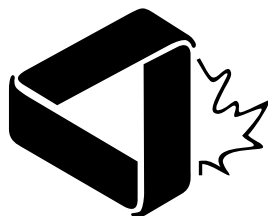# A Taste Of Mathematics

Aime–T–On les Mathématiques

Volume / Tome XIV

SEQUENCES AND SERIES

Margo Kondratieva
with Justin Rowsell

Memorial University

The ATOM series

The booklets in the series, **A Taste of Mathematics**, are published by the Canadian Mathematical Society (CMS). They are designed as enrichment materials for high school students with an interest in and aptitude for mathematics. Some booklets in the series will also cover the materials useful for mathematical competitions at national and international levels.

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Margo Kondratieva is an Associate Professor at the Memorial University (MUN). She was born in Moscow (Russia) and received her Master of Science degree in the area of Applied Mathematics before gaining her PhD in the area of Quantum Mechanics. After working at the University of Manitoba and University of Minnesota – Duluth (USA), she arrived at MUN, where she currently holds a joint appointment in the Department of Mathematics and Statistics and the Faculty of Education.

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Justin Rowsell was born in Newfoundland. He was a student of several calculus and analysis courses taught by Dr. Kondratieva in 2004-2005 at MUN. This booklet has resulted from a collaborative work during that period, aimed at collecting interesting and non-standard problems related to summations of infinite series and explained using mostly elementary techniques. Mr. Rowsell has completed a master's degree in Pure Mathematics from the University of Waterloo.

FOREWORD

Secondary school students are often familiar with finite arithmetic and geometric series. Those who attempt a more advanced level of study become introduced to infinite series and some formal techniques of their summation. However, many interesting, non-standard, and important examples remain outside of students' view and experience.

One reason is a cognitive difficulty related to dealing with infinite sequences of numbers and the notion of convergence. The formal 'epsilon-delta' definition of limit often presents a challenge even for university level students. Historically, there were different approaches and interpretations of the concept of convergent series before it received its modern interpretation. We found it important to refer to a few such historical facts, dates, and names for the reader' general information.

In this book, while maintaining a rigorous approach, we use a more intuitive treatment of the topic. We refer to mostly elementary techniques involving solving algebraic inequalities, linear and quadratic equations. We believe that the ideas we explain and illustrate with many examples can be understood at the secondary school level and help to develop a genuine understanding of the topic. An advanced familiarity with the topic may foster a deeper study of mathematics at the university level.

Some of our problems are connected to Euclidean geometry or reveal other links with topics studied at the secondary school level. We also illustrate how infinite sums may appear while solving some word problems that do not explicitly refer to series and convergence. We talk about some practical applications, such as calculations with an approximation. As well, we introduce some notions and objects that are extremely important in modern mathematics, for example, the Riemann zeta function and the Dirichlet kernel. We hope that reading this book and solving the exercises will stimulate students' interest and fascination with this amazing area of mathematics.

"It is always possible to think of a larger number..."
Aristotle.

INTRODUCTION

The word *infinity* comes from the Latin infinitus, "unbounded", or unlimited in quantity. Infinity is denoted by ∞, and is considered to be not a number but a concept of increase beyond bounds. When we write $n \to \infty$ we mean that variable n eventually grows beyond any assigned value or exceeds any finite restriction.

It is the 5th century B.C., Greece. Zeno thinks about paradoxes related to the notion of infinity, and asks "How can one get from point A to point B at all, if to do that one needs to go from A to the mid-point C and then to the mid-point between C and B, and so on, and thus to make an infinite number of iterations?"
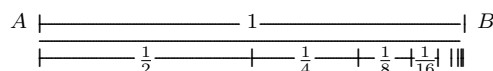
Figure 1. Zeno's paradox.

In a response, Diogenes stands up and simply walks from A to B. In modern mathematical language this story looks as follows:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

Here, the dots symbolize the addition of an infinite number of terms, where each term is one half as large as the previous one.

The left hand side of the formula is an example of what is called an *infinite series*. As we see, despite the fact that it has an infinite number of terms, the result is a finite number. This formula shows that the distance between A and B is finite, but it can be broken into an infinite number of parts. If an infinite series has a finite sum, as in our example, it is called a *convergent* series.

Starting from the first term of the infinite series and adding the subsequent terms one by one, we will obtain numbers —

$$S_1 = \frac{1}{2}, \qquad S_2 = \frac{1}{2} + \frac{1}{4}, \qquad S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \qquad S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \quad \cdots$$

— none of which can be regarded as a final result, but which approach the final result closer and closer as we sum more and more terms of the series. We can see it better if we rewrite the previous line as

$$S_1 = 1 - \frac{1}{2}, \qquad S_2 = 1 - \frac{1}{4}, \qquad S_3 = 1 - \frac{1}{8}, \qquad S_4 = 1 - \frac{1}{16}, \quad \ldots.$$

The number which is approached in this infinite process is called the *sum* of the infinite series.

There are some obvious situations where the process of accumulating more and more summands does not lead to a number. For example, if you add the same number infinitely many times

$$1 + 1 + 1 + 1 + \cdots .$$

Adding terms one by one —

$$S_1 = 1, \quad S_2 = 1 + 1, \quad S_3 = 1 + 1 + 1, \quad S_4 = 1 + 1 + 1 + 1, \ldots,$$

— S_n becomes arbitrarily large. In this case we say that the infinite series *diverges to infinity*.

Another example is the infinite series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots .$$

One can see that adding any odd number of terms we obtain 1, while the sum of any even number of terms gives 0. That is, in the process of adding more terms neither of the numbers is approached (in contrast with the first example). Such an infinite series is also said to be *divergent*.

There are many examples, such as

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots ,$$

in which the answer to the question about convergence is not easy to find.

In this book we talk about infinite series; under what conditions they can be represented by a finite number (are convergent), and how to find that number. This subject requires careful treatment. Infinite series differ from finite series, and thus you may not always trust your intuition based on your experience with finite algebraic sums. Formal manipulations can easily lead to nonsense. Treating our third example incorrectly, one can "prove" that $0 = 1$ as follows

$$0 = (1 - 1) + (1 - 1) + (1 - 1) + \cdots = 1 + (-1 + 1) + (-1 + 1) + \cdots = 1.$$

This paradoxical regrouping is possible because the number of terms is infinite. Notice that the expressions in parentheses are all zeros. Historically, this example was given at the very beginning of 18th century by the Italian scholar Guido Grandi to demonstrate the possibility of creating *everything* from *nothing*.

Another story from the past *assigns* the value 1/2 to the series

$$1 - 1 + 1 - 1 + 1 - 1 \cdots = \frac{1}{2} \quad \text{(false)}$$

symbolizing that if two people share something indivisible (like a diamond ring), to be fair they should take turns owning it; one year the first person owns the ring, next year the other one, and so on, forever. Interestingly, Gottfried Leibniz mentioned this incorrect result in a paper of 1674, in connection with geometric series (which we will discuss in Section 2.3).

We will have a few more examples of infinite series which are contradictory at first glance, and we will explain how to deal with them to avoid fallacies.

But, before starting our discussion of infinite series, we need to talk a little about infinite *sequences of numbers*, the terms of which may be summed to form series.

CHAPTER 1: SEQUENCES

§1.1 What is a sequence?

Definition 1. A *sequence* is an ordered list of numbers. The numbers are called *terms* (or *elements*) of the sequence.

The numbers could represent measurements of temperature made day by day, the height of students in your class, or something else.

In mathematics, a sequence is often given by a formula for its n-th term, this is called the *explicit formula* for the sequence.

Example 1. Let the n-th term be $a_n = n^2$, where n runs through the familiar set of natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

We say that the first term of the sequence (corresponding to $n = 1$) is $a_1 = 1^2 = 1$, the second term (corresponding to $n = 2$) being $a_2 = 2^2 = 4$, the third term being $a_3 = 3^2 = 9$, and so on. Thus, the sequence given by the general term $a_n = n^2$ is $\{1, 4, 9, 16, 25, 36, \ldots\}$.

