national mathematical Olympiad to regional and national competitions outside the Olympiad to participation in journal problem solving competitions and proposing problems himself. Mathematical Olympiads and mathematical competitions played an important role in his decision to pursue studies in mathematics. His involvement in mathematical competitions continued after he graduated with a degree in mathematics and started working as a teacher in a specialized science school; naturally, he taught mathematics there and participated in extracurricular activities in mathematics, which involved working with students on preparation for mathematical competitions, proposing problems for these competitions and participation in the marking of the students' papers. He has worked on organizing such competitions, accompanying teams for various stages of the national Olympiad and other national competitions. He has participated in various problem committees for local, regional and national competitions, and created some of the problems. His involvement with competitive mathematics continued after his coming to Canada for further studies in mathematics and work. At various stages of this part of his career he continued to be involved with the preparation of students for competitions (organizing preparation session for participants in the American High School Mathematics Examination (AHSME) and participating in preparation sessions for students from SFU and UNBC who were planning to write the Putnam Exam) and with selection of problems and marking papers for regional and national mathematical competitions, including the Canadian Mathematical Olympiad (CMO), and the British Columbia Secondary School Mathematics Contest (BCSSMC). In addition, he served on the Editorial Board of the problem solving journal Crux Mathematicorum with Mathematical Mayhem (now named Crux Mathematicorum), where he was one of the problem editors, going over dozens of proposed problems and writing up part of the solutions section of each issue of the journal for over ten years. Dr. Bluskov has also published many competition problems and solutions in various educational and competition oriented journals such as the aforementioned Crux Mathematicorum (Canada), The American Mathematical Monthly and The College Journal of Mathematics (USA), Mathematics in School (USSR), and The Education in Mathematics and Informatics, and Mathematics (Bulgaria).

FOREWORD

Over the years I have collected a huge number of nice problems and solutions on various topics in Mathematics and this publication represents a part of this collection. The publication is written partly as an introductory text, and partly as a book about solving problems. The problems are organized by some common idea, some common method of solving, and generally ordered in increasing level of difficulty. The book should be fully accessible to high school students, and parts of it to even younger students and can be used for preparation for mathematical competitions, but it can be useful in preparation for any future work in mathematics. It can be also used by teachers who work on preparation of students for competitions, and by instructors who teach any course that covers recurrence relations.

I will continue with a brief discussion about the philosophy of preparing for competitions; basically it is the philosophy of problem solving and can be applied in your school work, and later in your course work at the university level. Can you really prepare for mathematical competitions? After all, the problems there are difficult and tricky, and many students, even the best ones, fail to solve some of the problems. Nevertheless, preparation can help. Just like in any other activity, if you train, you get better. In a sense, it is no different than preparing for a test in school. The more you train, the better you become, up to a point. After that point, you need to be very smart... or very lucky, or both! The idea behind preparation for math competition, or the so called Eastern model, was quite simple: A question might look tricky, might have a tricky solution, some neat idea that remains hidden for most of the people. Well, once you see a couple of those, it cease to be a trick, it becomes a routine! Can a preparation win you a competition? I do not think so, you have to be smarter than most of the other participants, or way more prepared, or both, and that is, generally, not easy. Can you get good scores though? Yes you can! By being exposed to many types of problems, when you see these problems in a competition, you will not be deterred, you will not be thinking: Oh, I have never seen this, how do I start this problem, etc. For most of the problems, you will be able to jump right onto a reasonable path for solution, get to the details and most likely succeed. There are different thoughts on whether reading solutions written by other people helps as much as trying to solve the problems yourself. Well, having a problem with its solution allow you to do both: You can try to solve it for some time, and if you cannot, then look at the solution, or if you succeed, you can still look at the solution - it might be shorter than yours, it might be different, so reading might still make sense, might still bring some new knowledge. I see two major benefits from reading solutions. In a sense, it develops certain knowledge faster (one might argue that by trying to solve the problems yourself you will be able to acquire deeper knowledge, and this might be true, although it comes with the cost of spending somewhat more time). Another benefit from reading “book solutions” is the development of good mathematical writing style, good writing skills, in general. This naturally leads us to the important issue of how we actually read mathematics, in this case, the theory needed to solve certain type of problems and the solutions. Most people (including many of my students) make the mistake of only reading the solution and verifying it is correct. Well, this is a good thing, it is needed for understanding the solution, you have to be able to understand every line of it, every sentence of it; you have to be able to explain to yourself why this line follows from the previous one, or where it follows from: Would that be a known definition, theorem, or method, or, perhaps, a rule of logic? The best approach, however, will be to go a step further: For every step of the solution, ask and try to answer questions like: Why did they do that? What was the reason to take that step?

Can this step be performed differently? Can the entire problem be solved differently? Doing this on a regular basis will allow you to solve not only problems similar to the ones you have seen, but also problems which are quite different, and/or more difficult, because you will be accustomed to finding the answers by yourself, rather than being guided by a model applicable only in some restricted settings. I also always suggest reading with a pen and paper at your side; work out the details until every line of the solution is perfectly clear to you and you have all the explanations up to your level of understanding and knowledge.

Now, having said all that, no matter how prepared you are, you will encounter problems that require more than preparation, more than routine. What do you do with these problems? Well, pretty much what I and my colleagues mathematicians do when solving problems, and what I tell my students to do: Think about the condition. Ask yourself questions such as: What do I know about the objects in this question, what do I know about the mathematics needed to solve it? Recall pertinent theorems, definitions, formulas, known approaches, known tricks, things that we usually refer to as “the ingredients”. Apparently, given any problem, you need a finite number of those. Once you have the possible ingredients for a solution, you have to “mix” them together so the result constitutes a valid solution. You have to combine the ingredients; perhaps, using some good logic in the process, to produce the solution which will be read by the scrutinizing eye of the marker and will be expected to bring you the high score you need.

Teaching you a few tricks and giving you good practice is what the structure of this book is trying to achieve. In order to be good problem solvers, we need to have the basics, some solid background, and then we need to be exposed to problems of increasing levels of difficulty, up to the level of competition problems, and then to the level of the most difficult competitions, such as national and international Olympiads, and the Putnam exam. Now, the Putnam exam is a competition for undergraduate students, there are a couple of problems from it in this text. It is good to know that many problems on the Putnam exam are just difficult problems that often require nothing more than good logic, and some high school mathematics; and sometimes, even less than that. So, including some problems from it in a high school math competition preparation book seems reasonable to me. It is probably worth mentioning that during the long history of the Putnam exam, many high school students were given permission to write it, and some even managed to get to the list of winners, outperforming thousands of university students in the process!

The problems in this text are from a vast number of sources; ranging from problems proposed by me for various competitions or for my class work in relevant courses, problems that were proposed but not used for competitions, or problems from actual competitions, again, from a very broad range of sources - Olympiads, regional, national and international competitions, and also journal competition problems or proposals, again, ranging from quite obscure and unknown to some classic problems from texts or well-known competition problems.

Introduction

In this text we talk about the concept of a recurrence relation and learn how to solve various types of recurrence relations. We then proceed with some interesting competition problems involving recurrence relations. The more difficult problems for self-preparation are marked with an asterisks.

Recurrence relations have many applications. These include (but are not restricted to) counting problems, iterative algorithms for numerical approximation, computational complexity issues, and combinatorial generation.

A standard question about a sequence defined recursively is how to express the general term a_n as a function of n. For example, in computational complexity, quite often, the number of iterations of a particular algorithm can be determined as a sequence $\{a_n\}$, where n is the size of the problem. Knowing an explicit formula for a_n (as a function of n) allow us to compute the time necessary to perform the algorithm. This knowledge also allow us to compare different algorithms for solving the same problem.

We start with a simple example of a sequence defined recursively.

Example Let $\{a_k\}_{k=0}^{\infty}$ be a sequence defined by $a_0 = 3$, $a_{k+1} = 2a_k$ for $k \geq 0$. The first several terms of the sequence are $3, 6, 12, 24, ...$ Clearly, the sequence is a **geometric progression**, that is, a sequence in which every term is obtained from the preceding one by multiplying it by a fixed number r, called the **common ratio**. In our case, $r = 2$. The following approach allows us to express a_n as a function of n. The defining equality of the sequence, $a_{k+1} = 2a_k$, is true for every $k \geq 0$. Writing it out for $k = 0, 1, 2, ..., n$, we obtain

$$\begin{aligned} a_{n+1} &= 2a_n \\ a_n &= 2a_{n-1} \\ ... &= ... \\ a_2 &= 2a_1 \\ a_1 &= 2a_0 \end{aligned}$$

Multiplying all these equalities, we obtain

$$\begin{aligned} a_{n+1}(a_n...a_2a_1) &= 2^{n+1}(a_n...a_2a_1)a_0 \\ a_{n+1} &= 2^{n+1}a_0 \\ a_{n+1} &= 3 \cdot 2^{n+1}. \end{aligned}$$

Using the same approach, we can prove the following, more general result, giving explicitly the general term of a geometric progression: If the sequence $\{a_k\}_{k=0}^{\infty}$ is defined by $a_0 = a$, $a_{k+1} = ra_k$ for $k \geq 0$, then $a_n = ar^n$.

The next result gives an explicit formula for the sum of the first $n + 1$ terms of a geometric progression.

Example The sequence $\{S_n\}$ is defined by $S_0 = a$, $S_n = S_{n-1} + ar^n$. Show that

$$S_n = a\,\frac{r^{n+1} - 1}{r - 1}\,.$$

Solution Writing the recurrence for $k = 1, 2, \ldots, n$, then adding the results and canceling, we obtain

$$\begin{array}{rcl} S_n & = & S_{n-1} + ar^n \\ S_{n-1} & = & S_{n-2} + ar^{n-1} \\ ... & = & ... \\ S_2 & = & S_1 + ar^2 \\ S_1 & = & S_0 + ar^1, \end{array}$$

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} + ar^n,$$

which is the sum of the first $n+1$ terms of a geometric progression with first term a and common ratio r. Multiplying both sides by r, we get

$$rS_n = ar + ar^2 + \cdots + ar^n + ar^{n+1} = S_n - a + ar^{n+1}.$$

Thus $rS_n - S_n = a\left(r^{n+1} - 1\right)$, and then

$$S_n = a\,\frac{r^{n+1} - 1}{r - 1}\,,$$

as claimed.

Example It is known that the number of comparisons a_n needed to sort n numbers using the "bubble sort" algorithm is given by the following recurrence relation: $a_1 = 0$, $a_n = (n-1) + a_{n-1}$ for $n \geq 2$. Express a_n as a function of n.

Solution We use an approach similar to the one used in the preceding example. We have

$$\begin{array}{rcl} a_n & = & (n-1) + a_{n-1} \\ a_{n-1} & = & (n-2) + a_{n-2} \\ ... & = & ... \\ a_3 & = & a_2 + 2 \\ a_2 & = & a_1 + 1 \end{array}$$

Adding all these equalities, we obtain

$$a_n + (a_{n-1} + a_3 + a_2) = (a_{n-1} + a_3 + a_2) + \underbrace{a_1}_{=0} + 1 + 2 + ... + (n-1).$$

Hence

$$a_n = 1 + 2 + ... + (n-1).$$

The right-hand side represents the sum of an arithmetic progression, so that

$$a_n = \frac{1 + (n-1)}{2}(n-1) = \frac{n(n-1)}{2}.$$

Both of our examples so far,

$$a_{n+1} = 2a_n$$

and

$$a_n = (n-1) + a_{n-1}$$

represent **first-order linear recurrence relations**. The first one is **homogeneous**. The second is **non-homogeneous**.

The general **first-order linear recurrence relation with constant coefficients** has the form

$$a_{n+1} + ca_n = f(n), \quad n \geq 0,$$

where c is a constant and $f(n)$ is a function defined for every integer $n \geq 0$. If $f(n) = 0$ for every $n \geq 0$, then the relation is **homogeneous**; otherwise, it is **non-homogeneous**.

Example Find a_n as a function of n if $a_0 = 2$ and $a_n = (n+1)a_{n-1}$, $n \geq 0$.

Solution Note that this is not a recurrence relation with *constant* coefficients! However, we can solve it by using an approach similar to the one used in the preceding examples.

$$\begin{aligned}
a_n &= (n+1)a_{n-1} \\
a_{n-1} &= na_{n-2} \\
... &= ... \\
a_3 &= 4a_2 \\
a_2 &= 3a_1 \\
a_1 &= 2a_0
\end{aligned}$$

Multiplying all these equalities, we obtain

$$a_n(a_{n-1}a_{n-2}...a_2a_1) = (n+1)!(a_{n-1}a_{n-2}...a_2a_1)\underbrace{a_0}_{=2}.$$

Therefore, $a_n = 2(n+1)!$.

Linear Homogeneous Recurrence Relations of Order 2 with Constant Coefficients

We saw an example of a linear homogeneous recurrence of order 1, the geometric progression with first term a and common ratio r (defined by $a_0 = a$, $a_{k+1} = ra_k$ for $k \geq 0$); its solution was $a_k = ar^k$. If we just focus on the recurrence, it is easy to see one solution, namely $a_k = r^k$; then it is not much more difficult to notice that $a_k = cr^k$ is also a solution, for any constant c. This constant can now be found if we use the initial condition $a_0 = a$; we have $cr^0 = a$, and therefore, $c = a$, as expected. We can try to find a solution of the same type for a linear homogeneous recurrence of order 2, say, defined by

$$a_{k+2} = pa_{k+1} + qa_k \tag{1}$$

Suppose this recurrence has a solution of the type $a_k = r^k$. We can assume $r \neq 0$; the case $r = 0$ is trivial. Apparently, just any $r \neq 0$ would not work; we see that if r^k is a solution, the equality

$$r^{k+2} = pr^{k+1} + qr^k$$

must hold, and then upon dividing both sides by r^k, we see that r must satisfy the equation

$$r^2 = pr + q. \tag{2}$$

This equation is called the characteristic equation of the given recurrence (1). Suppose it has two real roots r_1 and r_2 and $r_1 \neq r_2$. Then both r_1^k and r_2^k are solutions to the given recurrence. The next two observations tell us how to find more solutions:

Observation 1: If $\{\alpha_k\}$ and $\{\beta_k\}$ are solutions of the recurrence (1), then $\{\alpha_k + \beta_k\}$ is also a solution.

Proof We have

$$\alpha_{k+2} = p\alpha_{k+1} + q\alpha_k$$
$$\beta_{k+2} = p\beta_{k+1} + q\beta_k$$

Adding these two equalities,

$$\alpha_{k+2} + \beta_{k+2} = p(\alpha_{k+1} + \beta_{k+1}) + q(\alpha_k + \beta_k),$$

that is, $\{\alpha_k + \beta_k\}$ is a solution.

Observation 2: If $\{\alpha_k\}$ is a solution of the recurrence (1), then $\{c\alpha_k\}$ is also a solution.

Proof Multiplying the equality $\alpha_{k+2} = p\alpha_{k+1} + q\alpha_k$ by c, we see that $\{c\alpha_k\}$ is a solution.

From these two observations it follows that

$$C_1 r_1^k + C_2 r_2^k \tag{3}$$

is also a solution of the recurrence (1). We now observe that if the first two terms of the sequence defined by (1) are given, then the sequence is completely determined. This suggests that the values of the constants C_1 and C_2 in the general solution (3) can be uniquely determined so that that the general solution (3) represents the so defined sequence. If say, a_0 and a_1 are the first two terms of the sequence, then we have a system for C_1 and C_2,

$$\begin{aligned} C_1 r_1^0 + C_2 r_2^0 &= a_0 \\ C_1 r_1^1 + C_2 r_2^1 &= a_1 \end{aligned}$$

It can be shown that this system always has a unique solution for C_1, C_2.

Unfortunately, this is not the case when the roots of the characteristic equation (2) are equal. For example, consider the sequence $a_0 = 0$, $a_1 = 1$, $a_{n+2} = 2a_{n+1} - a_n$, $n \geq 0$. The characteristic equation is $r^2 - 2r + 1 = 0$, or $(r-1)^2 = 0$, so that $r_1 = r_2 = 1$. If we continue as before, we will get $a_n = C_1(1^n) + C_2(1^n)$, that is, $a_n = C$ for some C. On the other hand, we can compute the first several terms, and see that the sequence is $0, 1, 2, 3, 4, \ldots$. Clearly, there is something wrong. Alternatively, the system for the constants C_1 and C_2,

$$\begin{aligned} C_1 1^0 + C_2 1^0 &= a_0 = 0 \\ C_1 1^1 + C_2 1^1 &= a_1 = 1, \end{aligned}$$

does not have a solution.

As another example, consider $a_0 = 0$, $a_1 = 1$, $a_{n+2} = 4a_{n+1} - 4a_n$, $n \geq 0$. The characteristic equation is $r^2 - 4r + 4 = 0$, or $(r-2)^2 = 0$, so that $r_1 = r_2 = 2$. Again, if we continue as before, we get $a_n = C_1(2^n) + C_2(2^n)$, and the system for the constants C_1 and C_2 is

$$\begin{aligned} C_1 2^0 + C_2 2^0 &= a_0 = 0 \\ C_1 2^1 + C_2 2^1 &= a_1 = 1, \end{aligned}$$

which does not have a solution, while the sequence is completely determined from the given information about it; the first terms being $0, 1, 4, 12, \ldots$.