We will employ the notation

$$n \in \mathbb{N},$$

which is a short way of saying that a number n is an element of the set of natural numbers $\mathbb{N}$. In general, $x \in X$ means that x is an element of the set X. We will use this notation regularly from now on.

Example 2. The sequence with n-th term $a_n = n$, where $n \in \mathbb{N}$, is obviously the sequence of natural numbers $\{1, 2, 3, 4, 5, \ldots\}$. [1]

Example 3. The sequence with n-th term $a_n = \frac{100}{n}$, where $n \in \mathbb{N}$, is

$$\{100, 50, 100/3, 25, 20, \ldots\}.$$

Example 4. The sequence with n-th term $a_n = 1$, $n \in \mathbb{N}$, is an example of a *constant sequence*. All its terms have the same value.

Example 5. The sequence with n-th term $a_n = (-1)^n$, $n \in \mathbb{N}$, belongs to the class of *alternating sequences* since its terms alternate in sign: $\{-1, 1, -1, 1, -1, \ldots\}$.

We will talk mostly about sequences with an infinite number of terms, which can be denoted by $\{a_n\}_{n=1}^{\infty}$. We summarize the above examples and include some

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[1] We follow the tradition of using the same notation $\{1, 2, 3, 4, 5, \ldots\}$ for both the set and the sequence of natural numbers.

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more in the table, and will use them in our future discussion.

Sequence $\{a_n\}_{n=1}^{\infty}$	n-th term a_n	1st term a_1	2nd term a_2	3rd term a_3	4th term a_4	5th term a_5
Example 1	n^2	1	4	9	16	25
Example 2	n	1	2	3	4	5
Example 3	$\dfrac{100}{n}$	100	50	$\dfrac{100}{3}$	25	20
Example 4	1	1	1	1	1	1
Example 5	$(-1)^n$	-1	1	-1	1	-1
Example 6	$\dfrac{(-1)^n}{n}$	-1	$\dfrac{1}{2}$	$-\dfrac{1}{3}$	$\dfrac{1}{4}$	$-\dfrac{1}{5}$
Example 7	$\dfrac{n-1}{n+1}$	0	$\dfrac{1}{3}$	$\dfrac{2}{4} = \dfrac{1}{2}$	$\dfrac{3}{5}$	$\dfrac{4}{6} = \dfrac{2}{3}$
Example 8	$\dfrac{1}{3^n}$	$\dfrac{1}{3}$	$\dfrac{1}{9}$	$\dfrac{1}{27}$	$\dfrac{1}{81}$	$\dfrac{1}{243}$

Exercises

1. In each case list the first 10 terms of the sequence[2] $\{a_n\}_{n=1}^{\infty}$:

i) $a_n = -n$ ii) $a_n = n^3$ iii) $a_n = \dfrac{1}{n^2}$ iv) $a_n = 2^n$ v) $a_n = n!$

2. Find n-th term of the sequence

i) $\{2, 5, 10, 17, 26, ...\}$ ii) $\{-1, \dfrac{1}{2}, -\dfrac{1}{6}, \dfrac{1}{24}, -\dfrac{1}{120}, ...\}$

iii) $\{1, 3, 7, 15, 31, ...\}$ iv) $\{3, 7, 11, 15, 19, ...\}$

v) $\{-\dfrac{1}{2}, \dfrac{1}{5}, -\dfrac{1}{10}, \dfrac{1}{17}, -\dfrac{1}{26}, ...\}$ vi) $\{\dfrac{1}{3}, \dfrac{1}{2}, \dfrac{3}{4}, \dfrac{9}{8}, \dfrac{27}{16}, \dfrac{81}{32}...\}$

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[2]In (v) n-factorial denotes the product of all integers from 1 to n: $n! = n(n-1)(n-2)\cdots 1$. For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

§1.2 Sequences which almost fit in a given interval

We use square bracket notation $[A, B]$ for *closed* intervals. That is, the interval from A to B including the endpoints: $x \in [A, B]$ means $A \leq x \leq B$. An *open* interval, one which does not include its endpoints, is denoted by (A, B). Thus, $x \in (A, B)$ means $A < x < B$.

In this section we become familiar with the situation in which a sequence *almost* fits in a given open interval.

Definition 2. We say that *almost all* of the elements of a sequence $\{a_n\}_{n=1}^{\infty}$ belong to an interval (A, B) if all but a finite number of terms of the sequence satisfy the inequality $A < a_n < B$.

Problem 1. In the following cases say whether almost all of the terms of $\{a_n\}_{n=1}^{\infty}$ are in (A, B):

i) $(A, B) = (0, 200)$, $a_n = \dfrac{100}{n}$;

Answer: Yes. All terms of the sequence are in the interval: $0 < \dfrac{100}{n} < 200$ for all $n \geq 1$.

ii) $(A, B) = (0, 20)$, $a_n = \dfrac{100}{n}$;

Answer: Yes. Only the first five terms are outside the interval. To see that we solve inequality $\dfrac{100}{n} < 20$ and get $n > 5$. Thus, almost all of the terms of the sequence are in the interval.

iii) $(A, B) = (1, 200)$, $a_n = \dfrac{100}{n}$;

Answer: No. Only the first 99 terms belong to the interval, while the rest do not. Solving $1 < \dfrac{100}{n}$ we get $n < 100$. Thus, if $n \geq 100$, then $a_n \leq 1$, which places a_n outside the given interval.

iv) $(A, B) = (0, 2000)$, $a_n = \sqrt{n}$;

Answer: No. The inequality $\sqrt{n} < 2000$ is true only for $n < 4,000,000$. Thus only the first 3,999,999 terms of this sequence belong to the interval, while the rest do not.

v) $(A, B) = (0, 10)$, $a_n = (-1)^n$;

Answer: No. All terms of this sequence with an even index $a_2 = a_4 = a_6 = \cdots = 1$, and they do belong to the interval $(0, 10)$. However, all terms of this sequence with an odd index $a_1 = a_3 = a_5 = \cdots = -1$, and they do not belong to the interval $(0, 10)$.

Problem 2. Given a sequence, find an *example* (if possible) of an interval to which almost all of the terms of the sequence belong.

Example	Answer
E1: $a_n = n^2$, $n \in \mathbb{N}$	no finite interval contains almost all terms
E2: $a_n = n$, $n \in \mathbb{N}$	no finite interval contains almost all terms
E3: $a_n = \dfrac{100}{n}$, $n \in \mathbb{N}$	$(0, 0.2)$ but not $(0.1, 0.2)$
E4: $a_n = 1$, $n \in \mathbb{N}$	$(0.5, 1.4)$
E5: $a_n = (-1)^n$, $n \in \mathbb{N}$	$(-2, 2)$ but not $(0, 2)$ or $(-2, 1)$
E6: $a_n = \dfrac{(-1)^n}{n}$, $n \in \mathbb{N}$	$(-0.1, 0.22)$
E7: $a_n = \dfrac{n-1}{n+1}$, $n \in \mathbb{N}$	$(0.5, 1.01)$ but not $(0.5, 0.99)$

Instead of giving just an example, we can attempt to give a complete description of the intervals in Problem 2 as is required in the following problem.

Problem 3. Given a sequence, describe the intervals to which almost all of the terms of the sequence belong.

Example	Answer
E3: $a_n = \dfrac{100}{n}$, $n \in \mathbb{N}$	$(A,\ B)$ for any numbers $A \leq 0$ and $B > 0$.
E4: $a_n = 1$, $n \in \mathbb{N}$	$(A,\ B)$ with $A < 1$ and $B > 1$.
E5: $a_n = (-1)^n$, $n \in \mathbb{N}$	$(A,\ B)$ for any numbers $A < -1$ and $B > 1$.
E6: $a_n = \dfrac{(-1)^n}{n}$, $n \in \mathbb{N}$	$(A,\ B)$ for any numbers $A < 0$ and $B > 0$.
E7: $a_n = \dfrac{n-1}{n+1}$, $n \in \mathbb{N}$	$(A,\ B)$ for any numbers $A < 1$ and $B \geq 1$.

Note that the general description given in the answers for Problem 3 provides all possible particular examples which can serve as answers in Problem 2. The general answer comes from careful analysis of all possible particular answers.

Let us consider Example 7 in more details. First note that all terms of the sequence with n-th term $a_n = \dfrac{n-1}{n+1} = 1 - \dfrac{2}{n+1}$, where $n \geq 1$, are less than 1 and greater or equal to zero : $0 \leq a_n < 1$. Thus interval $(0,1)$ contains almost all terms of the sequence. Obviously, any interval larger than that will also work, so any interval (A, B) with $A \leq 0$ and $B \geq 1$ contains almost all terms of the sequence. Now we note that even if A is greater than 0, the interval $(A, 1)$ still works as long as A is less than 1. That is, almost all terms of the sequence from Example 7 belong to interval (A, B) for any $A < 1$ and $B \geq 1$.

Indeed, given $0 < C < 1$ we see that term a_n belongs to the interval $(1 - C, 1)$ for any $n > \dfrac{2 - C}{C}$. To confirm that we simply solve the inequality

$$1 - C < 1 - \frac{2}{n+1}.$$

We can try some concrete numbers to make sense of this statement. For example, take $C = 0.1$ and observe that $0.9 < a_n < 1$ for all $n \geq 20$; similarly, take $C = 0.01$ and note that $0.99 < a_n < 1$ for all $n \geq 200$.

Exercises

1. Decide whether or not almost all the terms of $\{a_n\}_{n=1}^{\infty}$ belong to the given interval:

   i) $a_n = \dfrac{1}{n}$, $(0.05, 0.1)$ ii) $a_n = \pi^{-n}$, $(0, 1)$ iii) $a_n = \dfrac{1}{2^{-n}}$, $(0, 5)$

2. For the sequence $a_n = \dfrac{1}{3^n}$, $n \in \mathbb{N}$, (see Example 8) give an example of an interval to which almost all (but not all) of the terms belong (as in Problem 2). Then describe all the intervals to which almost all of the terms of this sequence belong (as in Problem 3).

3. For each sequence $\{a_n\}_{n=1}^{\infty}$ either find an interval to which almost all (but not all) of the terms belong or show that there is no finite interval containing almost all terms.

   i) $a_n = \dfrac{1}{n}$ ii) $a_n = n^5$ iii) $a_n = \sqrt{n}$ iv) $a_n = \sin \sqrt{n}$

4. Justify the answers given in Problem 3 for sequences in Examples 3, 4, 5, 6.

§1.3 What is the limit of a sequence?

As n increases, the terms a_n may approach a certain number. If this is the case, we say that the *sequence has a limit* or, equivalently, that the *sequence converges*. Intuitively, a limit is a target or a destination point for the sequence. Some sequences reach their target; for example $a_n = 1$, $n \in \mathbb{N}$, is always there. Some other sequences never reach their limits; for example $a_n = \frac{100}{n}$, $n \in \mathbb{N}$, never reaches the value zero – it gets extremely close to zero but no term of the sequence actually takes that value. A sequence can also approach its target not only from one side but from both sides: $a_n = \frac{(-1)^n}{n}$, $n \in \mathbb{N}$, approaches zero while the values a_n are alternately positive and negative.

Following the approach of German mathematician Heinrich Eduard Heine, who lived in the 19-th century, we will employ the following definition of the limit.

Definition 3. A number L is called the *limit* of a sequence $\{a_n\}_{n=1}^{\infty}$ as $n \to \infty$ if **every** open interval containing L also contains almost all terms of the sequence. Then we write

$$\lim_{n \to \infty} a_n = L.$$

Comments:

1. Note that at most one number L can be the limit of $\{a_n\}_{n=1}^{\infty}$ as $n \to \infty$.

2. The word **every** in the definition is significant and refers to *all without exception*. For example, suppose that someone tries to argue that $a_n = \frac{100}{n}$ has limit 1 because, for all $B > 1$, the interval $(0, B)$ contains 1 as well as almost all terms of the sequence. The inadequacy of this argument is clear when one points out that $(0.5, 1.5)$ contains 1 but only 133 terms of the sequence (for $67 \leq n \leq 199$). Thus 1 is not the limit of the sequence.

Problem 4. Show that the limit as $n \to \infty$ of the sequence $a_n = \frac{100}{n}$ is zero:

$$\lim_{n \to \infty} \frac{100}{n} = 0.$$

Solution: Note that **every** open interval containing zero has the form (A, B), where $A < 0$ and $B > 0$. We need to show that, for any $A < 0$ and any $B > 0$, almost all terms of the sequence under consideration satisfy the inequality:
$A < a_n < B$.

First we note that $a_n = \frac{100}{n}$ is positive for any $n \in \mathbb{N}$, thus $A < 0 < a_n$.

Second, solving the inequality $\frac{100}{n} < B$ for n, we find that given number B, terms $a_n < B$ for all $n > \frac{100}{B}$. For instance, if $B = 1$ then $a_n < 1$ for all $n > 100$; if $B = 0.1$ then $a_n < 0.1$ for all $n > 1000$, etc. Thus no matter how small the length of the interval (A, B) is, only a finite number of the terms a_n are outside (A, B).

According to the definition, this means that 0 is the limit of this sequence.

Note that every open interval containing the number L has a form (A, B) with $A < L$ and $B > L$. Thus to show that $\lim_{n\to\infty} a_n = L$ we take an open interval (A,B), with $A < L$ and $B > L$, and demonstrate that almost all of the terms of the sequence are in this interval no matter how small the length of the interval is. If we fail to do so, i.e. we can justify that for a given sequence it is impossible to find such a number L, we say that *the sequence does not have a limit* or, equivalently, that *the sequence diverges*.

Problem 5. In each case use the definition to determine whether the given sequence has a limit.

i) $a_n = n^2$, $n \in \mathbb{N}$ (Example 1).
Solution: In accordance with the definition, if a sequence has a limit then we must be able to find an interval to which almost all of it terms belong. But the sequence $a_n = n^2$ runs away from any given interval as n becomes large. Thus, it does not have a limit. In a case like this, when the terms of a sequence become arbitrarily large, we also say that the sequence *diverges to infinity* and write

$$\lim_{n\to\infty} a_n = \infty.$$

ii) $a_n = (-1)^n$, $n \in \mathbb{N}$ (Example 5).
Solution: In accordance with the definition, if a sequence has a limit then we must be able to find an arbitrarily small interval to which almost all of it terms belong. The sequence $a_n = (-1)^n$ cannot fit inside any interval of length less then 2. Thus it does not have a limit.

iii) $a_n = \frac{n-1}{n+1}$, $n \in \mathbb{N}$ (Example 7).
Solution: From our previous discussion of this sequence in Problem 3 we know that any interval (A, B) with $A < 1$ and $B \geq 1$ contains almost all terms of the sequence. The only problem is that there is no such a number L that belongs to all of the intervals. For that reason we will consider a slightly different set of intervals, namely (A, B) with $A < 1$ and $B > 1$. Note that we just changed the condition for B from $B \geq 1$ to $B > 1$, but it makes a big difference! Interval (A, B) with $A < 1$ and $B > 1$ is a general description of an interval containing 1. Thus now every interval in the set contains almost all terms of the sequence as well as the number $L = 1$. Therefore 1 is the limit of the sequence by definition. We will consider another way of getting this result in Section 1.4.

Exercises

1. Does the sequence $\{\cos(\pi n)\}_{n=1}^{\infty}$ converge?
   What about the sequence $\{\sin(\pi n)\}_{n=1}^{\infty}$?
   Explain your answers.