The last two examples show that we need a different approach if the roots of the characteristic equation are equal, say, $r_1 = r_2 = s$. Fortunately, the recurrence can still be solved, but the general solution will have a different form. We can find this form by observing, just as before, that s^k is a solution to the given recurrence, but then, another solution is ks^k. Indeed, since s is the double root of the equation (2), we can write

$$r^2 - pr - q = (r-s)(r-s) = r^2 - 2sr + s^2,$$

so that $p = 2s$ and $q = -s^2$ (this also follows by Vietta's formulas). The recurrence (1) becomes

$$a_{k+2} - 2sa_{k+1} + s^2 a_k = 0.$$

It is easy to see that $a_k = s^k$ is a solution, and perhaps, just slightly more difficult that $a_k = ks^k$ is also a solution; it follows from

$$(k+2)s^{k+2} - 2s(k+1)s^{k+1} + s^2 ks^k = 0.$$

Now, using Observations 1 and 2, we conclude that

$$C_1 s^k + C_2 ks^k \tag{4}$$

is a solution in this case. Just like in the previous case, if a_0 and a_1 are the first two terms of the sequence, then C_1 and C_2 can be uniquely determined from the system

$$\begin{aligned} C_1 s^0 + C_2(0)s^0 &= a_0 \\ C_1 s^1 + C_2(1)s^1 &= a_1 \end{aligned}$$

Note that in determining C_1 and C_2 we could have used any two consecutive terms of the sequence instead of a_0 and a_1, no matter what the roots of the characteristic equation (2) are.

The next example shows another way of solving a linear recurrence of order two, by reducing it to something we know at this point, namely, how to solve a recurrence of order one, or to recognize a geometric progression. It also show how the geometric progressions r_1^n and r_2^n (where r_1 and r_2, $r_1 \neq r_2$, are the roots of the characteristic polynomial) show up as solutions, and why the general solution is a particular linear combination of these two. It also shows where the coefficients of this linear combination of r_1^n and r_2^n come from. In a sense, the next example shows a way to remove the guesswork in the previous solution.

Example Solve the recurrence relation $a_{n+2} - 5a_{n+1} + 6a_n = 0$, $n \geq 0$, $a_0 = -4$, $a_1 = -7$.

Solution Write the recurrence as

$$a_{n+2} - 2a_{n+1} - 3(a_{n+1} - 2a_n) = 0,$$

and let $b_n = a_{n+1} - 2a_n$. Clearly, $b_{n+1} = 3b_n$ with $b_0 = a_1 - 2a_0 = 1$, and this defines a geometric progression with first term 1 and common ratio 3. Hence $b_n = 3^n$, and we have

$$a_{n+1} - 2a_n = 3^n. \tag{5}$$

Similarly, write the given relation as

$$a_{n+2} - 3a_{n+1} - 2(a_{n+1} - 3a_n) = 0,$$

and let $c_n = a_{n+1} - 3a_n$. Then $c_{n+1} = 2c_n$ with $c_0 = a_1 - 3a_0 = 5$, and this is a geometric progression with first term 5 and common ratio 2. Hence $c_n = 5.2^n$, and we have

$$a_{n+1} - 3a_n = 5.2^n \tag{6}$$

Subtracting (6) from (5), we obtain

$$a_n = 3^n - 5.2^n.$$

We can write a more general recurrence in the way we wrote the recurrence in the preceding example in the forms (5) and (6), and solve it, as shown next.

Example Solve the recurrence relation $a_{n+2} + sa_{n+1} + ta_n = 0$, $n \geq 0$, if a_0 and a_1 are given and $s^2 - 4t > 0$.

Solution We can find a quadratic with roots p and q so that

$$-(p+q) = s \text{ and } pq = t;$$

that will be $r^2 + sr + t$. The recurrence can then be written as

$$a_{n+2} - (p+q)a_{n+1} + pqa_n = 0,$$

and then as

$$a_{n+2} - pa_{n+1} - q(a_{n+1} - pa_n) = 0,$$

and

$$a_{n+2} - qa_{n+1} - p(a_{n+1} - qa_n) = 0.$$

Now, letting $b_n = a_{n+1} - pa_n$, and $c_n = a_{n+1} - qa_n$, we find $b_n = q^n b_0$ and $c_n = p^n c_0$, so that

$$a_{n+1} - pa_n = q^n b_0$$

and

$$a_{n+1} - qa_n = p^n c_0.$$

Subtracting the former from the latter, we get

$$(p-q)a_n = c_0 p^n - b_0 q^n.$$

Note that the condition $s^2 - 4t > 0$ guarantees p and q are distinct real numbers. (This condition can be relaxed to $s^2 - 4t \neq 0$ to allow p and q complex and distinct.) Since $b_0 = a_1 - pa_0$ and $c_0 = a_1 - qa_0$, we obtain

$$a_n = \frac{1}{p-q}\left[(a_1 - qa_0)p^n - (a_1 - pa_0)q^n\right].$$

Linear Recurrence Relations of Order k with Constant Coefficients

Let k be a positive integer and C_i real, $i = n-k, n-k+1, ..., n$, where $C_n \neq 0$, $C_{n-k} \neq 0$ and $n \geq k$. Then

$$C_n a_n + C_{n-1} a_{n-1} + ... + C_{n-k} a_{n-k} = f(n)$$

is a **linear recurrence relation of order k with constant coefficients**. Note that the defining equality says that a_n can be expressed as a function of the preceding k terms of the sequence.

If $f(n) = 0$ for every $n \geq k$, then the relation is **homogeneous**; otherwise, it is **non-homogeneous**.

We now focus on **homogeneous** relations, that is, relations of the form

$$C_n a_n + C_{n-1} a_{n-1} + ... + C_{n-k} a_{n-k} = 0, \quad C_n \neq 0, \quad C_{n-k} \neq 0, \tag{1}$$

where C_i is real. This is a recurrence relation of order k. A sequence $\{a_n\}_{n=0}^{\infty}$ which satisfies (1) for all $n \geq k$ is called a **solution** of the recurrence relation. It can be shown that assigning arbitrary values to k consecutive terms determines a unique solution, and that, for any such solution, a_n can be expressed explicitly as a function of n.

The **characteristic polynomial** of the recurrence (1) is

$$P(r) = C_n r^k + C_{n-1} r^{k-1} + ... + C_{n-k+1} r + C_{n-k}.$$

The degree of $P(r)$ is k. Consider the case in which all the roots of $P(r) = 0$ are real, that is, there are roots $r_1, r_2, ..., r_l$ with respective multiplicities $s_1, s_2, \ldots, s_l$, such that $s_1 + s_2 + ... + s_l = k$. Then the general term a_n of the sequence is given by

$$\begin{aligned} a_n \quad = \quad & A_1 r_1^n + A_2 n r_1^n + ... + A_{s_1} n^{s_1 - 1} r_1^n \\ & + B_1 r_2^n + B_2 n r_2^n + ... + B_{s_2} n^{s_2 - 1} r_2^n \\ & ... \\ & + M_1 r_l^n + M_2 n r_l^n + ... + M_{s_l} n^{s_l - 1} r_l^n. \end{aligned}$$

Example If the characteristics polynomial of a recurrence relation factors as

$$P(r) = (2r-1)^2 (r-4)^3,$$

then

$$r = 1/2 \text{ is a root of multiplicity } 2$$

and

$$r = 4 \text{ is a root of multiplicity } 3.$$

Hence the general term has the form

$$a_n = A_1 \left(\frac{1}{2}\right)^n + A_2 n \left(\frac{1}{2}\right)^n + B_1 4^n + B_2 n 4^n + B_3 n^2 4^n. \tag{2}$$

The coefficients $A_1, A_2, ..., A_{s_1}, B_1, B_2, ..., B_{s_2}, ..., M_1, M_2, ..., M_{s_l}$, can be determined once sufficient information about $\{a_n\}_{n=0}^{\infty}$ is given. For example, if a_0, a_1, a_2, a_3, a_4 are known, then the coefficients can be found from the equation (2) for $n = 0, 1, 2, 3, 4$.

Example Solve the recurrence relation $a_{n+2} - 5a_{n+1} + 6a_n = 0$, $n \geq 0$, $a_0 = 2$, $a_1 = 5$.

Solution The characteristic equation is

$$r^2 - 5r + 6 = 0$$

with roots 2 and 3, so that

$$a_n = C_1 2^n + C_2 3^n.$$

Writing out the general form for $n = 0$ and $n = 1$ and using the conditions $a_0 = 2$ and $a_1 = 5$, we obtain the system

$$\begin{aligned} 2 = a_0 &= C_1 + C_2 \\ 5 = a_1 &= 2C_1 + 3C_2 \end{aligned}$$

Solving the system gives $C_1 = C_2 = 1$. Therefore,

$$a_n = 2^n + 3^n.$$

Example Find a_n as a function of n, if $a_0 = a_2 = 0$, $a_1 = 9$ and

$$a_{n+3} - 3a_{n+1} - 2a_n = 0.$$

Solution The characteristic equation is

$$\begin{aligned} &r^3 - 3r - 2 = 0 \\ &r^3 - r - 2r - 2 = 0 \\ &r(r-1)(r+1) - 2(r+1) \\ &(r+1)(r^2 - r - 2) = 0 \\ &(r+1)^2(r-2) = 0 \end{aligned}$$

Thus $r = -1$ is a root of multiplicity 2 and $r = 2$ is a root of multiplicity 1. Hence, the general solution must have the form

$$a_n = A_1(-1)^n + A_2 n(-1)^n + B2^n.$$

Using the initial conditions $a_0 = a_2 = 0$ and $a_1 = 9$, we obtain

$$\begin{aligned} 0 = a_0 &= A_1(-1)^0 + A_2 0(-1)^0 + B2^0 \\ 9 = a_1 &= A_1(-1)^1 + A_2 1(-1)^1 + B2^1 \\ 0 = a_2 &= A_1(-1)^2 + A_2 2(-1)^2 + B2^2 \end{aligned}$$

or,

$$\begin{aligned}0 &= A_1 + B\\ 9 &= -A_1 - A_2 + 2B\\ 0 &= A_1 + 2A_2 + 4B\end{aligned}$$

Solving this system, we get $A_1 = -2$, $A_2 = -3$ and $B = 2$. Therefore, the solution to the recurrence relation is

$$a_n = (-2)(-1)^n + (-3)n(-1)^n + 2 \cdot 2^n,$$

or,

$$a_n = (3n + 2)(-1)^{n+1} + 2^{n+1}.$$

Transforming a Non-homogeneous Recurrence Relation into a Homogeneous Relation of Higher Order

Example Solve the recurrence relation defined by $a_{n+1} - 2a_n = 1$ for $n \geq 1$, and $a_1 = 1$.

Solution The defining equality is true for every $n \geq 1$. Writing it out for $n+1$ and n, we obtain

$$a_{n+2} - 2a_{n+1} = 1 = a_{n+1} - 2a_n,$$

from which we get the recurrence

$$a_{n+2} - 3a_{n+1} + 2a_n = 0.$$

The characteristic equation is

$$r^2 - 3r + 2 = 0$$

with roots $r = 1$ and $r = 2$. Hence the general form of the n-th term of the sequence is

$$a_n = A1^n + B2^n.$$

We need (at least) two equations in order to determine A and B. We know that $a_1 = 1$, so by applying the defining equation, we obtain $a_2 - 2a_1 = 1$, so that $a_2 - 2a(1) = 1$, whence $a_2 = 3$. Writing out the general form for $n = 1$ and $n = 2$ and using the initial conditions $a_1 = 1$ and $a_2 = 3$, we obtain the system

$$\begin{aligned} 1 &= a_1 = A + 2B \\ 3 &= a_2 = A + 4B \end{aligned}$$

Solving this system gives $A = -1$ and $B = 1$. Therefore, the general term of the sequence is

$$a_n = (-1)1^n + (1)2^n = 2^n - 1.$$

Example Solve the recurrence relation $a_{n+1} - 2a_n = n$, $n \geq 1$, $a_1 = 1$.

Solution The defining equality is true for every $n \geq 1$. Writing it out for $n+1$ and n, we obtain

$$\begin{aligned} a_{n+2} - 2a_{n+1} &= n + 1 \\ a_{n+1} - 2a_n &= n \end{aligned}$$

Subtracting the second equality from the first one, we obtain

$$a_{n+2} - 3a_{n+1} + 2a_n = 1.$$

We can now use the approach from the preceding example. Writing out the new recurrence relation for $n+1$ and n, we obtain

$$a_{n+3} - 3a_{n+2} + 2a_{n+1} = 1 = a_{n+2} - 3a_{n+1} + 2a_n,$$

which yields the homogeneous recurrence relation

$$a_{n+3} - 4a_{n+2} + 5a_{n+1} - 2a_n = 0.$$

The characteristic equation is

$$r^3 - 4r^2 + 5r - 2 = 0$$

or

$$(r-1)^2(r-2) = 0,$$

with roots $r = 1$ of multiplicity 2 and $r = 2$ of multiplicity 1. Hence the general form of the n-th term of the sequence is

$$a_n = A_1 1^n + A_1 n 1^n + B 2^n.$$

We need three equations in order to determine A_1, A_2 and B. We know that $a_1 = 1$, so by applying the defining equation, we obtain $a_2 = 3$ and $a_3 = 8$. Writing out the general form for $n = 1, 2, 3$, we obtain the system

$$\begin{aligned} 1 = a_1 &= A_1 + A_2 + 2B \\ 3 = a_2 &= A_1 + 2A_2 + 4B \\ 8 = a_3 &= A_1 + 3A_2 + 8B \end{aligned}$$

Solving this system gives $A_1 = A_2 = -1$ and $B = 3/2$. Therefore, the general term of the sequence is

$$a_n = (-1)1^n + (-1)n1^n + \frac{3}{2}2^n = (3)2^{n-1} - n - 1.$$

Example Solve the recurrence relation $x_{n+2} - 2x_{n+1} + x_n = 2^n$, $n \geq 1$, $x_1 = 2$, $x_2 = 0$.

Solution Using the relation for $n+1$ and n, we obtain

$$x_{n+3} - 2x_{n+2} + x_{n+1} = 2^{n+1} = 2 \cdot 2^n = 2(x_{n+2} - 2x_{n+1} + x_n).$$

This gives the homogeneous recurrence relation

$$x_{n+3} - 4x_{n+2} + 5x_{n+1} - 2x_n = 0,$$

which happens to be the same as the one from the preceding example. Hence the general form of the n-th term of the sequence is

$$x_n = A_1 1^n + A_1 n 1^n + B 2^n.$$

We need three equations in order to determine A_1, A_2 and B. We know that $x_1 = 2$ and $x_2 = 0$, so by applying the defining equation, we obtain $x_3 = 0$. Writing out the general form for $n = 1, 2, 3$, we obtain the system

$$\begin{aligned} 2 = x_1 &= A_1 + A_2 + 2B \\ 0 = x_2 &= A_1 + 2A_2 + 4B \\ 0 = x_3 &= A_1 + 3A_2 + 8B \end{aligned}$$

Solving this system gives $A_1 = 4$, $A_2 = -4$ and $B = 1$. Therefore, the general term of the sequence is

$$x_n = (4)1^n + (-4)n1^n + (1)2^n = 2^n - 4n + 4.$$

The next recurrence frequently occurs in applications.

Example The **Fibonacci sequence** is the sequence defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Express F_n as a function of n.

Solution The characteristic equation is

$$r^2 - r - 1 = 0$$

with roots $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$, so that

$$F_n = C_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Writing out the general form for $n = 0$ and $n = 1$ and using the conditions $F_1 = 1$ and $F_2 = 1$, we obtain the system

$$\begin{aligned} 1 = F_1 &= C_1 \tfrac{1+\sqrt{5}}{2} + C_2 \tfrac{1-\sqrt{5}}{2} \\ 1 = F_2 &= C_1 \left(\tfrac{1+\sqrt{5}}{2}\right)^2 + C_2 \left(\tfrac{1-\sqrt{5}}{2}\right)^2 \end{aligned}$$

Solving this system gives $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$. Therefore,

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right].$$

We now proceed with several application of recurrence relations in counting problems. Our goal will be to find a recurrence relation that gives the solution to a counting problem.

Example Let a_n be the number of binary n-digit numbers with no consecutive 1's. Find a recurrence relation for the numbers a_n.

Solution Let a_n be the number of binary n-digit numbers with no consecutive 1's. We start by computing some values of a_n for small n. There might be some benefits of such an exercise: First, we might need the first several terms of the sequence, and second, we might get some insights on how to solve the problem. If $n = 1$, we have two such numbers, 0 and 1, so that, $a_1 = 2$. If $n = 2$, we have three such numbers, 00, 01 and 10, so that, $a_2 = 3$. If $n = 3$, we have five such numbers, 000, 001, 010, 100 and 101, so that, $a_3 = 5$. In the process, you might nave noticed that prefixing any valid 2-digit number with a zero produces a valid number with three digits. Also, prefixing any 1-digit valid number with 10 produces a valid number with three digits. Moreover, these two operations produce all of the valid 3-digit numbers. This observation works in general.

Let A be the set of all binary n-digit numbers with no consecutive 1's and let us assume that $n \geq 3$. Let B be the subset of those starting with 0 and C the subset of those starting with 1. Then $a_n = |A| = |B| + |C|$. (Here $|X|$ denotes the number of elements of the set X.)

If $b \in B$, then b starts with 0 and the remaining $n-1$ digits of b can form any of the a_{n-1} binary $(n-1)$-digit numbers with no consecutive 1's. Hence

$$|B| = a_{n-1}.$$

If $c \in C$, then c starts with 1, so that, the next digit can only be 0, but then the remaining $n-2$ digits can form any of the a_{n-2} binary $(n-2)$-digit numbers with no consecutive 1's. Hence

$$|C| = a_{n-2}.$$

Thus,

$$a_n = |A| = |B| + |C| = a_{n-1} + a_{n-2}$$

with $a_1 = 2$ and $a_2 = 3$. (Clearly, $a_n = F_{n+2}$, where F_n is the n-th Fibonacci number.)