2. Justify the following formulas $\lim_{n\to\infty} \dfrac{n^2 - 2}{n^2} = 1$ and $\lim_{n\to\infty} \dfrac{3n + 2}{n} = 3$.

§1.4 Properties of limits and some ways of finding them

It is often easier to find a limit using the properties which follow from the definition, rather than using the definition itself. In this section we give some examples of this.

From our previous discussion we have the following:

P1. $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.

P2. $\lim\limits_{n\to\infty} 1 = 1$.

P3. $\lim\limits_{n\to\infty} n = \infty$.

Provided $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist, in the equalities P4, P5, P6, and P7 given below, the limit on the left exists and equals the expression on the right:

P4. $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot \lim\limits_{n\to\infty} a_n$ for any constant c.

P5. $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$.

P6. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \cdot \left(\lim\limits_{n\to\infty} b_n\right)$.

P7. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, assuming that $b_n \neq 0$, $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} b_n \neq 0$.

From the above properties one can also derive:

P8. $\lim\limits_{n\to\infty} \dfrac{1}{n^k} = 0$, for any integer $k > 0$.

P9. $\lim\limits_{n\to\infty} c = c$, with c being a constant.

P10. $\lim\limits_{n\to\infty} n^k = \infty$, for any integer $k > 0$.

Let us now calculate the limit from the previous subsection (Problem 5 (iii)). It could be done as follows:

$$\lim_{n\to\infty} \frac{n-1}{n+1} = \lim_{n\to\infty} \frac{1 - \frac{1}{n}}{1 + \frac{1}{n}} = \frac{\lim\limits_{n\to\infty} \left(1 - \frac{1}{n}\right)}{\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)} = \frac{\lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty} \frac{1}{n}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n}} = \frac{1 - 0}{1 + 0} = 1. \quad (1)$$

Here we first divided the top and bottom of the fraction by n, (the highest power of n in this case) then used property P7, then P5, and finally P1 and P2.

Note that the first step is necessary, because otherwise by properties P2, P3, P4 and P5 both numerator and denominator of the fraction diverge to infinity, so we cannot use property P7. But even if we could apply P7 formally, we would have obtained

$$\lim_{n\to\infty} \frac{n-1}{n+1} = \frac{\lim\limits_{n\to\infty} (n-1)}{\lim\limits_{n\to\infty} (n+1)} = \frac{\infty}{\infty}.$$

The result is uncertain and the situation needs to be resolved, which we accomplish by the division. This $\frac{\infty}{\infty}$ is what mathematicians call an *indeterminate form.* Some other indeterminate forms are

$$\frac{0}{0}, \qquad 0^0, \qquad 1^\infty.$$

We will have an example of the last of these cases when we talk about the Euler sequence (see Section 1.9.2). If one arrives at an indeterminate form, one needs to resolve it. A useful technique for doing this called L'Hospital's rule is given in Section 2.9.

Note that in some cases we reach expressions which look like indeterminate forms but are not. For instance, for a constant $c \neq 0$ you can safely proceed as follows:

$$\frac{0}{c} = 0, \quad c^0 = 1, \quad 0^{|c|} = 0.$$

Formally you can also write $\frac{c}{0} = \infty$ if $c > 0$, and $\frac{c}{0} = -\infty$ if $c < 0$. However in contrast with previous formulas, here we do not have equality of two numbers as both values are a short way of describing unboundedness.

The result in (1) may be generalized as follows. Consider the case in which a_n is a rational function of n, i.e., the ratio of two polynomials of n of degrees m and k

$$a_n = \frac{B_m n^m + \cdots + B_1 n + B_0}{C_k n^k + \cdots + C_1 n + C_0}.$$

Then

$$\lim_{n \to \infty} a_n = \begin{cases} 0 & \text{if } m < k \\ \text{sign}\left(\frac{B_m}{C_k}\right) \infty & \text{if } m > k \\ \frac{B_m}{C_k} & \text{if } m = k. \end{cases}$$

Examples:

Example 9. $a_n = \dfrac{3n^5 + 1}{4n^7 + n + 1}$. Here $m = 5$, $k = 7$. Thus $\lim\limits_{n \to \infty} a_n = 0$.

Example 10. $a_n = \dfrac{3n^8 + 1}{4n^7 + n + 1}$. Here $m = 8$, $k = 7$ and $B_8 = 3$, $C_7 = 4$. Thus $\lim\limits_{n \to \infty} a_n = \infty$.

Example 11. $a_n = \lim\limits_{n \to \infty} \dfrac{3n^5 + 1}{4n^5 + n + 1}$. Here $m = k = 5$ and $B_5 = 3$, $C_5 = 4$. Thus $\lim\limits_{n \to \infty} a_n = \dfrac{3}{4}$.

Exercises

1. Provide detailed calculations for Examples 9, 10, and 11. Follow the method given in (1) starting with the division of numerator and denominator by an appropriate power of n. At each step show which property of limits you use.

2. In each case find the limit as $n \to \infty$ of given sequence either by using the definition of limit directly or the properties P1 - P10:

$$\text{i) } a_n = \frac{1}{n^3 - 3n^2 - 3} \qquad \text{ii) } a_n = \frac{20 - n}{2n + 12} \qquad \text{iii) } a_n = \frac{\sqrt{n}}{n + 1}$$

§1.5 Recursive formulas

A sequence can be defined recursively as well as explicitly. A recursive definition of a sequence is a definition which describes a term by using one or more previous terms. The recursive definition of the natural numbers is $a_n = a_{n-1} + 1$, $n > 1$, with initial condition $a_1 = 1$. The explicit formula is $a_n = n$, $n \geq 1$.

When a sequence is defined recursively, there must always be initial conditions given. The number of initial conditions needed to define a sequence uniquely is the *order of the recurrence relation*. For example, the sequence $a_n = 3a_{n-1}$, $a_1 = 1$ is of the first order, while $a_n = 3a_{n-1} + a_{n-2}$, $a_1 = 1$, $a_2 = 3$ is of the second order.

Finding the limit of a sequence from its recursive definition is straightforward in some cases while in others it requires some thought. A straightforward example is the following.

Problem 6. Find the limit of the sequence (asuming that the limit exists)

$$a_n = \frac{2}{1 + a_{n-1}}, \quad a_1 = \frac{1}{2}.$$

Solution: If we know that the sequence has a limit, we can derive the equation that it must satisfy. From Definition 3 it is clear that, if a_0 is assigned an arbitrary value, and $\lim_{n \to \infty} a_n$ is defined, equal L say, then $\lim_{n \to \infty} a_{n-1} = L$ also.

So, using properties P5, P7 and P9 of the limit, we can say that:

$$\lim_{n \to \infty} a_n = \frac{2}{1 + \lim_{n \to \infty} a_{n-1}}$$
$$L = \frac{2}{1 + L}$$
$$L^2 + L = 2$$
$$L^2 + L - 2 = 0$$
$$L = -2 \quad \text{or} \quad L = 1$$

We know that a sequence can only have one limit so we want to find out which one of these is the right value. By looking at our recursive formula, we know that as long as a term is positive, the term after it must be positive. This is because the term after it would be a ratio of two positive numbers. Since the first term is positive, the term after it must be positive, as is the term after that one and so on. Therefore, the limit must be non-negative. Thus, $\lim_{n \to \infty} a_n = 1$.

Let us justify this result using the definition of the limit discussed earlier. We need to show that any open interval that contains 1 will also contain an infinite number of terms of the sequence $\{a_n\}$. For that we estimate the distance between a_n and 1 as n increases. We find

$$1 - a_{n+1} = \frac{(1 + a_n) - 2}{1 + a_n} = \frac{a_n - 1}{a_n + 1}, \quad \text{so} \quad |1 - a_{n+1}| = |1 - a_n| \cdot |(a_n + 1)^{-1}|.$$

Now, if we could show that $|(a_n + 1)^{-1}| \leq r$, for all $n \geq 1$, then it would imply that $|1 - a_{n+1}| \leq |1 - a_1| r^n < r^n$ (since $1 - a_1 = 0.5$). Note that if $0 < r < 1$ then $r^n < 1/n$ for all $n \geq n_0$ starting from some positive n_0. In this case we would have $|1 - a_{n+1}| < 1/n$ and conclude that every interval $(1 - A, 1 + A)$, where $A > 0$ contains all terms a_n for $n > 1/A$.

Now let us find $r < 1$ such that $|(a_n + 1)^{-1}| \leq r$. Calculate the few first terms of the sequence: $a_1 = 1/2$, $a_2 = 4/3$, $a_3 = 6/7$, $a_4 = 14/13$, etc. Observe that if $a_n \geq 1/2$ then $a_n + 1 \geq 3/2$ and thus $(a_n + 1)^{-1} \leq 2/3$. Consequently, $a_{n+1} \leq 4/3$. As well, if $a_n \leq 4/3$ then $a_{n+1} \geq 6/7 > 1/2$. Thus if we start with a term in the interval $[1/2, 4/3]$, all terms of the sequence will remain in this interval, and for all of them we have $(a_n + 1)^{-1} \leq r$, where $r = 2/3 < 1$. Since $(2/3)^n < 1/n$ for all $n \geq 1$, we conclude that the limit of the sequence is 1.

This method of solving for the limit will not always work. **We need to know before using this method that the sequence does have a limit**. The following example illustrates that otherwise the method can lead to a wrong conclusion.

Example 12. Let us study the sequence $a_{n+1} = 3a_n$, $a_1 = 1$. If we just consider what this means, it is easy to give the answer. Every term is 3 times as large as the term before it. If we take a positive number, and keep tripling it infinitely many times, the value is going to go towards infinity, and we say that the sequence diverges to infinity. If we had assumed that this sequence had a limit and tried to solve for it using the method explained, we would have obtained

$$L = 3L.$$

Technically, infinity satisfies the above relation. But since we assumed that L is a number (not infinity), we get $L = 0$, which is clearly false.

Note also that a sequence defined by recursive formula $a_{n+1} = 3a_n$ has limit $L = 0$ if and only if the initial term $a_1 = 0$.

Exercises

1. Find the limit of each recursively defined sequence given that it exists.

$$\text{i) } a_{n+1} = \frac{a_n}{a_n - 2}, \ a_1 = 3 \qquad \text{ii) } a_{n+1} = a_n^2, \ a_1 = \frac{1}{2}$$

§1.6 A criterion for convergence

Example 12 from previous the section shows that it is helpful to know whether the limit of a sequence exists before attempting to evaluate it. The theorem given in this section helps to determine when a sequence has a limit.

Definition 4. A sequence is called *non-decreasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is called *non-increasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is called *monotone* if it is either non-decreasing or non-increasing.

Definition 5. A sequence is called *bounded* if there are numbers K and M such that $K \leq a_n \leq M$ for all $n \in \mathbb{N}$.

The following is a very important property of the real numbers whose proof is beyond the scope of this work.

Theorem 1. If a sequence is monotone and bounded then it has a limit.

Neither hypothesis by itself is sufficient. A sequence which is monotone and unbounded will diverge, like $a_n = 3^n$. Also, a sequence which is bounded but not monotone may not have a limit, like $a_n = (-1)^n$. On the other hand, a convergent sequence may not be monotone, $a_n = (-1)^n/n$ for example, but it must be bounded.

Problem 7. Show that the sequence

$$a_n = \frac{1 + a_{n-1}}{4}, \qquad a_1 = 1$$

is monotone and bounded, and thus has a limit.
Solution: We can show both properties using the method of mathematical induction. Looking at the first terms of the sequence, $\{1, 1/2, 3/8, ...\}$, we guess that the sequence is non-increasing and bounded by $K = 0$ and $M = 1$.
First, we prove that the sequence is non-increasing. We see that $a_1 \geq a_2$. Now assume that $a_{k-1} \geq a_k$. Then $1 + a_{k-1} \geq 1 + a_k$, and $\frac{1 + a_{k-1}}{4} \geq \frac{1 + a_{k+1}}{4}$. Using the recursive formula which defines the sequence, we rewrite the last formula as $a_k \geq a_{k+1}$. Thus, we have shown that the inequality $a_{k-1} \geq a_k$ implies $a_k \geq a_{k+1}$ for any $k \geq 2$. Since this inequality is true for $k = 1$, we can conclude that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$, and by definition the sequence is non-increasing.
Second we prove that the sequence is bounded: $0 < a_n \leq 1$. Clearly, $0 < a_1 \leq 1$. Now assume that $0 < a_k \leq 1$. Since the terms are non-increasing, $a_{k+1} \leq a_k \leq 1$. Also, $a_{k+1} = (1 + a_k)/4 > 0$ since $a_k > 0$. Thus $0 < a_k \leq 1$ implies $0 < a_{k+1} \leq 1$ for all $k \geq 1$. We conclude that $0 < a_n \leq 1$ for all $n \geq 1$.
Now, using Theorem 1, we conclude that the sequence has a limit. To find the limit we use the method from the previous section and obtain $L = 1/3$.

Problem 8. Show that the sequence $a_n = \frac{n-1}{n+1}$ is monotone.
Solution: We write

$$a_{n+1} = \frac{(n+1) - 1}{(n+1) + 1} = \frac{n}{n+2} > \frac{n-1}{n+1} = a_n.$$

The inequality follows from $n(n+1) > (n+2)(n-1)$ which is equivalent to $0 > -2$. Recall that this sequence is bounded and has 1 as its limit.

The question of whether a sequence is monotone or not can also be addressed by employing one rather useful tool which we now introduce. If you are not familiar with derivatives, you should read Appendix A before continuing.

§1.7 Monotonicity check using the derivative

Let us look at a sequence simply as a function whose domain is the positive integers. So we will say that a sequence $\{a_n\}_{n=1}^{\infty}$ and a function $f(x)$, $x \geq 1$, correspond to each other if and only if $a_n = f(n)$ for all positive integers. For example, the sequence $a_n = n^2$ corresponds to the function $f(x) = x^2$ and the sequence $a_n = \sin n$ corresponds to the function $f(x) = \sin x$.

This point of view is convenient because instead of checking the monotonicity of a sequence, we now can ask this question about its corresponding function. In Appendix A we tell you how to determine the monotonicity of a function by using its derivative. This process is straightforward, and therefore applicable in a large number of cases. Assuming that by now the reader is familiar with the Appendix, let us consider the following example.

Problem 9. Is the sequence $a_n = n^2 - 50n$, $n \geq 1$ monotone?
Solution: The first few terms of the sequence are:
$\{-49, -96, -141, -184, -225, -264, -301\}$. From this, it seems that the sequence is monotonically decreasing. But having written down any finite number of terms, you still cannot be sure of what will happen to a sequence next, so the conclusion cannot be made yet.

Let us consider the corresponding function. For that we just replace n with x. This gives $f(x) = x^2 - 50x$. The derivative of the function is $f'(x) = 2x - 50$. Note that the derivative is negative for $x < 25$ and it is positive for $x > 25$. This means that the function $f(x) = x^2 - 50x$ is decreasing on the interval $x < 25$ and is increasing for $x > 25$. Consequently the sequence $a_n = n^2 - 50n$ is decreasing for $1 \leq n \leq 24$ and is increasing for $n \geq 26$.

For the purpose of a convergency check, we are interested in the behavior of the sequence as $n \to \infty$. Therefore, when checking monotonicity, we can disregard any finite number of terms. From our analysis we can conclude that the sequence is monotonically increasing for $n \geq 26$.