Example Find a recurrence relation for the number of ways to fill a row of n motorcycle parking spaces with cars and motorcycles if each motorcycle requires one space and each car requires two spaces.

Solution Let a_n be the required number. We start by writing out the values of a_n for several small values of n with the hope that we can observe how these numbers relate to each other. We represent the possible arrangements by words consisting of M's and C's, keeping in mind that M and C require one and two spaces, correspondingly.

$n=1$	$a_1=1$	M		
$n=2$	$a_2=2$	MM	C	
$n=3$	$a_3=3$	MMM	MC	CM
$n=4$	$a_4=5$	M|MMM	M|MC	M|CM
		C|MM	C|C	

We can now observe (look at $n = 2, 3, 4$, for example) the following. If the first vehicle in the row is a motorcycle, then there are a_{n-1} ways to place the remaining vehicles. If the first vehicle is a car, then there are a_{n-2} ways to place the remaining vehicles. This gives the recurrence

$$a_n = a_{n-1} + a_{n-2}$$

for $n \geq 3$, which, together with the initial values $a_1 = 1$ and $a_2 = 2$, completely determines the number a_n of ways to park vehicles as required. (This is another variation of the Fibonacci sequence: $a_n = F_{n+1}$, where F_n is the n-th Fibonacci number.)

Example Let a_n be the number of ternary n-digit numbers (a ternary number is a number composed of the digits $0, 1$ and 2 only) with no 0 and 1 adjacent. Find a recurrence relation for the numbers a_n.

Solution First we note that $a_1 = 3$ (the 1-digit numbers that meet the condition are 0, 1 and 2) and $a_2 = 7$, because the 2-digit numbers that meet the condition are

20	00	11
21	02	12
22		

Let $t = d_1 d_2 ... d_n$ be a ternary n-digit number that meets the condition (here $d_1, d_2, ..., d_n$ are the digits of t). If $d_1 = 2$, then $d_2, ..., d_n$ could form any of the a_{n-1} ternary $(n-1)$-digit numbers with no 0 and 1 adjacent. Note that, if $d_1 = 0$ or $d_1 = 1$, then the argument from the preceding example does not work: Suppose $d_1 = 0$. Then d_2 can only be 2 or 0. If $d_2 = 2$, then the remaining $n-2$ digits can form any of the a_{n-2} valid $(n-2)$-digit numbers. However, if $d_2 = 0$, we encounter new branching: d_3 can either be 2 or 0 and the procedure does not close.

So, let us do something else. Let A_i be the set of ternary n-digit numbers with no adjacent 0 and 1 so that $d_1 = i$, $i = 0, 1, 2$. If A is the set of all ternary n-digit numbers with no adjacent 0 and 1, then

$$|A| = |A_0| + |A_1| + |A_2|.$$

Note also that $|A_0| = |A_1|$, because of the symmetry. Let $|A_0| = |A_1| = b_n$. Then $b_1 = 1$ and $b_2 = 2$. We can find a system of recurrence relations for the sequences $\{a_n\}$ and $\{b_n\}$. We have already established that $|A_2| = a_{n-1}$; also, $|A| = a_n$, so that

$$a_n = |A| = |A_0| + |A_1| + |A_2| = 2b_n + a_{n-1}.$$

On the other hand, if, say, $d_1 = 0$, then $d_2 = 0$ or $d_2 = 2$. There are b_n n-digit numbers starting with 0, b_{n-1} n-digit numbers starting with 00 and a_{n-2} n-digit numbers starting with 02, so that

$$b_n = b_{n-1} + a_{n-2}.$$

Now we have to solve the system

$$\begin{cases} a_n = 2b_n + a_{n-1} \\ b_n = b_{n-1} + a_{n-2} \end{cases}$$

From the first equation, $2b_n = a_n - a_{n-1}$, so that $2b_{n-1} = a_{n-1} - a_{n-2}$. Substitute these in $2b_n = 2b_{n-1} + 2a_{n-2}$ (which is just the second equation multiplied by 2) to obtain

$$a_n - a_{n-1} = (a_{n-1} - a_{n-2}) + 2a_{n-2},$$

which simplifies to

$$a_n - 2a_{n-1} - a_{n-2} = 0.$$

This recurrence, together with the conditions $a_1 = 3$ and $a_2 = 7$, completely determines the answer a_n to our counting problem.

Exercises

(Solutions and answers are given at the end of this text, starting on page 52)

1. Choose the correct answer: The recurrence relation $5a_{n+4} + 4a_{n+2} + 3a_n = 3$ is:

   (1) homogeneous of order 4

   (2) non-homogeneous of order 3

   (3) linear of order 5

   (4) non-homogeneous of order 4

   (5) linear of order 3

   (6) non-homogeneous of order 2

2. Solve the recurrence relation $2a_{n+2} - 5a_{n+1} + 2a_n = 0$, $n \geq 0$, $a_0 = 2$, $a_1 = \frac{5}{2}$.

3. Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$, $n \geq 1$, $a_1 = 16$, $a_2 = 52$.

4. Solve the recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$, $a_0 = -2$, $a_1 = 6$.

5. Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$, $a_0 = 7$, $a_1 = 4$.

6. Solve the recurrence relation $a_n + 6a_{n-1} - 7a_{n-2} = 0$ for $n \geq 2$, $a_0 = 1$, $a_1 = 2$.

7. Solve the recurrence relation $a_{n+3} - 4a_{n+2} + 5a_{n+1} - 2a_n = 0$ for $n \geq 0$, $a_0 = 4$, $a_1 = 7$, $a_2 = 17$.

8. Solve the recurrence relation $a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 0$ for $n \geq 3$, $a_0 = 0$, $a_1 = 2$, $a_2 = 4$.

9. Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2$, $n \geq 0$, $a_0 = 3$, $a_1 = 6$.

10. Solve the recurrence relation $a_n - 3a_{n-1} = 5^{n-1}$, $n \geq 1$, $a_0 = 1$.

11. Solve the recurrence relation $a_{n+1} = 5a_n + 16$, $n \geq 1$, $a_1 = 1$.

12. $^{(*)}$ Solve the recurrence $d_n = (n-1)(d_{n-1} + d_{n-2})$, $n \geq 2$, $d_0 = 1$, $d_1 = 0$.

13. Solve the following system of recurrence relations

$$\begin{aligned} a_{n+1} &= -2a_n - 4b_n \\ b_{n+1} &= 4a_n + 6b_n \end{aligned}$$

   where $n \geq 0$, $a_0 = 1$ and $b_0 = 0$.

14. Solve the system of recurrence relations

$$\begin{aligned} a_{n+1} &= a_n + 3b_n \\ b_{n+1} &= a_n + b_n \end{aligned}$$

where $n \geq 1$, $a_1 = 1$ and $b_1 = 1$.

15. Solve the system of recurrence relations

$$\begin{aligned} a_{n+1} &= 2a_n + 2b_n \\ b_{n+1} &= a_n + 2b_n \end{aligned}$$

where $n \geq 1$, $a_1 = 2$ and $b_1 = 1$.

16. Let a_n be the number of strings $b_1 b_2 ... b_k$, where $b_i \in \{1, 2\}$ and

$$\sum_{i=1}^{k} b_i = n.$$

(For example, $a_4 = 5$, because there are 5 strings that meet the condition: 1111, 112, 121, 211 and 22.) Show that $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, $a_1 = 1$, $a_2 = 2$.

17. Let S_n denote the number of n-digit binary numbers that do not contain three consecutive 1's. Develop a recurrence relation and initial conditions that uniquely define the sequence $\{S_n\}_{n=1}^{\infty}$. (Do not solve the recurrence.)

18. Let a_n be the number of n-digit ternary numbers that have no double 0. (A ternary number is one formed by the digits $0, 1$ and 2 only.) Find a recurrence relation and initial conditions that uniquely define the sequence $\{a_n\}_{n=1}^{\infty}$. (Do not solve the recurrence.)

19. $^{(*)}$ Let a_n denote the number of n-digit ternary numbers that have no 1 immediately to the right of any 0. Develop a recurrence relation and initial conditions that uniquely define the sequence $\{a_n\}_{n=1}^{\infty}$. (Do not solve the recurrence.)

20. $^{(*)}$ Find a recurrence plus initial conditions for the number a_n of ternary n-digit numbers with neither 0's nor 1's adjacent.

21. $^{(*)}$ For $n \geq 1$, let a_n be the number of ways to write n as an ordered sum of odd positive integers. (For example, $a_4 = 3$, because $4 = 3 + 1 = 1 + 3 = 1 + 1 + 1 + 1$, and there are no other ways to represent 4 as a sum of odd positive integers.) Find a recurrence relation for a_n.

22. $^{(*)}$ Let x_n be the number of words of length n formed by the letters a, b, c with an even number of a's. Express x_n as a function of n.

We now continue with less standard problems, grouped by some common idea. This common idea is stated in the title of each section.

Reverse Engineering

In these problems we do not explicitly have a recurrence relation. What we see might look like the solution of a recurrence relation, so we "reverse engineer" to the original recurrence and use it to argue toward a proof of the problem statement.

1. Show that $b_n = 19.8^n + 17$ is a composite number for any positive integer n.

   Solution We can write $b_n = 19.8^n + 17.1^n$. Clearly, $\{b_n\}$ is a sequence, which, if defined recursively, would have a characteristic equation with roots 1 and 8, that is,
   $$(q-1)(q-8) = q^2 - 9q + 8 = 0.$$
   Therefore, its recurrence relation is
   $$b_{n+2} = 9b_{n+1} - 8b_n.$$
   Now, $b_0 = 36$ is divisible by 3, so that $b_2 = 9b_1 - 8b_0$ is also divisible by 3. By an easy induction, b_{2k} is divisible by 3 for $k = 0, 1, 2, \ldots$, and thus b_{2k} is a composite number.

   Now, consider the sequence $c_k = b_{2k+1} = 19.8^{2k+1} + 17.1^{2k+1}$. It can be written as
   $$c_k = b_{2k+1} = 19(8)64^k + 17.1^k.$$
   The characteristic equation is $(q-1)(q-64) = 0$, or $q^2 = 65q - 64$. Hence a recurrence relation for the sequence $\{c_k\}$ is
   $$c_{k+2} = 65c_{k+1} - 64c_k.$$
   Since $c_0 = b_1 = 169$ and 65 are divisible by 13, then c_2 is divisible by 13. By induction, $c_{2k} = b_{4k+1}$ is divisible by 13, and so b_{4k+1} is composite. Also, $c_1 = b_3 = 9745$ and 65 are divisible by 5, so $c_3 = b_7$ is divisible by 5. By induction, $c_{2k+1} = b_{4k+3}$ is divisible by 5. Hence b_{4k+3} is composite. Since
   $$\{b_{2k}\}_1^\infty \cup \{b_{4k+1}\}_1^\infty \cup \{b_{4k+3}\}_1^\infty = \{b_n\}_1^\infty,$$
   the proof is complete.

2. Let n be a positive integer. Show that the fractional part of the decimal expansion of $(5+\sqrt{26})^n$ starts with n identical digits.

   Solution Let $a_n = (5+\sqrt{26})^n$. Consider also $b_n = (5-\sqrt{26})^n$ and note that $x_n = a_n + b_n$ is an integer which is very close to a_n, because $|5-\sqrt{26}| < 1$ and $(5-\sqrt{26})^n \to 0$ as $n \to \infty$. The fact that x_n is an integer follows from the Binomial Formula, or from the following argument: Note that the characteristic equation of
   $$x_n = (5+\sqrt{26})^n + (5-\sqrt{26})^n$$

has roots $5+\sqrt{26}$ and $5-\sqrt{26}$, that is, it has the form

$$[q-(5+\sqrt{26})][q-(5-\sqrt{26})]=0,$$

or

$$q^2-10q-1=0,$$

which means that a recurrence relation for $\{x_n\}$ is

$$x_{n+2}=10x_{n+1}+x_n.$$

Since $x_0=2$ and $x_1=10$, then, inductively, x_n is an integer for all $n\geq 0$. Note also that

$$b_n=(5-\sqrt{26})^n=\left(\frac{-1}{5+\sqrt{26}}\right)^n.$$

Notice that $a_n-x_n=-b_n$ is positive if n is odd and negative if n is even, and its absolute value is less than $\left[1/(5+\sqrt{25})\right]^n=10^{-n}$. Since as noted above x_n is a positive integer, the fractional part of a_n begins with n 0s if n is odd, and with n 9s if n is even.

Different Representations

In this selection of problems finding a new recurrence relation which defines the given sequence helps us prove the desired statement.

1. The sequence $\{a_n\}$ is defined by $a_1 = a_2 = 1$ and

$$a_{n+1} = \frac{a_n^2 + 2}{a_{n-1}}.$$

Show that a_n is an integer for every $n \geq 2$.

Solution We start with a short experiment/observation. Writing the first several terms of the sequence let us see that they differ approximately by a factor of 4; more precisely:

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= 1 \\
a_3 &= 3 = 4(1) - 1 = 4a_2 - a_1 \\
a_4 &= 11 = 4(3) - 1 = 4a_3 - a_2 \\
a_5 &= 41 = 4(11) - 3 = 4a_4 - a_3 \\
a_6 &= 153 = 4(41) - 11 = 4a_5 - a_4 \\
&\dots
\end{aligned}$$

Based on this observation, we now have a hypothesis: The sequence $\{a_n\}$ can also be defined by $a_1 = a_2 = 1$ and

$$a_{n+1} = 4a_n - a_{n-1} \quad \text{for } n \geq 2.$$

Indeed, if $\{a_n\}$ is the newly defined sequence, then

$$\begin{aligned}
a_{n+1}a_{n-1} - a_n^2 &= (4a_n - a_{n-1})a_{n-1} - (4a_{n-1} - a_{n-2})a_n \\
&= a_n a_{n-2} - a_{n-1}^2.
\end{aligned}$$

But then, inductively,

$$a_{n+1}a_{n-1} - a_n^2 = a_n a_{n-2} - a_{n-1}^2 = \cdots = a_3 a_1 - a_2^2 = 2.$$

Thus $a_{n+1}a_{n-1} - a_n^2 = 2$, or

$$a_{n+1} = \frac{a_n^2 + 2}{a_{n-1}},$$

which shows that the newly defined sequence (which is linear of order two) is the same as the one given in the condition. Now the claim follows immediately: Since all of the terms of the newly defined sequence are integers, then so are the terms of the sequence defined in our problem.

2. Let $\{a_n\}$ be the sequence defined by $a_0 = 0$ and
$$a_{n+1} = 5a_n + \sqrt{24a_n^2 + 1}. \tag{3}$$
Show that a_n is an integer for every $n \geq 0$.

Solution We have
$$(a_{n+1} - 5a_n)^2 = 24a_n^2 + 1$$
or
$$a_n^2 - 10a_{n+1}a_n + (a_{n+1}^2 - 1) = 0.$$
Solving this quadratic for a_n, we get
$$a_n = 5a_{n+1} \pm \sqrt{24a_{n+1}^2 + 1}. \tag{4}$$
From (3) and (4) we have
$$a_{n+2} - 5a_{n+1} = \sqrt{24a_{n+1}^2 + 1} = \pm\left(5a_{n+1} - a_n\right).$$
By an easy induction it follows that all terms are integers.

3. The sequence $\{a_n\}$ is defined by $3a_k = a_{k-1} + a_{k+1}$. Show that
$$5a_n^2 + 4(a_0^2 + a_1^2 - 3a_0a_1)$$
is a square of an integer.

Solution We have $a_{k-1} = 3a_k - a_{k+1}$ and $a_{k+1} = 3a_k - a_{k-1}$. Then
$$\begin{aligned}
a_0^2 + a_1^2 - 3a_0a_1 &= a_1^2 + a_0(a_0 - 3a_1) \\
&= a_1^2 + (3a_1 - a_2)(-a_2) \\
&= a_1^2 + a_2^2 - 3a_1a_2 \\
&= a_2^2 + a_1(a_1 - 3a_2) \\
&= a_2^2 + (3a_2 - a_3)(-a_3) \\
&= a_2^2 + a_3^2 - 3a_2a_3 \\
&= \ldots \quad \text{(inductively)} \\
&= a_{n-1}^2 + a_n^2 - 3a_{n-1}a_n.
\end{aligned}$$
Thus $a_0^2 + a_1^2 - 3a_0a_1 = a_{n-1}^2 + a_n^2 - 3a_{n-1}a_n$ and then
$$\begin{aligned}
&5a_n^2 + 4(a_0^2 + a_1^2 - 3a_0a_1) \\
= \;&5a_n^2 + 4(a_{n-1}^2 + a_n^2 - 3a_{n-1}a_n) \\
= \;&(3a_n - 2a_{n-1})^2,
\end{aligned}$$
which completes the proof.