Problem 10. Apply the same approach for $a_n = \frac{n-1}{n+1}$, $n \in \mathbf{N}$, considered previously.
Solution: The corresponding function is $f(x) = \frac{x-1}{x+1}$. The derivative, calculated using the quotient rule, is $f'(x) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$. This expression is always positive for $x \geq 1$ and thus the sequence is monotonically increasing for $n \in \{1, 2, 3, \ldots\}$, which confirms the result obtained before by direct calculations.

Exercises

1. Give an example of a convergent and non-monotone sequence.
2. Give an example of a divergent and monotone sequence.
3. Give an example of a divergent and bounded sequence.
4. Show that $a_n = \dfrac{n^2}{n^2 + 1}$ is monotone and bounded. Find its limit.

§1.8 Converting recursive formulas to explicit formulas

It can often be difficult to find the limit of a sequence when we only know its recursive definition. It is generally much easier to work with a sequence if we have an explicit formula for a_n. For example:

Problem 11. Find the limits of the following sequences:

i) $a_n = (1/2)^n$, $n \geq 0$.

ii) $b_n = 2b_{n-1} - \dfrac{3}{4}b_{n-2}$, $n \geq 2$ with $b_0 = 1, b_1 = \frac{1}{2}$

Comment: In all previous examples the index n was a natural number, $n \geq 1$. However, sometimes we start counting from another number, for instance in this problem from 0. This is a common practice, so in the future please pay attention to this aspect of the problems as well.

Solution:

i) This sequence is decreasing since each term is half the previous one: $a_{n+1} = (1/2)^{n+1} = (1/2)a_n < a_n$. It is also bounded from below since all terms are positive. Thus the limit exists. Since $a_n = a_{n-1}/2$ for all $n \geq 1$, the limit of the sequence is 0 by the method of Section 1.5.

ii) If we knew that the sequence had a limit, it would be straightforward to show that it is 0 by the method of Section 1.5. The problem is to justify that the limit exists. To accomplish this we will show that the two formulas, (i) and (ii), actually describe the same sequence, that is $a_n = b_n$ for all $n \geq 0$. To support this we begin by writing down the first six terms of each sequence,

$$\{a_n\} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\}$$
$$\{b_n\} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\}$$

and observing that they are the same (note that indexes start with zero for both sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ in this example). Next we will show that $a_n = b_n$ for all $n \geq 0$. Thus, we will convert the definition of b_n from recursive to explicit form. We can do it by method of mathematical induction. First, observe that $b_0 = 1 = (1/2)^0$ and $b_1 = 1/2 = (1/2)^1$. Take $n \geq 1$ and assume that $b_k = (1/2)^k$ for $0 \leq k \leq n$. We need to show that these conditions imply $b_{n+1} = (1/2)^{n+1}$. Indeed, using the recursive formula we have

$$b_{n+1} = 2\left(\frac{1}{2}\right)^n - \frac{3}{4}\left(\frac{1}{2}\right)^{n-1} = \left(1 - \frac{3}{4}\right)\left(\frac{1}{2}\right)^{n-1} = \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n+1}.$$

Thus, by principle of mathematical induction we have $b_k = (1/2)^k = a_k$ for $k \geq 0$.

However, if we did not know ahead of time to which explicit formula we need to compare a given recursive equation, we could use a different method. For doing that we need a little bit of theory.

Definition 6. A *second order linear homogeneous recurrence relation* is a recurrence relation of the form

$$a_n = ra_{n-1} + qa_{n-2}, \tag{2}$$

where $r, q \neq 0$ are constants. The quadratic equation

$$x^2 - rx - q = 0 \tag{3}$$

is called the *characteristic equation* of the recurrence relation (2).

We will assume that the constants r, q satisfy the inequality $r^2 + 4q > 0$, and thus the quadratic equation (3) has two distinct real solutions.

Theorem 2. Let x_1 and x_2 be solutions of the quadratic equation (3). Let A and B be arbitrary constants. Then, the formula

$$a_n = Ax_1^n + Bx_2^n, \qquad n \geq 0 \tag{4}$$

defines explicitly the same sequence as the one given by the recurrence relation $a_n = ra_{n-1} + qa_{n-2}$ with initial conditions $a_0 = A + B$, $a_1 = Ax_1 + Bx_2$.

Let us apply Theorem 2 to sequence $\{b_n\}$ defined in Problem 11. We have $r = 2$ and $q = -3/4$. The corresponding characteristic equation is $x^2 - 2x + 3/4 = 0$ which has roots $x_1 = 1/2$ and $x_2 = 3/2$. From the initial conditions $b_0 = 1$, $b_1 = 1/2$, we have $A + B = 1$, $A/2 + 3B/4 = 1/2$. Thus $A = 1$, $B = 0$, and the explicit formula is $b_n = Ax_1^n + Bx_2^n = (1/2)^n$. This is exactly what we have claimed above.

Now we give the proof of the theorem.

Proof: First, we show that for any numbers A, B the n-th term given by (4) will satisfy the recurrence relation (2). Suppose that $a_n = Ax_1^n + Bx_2^n$ then $a_{n-1} = Ax_1^{n-1} + Bx_2^{n-1}$ and $a_{n-2} = Ax_1^{n-2} + Bx_2^{n-2}$. In order for (2) to hold, we must have

$$Ax_1^n + Bx_2^n = r(Ax_1^{n-1} + Bx_2^{n-1}) + q(Ax_1^{n-2} + Bx_2^{n-2}).$$

That is, we must have

$$A(x_1^n - rx_1^{n-1} - qx_1^{n-2}) + B(x_2^n - rx_2^{n-1} - qx_2^{n-2}) \quad = \quad 0,$$

which is equivalent to,

$$x^{n-2}\left[A(x_1^2 - rx_1 - q) + B(x_2^2 - rx_2 - q)\right] \quad = \quad 0.$$

This is clearly true for all n since both of the expressions in parentheses are equal to zero because x_1 and x_2 are solutions to the characteristic equation (3).

The initial conditions for (2) follow from (4) with $n = 0, 1$.

We will need this theorem again to obtain the explicit formula for the Fibonacci numbers (Section 1.9.1).

§1.8.1 Geometric sequences

Setting $q = 0$ in Theorem 2 we obtain the following:

Theorem 3. The recurrence relation

$$a_n = ra_{n-1}, \qquad a_0 = A,$$

where r, A are constants, defines the sequence given explicitly by

$$a_n = Ar^n, \qquad n \geq 0.$$

Problem 12. Find the explicit formula for the sequence defined by the recurrence relation $a_n = 2a_{n-1}$ with $a_0 = 3$.
Solution: This is the first order linear homogeneous recurrence relation with $r = 2$ and $A = 3$. Thus, in the explicit form, $a_n = 3 \cdot 2^n$.

The sequences of the form $a_n = Ar^n$, $A \neq 0$ are called *geometric*. In other words, a geometric sequence is a sequence where consecutive terms have a common ratio, r. Some geometric sequences have a finite limit while some other ones do not depending on the value of the common ratio r. There are four different cases and they are described in the following table.

r	$\lim\limits_{n \to \infty} Ar^n$
$r \leq -1$	no limit
$-1 < r < 1$	0
$r = 1$	A
$r > 1$	∞ or $-\infty$ according as $A > 0$ or $A < 0$

§1.8.2 Arithmetic sequences

Now we consider the *non-homogeneous linear recurrence relations* of the form

$$a_n = a_{n-1} + d, \qquad a_0 = A.$$

For the sequence defined by these equations the difference between any term (after the first) and the one before is a constant d, called the *common difference*. Such a sequence is called *arithmetic*. The equivalent explicit definition is

$$a_n = A + nd, \quad n \geq 0.$$

An example of an arithmetic sequence is $\{8, 5, 2, -1, -4, -7\}$. Its explicit definition is $a_n = 8 - 3n$, $n \geq 0$. The recursive definition is $a_n = a_{n-1} - 3$, $a_0 = 8$. The common difference is -3.

All arithmetic sequences (with $d \neq 0$) diverge. If $d > 0$ then $\lim\limits_{n \to \infty} a_n = \infty$, and if $d < 0$ then $\lim\limits_{n \to \infty} a_n = -\infty$.