Fibonacci Sequence Problems

Earlier we introduced the Fibonacci sequence; we will denote it by $\{a_n\}$ here; it is defined by $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. The general term of the Fibonacci sequence was found to be

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

In this section we look at some more problems and facts about this sequence. Note that the general term can be written as

$$a_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \tag{5}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the polynomial $q^2 - q - 1$. Note that α and β are also roots of the polynomial $q^3 - 2q - 1$, because $q^3 - 2q - 1 = (q+1)(q^2 - q - 1)$. Thus $\alpha^3 - 1 = 2\alpha$ and $\beta^3 - 1 = 2\beta$. We can also establish such equalities as follows: Multiply both sides of $\alpha^2 = \alpha + 1$ by α to obtain $\alpha^3 = \alpha^2 + \alpha$ and then

$$\alpha^3 - 1 = \alpha^2 + \alpha - 1 = \underbrace{\alpha^2 - \alpha - 1}_{=0} + 2\alpha = 2\alpha.$$

1. Show that $\sum_{i=1}^n a_{3i} = \frac{1}{2}(a_{3n+2} - 1)$.

   Solution 1 We will use the representation (5), the formulas $\alpha^3 - 1 = 2\alpha$ and $\beta^3 - 1 = 2\beta$ (that we just established above), and the formula for the sum of a geometric progression:

$$\sum_{i=1}^n x^i = x + x^2 + \cdots + x^n = \frac{x^{n+1} - x}{x - 1}.$$

We now have

$$\begin{aligned}
\sum_{i=1}^{n} a_{3i} &= a_3 + a_6 + \cdots + a_{3n} \\
&= \frac{\alpha^3 - \beta^3}{\sqrt{5}} + \frac{\alpha^6 - \beta^6}{\sqrt{5}} + \cdots + \frac{\alpha^{3n} - \beta^{3n}}{\sqrt{5}} \\
&= \frac{1}{\sqrt{5}} \left(\sum_{i=1}^{n} \alpha^{3i} - \sum_{i=1}^{n} \beta^{3i} \right) \\
&= \frac{1}{\sqrt{5}} \left(\frac{\alpha^{3n+3} - \alpha^3}{\underbrace{\alpha^3 - 1}_{=2\alpha}} - \frac{\beta^{3n+3} - \beta^3}{\underbrace{\beta^3 - 1}_{=2\beta}} \right) \\
&= \frac{1}{\sqrt{5}} \left(\frac{\alpha^{3n+3} - \alpha^3}{2\alpha} - \frac{\beta^{3n+3} - \beta^3}{2\beta} \right) \\
&= \frac{1}{2} \left(\frac{\alpha^{3n+2} - \beta^{3n+2}}{\sqrt{5}} - \frac{\alpha^2 - \beta^2}{\sqrt{5}} \right) \\
&= \frac{1}{2}(a_{3n+2} - a_2) = \frac{1}{2}(a_{3n+2} - 1),
\end{aligned}$$

as claimed.

Solution 2 (Suggested by A. Lachlan) Let S_n denote $\sum_{i=1}^{n} a_i$. Note that $a_{3i} = a_{3i-1} + a_{3i-2}$ and so

$$2\sum_{i=1}^{n} a_{3i} = \sum_{i=1}^{n} (a_{3i} + a_{3i-1} + a_{3i-2}) = \sum_{i=1}^{3n} a_i = S_{3n}.$$

This means that what we need to show is that $S_{3n} - a_{3n+2} = -1$. Next observe that

$$\begin{aligned}
(S_{3n+3} - a_{3n+5}) - (S_{3n} - a_{3n+2}) &= (S_{3n+3} - S_{3n}) + a_{3n+2} - a_{3n+5} \\
&= a_{3n+1} + 2a_{3n+2} + a_{3n+3} - a_{3n+5}
\end{aligned} \tag{6}$$

If we add together the following three instances of the recurrence relation

$$a_{3n+5} = a_{3n+4} + a_{3n+3}, \quad a_{3n+4} = a_{3n+3} + a_{3n+2}, \quad a_{3n+3} = a_{3n+2} + a_{3n+1},$$

and rearrange, we get

$$a_{3n+5} - a_{3n+3} - 2a_{3n+2} - a_{3n+1} = 0. \tag{7}$$

From (6) and (7) it follows that

$$S_{3n+3} - a_{3n+5} = S_{3n} - a_{3n+2}.$$

Since $S_3 - a_5 = -1$, by induction it follows that $S_{3n} - a_{3n+2} = -1$ for all n, which completes the proof.

2. Show that

(a) $\displaystyle\sum_{i=1}^{n} a_i = a_{n+2} - 1$ (b) $\displaystyle\sum_{i=1}^{n} a_{2i-1} = a_{2n}$ (c) $\displaystyle\sum_{i=1}^{n} a_i^2 = a_n a_{n+1}$

Solution (a) Write

$$\begin{aligned} a_n &= a_{n+2} - a_{n+1} \\ a_{n-1} &= a_{n+1} - a_n \\ \ldots &= \ldots \\ a_2 &= a_4 - a_3 \\ a_1 &= a_3 - a_2. \end{aligned}$$

Add all these to obtain

$$\sum_{i=1}^{n} a_i = a_{n+2} - a_2 = a_{n+2} - 1.$$

(b) Write

$$\begin{aligned} a_{2n-1} &= a_{2n} - a_{2n-2} \\ a_{2n-3} &= a_{2n-2} - a_{2n-4} \\ \ldots &= \ldots \\ a_3 &= a_4 - a_2 \\ a_1 &= a_2. \end{aligned}$$

Add all these to obtain

$$\sum_{i=1}^{n} a_{2i-1} = a_{2n}.$$

(c) We have

$$a_k a_{k+1} - a_{k-1} a_k = a_k(a_{k+1} - a_{k-1}) = a_k^2.$$

Now, write

$$\begin{aligned} a_n^2 &= a_n a_{n+1} - a_{n-1} a_n \\ a_{n-1}^2 &= a_{n-1} a_n - a_{n-2} a_{n-1} \\ \ldots &= \ldots \\ a_3^2 &= a_3 a_4 - a_2 a_3 \\ a_2^2 &= a_2 a_3 - a_1 a_2 \\ a_1^2 &= a_1 a_2. \end{aligned}$$

Add all these to obtain

$$\sum_{i=1}^{n} a_i^2 = a_n a_{n+1}.$$

3. Show that

   (a) $a_{n+m} = a_{n-1}a_m + a_n a_{m+1}$,

   (b) a_n divides a_{2n}.

   Solution (a) Induction on m:

   If $m = 1$, then the equality is $a_{n+1} = a_{n-1}a_1 + a_n a_2$ or $a_{n+1} = a_n + a_{n-1}$, which is true.

   If $m = 2$, then the equality is $a_{n+2} = a_{n-1}a_2 + a_n a_3$,

$$\begin{aligned} \text{or} \;\; a_{n+2} &= a_{n-1} + 2a_n, \\ \text{or} \;\; a_{n+2} &= \underbrace{a_{n-1} + a_n}_{=a_{n+1}} + a_n, \\ \text{or} \;\; a_{n+2} &= a_{n+1} + a_n, \end{aligned}$$

   which is true.

   Inductive step: We show that if the claim is true for $m = k$ and $m = k + 1$, that is, if

$$a_{n+k} = a_{n-1}a_k + a_n a_{k+1}, \tag{8}$$

$$a_{n+k+1} = a_{n-1}a_{k+1} + a_n a_{k+2}, \tag{9}$$

   then the claim is also true for $m = k + 2$, that is,

$$a_{n+k+2} = a_{n-1}a_{k+2} + a_n a_{k+3}. \tag{10}$$

   Using the fact that a_{n+k}, a_{n+k+1} and a_{n+k+2} are consecutive terms of the Fibonacci sequence, and applying (8) and (9), we get

$$\begin{aligned} a_{n+k+2} &= a_{n+k+1} + a_{n+k} \\ &= (a_{n-1}a_{k+1} + a_n a_{k+2}) + (a_{n-1}a_k + a_n a_{k+1}) \\ &= a_{n-1}(a_{k+1} + a_k) + a_n(a_{k+2} + a_{k+1}) \\ &= a_{n-1}a_{k+2} + a_n a_{k+3}, \end{aligned}$$

   so that (10) holds. Therefore, the claim is true for all m.

   (b) This part follows from part (a): Let $m = n$. Then

$$\begin{aligned} a_{2n} &= a_{n-1}a_n + a_n a_{n+1} \\ &= a_n(a_{n-1} + a_{n+1}), \end{aligned}$$

   which shows that a_n divides a_{2n}.

In the next sections we will need some results about the limit of a sequence; these are summarized in what follows.

THEOREM If r is a real number such that $|r| < 1$, then

$$\lim_{n\to\infty} r^n = 0.$$

Using the preceding result and the formula for sum of geometric progression (example on page 1) we obtain the following.

THEOREM (Sum of Infinite Geometric Progression) If S_n is the sum of the first $n+1$ terms of the geometric progression $\{ar^k\}_{k=0}^{\infty}$ with $|r| < 1$, then

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}.$$

This can also be written as

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}, \quad \text{if } |r| < 1.$$

THEOREM If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$ as well.

THEOREM (The Squeeze Theorem) If $a_n \le b_n \le c_n$ for all $n \ge k$, where k is some integer, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$ as well.

Definition The sequence $\{a_n\}$ is **increasing** if there exists k such that $a_n \le a_{n+1}$ for all $n \ge k$, that is, if

$$a_k \le a_{k+1} \le a_{k+2} \le \cdots$$

If there exists k such that $a_n \ge a_{n+1}$ for all $n \ge k$, then the sequence is **decreasing**.

Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \le M$ for all $n \ge 1$. A sequence $\{a_n\}$ is **bounded below** if there is a number m such that $a_n \ge m$ for all $n \ge 1$.

THEOREM If a sequence is increasing and bounded above, then it is convergent.

THEOREM If a sequence is decreasing and bounded below, then it is convergent.

THEOREM If $\{a_n\}$ and $\{b_n\}$ are two sequences such that $a_n \le b_n$ for all $n \ge k$, where k is some integer, then

$$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

Recurrence Relations and Limits

1. The sequence $\{a_n\}$ is defined by $a_n = a_{n-1} + \frac{1}{n^2+n}$, $a_1 = \frac{1}{2}$. Find out if the sequence is convergent and if so, find its limit.

   Solution Writing the defining equality as $a_k = a_{k-1} + \frac{1}{k(k+1)}$, then writing it for $k = 1, 2, \ldots, n$ and adding the resulting n equalities, we obtain

   $$a_n = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)}.$$

   We can now apply an approach, known as telescoping, to finish the solution. Using the identity $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, we can rewrite the expression for a_n as

   $$a_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right),$$

   and after cancellations, we get $a_n = 1 - \frac{1}{n}$. It is now clear that

   $$\lim_{n\to\infty} a_n = 1.$$

   Note that assuming the sequence has a limit, and taking limit on both sides of the defining equality $a_n = a_{n-1} + \frac{1}{n^2+n}$ does not give us any information about the value of this limit.

2. The sequence $\{a_n\}$ is defined by $a_n = \frac{n^3-1}{n^3+1}\, a_{n-1}$, $a_1 = 1$. Find out if the sequence is convergent and if so, find its limit.

   Solution We will apply the idea of telescoping again, but we will need an extra observation:

   $$\begin{aligned} a_k = \frac{k^3-1}{k^3+1}\, a_{k-1} &= \frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)}\, a_{k-1} \\ &= \frac{(k-1)(k^2+k+1)}{(k+1)[(k-1)^2+(k-1)+1]}\, a_{k-1}. \end{aligned}$$

   Writing this representation of a_k for $k = 2, 3, \ldots, n$, then multiplying the resulting $n-1$ equalities and simplifying, based on the above observation, we obtain

   $$\begin{aligned} a_n &= \frac{1.7}{3.3} \cdot \frac{2.13}{4.7} \cdot \frac{3.21}{5.13} \cdot \frac{4.34}{6.21} \cdots \frac{(n-1)(n^2+n+1)}{(n+1)[(n-1)^2+(n-1)+1]}\, a_1 \\ &= \frac{2}{3} \cdot \frac{n^2+n+1}{n(n+1)}. \end{aligned}$$

   Now, it is easy to see that $\lim_{n\to\infty} a_n = \frac{2}{3}$.

3. Let $a_n = \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n\ times}$. Show that the sequence $\{a_n\}$ is convergent and find its limit.

Solution We use the fact that a bounded above and increasing sequence has a limit. The sequence $\{a_n\}$ can be defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$ for $n \geq 1$. The bounded above property can be proved by induction: $a_1 = \sqrt{2} < 2$, and then, if $a_n < 2$, then

$$a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2.$$

To show that $\{a_n\}$ is increasing, we write

$$\begin{aligned} a_{n+1}^2 &= 2 + a_n \\ a_n^2 &= 2 + a_{n-1} \end{aligned}$$

Subtracting the second equality from the first one, we get

$$(a_{n+1} - a_n)(a_{n+1} + a_n) = a_n - a_{n-1} \tag{11}$$

Now, $a_{n+1} + a_n > 0$, because $a_n > 0$ for all $n \geq 1$. Since

$$a_2 - a_1 = \sqrt{2+\sqrt{2}} - \sqrt{2} > 0,$$

induction on n (using (11)) shows $a_{n+1} - a_n > 0$ for all $n \geq 1$. Hence the sequence $\{a_n\}$ is increasing. We have already shown $\{a_n\}$ is bounded above, so that $\{a_n\}$ is convergent, and therefore, it must have a limit, say,

$$\lim_{n\to\infty} a_n = l.$$

Taking limits on both sides of

$$a_{n+1}^2 = 2 + a_n$$

we get

$$l^2 = 2 + l,$$

which gives $l = -1$ or $l = 2$. Clearly, $l = -1$ does not work (because $a_n > 0$ for all $n \geq 1$), so that we must have

$$\lim_{n\to\infty} a_n = 2.$$

4. Show that the sequence $\{a_n\}$ defined by $a_1 = 2$ and

$$a_{n+1} = 4 - \frac{3}{a_n}$$

is convergent and find its limit.

Solution We start by a short experiment, just to get some feeling about how the sequence behaves: $a_1 = 2$, $a_2 = 2.5$, $a_3 = 2.8$, etc.; computing several more terms will not hurt. Based on this experiment we can hypothesize that

$$2 \le a_n < a_{n+1} < 3 \;\; (n \ge 1). \tag{12}$$

Let us try to prove it.

Consider the function $4 - \frac{3}{x}$ on the interval $[2, 3]$. The function is strictly increasing, because $\frac{3}{x}$ decreases as x increases for $x > 0$.

At the endpoints, the function values are

$$4 - \frac{3}{2} = 2.5 \quad \text{and} \quad 4 - \frac{3}{3} = 3.$$

Thus the range of $4 - \frac{3}{x}$ on $[2, 3)$ is a subset of $[2, 3)$. Since a_1 is in $[2, 3)$, it follows by induction that $2 \le a_n < 3$ for all $n \ge 1$.

Since $a_1 < a_2$ and the function $4 - \frac{3}{x}$ is increasing, it follows by induction that $a_n < a_{n+1}$ for all $n \ge 1$. This proves (12). Since $\{a_n\}$ is bounded above and increasing, it has a limit l, say,

$$\lim_{n\to\infty} a_n = l.$$

Taking limits on both sides of the defining relation $a_{n+1} = 4 - \frac{3}{a_n}$, we get

$$l = 4 - \frac{3}{l},$$

which gives $l = 1$ or $l = 3$. Since $\{a_n\}$ is increasing and $a_n > 1$, the limit of a_n cannot be 1. Consequently, $\lim a_n = 3$.

5. Let the sequence $\{x_n\}$ be defined by $x_1 = 5$ and $x_{n+1} = x_n^2 - 2$ for $n \ge 1$. Find

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_1 x_2 \dots x_n}.$$

Solution We will start with a different representation; we have

$$x_{n+1}^2 = (x_n^2 - 2)^2 = x_n^4 - 4x_n^2 + 4,$$

so that

$$x_{n+1}^2 - 4 = x_n^2(x_n^2 - 4),$$

and we can write

$$\begin{aligned} x_{n+1}^2 - 4 &= x_n^2(x_n^2 - 4) \\ x_n^2 - 4 &= x_{n-1}^2(x_{n-1}^2 - 4) \\ \dots &= \dots \\ x_2^2 - 4 &= x_1^2(x_1^2 - 4) \end{aligned}$$

Multiplying all these equalities, we get

$$x_{n+1}^2 - 4 = x_n^2 x_{n-1}^2 \dots x_1^2 (x_1^2 - 4),$$

or

$$x_{n+1}^2 - 4 = 21(x_1 x_2 \dots x_n)^2 \tag{13}$$

Let

$$A_n = \left(\frac{x_{n+1}}{x_1 x_2 \dots x_n} \right)^2 .$$

Using (13), we get

$$A_n = 21 + \frac{4}{(x_1 x_2 \dots x_n)^2} \tag{14}$$

Now, note that $x_n > 2$. (This follows by induction: $x_1 = 5 > 2$, and if $x_n > 2$, then $x_{n+1} = x_n^2 - 2 > 2^2 - 2 = 2$.) Then, from (14),

$$21 < A_n = 21 + \frac{4}{(x_1 x_2 \dots x_n)^2} < 21 + \frac{4}{2^n}.$$

By the Squeeze Theorem,

$$\lim_{n \to \infty} A_n = 21.$$

By the continuity of the function $\sqrt{x}$,

$$\lim_{n \to \infty} \sqrt{A_n} = \sqrt{\lim_{n \to \infty} A_n} = \sqrt{21},$$

and therefore,

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_1 x_2 \dots x_n} = \sqrt{21}.$$

6. The sequence $\{a_n\}$ is defined by $a_1 = a_2 = 1$ and

$$a_{n+1} = a_n + \frac{1}{n(n+1)} a_{n-1} \quad \text{for} \quad n \geq 2.$$

Show that the sequence is convergent.

Solution Clearly, $a_n > 0$, and $a_{n+1} \geq a_n$ for $n \geq 1$, so the sequence is increasing. We will show it is also bounded above; in particular, we will show by induction that $a_n < \frac{5}{3}$ for $n \geq 1$. We have $a_3 = \frac{7}{6}$, so the claim is true for $n = 1, 2, 3$. Assuming the claim is true for $n = 1, 2, \dots, k$, we want to show it is also true for $n = k + 1$. Writing the defining equality for $n = 2, 3, \dots k$ and adding the resulting $k - 1$ equalities, we obtain

$$a_{k+1} = \frac{a_{k-1}}{k(k+1)} + \frac{a_{k-2}}{(k-1)k} + \dots + \frac{a_2}{3(4)} + \frac{a_1}{2(3)} + a_2.$$

Let $k \geq 3$. Using the induction assumption, we obtain

$$\begin{aligned}
a_{k+1} &< \frac{5}{3}\left[\frac{1}{k(k+1)} + \frac{1}{(k-1)k} + \cdots + \frac{1}{4(5)}\right] + \frac{1}{3(4)} + \frac{1}{2(3)} + 1 \\
&= \frac{5}{3}\left[\left(\frac{1}{k} - \frac{1}{k+1}\right) + \left(\frac{1}{k-1} - \frac{1}{k}\right) + \cdots + \left(\frac{1}{4} - \frac{1}{5}\right)\right] + \frac{5}{4} \\
&= \frac{5}{3}\left[\frac{1}{4} - \frac{1}{k+1}\right] + \frac{5}{4} = \frac{5}{3} \cdot \frac{(k-3)}{4(k+1)} + \frac{5}{4} \\
&= \frac{5}{4}\left[\frac{k-3}{3(k+1)} + 1\right] < \frac{5}{4}\left[\frac{k+1}{3(k+1)} + 1\right] = \frac{5}{3}.
\end{aligned}$$

This completes the induction. Thus the sequence is increasing and bounded above, and therefore convergent.

7. The sequence $\{a_n\}$ is defined by $a_k = \frac{2k-1}{2k}\, a_{k-1}$, $a_1 = \frac{1}{2}$. Find out if the sequence is convergent and if so, find its limit.

   Solution Writing the defining equality for $k = 1, 2, \ldots n$ and multiplying the results, we get

$$a_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}.$$

None of the previous approaches seems to work here, but we can still apply the Squeeze Theorem; we just need to consider a_n^2 instead of a_n, and apply the observation

$$n(n+2) = n^2 + 2n < (n+1)^2.$$

We have

$$\begin{aligned}
a_n^2 &= \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdots \frac{(2n-1)^2}{(2n)^2} \\
&= \frac{1.(1.3)(3.5)(5.7)\ldots[(2n-3)(2n-1)](2n-1)}{2^2.4^2.6^2\ldots(2n-2)^2.(2n)^2} \\
&< \frac{1.2^2.4^2.6^2\ldots(2n-2)^2(2n-1)}{2^2.4^2.6^2\ldots(2n-2)^2.(2n)^2} = \frac{2n-1}{4n^2} < \frac{1}{2n},
\end{aligned}$$

and therefore, $0 < a_n < \sqrt{\frac{1}{2n}}$. Since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \sqrt{\frac{1}{2n}}$, it follows that $\lim_{n\to\infty} a_n = 0$ as well.

Readers familiar with series might appreciate the following alternative solution suggested by the editor A. Lachlan:

$$\begin{aligned}
a_n &= \frac{1}{2} \cdot \frac{1}{\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right) + \ldots + \left(1+\frac{1}{2n-1}\right)} \\
&< \frac{1}{2} \cdot \frac{1}{1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}} \\
&< \frac{1}{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots + \frac{1}{n}}
\end{aligned}$$

Since the harmonic series diverges, $\lim_{n\to\infty} a_n = 0$.

Systems of Recurrence Relations and Limits

We have already solved some systems of linear recurrence relations, so this knowledge might come handy; we will also look at some alternative approaches in the case when limits of sequences are involved.

1. Given the system of recurrence relations

$$\begin{aligned} a_{n+1} &= a_n + 3b_n \\ b_{n+1} &= a_n + b_n \end{aligned}$$

where $n \geq 1$, and $a_1 = b_1 = 1$, show that the sequence $\left\{\frac{a_n}{b_n}\right\}$ is convergent and find its limit.

Solution This system was already suggested as an exercise (on page 17; the answer is on page 56). We will give a brief solution here anyway.

From the second equation, $a_n = b_{n+1} - b_n$, so that $a_{n+1} = b_{n+2} - b_{n+1}$. Substitute these into the first equation to obtain

$$b_{n+2} - b_{n+1} = b_{n+1} - b_n + 3b_n,$$

that is,

$$b_{n+2} - 2b_{n+1} - 2b_n = 0.$$

Similarly,

$$a_{n+2} - 2a_{n+1} - 2a_n = 0.$$

We can now determine $a_2 = 4$ and $b_2 = 2$. The characteristic equation for the sequence $\{a_n\}$ is $r^2 - 2r - 2 = 0$ with roots $1 \pm \sqrt{3}$ (it is the same for the sequence $\{b_n\}$). Thus

$$a_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n.$$

We can now determine the coefficients C_1 and C_2 from the system

$$\begin{aligned} 1 = a_1 = C_1(1+\sqrt{3}) + C_2(1-\sqrt{3}) \\ 4 = a_2 = C_1(1+\sqrt{3})^2 + C_2(1-\sqrt{3})^2 \end{aligned}$$

Solving for C_1 and C_2 gives $C_1 = C_2 = \frac{1}{2}$. Hence

$$a_n = \frac{1}{2}\left[(1+\sqrt{3})^n + (1-\sqrt{3})^n\right].$$

Similarly,

$$b_n = \frac{1}{2\sqrt{3}}\left[(1+\sqrt{3})^n - (1-\sqrt{3})^n\right].$$

Now, let $x_n = (1+\sqrt{3})^n$ and $y_n = (1-\sqrt{3})^n$. Then

$$\frac{a_n}{b_n} = \sqrt{3}\,\frac{x_n + y_n}{x_n - y_n} = \sqrt{3}\,\frac{1 + \frac{y_n}{x_n}}{1 - \frac{y_n}{x_n}}.$$

But

$$\lim_{n\to\infty}\frac{y_n}{x_n}=\lim_{n\to\infty}\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)^n=0,$$

because $\left|\frac{1-\sqrt{3}}{1+\sqrt{3}}\right|<1$. Therefore,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\sqrt{3}\,\frac{1+0}{1-0}=\sqrt{3}.$$

2. Let a_n and b_n be the unique integers such that $(2+\sqrt{2})^n=a_n+b_n\sqrt{2}$ (a_n and b_n are unique, because $\sqrt{2}$ is irrational). Does $\lim_{n\to\infty}\frac{a_n}{b_n}$ exist? If so, find it.

Solution We note that $a_1=2$ and $b_1=1$. Also,

$$\begin{aligned} a_{n+1}+b_{n+1}\sqrt{2} &= (2+\sqrt{2})^{n+1}\\ &= (2+\sqrt{2})^n(2+\sqrt{2})\\ &= (a_n+b_n\sqrt{2})(2+\sqrt{2}) \end{aligned}$$

Then

$$a_{n+1}+b_{n+1}\sqrt{2}=2a_n+2b_n+a_n\sqrt{2}+2b_n\sqrt{2},$$

or

$$(a_{n+1}-2a_n-2b_n)+(b_{n+1}-2b_n-a_n)\sqrt{2}=0.$$

This can only happen if

$$\begin{aligned} a_{n+1}-2a_n-2b_n=0\\ b_{n+1}-2b_n-a_n=0, \end{aligned}$$

which reduces the given problem to a problem similar to the preceding one (see also Exercise 17 on page 17), so we can solve the system, express the ratio $\frac{a_n}{b_n}$ as a function of n and find its limit. Rather than continuing with the details, we will leave these to the reader, and suggest alternative solution: Note that the Binomial Formula shows that

$$a_n+b_n\sqrt{2}=(2+\sqrt{2})^n$$

implies

$$a_n-b_n\sqrt{2}=(2-\sqrt{2})^n$$

Solving this system for a_n and b_n gives

$$\begin{aligned} a_n &= \frac{1}{2}\left[(2+\sqrt{2})^n+(2-\sqrt{2})^n\right]\\ b_n &= \frac{1}{2\sqrt{2}}\left[(2+\sqrt{2})^n-(2-\sqrt{2})^n\right] \end{aligned}$$

Now continue as in the preceding question.
Answer:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\sqrt{2}.$$

Look at the Limit First

In some problems of finding a limit of a sequence defined recursively we might start by assuming the limit exists and find its possible value, and then using this information prove that the sequence is actually convergent, thus justifying our computation of the limit. (Computing a possible value of the limit would not be useful if the sequence is not convergent...) Looking at the possible value of the limit might be useful even if the problem is not to find the limit. We start by taking a second look at the problem from the previous section

1. Given the system

$$\begin{aligned} a_{n+1} &= a_n + 3b_n \\ b_{n+1} &= a_n + b_n \end{aligned}$$

where $n \geq 1$, and $a_1 = b_1 = 1$, show that the sequence $\left\{\frac{a_n}{b_n}\right\}$ is convergent and find its limit.

Solution Let $x_n = \frac{a_n}{b_n}$. Then

$$x_{n+1} = \frac{a_{n+1}}{b_{n+1}} = \frac{a_n + 3b_n}{a_n + b_n} = \frac{\frac{a_n}{b_n} + 3}{\frac{a_n}{b_n} + 1} = \frac{x_n + 3}{x_n + 1}.$$

Now, note that $a_n > 0$ and $b_n > 0$, so that $x_n > 0$ for all $n \geq 1$. In fact, $x_1 = 1$, and

$$x_{n+1} = \frac{x_n + 3}{x_n + 1} = 1 + \frac{2}{x_n + 1} > 1,$$

so that $x_n \geq 1$ for all $n \geq 1$.

If $\lim_{n\to\infty} x_n$ exists, say, $\lim_{n\to\infty} x_n = l$, then taking limits on both sides of $x_{n+1} = \frac{x_n+3}{x_n+1}$, we get

$$l = \frac{l+3}{l+1},$$

which gives $l = \pm\sqrt{3}$. The case $l = -\sqrt{3}$ is impossible, because $x_n \geq 1$, so we must have $l = \sqrt{3}$ (if the sequence is convergent). We will now show that the sequence $\{x_n\}$ is convergent. Consider $|x_{n+1} - \sqrt{3}|$. We have

$$\begin{aligned}
0 &\leq |x_{n+1} - \sqrt{3}| \\
&= \left| \frac{x_n + 3}{x_n + 1} - \sqrt{3} \right| \\
&= \frac{(\sqrt{3} - 1)\,|\sqrt{3} - x_n|}{x_n + 1} \\
&\leq \frac{\sqrt{3} - 1}{2}\,|x_n - \sqrt{3}| \quad \text{(because } x_n \geq 1\text{)} \\
&\leq \left(\frac{\sqrt{3} - 1}{2} \right)^2 |x_{n-1} - \sqrt{3}| \quad \text{(by a similar argument)} \\
\cdots & \cdots \\
&\leq \left(\frac{\sqrt{3} - 1}{2} \right)^n |x_1 - \sqrt{3}| \quad \text{(continuing the process)} \\
&= \left(\frac{\sqrt{3} - 1}{2} \right)^n |1 - \sqrt{3}|
\end{aligned}$$

Thus

$$0 \leq |x_{n+1} - \sqrt{3}| \leq |1 - \sqrt{3}| \left(\frac{\sqrt{3} - 1}{2} \right)^n.$$

Since

$$\lim_{n \to \infty} \left(\frac{\sqrt{3} - 1}{2} \right)^n = 0,$$

then, by the Squeeze Theorem,

$$\lim_{n \to \infty} |x_{n+1} - \sqrt{3}| = 0,$$

so that $\lim_{n \to \infty} x_{n+1} = \sqrt{3}$, or $\lim_{n \to \infty} x_n = \sqrt{3}$, as expected.

2. The sequence $\{x_n\}$ defined by $x_1 = 1$ and

$$x_{n+1} = \frac{4x_n^2 + 1}{5x_n + 1}$$

for $n \geq 1$. Show that $\{x_n\}$ converges and find $\lim_{n \to \infty} x_n$.

Solution (outline).

(1) Show that if $\lim_{n\to\infty} x_n$ exists, say, $\lim_{n\to\infty} x_n = l$, then

$$l = \frac{\sqrt{5}-1}{2}.$$

(2) Show that

$$x_n > \frac{\sqrt{5}-1}{2} \quad \text{for all} \quad n \geq 1.$$

(3) Show that $\{x_n\}$ is decreasing.

Then (2) and (3) show that $\{x_n\}$ is convergent, and therefore the limit found in (1) is the correct one.

3. (Crux 1686, 9/1991) The sequence $\{a_n\}$ is defined by $a_0 = \frac{4}{3}$ and

$$a_{n+1} = \frac{3(5-7a_n)}{2(10a_n+17)}$$

for $n \geq 0$. Find a formula for a_n in terms of n.

Solution This question asks us to do what we referred to earlier as "solve" the recurrence; the only problem is the recurrence is not linear. What we might try is to somehow reduce it to a linear one. Convergence and limit are not explicit in this question, but we can start with some investigation on that as a part of getting some more information about the behaviour of this sequence. A brief experiment (writing the first several terms) suggests that the sequence might be converging to $\frac{1}{4}$. Moreover, if it has limit, say $\lim_{n\to\infty} a_n = l$, then taking limits on both sides of the defining equation, we get

$$l = \frac{3(5-7l)}{2(10l+17)},$$

which gives $l = -3$ or $l = \frac{1}{4}$. The experiment suggests that the limit (if it exists) would be $\frac{1}{4}$. Hence the substitution

$$b_n = a_n - \frac{1}{4}$$

gives a promise of simplifying the defining equation. Then $a_n = b_n + \frac{1}{4}$ and we obtain

$$b_{n+1} + \frac{1}{4} = \frac{3\left[5-7\left(b_n+\frac{1}{4}\right)\right]}{2\left[10\left(b_n+\frac{1}{4}\right)+17\right]},$$

which simplifies to

$$b_{n+1} = -\frac{26b_n}{20b_n+39}.$$

We can also compute $b_0 = a_0 - \frac{1}{4} = \frac{13}{12}$. Now, the defining equation for b_n can be written as

$$\frac{1}{b_{n+1}} = -\left(\frac{10}{13} + \frac{3}{2}\,\frac{1}{b_n}\right),$$

so that we can next substitute $c_n = \frac{1}{b_n}$, thereby obtaining

$$c_{n+1} = -\frac{3}{2}c_n - \frac{10}{13}, \text{ with } c_0 = \frac{13}{12}\,.$$

We are now in familiar territory:

$$c_{n+1} + \frac{3}{2}c_n = -\frac{10}{13} = c_n + \frac{3}{2}c_{n-1},$$

so that

$$c_{n+1} + \frac{1}{2}c_n - \frac{3}{2}c_{n-1} = 0, \text{ with } c_0 = \frac{13}{12},\ c_1 = -\frac{28}{13}\,,$$

which is a homogeneous linear recurrence relation of order 2. Solving it gives

$$c_n = \frac{16}{13}\left(-\frac{3}{2}\right)^n - \frac{4}{13}.$$

Then, performing back-substitutions,

$$b_n = \frac{13}{16\left(-\frac{3}{2}\right)^n - 4},$$

and finally,

$$a_n = \frac{\left(-\frac{3}{2}\right)^n + 3}{4\left(-\frac{3}{2}\right)^n - 1} = \frac{3^n + 3(-2)^n}{4(3^n) - (-2)^n} = \frac{1 + 3\left(-\frac{2}{3}\right)^n}{4 - \left(-\frac{2}{3}\right)^n}\,,$$

which completes the solution. From the last expression for a_n we can see that, indeed, $\lim_{n\to\infty} a_n$ exists, and it is $\frac{1}{4}$, as conjectured.

4. Find all sequences $\{x_n\}_{n=0}^{\infty}$ such that $0 < x_0 \le 1$ and $0 < x_{n+1} \le 2 - \frac{1}{x_n}$ for $n \ge 0$.

Solution Since $2 - \frac{1}{x_n} > 0$ and $x_n > 0$, we obtain $x_n > \frac{1}{2}$, so that the sequence is bounded below. We also have

$$x_{n+1} - x_n \le 2 - \frac{1}{x_n} - x_n \le 0,$$

because $x_n + \frac{1}{x_n} \ge 2$ when $x_n > 0$. Thus the sequence is also decreasing, and therefore convergent. Let $\lim_{n\to\infty} x_n = l$. Since $x_{n+1} \le 2 - \frac{1}{x_n}$ and $x_n > \frac{1}{2}$ we have $l \ge \frac{1}{2} > 0$ and $l \le 2 - \frac{1}{l}$, which gives $(l-1)^2 \le 0$, so that l must be 1. Since $x_0 \le 1$ and the sequence is decreasing, we must have

$$1 \ge x_0 \ge x_1 \ge x_2 \ge \cdots$$

On the other hand, the limit of this sequence is 1. Clearly, this can only happen if all the terms of the sequence are equal to 1. Therefore, the only sequence that meets the condition is $1, 1, 1, \ldots$.

5. (Crux 1378, 8/1988) Suppose $a_0, a_1, a_2, \ldots$ is a sequence of positive real numbers such that $a_0 = 1$ and $a_n = a_{n+1} + a_{n+2}$, $n \geq 0$. Find a_n.

   Solution It seems that we are one initial condition short of what we need to solve the recurrence. However, the fact that the sequence consists of *positive* real numbers might just provide enough information for us to find a_n. We have $a_n - a_{n+1} = a_{n+2} > 0$, so that the sequence is decreasing. Since the members of the sequence are positive, the sequence is bounded below and therefore it has a limit, say $\lim_{n\to\infty} a_n = l$. Taking limits on both sides of the defining recurrence, we get $l = l + l$, and then $l = 0$.