Exercises

1. In each case convert the given recursive definition to an explicit definition of the same sequence. Check your answer by verifying that the two definitions agree for the first few terms. In each case decide whether the sequence is convergent or divergent and explain your answer.

i) $a_n = \dfrac{a_{n-1}}{10}$, $a_0 = 100$

ii) $a_n = 10 + a_{n-1}$, $a_0 = 100$

iii) $a_n = a_{n-1} - \dfrac{3}{16}a_{n-2}$, $a_0 = 1$, $a_1 = \dfrac{1}{4}$

§1.9 Some interesting sequences

In this section we look at four well-known sequences. The first is notable because of the properties of its members. The next two are interesting because of their limits. The latter problem is attractive because it looks simple but is still unsolved.

§1.9.1 The Fibonacci sequence

The Fibonacci sequence is usually defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n > 2, \quad \text{with} \quad F_1 = 1 \quad \text{and} \quad F_2 = 1.$$

In other words, it is the sequence for which the first two terms are 1 and each term after that is the sum of the two terms before it. The first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots.$$

The sequence is named after the Italian mathematician of the Middle Ages, who used this example in his Book of Calculation (Liber Abaci) published in 1202. The sequence represented the growth of the number of pairs of rabbits on an island if each pair gives rise to a new pair every month starting from the second month of their own life, and never die. Now, the sequence is referred to as the first model of population dynamics.

If you were given the recursive formula for this sequence and were told to find the 1000-th Fibonacci number (F_{1000}), it would be a very tedious task. That is why we need an explicit formula for the n-th term of the sequence. First we will define the *golden ratio*. The golden ratio is equal to $\frac{1+\sqrt{5}}{2} \approx 1.61$ and is denoted by ϕ. Taking $q = r = 1$, $a_0 = 0$, and $a_1 = 1$, in Theorem 2 we see that the sequence referred to in the theorem is $\{0, F_1, F_2, F_3, \dots\}$. In this case the roots of the characteristic equation are ϕ and $1 - \phi$. We ensure that $a_1 = a_2 = 1$ by letting $A = 1/\sqrt{5}$ and $B = -1/\sqrt{5}$. It follows that the n-th Fibonacci number is:

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}. \tag{5}$$

Even though this equation contain radicals, every term of the sequence is an integer.

There are many interesting properties of the Fibonacci numbers. Some of them we list here.

1. The limit of the Fibonacci sequence is infinity, but the limit of the ratio of consecutive terms is the golden ratio

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi.$$

2. The sum of the first N Fibonacci numbers is

$$F_1 + F_2 + \cdots F_N = F_{N+2} - 1.$$

For example, if $N = 3$ we have $F_1 + F_2 + F_3 = 1 + 1 + 2 = 4 = 5 - 1 = F_5 - 1$.

3. The absolute value of the difference between the square of a Fibonacci number and the product of the previous and the next Fibonacci numbers is 1.

$$F_N^2 - F_{N-1}F_{N+1} = (-1)^{N+1}.$$

For example, if $N = 4$ then $F_4^2 - F_3 F_5 = 3^2 - 2 \cdot 5 = -1$.

4. The sum of the squares of two consecutive Fibonacci numbers is another Fibonacci number

$$F_N^2 + F_{N+1}^2 = F_{2N+1}.$$

For example, if $N = 4$ then $F_4^2 + F_5^2 = 3^2 + 5^2 = 34 = F_9$.

5. The difference of the squares of two consecutive Fibonacci numbers is the product of two other Fibonacci numbers

$$F_{N+1}^2 - F_N^2 = F_{N-1}F_{N+2}.$$

For example, if $N = 4$ then $F_5^2 - F_4^2 = 5^2 - 3^2 = 16 = 2 \cdot 8 = F_3 F_6$.

6. The only squares among Fibonacci numbers are $F_1 = 1$, $F_2 = 1$, $F_{12} = 144$.

7. Every 3rd Fibonacci number is even (i.e. divisible by $F_3 = 2$). Every 4th Fibonacci number is divisible by $F_4 = 3$. Every 5th Fibonacci number is divisible by $F_5 = 5$. More generally, the ratio $F_{n \cdot k}/F_n$ is always an integer.

8. There are prime numbers amongst the Fibonacci numbers, like $1, 2, 3, 5, 13$, and so on. The question of whether there are an infinite or finite number of primes amongst the Fibonacci numbers still remains open.

There are other interesting properties of the Fibonacci sequence. We will provide one more, with its simple proof.

Problem 13. Show that:

$$\frac{1}{F_{n-1} \cdot F_n} - \frac{1}{F_n \cdot F_{n+1}} = \frac{1}{F_{n-1} \cdot F_{n+1}}$$

Solution: Start by finding a common denominator

$$\frac{1}{F_{n-1} \cdot F_n} - \frac{1}{F_n \cdot F_{n+1}} = \frac{F_{n+1} - F_{n-1}}{F_{n-1} \cdot F_n \cdot F_{n+1}}$$

and then use the recursive definition of the Fibonacci sequence and substitute $F_n = F_{n+1} - F_{n-1}$ to allow cancellation:

$$\frac{F_{n+1} - F_{n-1}}{F_{n-1} \cdot F_n \cdot F_{n+1}} = \frac{F_n}{F_{n-1} \cdot F_n \cdot F_{n+1}}$$

$$= \frac{1}{F_{n-1} \cdot F_{n+1}}$$

which is our desired result.

§1.9.2 Euler sequence. The first remarkable limit

Some sequences have limits which are unexpected. In this section we give one such example, which is due Leonard Euler, a Swiss mathematician of the 18th century. Consider a sequence given by its n-th term

$$a_n = \left(1 + \frac{1}{n}\right)^n, \qquad n \geq 1.$$

We know that $\lim_{n \to \infty}(1 + (1/n)) = 1$. But, because $1 + (1/n)$ is raised to the power n, none of the discussion above gives us a way of knowing whether $\lim_{n \to \infty} a_n$ exists or what its value is if it does exist.

We will have to find the limit of this sequence using another method.

First, note that the expression $(1 + (1/n))^n$ can be rewritten using the binomial formula. Familiar examples of this formula are $(a + b)^2 = a^2 + 2ab + b^2$ and $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. In general for $n \geq 2$ we have the following sum of $n + 1$ terms:

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{k} a^{n-k} b^k + \cdots + b^n,$$

where the *binomial coefficients* $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ are defined for $1 \leq k < n$. Using the formula $n! = n(n-1)(n-2)\cdots 1$ one can also write

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Setting $a = 1$ and $b = 1/n$ we obtain

$$(1 + \frac{1}{n})^n = 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{k}\frac{1}{n^k} + \cdots + \frac{1}{n^n}.$$

Now observe what happens to each term on the right hand side as $n \to \infty$. First, $\lim_{n\to\infty} 1 = 1$. Second,

$$\lim_{n\to\infty} \binom{n}{1}\frac{1}{n} = \lim_{n\to\infty} \frac{n!}{(n-1)!1!}\frac{1}{n} = \lim_{n\to\infty} 1 = 1;$$

Third,

$$\lim_{n\to\infty} \binom{n}{2}\frac{1}{n^2} = \lim_{n\to\infty} \frac{n!}{(n-2)!2!}\frac{1}{n^2} = \lim_{n\to\infty} \frac{n(n-1)}{n^2 2!} = \frac{1}{2!};$$

Similarly, for any $1 \le k < n$ we have

$$\lim_{n\to\infty} \binom{n}{k} \cdot \frac{1}{n^k} = \lim_{n\to\infty} \frac{n!}{(n-k)!k!}\frac{1}{n^k} = \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k k!} = \frac{1}{k!};$$

Note that

$$\lim_{n\to\infty} (1+\frac{1}{n})^n = \lim_{n\to\infty} \left(1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{k}\frac{1}{n^k} + \cdots + \frac{1}{n^n} \right).$$

The limits of individual terms found above suggest that the righthand side tends to the infinite sum of inverse factorials

$$1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{k!} + \cdots . \tag{6}$$

This can be shown rigorously by demonstrating that the sequence $\{(1+1/n)^n\}_{n=1}^{\infty}$ is increasing and bounded and that it approaches the sum (6) as $n \to \infty$.