   The characteristic equation of the given recurrence relation is $q^2 + q - 1 = 0$ with roots $q_1 = \frac{\sqrt{5}-1}{2}$ and $q_2 = \frac{-\sqrt{5}-1}{2}$. Hence

$$a_n = C_1 q_1^n + C_2 q_2^n.$$

   Suppose that $C_2 \neq 0$. Since $|q_1| < 1$ and $|q_2| > 1$, $C_1 q_1^n \to 0$ and $|C_2 q_2^n| \to \infty$ as $n \to \infty$. It follows that, for large n, a_n has the same sign as $C_2 q_2^n$. But $q_n < 0$ and so the sign of $C_2 q_2^n$ alternates as n increases. This contradicts a_n being positive for all n. Therefore $C_2 = 0$. We can now determine C_1 from $1 = a_0 = C_1 + C_2$; $C_1 = 1$, and therefore,

$$a_n = \left(\frac{\sqrt{5}-1}{2} \right)^n .$$

Spot the Recurrence

This section deals with problems which are somewhat similar to the ones in the section Reverse Engineering, in the sense that no recurrence relation is explicitly mentioned in the condition. However, unlike in that section, nothing that looks like a solution to a recurrence relation appears in the condition either; yet a recurrence can be spotted/constructed and used to produce a solution.

1. Find the sum of the 11th powers of the roots of the equation $x^3+x+1=0$.

   Solution It is known that a polynomial equation of degree n has n roots, some of which might not be real. In any case, if ϵ is a root of the equation $x^3+x+1=0$, then $\epsilon^3+\epsilon+1=0$. Let α,β,γ be the roots of the equation, and let $S_k=\alpha^k+\beta^k+\gamma^k$, $k=1,2,\dots$ Multiplying the equalities

   $$\alpha^3+\alpha+1=0, \quad \beta^3+\beta+1=0, \quad \text{and} \quad \gamma^3+\gamma+1=0$$

   by α^n, β^n and γ^n, respectively, and adding the results, we obtain a recurrence for the sequence $\{S_k\}_{k=0}^{\infty}$, namely,

   $$S_{n+3}+S_{n+1}+S_n=0, \tag{15}$$

   which is a linear recurrence relation of order 3. Since α,β and γ are the roots of the equation $x^3+x+1=0$, we have

   $$\begin{aligned} & x^3+x+1 \\ = \ & (x-\alpha)(x-\beta)(x-\gamma) \\ = \ & x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma, \end{aligned}$$

   from which $\alpha+\beta+\gamma=0$ and $\alpha\beta+\alpha\gamma+\beta\gamma=1$. Thus $S_1=0$, and

   $$\begin{aligned} S_2 &= \alpha^2+\beta^2+\gamma^2 \\ &= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma) \\ &= 0-2(1)=-2 \end{aligned}$$

   Clearly, $S_0=3$. Thus we have the values of the first three terms of the sequence $\{S_k\}_{k=0}^{\infty}$, and we can find S_{11} by substituting $n=0,1,\dots,8$ in (15). Answer: $S_{11}=0$.

2. Let d_k be the largest odd divisor of the positive integer k. Show that

   $$\sum_{k=1}^{2^n} d_k = \frac{4^n+2}{3}.$$

   Solution Let S_n denote the sum in the left hand side of the desired equality. A part of this sum can be easily computed: If k is odd, then its largest odd divisor is k. Let S'_n be the sum of the largest odd divisors of the numbers $1,3,5,\dots,2^n-1$, and let S''_n be the sum of the largest odd divisors of the numbers $2, 4, 6, \ldots, 2^n$. Clearly, $S_n = S'_n + S''_n$. Also, observe that the sum of the largest odd divisors of the numbers $2, 4, 6, \ldots, 2^n$ is the same as the sum of the largest odd divisors of the numbers $1, 2, 3, \ldots, 2^{n-1}$, that is, $S''_n = S_{n-1}$, and therefore,

$$S_n = S'_n + S_{n-1}.$$

Let us compute S'_n.

$$\begin{aligned} S'_n &= 1 + 3 + 5 + \cdots + (2^n - 1) = 2 + 4 + 6 + \cdots + 2^n - 2^{n-1} \\ &= 2(1 + 2 + 3 + \cdots + 2^{n-1}) - 2^{n-1} \\ &= 2 \cdot \frac{2^{n-1}(2^{n-1} + 1)}{2} - 2^{n-1} = 4^{n-1}. \end{aligned}$$

Thus $S_n = S_{n-1} + 4^{n-1}$. We also have $S_1 = 2$, so we can finish the solution by solving this recurrence. We have

$$\begin{aligned} S_n &= S_{n-1} + 4^{n-1} \\ S_{n-1} &= S_{n-2} + 4^{n-2} \\ \ldots &= \ldots \\ S_2 &= S_1 + 4^1. \end{aligned}$$

Adding all these equalities, we obtain

$$\begin{aligned} S_n &= S_1 + 4^1 + 4^2 + \cdots + 4^{n-1} = 2 + 4^1 + 4^2 + \cdots + 4^{n-1} \\ &= 1 + 4^0 + 4^1 + 4^2 + \cdots + 4^{n-1} = 1 + \frac{4^n - 1}{4 - 1} = \frac{4^n + 2}{3}, \end{aligned}$$

as claimed. (Check Exercise 10 for a slightly different solution to a similar recurrence.)

3. Find the number of n-tuples $(x_1, x_2, \ldots, x_n)$, $n \geq 2$, such that $x_i \in \{a, b, c\}$, $i = 1, 2, \ldots, n$, $x_1 = x_n = a$, and $x_i \neq x_{i+1}$ for $i = 1, 2, \ldots, n-1$.

   Solution Let f_n be the number of all n-tuples that meet the condition. Clearly, $f_2 = 0$, $f_3 = f_4 = 2$. It is easy to compute $f_5 = 6$; this can give us some idea on how to solve the problem. Let $n \geq 5$. To find a recurrence for the sequence $\{f_n\}$, we need a second count for the number f_n. Each n-tuple has the form $(a, x_2, \ldots, x_{n-1}, a)$. If $x_i \neq a$ for all $i = 2, 3 \ldots, n-1$, then there are only two possibilities for the n-tuple $(a, x_2, \ldots, x_{n-1}, a)$; it is either $(a, b, c, b, c, \ldots, a)$ or $(a, c, b, c, b, \ldots, a)$. Now, let x_i, $i \geq 3$, be the first a after the starting a in the n-tuple ($i \neq 2$, because of the condition). Clearly, $i \neq n$ (we already considered this case); also, $i \neq n-1$, because of the condition. Thus the n-tuple must have the form

$$(a, x_2, x_3 \ldots, x_{i-1}, \underbrace{a, x_{i+1}, \ldots, x_{n-1}, a}_{n-(i-1)}), \quad 3 \leq i \leq n-2. \tag{16}$$

According to the condition, $(a, x_2, x_3 \ldots, x_{i-1})$ must either be $(a, b, c, b, c \ldots)$ or $(a, b, c, b, c \ldots)$. There are f_{n-i+1} ways to form the remaining part of the n-tuple (16). Hence there are $2f_{n-i+1}$ n-tuples of the kind (16). The first a after the starting one can be at any position from 3 to $n-2$ $(3 \leq i \leq n-2)$, so we get a second count for all n-tuples, and a recurrence relation for the sequence $\{f_n\}$:

$$f_n = 2 + 2(f_3 + f_4 + \cdots + f_{n-2}). \tag{17}$$

Technically, this is a linear recurrence relation of order $n-1$, but n varies; fortunately, we can easily find a better representation. Write the recurrence relation (17) for $(n-1)$ instead of n, and subtract the result from (17) to obtain a different looking recurrence for the sequence $\{f_n\}$

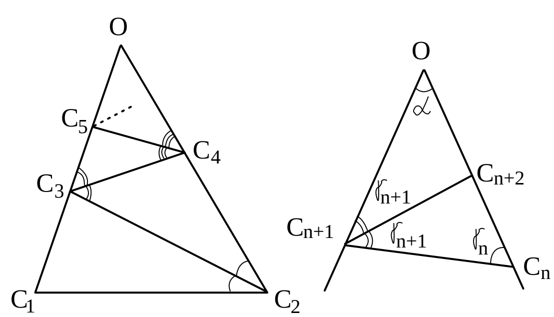
$$f_{n-1} = 2 + 2(f_3 + f_4 + \cdots + f_{n-3}),$$

$$f_n - f_{n-1} - 2f_{n-2} = 0.$$

Using $f_3 = 2$ and $f_4 = 2$, we can solve the recurrence and get an explicit formula for f_n. Answer: $f_n = \frac{1}{6}2^n - \frac{2}{3}(-1)^n$, $n \geq 2$.

4. Let C_1C_2O be a triangle. Form a sequence of points $C_3, C_4, C_5, \ldots$ alternating between the sides OC_1 and OC_2 of the triangle so that $C_{n+1}C_{n+2}$ is the bisector of the angle $C_nC_{n+1}O$ for $n = 1, 2, \ldots.$ Show that the sequence of angles $\gamma_n = \angle C_nC_{n+1}O$ is convergent and find its limit if $\angle C_1OC_2 = \alpha$.

Solution



Using the the condition, from triangle $C_nC_{n+1}O$ we get a recurrence for the sequence $\{\gamma_n\}$, namely, $2\gamma_{n+1} + \gamma_n + \alpha = \pi$. We will use the idea from the previous section and look at the possible limit first. If the sequence $\{\gamma_n\}$ has a limit l, then we must have $2l + l + \alpha = \pi$, so that $l = \frac{\pi-\alpha}{3}$. We now use this information to show that the sequence $\{\gamma_n\}$ is indeed convergent. We have

$$\begin{aligned} \left|\gamma_{n+1} - \frac{\pi-\alpha}{3}\right| &= \left|\frac{\pi - \gamma_n - \alpha}{2} - \frac{\pi-\alpha}{3}\right| \\ = \quad \left|\frac{\pi-\alpha}{6} - \frac{\gamma_n}{2}\right| &= \frac{1}{2}\left|\gamma_n - \frac{\pi-\alpha}{3}\right| \end{aligned}$$

Let $\beta = \frac{\pi-\alpha}{3}$. Then we can write the above result as

$$|\gamma_{n+1} - \beta| = \frac{1}{2}\,|\gamma_n - \beta|.$$

Applying it for $n-1, n-2, \ldots, 1$, in place of n, we get

$$|\gamma_n - \beta| = \frac{1}{2}\,|\gamma_{n-1} - \beta| = \frac{1}{2^2}\,|\gamma_{n-2} - \beta| = \cdots = \frac{1}{2^n}\,|\gamma_1 - \beta|\,.$$

Thus

$$0 \le \left|\gamma_n - \frac{\pi-\alpha}{3}\right| = \frac{1}{2^n}\left|\gamma_1 - \frac{\pi-\alpha}{3}\right|.$$

Since $\lim_{n\to\infty} \frac{1}{2^n} = 0$ and $\gamma_1 - \frac{\pi-\alpha}{3}$ is a constant, then

$$\lim_{n\to\infty} \frac{1}{2^n}\left|\gamma_1 - \frac{\pi-\alpha}{3}\right| = 0,$$

and the Squeeze Theorem tells us that the sequence $\{\gamma_n\}$ is convergent and its limit is $\frac{\pi-\alpha}{3}$.

5. Is the sequence $a_n = \tan n$ convergent?

Solution A recurrence can be obtained by using a trigonometric formula:

$$a_{n+1} = \tan(n+1) = \frac{\tan n + \tan 1}{1 - (\tan n)(\tan 1)} = \frac{a_n + \tan 1}{1 - a_n \tan 1}.$$

Now, if $\{a_n\}$ is convergent, then $\lim_{n\to\infty} a_n = l$, where l is some real number. Taking limits on both sides of the above recurrence, we get

$$l = \frac{l + \tan 1}{1 - l\tan 1},$$

or $(\tan 1)(l^2 + 1) = 0$, a contradiction with l being real. Therefore, the sequence $\{a_n\}$ is divergent.

There are some details to be filled in. It is known that π is transcendental which implies that it is irrational. Taking this for granted it follows that n is not an odd multiple of $\pi/2$ and so $\tan n$ is defined for all n. Now, if $(\tan n)(\tan 1) = 1$, then $\tan n = \tan((\pi/2) - 1)$ and so $n = k\pi + ((\pi/2) - 1)$ for some integer k. This contradicts the irrationality of π. So $(\tan n)(\tan 1) \neq 1$ for all n.

Readers who have the feeling that the problem can be solved without writing a recurrence might appreciate the following argument, suggested by the editor A. Lachlan: It is not hard to prove that if α is irrational, then the numbers $n\alpha - \lfloor n\alpha \rfloor$, where n runs through the positive integers, are dense in $(0, 1)$. Applying this with $\alpha = 1/\pi$ we see that there are infinitely many values of n for which n is close to a multiple of π, and infinitely many values of n for which n is close to an odd multiple of $\pi/2$.

6. (Crux 2337, 3/1998; slightly modified) Prove that for every integer $n > 0$

$$\left\lceil \frac{n^2+3n+1}{n^2+2n} \left\lceil \frac{n^2+3n}{n^2+2n-1} \cdots \left\lceil \frac{n^2+2n+2}{n^2+n+1} \right\rceil \cdots \right\rceil \right\rceil = n+1.$$

Note that $\lceil x \rceil$ denotes the least integer $\geq x$.

Solution Let

$$a_n = \left\lceil \frac{n^2+3n+1}{n^2+2n} \left\lceil \frac{n^2+3n}{n^2+2n-1} \cdots \left\lceil \frac{n^2+2n+2}{n^2+n+1} \right\rceil \cdots \right\rceil \right\rceil.$$

For every $n \geq 1$, consider the sequence

$$b_1 = \left\lceil \frac{n^2+2n+2}{n^2+n+1} \right\rceil = 1 + \left\lceil \frac{n+1}{n^2+n+1} \right\rceil = 2,$$

$$b_i = \left\lceil \frac{n^2+2n+2+(i-1)}{n^2+n+1+(i-1)} b_{i-1} \right\rceil, \quad 2 \leq i \leq n.$$

It is clear that $a_n = b_n$. We prove inductively that $b_n = n+1$. Assuming that $b_{i-1} = i$ for some i, $2 \leq i \leq n$, we obtain

$$b_i = \left\lceil \frac{n^2+2n+2+(i-1)}{n^2+n+1+(i-1)} b_{i-1} \right\rceil = \left\lceil \frac{n^2+2n+i+1}{n^2+n+i} i \right\rceil$$

$$= \left\lceil \frac{n^2 i+2ni+i^2+i}{n^2+n+i} \right\rceil = i + \left\lceil \frac{ni+i}{n^2+n+i} \right\rceil = i+1,$$

because $ni+i \leq n^2+n < n^2+n+i$. Thus $a_n = b_n = n+1$, which completes the proof.

Patterns and Repetition

We have seen that some experiment with the first terms of a sequence can help in observing some properties or some behavior that can be utilized in the solution. Problems of that type were considered in the section Different Representations. Experiments also helped us in getting some idea about whether a sequence is bounded, and in finding an appropriate bound. Writing the first several terms of a sequence could also help in noticing some kind of repetitive behaviour; this could be repeating terms of the sequence, or it could be that the sequence has different terms but something else might be repeating, like remainders on division by a particular integer, or a pattern that can be utilized.

1. The sequence $\{a_n\}$ is determined by $a_n = a_{n-1}a_{n-2}a_{n-4}$, with $a_1 = a_3 = 1$, $a_2 = a_4 = -1$. Find a_{2020}.

   Solution Clearly, $|a_n| = 1$. Writing the first several terms of the sequence, we observe periodicity. More precisely,

   $$\begin{aligned} a_n &= a_{n-1}a_{n-2}a_{n-4} = (a_{n-2}a_{n-3}a_{n-5})a_{n-2}a_{n-4} \\ &= a_{n-2}^2 a_{n-3}a_{n-4}a_{n-5} = a_{n-2}^2(a_{n-4}a_{n-5}a_{n-7})a_{n-4}a_{n-5} \\ &= a_{n-2}^2 a_{n-4}^2 a_{n-5}^2 a_{n-7} = a_{n-7}, \end{aligned}$$

   which shows that the sequence $\{a_n\}$ is periodic with period 7. Since $2020 = 7(288) + 4$, then $a_{2020} = a_4 = -1$.

2. The sequence $\{u_n\}$ is defined by $u_0 = 2$, $u_1 = \frac{5}{2}$, and $u_{n+1} = u_n(u_n^2 - 2) - \frac{5}{2}$. Show that

   $$\lfloor u_n \rfloor = 2^{\frac{2^n - (-1)^n}{3}}.$$

   Solution We can start by a little experiment, namely, writing the first several terms and observing a pattern:

   $$\begin{aligned} &u_0 = 2 = 2^0 + 2^0 \\ &u_1 = \frac{5}{2} = 2^1 + 2^{-1} \\ &u_2 = \frac{5}{2} = 2^1 + 2^{-1} \\ &u_3 = \frac{65}{8} = 2^3 + 2^{-3} \\ &u_4 = \frac{1025}{32} = 2^5 + 2^{-5} \\ &\dots \end{aligned}$$

   Let $\{a_n\}$ be the sequence defined by $a_n = \frac{2^n - (-1)^n}{3}$. Clearly, $a_0 = 0$, $a_1 = a_2 = 1$, $a_3 = 3$, $a_4 = 5$, etc.; and these are the exponents of 2 from our experiment, which allow us to form the hypothesis

   $$u_n = 2^{a_n} + 2^{-a_n}, \quad n \geq 0. \tag{18}$$

We will prove this claim by induction, but first we reverse engineer a recurrence for the sequence $\{a_n\}$. Since

$$a_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n,$$

the roots of the characteristic equation must be 2 and -1, so that the characteristic equation is $r^2 - r - 2 = 0$, and then a recurrence for $\{a_n\}$ is $a_{n+2} = a_{n+1} + 2a_n$, $n \geq 0$. The recurrence relation can be rewritten as $a_{n+2} - 2a_{n+1} = -(a_{n+1} - 2a_n)$. From this it follows easily by induction that

$$a_n - 2a_{n-1} = (-1)^{n+1}(a_1 - 2a_0) = (-1)^{n+1}. \tag{19}$$

Returning to the main induction, we observe that the statement (18) is correct for $n = 0$ and $n = 1$. Assuming the statement is correct for $n - 1$ and n we show it is correct for $n + 1$ as well. We have

$$\begin{aligned} u_{n+1} &= u_n(u_n^2 - 2) - \frac{5}{2} \\ &= \left(2^{a_n} + 2^{-a_n}\right)\left[\left(2^{a_{n-1}} + 2^{-a_{n-1}}\right)^2 - 2\right] - \frac{5}{2} \\ &= \left(2^{a_n} + 2^{-a_n}\right)\left(2^{2a_{n-1}} + 2^{-2a_{n-1}}\right) - \frac{5}{2} \\ &= 2^{a_n+2a_{n-1}} + 2^{-(a_n+2a_{n-1})} + \left(2^{a_n-2a_{n-1}} + 2^{-(a_n-2a_{n-1})} - \frac{5}{2}\right) \end{aligned}$$

From (19), $a_n - 2a_{n-1}$ and $-(a_n - 2a_{n-1})$ are 1 and -1 in some order, and so the expression in parentheses is 0. Finally, using the recurrence relation, we obtain

$$u_{n+1} = 2^{a_{n+1}} + 2^{-a_{n+1}},$$

which completes the induction. Thus (18) holds, and then

$$\lfloor u_n \rfloor = 2^{a_n} + \lfloor 2^{-a_n} \rfloor = 2^{a_n} = 2^{\frac{2^n - (-1)^n}{3}}.$$

3. The sequence $\{u_n\}$ is defined by $u_1 = 1$, $u_2 = 2$, $u_3 = 3$ and $u_{n+3} = u_n$ for $n \geq 1$. Find a formula for u_n in terms of n.

Solution The general theory presented earlier in the text applies in the case when the characteristic equation has complex roots, which is the case here; the characteristic equation of the given recurrence is $r^3 - 1 = 0$. We choose to proceed with an elementary solution.

First we observe that the given sequence

$$1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots,$$

can be written as $u_n = a_n + 2$, where $\{a_n\}$ is the sequence

$$-1, 0, 1, -1, 0, 1, -1, 0, 1, \ldots$$

The terms of $\{a_n\}$ can be written as

$$\sin\frac{3\pi}{2}, \sin\frac{4\pi}{2}, \sin\frac{5\pi}{2}, \sin\frac{7\pi}{2}, \sin\frac{8\pi}{2}, \sin\frac{9\pi}{2}, \sin\frac{11\pi}{2}, \ldots,$$

or $a_n = \sin\left(b_n \frac{\pi}{2}\right)$, where $\{b_n\}$ is the sequence

$$3, 4, 5, 7, 8, 9, 11, 12, 13, \ldots$$

Now, $b_n = c_n + 2$, where $\{c_n\}$ is the sequence

$$1, 2, 3, 5, 6, 7, 9, 10, 11, \ldots$$

Finally, we observe that $c_n = n + d_n$, where $\{d_n\}$ is the sequence

$$0, 0, 0, 1, 1, 1, 2, 2, 2, \ldots$$

Clearly, $d_n = \left\lfloor \frac{n-1}{3} \right\rfloor$, and then going backwards, we find

$$\begin{aligned} c_n &= n + \left\lfloor \frac{n-1}{3} \right\rfloor = \left\lfloor \frac{4n-1}{3} \right\rfloor \\ b_n &= c_n + 2 = \left\lfloor \frac{4n-1}{3} \right\rfloor + 2 = \left\lfloor \frac{4n+5}{3} \right\rfloor \\ a_n &= \sin\left(b_n \frac{\pi}{2}\right) = \sin\left(\left\lfloor \frac{4n+5}{3} \right\rfloor \frac{\pi}{2}\right) \\ u_n &= a_n + 2 = 2 + \sin\left(\left\lfloor \frac{4n+5}{3} \right\rfloor \frac{\pi}{2}\right). \end{aligned}$$

4. The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 2$ and $a_n = 4a_{n-1} - a_{n-2}$. Show that this sequence has infinitely many terms divisible by 7 and infinitely many terms divisible by 13.

   Solution We start by computing the first several term of the sequence and factoring them to find some divisible by 7 or by 13. We have $a_2 = 7$, $a_3 = 26 = 2(13)$, $a_4 = 97$, $a_5 = 362 = 2(181)$, $a_6 = 1351 = 7(193)$, $a_7 = 5042 = 2(2521)$, $a_8 = 18817 = 31(607)$, $a_9 = 70226 = (2)(13)(37)(73)$, $a_{10} = 262087 = 7(37441)$. We see that a_2, a_6, and a_{10} are divisible by 7 and a_3 and a_9 by 13, so we can form the hypotheses that a_{4k+2} is divisible by 7 and a_{6k+3} is divisible by 13 for $k = 0, 1, 2, \ldots$. Using the defining recurrence, we get

$$\begin{aligned} a_n &= 4a_{n-1} - a_{n-2} \\ &= 4(4a_{n-2} - a_{n-3}) - a_{n-2} \\ &= 15a_{n-2} - 4a_{n-3} \\ &= 15(4a_{n-3} - a_{n-4}) - 4a_{n-3} \\ &= 56a_{n-3} - 15a_{n-4}, \end{aligned}$$

from which it follows that if 7 divides a_{n-4}, then it also divides a_n, and now easy induction shows 7 divides a_{4k+2} for $k = 0, 1, 2, \ldots$. Continuing the above computation with two similar steps, we get

$$\begin{aligned} a_n &= 56(4a_{n-4} - a_{n-5}) - 15a_{n-4} \\ &= 209a_{n-4} - 56a_{n-5} \\ &= 209(4a_{n-5} - a_{n-6}) - 56a_{n-5} \\ &= 780a_{n-5} - 209a_{n-6}, \end{aligned}$$

form which the second hypothesis similarly follows.

5. The sequence $\{a_n\}$ is defined by $a_0 = 1$, $a_1 = 3$, $a_2 = 6$ and $a_n = a_{n-1}^3 + a_{n-2}^2 + a_{n-3}$ for $n \geq 3$. Let α_n denote the last digit of the integer a_n. Show that the number $r = 0.\alpha_0\alpha_1\alpha_2\alpha_3\ldots$ is rational.

Solution It suffices to show that the sequence $\{\alpha_n\}$ is periodic. Let $\beta_i = \{\alpha_i, \alpha_{i+1}, \alpha_{i+2}\}$. Since $0 \leq \alpha_k \leq 9$ for $k = i, i+1, i+2$, there are 10^3 possible triples β_i. Hence there must be two equal terms in the sequence $\beta_1, \beta_2, \ldots, \beta_{1001}$, say, $\beta_m = \beta_l$. Without loss of generality, we can assume $m < l$.

Now, α_{m+3} is the last digit of $a_{m+3} = a_{m+2}^3 + a_{m+1}^2 + a_m$, and also the last digit of $\alpha_{m+2}^3 + \alpha_{m+1}^2 + \alpha_m$.

Similarly, α_{l+3} is the last digit of $\alpha_{l+2}^3 + \alpha_{l+1}^2 + \alpha_l$.

Since $\beta_m = \beta_l$, then $\{\alpha_m, \alpha_{m+1}, \alpha_{m+2}\} = \{\alpha_l, \alpha_{l+1}, \alpha_{l+2}\}$, and then, using the recurrence, $\alpha_{m+3} = \alpha_{l+3}$. But then

$$\{\alpha_{m+1}, \alpha_{m+2}, \alpha_{m+3}\} = \{\alpha_{l+1}, \alpha_{l+2}, \alpha_{l+3}\},$$

and similarly, $\alpha_{m+4} = \alpha_{l+4}$. Continuing inductively, we obtain $\alpha_{m+s} = \alpha_{l+s}$ for $s = 3, 4, \ldots, l-m+2$, which shows that the sequence $\{\alpha_n\}$ is periodic with period of length $l-m$, and therefore, the number r is a repeating decimal, and therefore rational, as claimed. More precisely,

$$r = 0.\alpha_1\alpha_2\ldots\alpha_{m+2}\overline{\alpha_{m+3}\alpha_{m+4}\ldots\alpha_{m+(l-m+2)}}\ldots,$$

where the over-lined part is the repetend.

The following result generalizes the observations presented in the last two problems: Let $\{a_n\}$ be a sequence defined by

$$a_{n+1} = f(a_n, a_{n-1}, \ldots, a_{n-k+1}),$$

where $n \geq k \geq 1$, f is a polynomial of $a_n, a_{n-1}, \ldots, a_{n-k+1}$ with integer coefficients, and $a_1, a_2, \ldots a_k$ are integers. Let r_n be the remainder on division of a_n by the positive integer m (so that $a_n = mq + r_n$, where q is an integer and $0 \leq r_n \leq m-1$). Then the sequence $\{r_n\}$ is periodic.

More Problems

Here we list some problems without solutions; most of these can be solved by using ideas and approaches discussed in the previous sections.

1. Show that $2^n - 3$ is divisible by 5 for infinitely many values of the integer n.

2. $^{(*)}$ The sequence $\{a_n\}$ is defined by $a_1 = a_2 = a_3 = 1$ and
$$a_{n+1} = \frac{1 + a_{n-1}a_n}{a_{n-2}}.$$
Show that a_n is an integer for all $n \geq 1$.

3. $^{(*)}$ The sequence $\{a_n\}$ is defined by
$$a_n = \frac{a_{n-1}^2 + c}{a_{n-2}},$$
where c is a constant. If a_1, a_2 and $\frac{a_1^2+a_2^2+c}{a_1a_2}$ are integers, then show that a_n is an integer for all $n \geq 1$.

4. The sequence $\{a_n\}$ is defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1}a_n + 1$ for $n \geq 1$. Show that no term of this sequence is divisible by 4.

5. $^{(*)}$ The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 2$ and
$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n & \text{if } a_na_{n+1} \text{ is even,} \\ a_{n+1} - a_n & \text{if } a_na_{n+1} \text{ is odd.} \end{cases}$$
Show that this sequence contains infinitely many positive and infinitely many negative terms.

6. $^{(*)}$ (Putnam'1999) The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$ and
$$a_n = \frac{6a_{n-1}^2a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$
Show that a_n is an integer multiple of n for all $n \geq 1$.

7. Show that
$$\lfloor (45 + \sqrt{1975})^{2016} \rfloor$$
is odd. (Recall that $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

8. $^{(*)}$ Show that
$$\lfloor (\sqrt{3} + 1)^{2m} \rfloor + 1$$
is divisible by 2^{m+1} for every integer $m \geq 0$.

9. $^{(*)}$ Show that $(55 + \sqrt{3035})^{2k+1}$ has a decimal expansion starting with at least $3k + 2$ consecutive identical digits after the decimal point.

10. $^{(*)}$ (Putnam'1998) Let N be a positive integer with 1998 decimal digits, all of them 1; that is,
$$N = \underbrace{11\ldots1}_{1998}.$$
Find the thousandth digit after the decimal point of $\sqrt{N}$.
(Answer: 1)

11. Let $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Show that
$$\sum_{i=1}^{n} a_{2i} = a_{2n+1} - 1.$$

12. Show that the sequence $\{a_n\}$ defined by $a_1 = 1$ and
$$a_{n+1} = 3 - \frac{1}{a_n}$$
is convergent and find its limit. (Answer: $\lim_{n\to\infty} a_n = \frac{3+\sqrt{5}}{2}$.)

13. The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by $a_1 = 0$, $a_{n+1} = \frac{1}{2}(b + a_n^2)$ if $n \geq 1$, where $0 \leq b \leq 1$. Show that this sequence is convergent and find its limit. (Answer: The limit is $1 - \sqrt{1-b}$.)

14. The sequence $\{x_n\}$ is defined by $x_1 = a$, $x_2 = b$ and
$$x_n = \frac{x_{n-1} + x_{n-2}}{2} \quad \text{for} \quad n \geq 3.$$
Show that the sequence is convergent and find its limit.
(Answer: $\lim_{n\to\infty} x_n = \frac{a+2b}{3}$.)

15. The sequence $\{x_n\}$ is defined by $x_0 > 0$, and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$ for $n \geq 0$. Show that the sequence is convergent and find its limit.

16. The sequence $\{x_n\}$ is defined by $x_1 = x_2 = 1$, and
$$x_{n+1} = \frac{1}{x_{n-1} + \frac{1}{x_n}} \quad \text{for} \quad n \geq 3.$$
Show that the sequence is convergent and find its limit.

17. The sequence $\{x_n\}$ is defined by $x_1 = 4$, and
$$x_{n+1} = \frac{n+2}{2(n+1)}(x_n + 1) \quad \text{for} \quad n \geq 1.$$
Show that the sequence is convergent and find its limit.

18. The sequence $\{a_n\}$ is defined by $a_{n+2} - 2a_{n+1} + a_n = A$. Find $\lim_{n\to\infty} \frac{a_n}{n^2}$.
(Answer: The limit is $\frac{A}{2}$.)

19. The sequence $\{x_n\}$ is defined by $x_1 = a$, and $x_n = x_{n-1}^2 + 3x_n + 1$ for $n \geq 2$. Find the values of a for which this sequence is convergent.
(Answer: $a \in [-2, -1]$.)

20. The sequences $\{a_n\}$ and $\{b_n\}$ are defined by $a_1 = a$, $b_1 = b$, $a,\ b \geq 0$, $a_{n+1} = \frac{a_n + b_n}{2}$, and $b_{n+1} = \frac{b_n + a_{n+1}}{2}$. Show that both sequences are convergent and have the same limit. (Answer: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{a+2b}{3}$.)

21. $^{(*)}$ The sequences $\{x_n\}$ and $\{y_n\}$ are defined by $x_1 = a$, $y_1 = b$, $x_{n+1} = \sqrt{x_n y_n}$, and $y_{n+1} = \frac{x_n + y_n}{2}$. Show that both sequences are convergent and have the same limit.

22. The sequence $\{a_n\}_{n=1}^{\infty}$ satisfies the recurrence relation $a_n - a_{n+1} = a_n a_{n+1} + 1$. Find the first four terms of the sequence, if $a_{2017} = c$.

23. $^{(*)}$ The sequences $\{a_n\}$ and $\{b_n\}$ are defined by $1 < a_1 < b_1$ and

$$a_n = \frac{b_{n-1}}{a_{n-1}},\ b_n = \frac{b_{n-1} - 1}{a_{n-1} - 1} \quad \text{for } n \geq 2.$$

Find all values of a_1 and b_1 for which both sequences are convergent.
(Answer: $a_1 = \frac{1+\sqrt{5}}{2}, b_1 = \frac{3+\sqrt{5}}{2}$. Hint: Compute the first seven terms of each sequence. Experiment helps!)

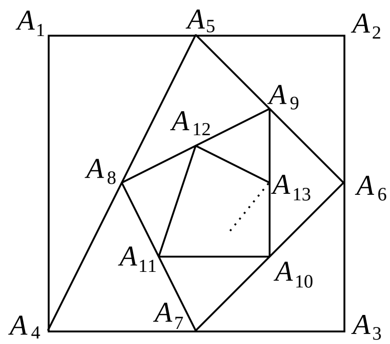
24. The sequence $\{a_n\}$ is defined by $a_2 = \frac{3}{4}$ and $a_n = a_{n-1}\left(1 - \frac{1}{n^2}\right)$ for $n \geq 3$. Find out if the sequence is convergent and if so, find its limit.

25. The sequence $\{s_n\}$ is defined by $s_1 = \frac{1}{2}$ and $s_n = s_{n-1} + \frac{2n-1}{2^n}$ for $n \geq 2$. Find out if the sequence is convergent and if so, find its limit.

26. The sequence $\{a_n\}$ is defined by $a_2 = \frac{2}{3}$ and $a_n = a_{n-1}\frac{n^2+n-2}{n^2+n}$ for $n \geq 3$. Find out if the sequence is convergent and if so, find its limit.

27. The sequence $\{x_n\}$ is defined by $x_0 = 1 + a$, $|a| < 1$ and $x_n = x_{n-1}(1+a)^{2^n}$. Find out if the sequence is convergent and if so, find its limit.

28. (Crux 1679, 8/1991) $A_1A_2A_3A_4$ is a unit square in the plane, with $A_1(0,1)$, $A_2(1,1)$, $A_3(1,0)$, $A_4(0,0)$. A_5 is the midpoint of the segment A_1A_2, A_6 is the midpoint of A_2A_3, A_7 is the midpoint of A_3A_4, A_8 is the midpoint of A_4A_5, and so on. This forms a spiral polygonal path $A_1A_2A_3A_4A_5A_6A_7A_8\ldots$ converging to a unique point inside the square. Find the coordinates of this point. (Answer: $\left(\frac{4}{7}, \frac{3}{7}\right)$.)



29. (*) (American Mathematical Monthly 10695, 9/1998) Find

$$\lim_{n\to\infty} \frac{1}{n} \left\lceil \frac{3n^2+2n}{3n^2} \left\lceil \frac{3n^2+2n-1}{3n^2-1} \cdots \left\lceil \frac{3n^2+1}{3n^2-2n+1} \right\rceil \cdots \right\rceil \right\rceil .$$

(Note that $\lceil x \rceil$ denotes the least integer $\geq x$)

30. (*) (Crux 1705, 1/1992) Let $n \geq 2$ and $b_0 \in [2, 2n-1]$ be integers, and consider the recurrence

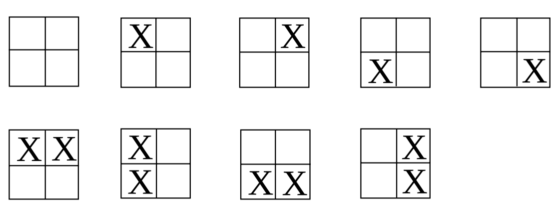
$$b_{i+1} = \begin{cases} 2b_i - 1 & \text{if } b_i \leq n, \\ 2b_i - 2n & \text{if } b_i > n. \end{cases}$$

Let $p = p(b_0, n)$ be the smallest positive integer such that $b_p = b_0$.

(a) Find $p(2, 2^k)$ and $p(2, 2^k+1)$ for all $k \geq 1$.

(b) Prove that $p(b_0, n)$ divides $p(2, n)$.

31. (*) (Crux 1709, 1/1992) Find the number of ways to choose cells from a $2 \times n$ "chessboard" so that no two chosen cells are next to each other **diagonally** (one way is to choose no cells at all). For example, for $n = 2$ the number of ways is 9, namely



Answer: $\frac{1}{5}\left[\left(\frac{3-\sqrt{5}}{2}\right)^{n+2} + \left(\frac{3+\sqrt{5}}{2}\right)^{n+2} - 2(-1)^{n+2}\right]$.

32. (*) (Crux 1853, 6/1993) Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers which satisfies the condition

$$3b_{n+2} \geq b_{n+1} + 2b_n$$

for every $n \geq 1$. Prove that either the sequence converges or $\lim_{n\to\infty} b_n = \infty$.

Solutions and Answers to the Exercises

1. The correct answer is (4).
2. The characteristic equation is
$$2r^2 - 5r + 2 = 0$$
with roots 2 and 1/2, so that
$$a_n = C_1 2^n + C_2 \left(\frac{1}{2}\right)^n .$$
Writing out the general form for $n = 0$ and $n = 1$ and using the conditions $a_0 = 2$ and $a_1 = \frac{5}{2}$, we obtain the system
$$\begin{aligned} 2 &= a_0 = C_1 + C_2 \\ \tfrac{5}{2} &= a_1 = 2C_1 + \tfrac{1}{2}C_2 \end{aligned}$$
Solving the system gives $C_1 = C_2 = 1$. Therefore,
$$a_n = 2^n + \left(\frac{1}{2}\right)^n .$$
3. The characteristic equation is
$$r^2 - 4r + 4 = 0$$
with $r = 2$ a root of multiplicity 2, so that
$$a_n = C_1 2^n + C_2 n 2^n.$$
Writing out the general form for $n = 1$ and $n = 2$ and using the conditions $a_1 = 16$ and $a_2 = 52$, we obtain the system
$$\begin{aligned} 16 &= a_1 = 2C_1 + 2C_2 \\ 52 &= a_2 = 2^2 C_1 + 2 \cdot 2^2 C_2 \end{aligned}$$
Solving the system gives $C_1 = 3$ and $C_2 = 5$. Therefore,
$$a_n = 3 \cdot 2^n + 5n2^n = (3 + 5n)2^n.$$
4. Answer: $a_n = (-2 + 4n)3^n$.
5. Answer: $a_n = 5 \cdot 2^n + 2(-3)^n$.
6. Answer: $a_n = \frac{1}{8}\left[9 - (-7)^n\right]$.
7. Answer: $a_n = 7 \cdot 2^n - 3 - 4n$.

8. Answer: $a_n = -4 + (4 - n)2^n$.

9. The defining equality is true for every $n \geq 0$. Writing it out for n and $n+1$, we obtain

$$a_{n+3} - 2a_{n+2} + a_{n+1} = 2 = a_{n+2} - 2a_{n+1} + a_n,$$

from which we get the recurrence

$$a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0.$$

The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = 0$$

or

$$(r-1)^3 = 0,$$

with $r = 1$ a root of multiplicity 3. Hence the general form of the n-th term of the sequence is

$$a_n = C_1 + C_2 n + C_3 n^2.$$

Writing out the general form for $n = 0$, $n = 1$ and $n = 2$ and using the initial conditions $a_0 = 3$, $a_1 = 6$ and $a_2 = 11$ (the last follows from $a_2 = 2a_1 - a_0 + 2 = 11$), we obtain the system

$$\begin{aligned} 3 &= a_0 = C_1 \\ 6 &= a_1 = C_1 + C_2 + C_3 \\ 11 &= a_2 = C_1 + 2C_2 + 4C_3 \end{aligned}$$

Solving this system gives $C_1 = 3$, $C_2 = 2$ and $C_3 = 1$. Therefore,

$$a_n = n^2 + 2n + 3.$$

10. The defining equality is true for every $n \geq 1$, Writing it out for n and $n+1$, we obtain

$$a_{n+1} - 3a_n = 5^n = 5 \cdot 5^{n-1} = 5(a_n - 3a_{n-1}),$$

from which we get the recurrence

$$a_{n+1} - 8a_n + 15a_{n-1} = 0.$$

The characteristic equation is

$$r^2 - 8r + 15 = 0$$

with roots $r = 3$ and $r = 5$. Hence the general form of the n-th term of the sequence is

$$a_n = C_1 3^n + C_2 5^n.$$

Writing out the general form for $n = 0$ and $n = 1$ and using the initial conditions $a_0 = 1$ and $a_1 = 4$ (the last follows from $a_1 = 3a_0 + 5^0 = 4$), we obtain the system

$$\begin{aligned} 1 &= a_0 = C_1 + C_2 \\ 4 &= a_1 = 3C_1 + 5C_2 \end{aligned}$$

Solving this system gives $C_1 = C_2 = \frac{1}{2}$. Therefore,

$$a_n = \frac{1}{2}3^n + \frac{1}{2}5^n = \frac{1}{2}(3^n + 5^n).$$

11. The defining equality is true for every $n \geq 1$, Writing it out for n and $n+1$, we obtain

$$a_{n+2} - 5a_{n+1} = 16 = a_{n+1} - 5a_n,$$

from which we get the recurrence

$$a_{n+2} - 6a_{n+1} + 5a_n = 0.$$

The characteristic equation is

$$r^2 - 6r + 5 = 0$$

with roots $r = 5$ and $r = 1$. Hence the general form of the n-th term of the sequence is

$$a_n = C_1 5^n + C_2 1^n.$$

Writing out the general form for $n = 1$ and $n = 2$ and using the initial conditions $a_1 = 1$ and $a_2 = 21$ (the last follows from $a_2 = 5a_1 + 16 = 21$), we obtain the system

$$\begin{aligned} 1 &= a_1 = 5C_1 + C_2 \\ 21 &= a_2 = 25C_1 + C_2 \end{aligned}$$

Solving this system gives $C_1 = 1$ and $C_2 = -4$. Therefore,

$$a_n = 5^n - 4.$$

12. Note that this recurrence is an example of a linear recurrence of order two with non-constant coefficients. In order to reduce the order of the recurrence, we rewrite the condition as

$$d_n - nd_{n-1} = -d_{n-1} + (n-1)d_{n-2}.$$

Let $a_n = d_n - nd_{n-1}$. The above recurrence becomes

$$a_n = (-1)a_{n-1}.$$

Note also that $a_2 = 1$. Now,

$$a_n = (-1)a_{n-1} = (-1)^2 a_{n-2} = ... = (-1)^{n-2} \underbrace{a_2}_{=1} = (-1)^n,$$

so that, $d_n - nd_{n-1} = (-1)^n$, which gives another defining recurrence for our sequence $\{d_n\}$. Note that the new recurrence has order one. We can now express d_n as a function of n by writing the defining recurrence for $n, n-1, ..., 2, 1$ and manipulating the n equations.

$$\begin{array}{rcll} d_n - nd_{n-1} & = & (-1)^n & \\ d_{n-1} - (n-1)d_{n-2} & = & (-1)^{n-1} & \mid \times n \\ d_{n-2} - (n-2)d_{n-3} & = & (-1)^{n-2} & \mid \times n(n-1) \\ \dots & & & \mid \dots \\ d_2 - 2 \cdot d_1 & = & (-1)^2 & \mid \times n(n-1)\dots 3 \\ d_1 - 1 \cdot \underbrace{d_0}_{=1} & = & (-1)^1 & \mid \times n(n-1)\dots 3 \cdot 2 \end{array}$$

Multiplying the i-th equation by $n(n-1)...(n-i+2)$, $i = 2, 3, ..., n$, adding all of the new equations and canceling, we obtain

$$\begin{aligned} d_n &= (-1)^n + n(-1)^{n-1} + n(n-1)(-1)^{n-2} + \cdots \\ &\quad + \frac{n!}{2!}(-1)^2 + n!(-1)^1 + n! \underbrace{d_0}_{=(-1)^0} \\ &= n! \left(\frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} + ... + \frac{(-1)^2}{2!} + \frac{(-1)^1}{1!} + \frac{(-1)^0}{0!} \right) \\ &= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - ... + (-1)^{n-1} \frac{1}{(n-1)!} + (-1)^n \frac{1}{n!} \right) \end{aligned}$$

13. From the first recurrence, we have $4b_n = -a_{n+1} - 2a_n$.
Substitute in $4b_{n+1} = 16a_n + 6(4b_n)$ (which is just the second recurrence multiplied by four) to obtain

$$-a_{n+2} - 2a_{n+1} = 16a_n + 6(-a_{n+1} - 2a_n)$$

and then

$$a_{n+2} - 4a_{n+1} + 4a_n = 0.$$

The characteristic equation of the last recurrence is

$$r^2 - 4r + 4 = 0$$

with $r = 2$ a root of multiplicity 2. It follows that the general solution of the recurrence is

$$a_n = C_1 2^n + C_2 n 2^n.$$

We have $a_1 = -2a_0 - 4b_0 = -2$, so that we can obtain a system for the coefficients C_1 and C_2:

$$\left. \begin{array}{r} 1 = a_0 = C_1 2^0 + C_2(0)2^0 \\ -2 = a_1 = C_1 2^1 + C_2(1)2^1 \end{array} \right\} \Longrightarrow C_1 = 1,\ C_2 = -2$$

Hence $a_n = 2^n - (2n)2^n = (1-2n)2^n$. Then

$$4b_n = -a_{n+1} - 2a_n = -[1-2(n+1)]2^{n+1} - 2(1-2n)2^n = 4n2^{n+1},$$

so that $b_n = n2^{n+1}$. Therefore, the solution to the system of recurrences is

$$a_n = (1-2n)2^n, \quad b_n = n2^{n+1}.$$

14. Answer:

$$\begin{aligned} a_n &= \frac{1}{2}\left[(1+\sqrt{3})^n + (1-\sqrt{3})^n\right] \\ b_n &= \frac{1}{2\sqrt{3}}\left[(1+\sqrt{3})^n - (1-\sqrt{3})^n\right] \end{aligned}$$

15. Answer:

$$\begin{aligned} a_n &= \frac{1}{2}\left[(2+\sqrt{2})^n + (2-\sqrt{2})^n\right] \\ b_n &= \frac{1}{2\sqrt{2}}\left[(2+\sqrt{2})^n - (2-\sqrt{2})^n\right] \end{aligned}$$

16. Let a_n be the number of required strings. The cases a_1 and a_2 are trivial. If $b_1 = 1$, then there are a_{n-1} ways to complete a string $b_1b_2...b_k$ such that $\sum_{i=1}^k b_i = n$. If $b_1 = 2$, then there are a_{n-2} ways to to complete a string $b_1b_2...b_k$ such that $\sum_{i=1}^k b_i = n$. Therefore, for all $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$, as claimed.

17. We count the number of the required n-digit binary numbers, $n \geq 4$, in two different ways. One count is S_n. For a second count, we observe that there are three types of n-digit binary numbers:

    (a) those beginning with 0; there are S_{n-1} of them;

    (b) those beginning with 10; there are S_{n-2} of them;

    (c) those beginning with 11; there are S_{n-3} of them, because these numbers must start with 110 (the choice of the third digit is forced by the condition).

    Using the two counts, we obtain

    $$S_n = S_{n-1} + S_{n-2} + S_{n-3}, \quad n \geq 4.$$

    The sequence $\{S_n\}_{n=1}^{\infty}$ will be completely determined if we know S_1, S_2 and S_3. These are easily computed to be $S_1 = 2$, $S_2 = 4$ and $S_3 = 7$.

18. We count the number of the required n-digit ternary numbers, $n \geq 3$, in two different ways. One count is a_n. For a second count, we observe that there are four types of n-digit ternary numbers:

(a) those beginning with 1; there are a_{n-1} of them;

(b) those beginning with 2; there are a_{n-1} of them;

(c) those beginning with 01; there are a_{n-2} of them;

(d) those beginning with 02; there are a_{n-2} of them.

Using the two counts, we obtain

$$a_n = 2a_{n-1} + 2a_{n-2}, \quad n \geq 3.$$

The sequence $\{a_n\}_{n=1}^{\infty}$ will be completely determined if we know a_1 and a_2. These are easily computed to be $a_1 = 3$ and $a_2 = 8$.

19. Let a_n be the number of n-digit ternary numbers that have no 1 immediately to the right of any 0. Let A be the set of all such numbers. Then $A = A_0 \cup A_1 \cup A_2$, where A_i is the number of those starting with i, $i = 0, 1, 2$. Clearly, $|A| = |A_0|+|A_1|+|A_2|$, because A_0, A_1 and A_2 are pairwise disjoint. Also, $|A_1| = |A_2| = a_{n-1}$, because an n-digit valid number can be obtained by adding a first digit 1 or 2 to any $(n-1)$-digit valid ternary number. There are a_{n-1} n-digit ternary numbers starting in 0. However, those starting in 01 are not valid; there are a_{n-2} of them. Hence $|A_0| = a_{n-1} - a_{n-2}$. Thus

$$a_n = |A| = |A_0| + |A_1| + |A_2| = a_{n-1} - a_{n-2} + 2a_{n-1},$$

so that, $a_n = 3a_{n-1} - a_{n-2}$. It is easy to check directly that $a_1 = 3$ and $a_2 = 8$. These initial conditions and the recurrence completely determine the sequence $\{a_n\}_{n=1}^{\infty}$.

20. It is easy to check that $a_1 = 3$ and $a_2 = 7$. For each of the desired $(n-1)$-digit ternary numbers σ generate two n-digit ternary numbers as follows:

    if σ begins with 0, generate 1σ and 2σ;

    if σ begins with 1, generate 0σ and 2σ;

    if σ begins with 2, generate 0σ and 1σ. Clearly, we have generated $2a_{n-1}$ n-digit ternary numbers of the desired kind. The only n-digit ternary numbers we missed are those beginning with 22 of which there are a_{n-2}. Therefore, $a_n = 2a_{n-1} + a_{n-2}$. This recurrence, together with the conditions $a_1 = 3$ and $a_2 = 7$, completely describe the answer a_n to our counting problem.

21. It is easy to notice that $a_1 = 1$ and $a_2 = 2$. Let $n = q_1 + q_2 + ... + q_k$ be a representation of n as a sum of odd positive integers. There are a_n such representations.

    Case 1. If $q_1 = 1$, then we have

$$n - 1 = q_2 + q_3 + ... + q_k,$$

a representation of $n - 1$ as a sum of odd positive integers; there are a_{n-1} such representations.

Case 2. If $q_1 \geq 3$, then we have

$$n - 2 = (q_1 - 2) + q_2 + ... + q_k,$$

a representation of $n - 2$ as a sum of odd positive integers; there are a_{n-2} of these.

Therefore, $a_n = a_{n-1} + a_{n-2}$. This recurrence, together with the conditions $a_1 = 1$ and $a_2 = 2$, completely determine the answer a_n to our counting problem.

22. Let E be the set of nonempty words on $\{a, b, c\}$ containing an even number of as, and Ω the set of nonempty words on $\{a, b, c\}$ containing an odd number of as. Let x_n and y_n denote the numbers of words of length n in E and Ω, respectively.

Every word in E of length $n > 1$ can be written uniquely in one of the forms σb, σc, or τa, where σ is in E, or τ is in Ω as the case may be. Conversely, for any σ in E of length $n - 1$, and any τ in Ω of length $n - 1$, the words σb, σc, and τa are in E, and have length n. Therefore, for all $n > 1$,

$$x_n = 2x_{n-1} + y_{n-1},$$

and similarly,

$$y_n = 2y_{n-1} + x_{n-1}.$$

Now we have to solve the system

$$\begin{cases} x_n = 2x_{n-1} + y_{n-1} \\ y_n = 2y_{n-1} + x_{n-1} \end{cases}$$

From the first equation, $y_{n-1} = x_n - 2x_{n-1}$, and therefore, $y_n = x_{n+1} - 2x_n$. Substituting into the second equation of the system, we obtain

$$x_{n+1} - 2x_n = 2(x_n - 2x_{n-1}) + x_{n-1},$$

which simplifies to

$$x_{n+1} - 4x_n + 3x_{n-1} = 0.$$

It is easy to find $x_1 = 2$ (the words of length 1 are b and c) and $x_2 = 5$ (the words of length 2 are aa, bc, cb, bb and cc). The characteristic equation is $r^2 - 4r + 3 = 0$ with roots 3 and 1, so that,

$$x_n = A3^n + B.$$

Using $x_1 = 2$ and $x_2 = 5$, we can write a system for the coefficients A and B:

$$\begin{cases} 2 = x_1 = 3A + B \\ 5 = x_2 = 9A + B \end{cases},$$

from which $A = B = 1/2$. Therefore,

$$x_n = \frac{3^n + 1}{2}.$$

ATOM

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