As we will see in the next chapter, the infnite sum of inverse factorials (6) is equal to the number $e \approx 2.71$, the base of the natural logarithm. [3]

$$1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{k!} + \cdots = e.$$

Thus, we claim:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e. \tag{7}$$

The limit is known as the *first remarkable limit*. This limit can be also found using logarithms, and L'Hospital's rule (see Sec. 2.9 Example 3).

In fact, we have a more general formula. For any constant M,

$$\lim_{n\to\infty} \left(1 + \frac{M}{n}\right)^n = e^M. \tag{8}$$

You will be able to prove (8) after reading section 2.9.

---

[3] The natural logarithm $\log_e(x)$ of a number $x > 0$ is the power to which e would have to be raised to equal x, and it also equals the area under the hyperbola $y = 1/x$ on the segment from 1 to x. Since $\log_e e = 1$, the number e can be defined as the unique number greater than 1 such that the area under the hyperbola y=1/x from 1 to e equals 1.

§1.9.3 The second remarkable limit

Let us consider the sequence $a_n = n \sin\left(\dfrac{1}{n}\right)$, $n \in \mathbb{N}$. It is not immediately clear what the limit of this sequence is, since $\lim\limits_{n \to \infty} n = \infty$, $\lim\limits_{n \to \infty} \sin(\dfrac{1}{n}) = 0$ and $0 \cdot \infty$ is an indeterminate form.

Let us introduce the notation $x_n = \dfrac{1}{n}$. Then

$$\lim_{n \to \infty} n \sin\left(\frac{1}{n}\right) = \lim_{n \to \infty} \frac{\sin(x_n)}{x_n}.$$

Note that $0 < x_n < \pi/2$ for all $n \geq 1$. Consider an acute positive angle of measure x_n radians, and consider a sector of a unit circle with inner angle x_n. Let O be the centre of the circle, and let A and B be the points where the angle meets the arc, with A below, as shown in the Figure 2. Also, let us draw a line perpendicular to OA which passes through B and let its intersection with OA be called C. Similarly, extend OB, draw a line perpendicular to OA which runs through A and let its intersection with OB be called D. We have,
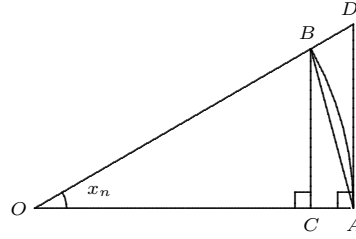
Figure 2. A geometric proof of the second remarkable limit.

Now, $|OA| = |OB| = 1$, because they are both radii of the unit circle. So, by the definition of sin and tan, we have $|BC| = \sin(x_n)$ and $|DA| = \tan(x_n)$. It is easy to see from the figure that the area of the sector lies strictly between the areas $[OAB]$ and $[OAD]$ of triangles $\triangle OAB$ and $\triangle OAD$, respectively. Now, area $[OAB] = \dfrac{1}{2}|OA| \cdot |BC| = \dfrac{\sin(x_n)}{2}$, area $[OAD] = \dfrac{1}{2}|OA| \cdot |AD| = \dfrac{\tan(x_n)}{2}$.

Since the area of the sector is $\pi \cdot \dfrac{x_n}{2\pi} = \dfrac{x_n}{2}$, we have

$$\frac{\sin(x_n)}{2} < \frac{x_n}{2} < \frac{\tan(x_n)}{2}.$$

Because $\dfrac{2}{\sin(x_n)} > 0$ for $0 < x_n < \pi/2$, we can multiply throughout the inequality by $\dfrac{2}{\sin(x_n)}$ to obtain

$$1 < \frac{x_n}{\sin(x_n)} < \frac{1}{\cos(x_n)}$$

which is equivalent to

$$1 > \frac{\sin(x_n)}{x_n} > \cos(x_n).$$

Note that $\cos(x_n) = |OC|$ in Figure 2 and $|OA| = 1$. As n increases C moves to the right and becomes closer and closer to A. We claim that for every point E in the segment OA, for all but a finite number of n, C lies to the right of E and so $\lim_{n\to\infty} \cos(x_n) = |OA| = 1$.

Now we can apply the **squeeze principle**, which says that if $a_n \geq b_n \geq c_n$ for all $n \geq 1$ and both sequences $\{a_n\}$ and $\{c_n\}$ have the same limit L, then $\lim_{n\to\infty} b_n = L$. In our case both $a_n = 1$ and $c_n = \cos(\frac{1}{n})$ have the limit 1 as $n \to \infty$, and $b_n = \dfrac{\sin(x_n)}{x_n} = n\sin(\frac{1}{n})$. Consequently,

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = 1. \tag{9}$$

This result is often referred to as the *second remarkable limit*.

§1.9.4 The "half or triple plus one" sequence

Let us consider the following recursive process. Take any odd number n. Multiply it by 3 and add 1; divide the result, $3n + 1$, by 2 as many times as necessary to get an odd number. Repeat the process indefinitely.

For example, if $a_1 = 1$ then $3 \times 1 + 1 = 4$ and after division by 2 twice we get $a_2 = 1$. Thus for the initial value 1, all terms of this sequence are 1. Other examples are given in the table below:

1st term a_1	2nd term a_2	3rd term a_3	4th term a_4	5th term a_5	6th term a_6	7th term a_7
3	5	1	1	1	1	1
7	11	17	13	5	1	1
19	29	11	17	13	5	1

As you see, in all considered examples we eventually get a term equal to 1, after which all following terms are 1's. In other words, in all our examples the sequence converges to 1. You are encouraged to try some other numbers as well. If you start with an even number then first divide it by 2 as many times as necessary to get an odd number and then continue as described above.

In 1937 Lothar Collatz proposed that no matter what number you start with, you will always eventually reach 1. However, this statement has not yet been either proved or disproved.

This situation demonstrates that even if some problems look simple, modern mathematics may not be powerful enough to solve them.

Exercises

1. i) Find the characteristic equation of the Fibonacci sequence and show that the golden ratio ϕ is one of its roots. What is the other root?

   ii) Derive the explicit formula (5) for the n-th term of the Fibonacci sequence.

   iii) What do you know about the golden ration ϕ from geometry?

2. Prove property 5 of the Fibonnaci numbers.

3. Prove property 7 of the Fibonnaci numbers.

4. Calculate $\binom{5}{k}$ for $k = 1, 2, 3, 4$ and use the binomial formula to expand $(a + b)^5$ as a sum of six terms. Verify your result by direct multiplication.

5. Find the sum $\binom{4}{1} + \binom{4}{2} + \binom{4}{3}$. Prove that the sum of the binomial coefficients

$$\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} = 2^n - 2$$

   for all $n \geq 2$.

6. Prove using the binomial formula that the sequence

$$\left\{ \left(1 + \frac{1}{n} \right)^n \right\}_{n=1}^{\infty}$$

   is increasing.

7. i) Show that $x \sin \frac{1}{x}$ is increasing for $x > 1$.

   ii) Use (9) to show that for any number a

$$\lim_{n \to \infty} n \sin \frac{a}{n} = a.$$

8. Justify the squeeze principle used in 1.9.3 from Definition 3.

9. Fix a positive integer k. Show that, for all but a finite number of n,

$$\left(1 + \frac{1}{n} \right)^n \geq 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!}.$$

CHAPTER 2: SERIES

§2.1 Definitions and sigma notation

Definition 7. A *series* is an expression of the form $a_1 + a_2 + a_3 + \cdots$ which is the sum of an infinite number of terms of the sequence $\{a_n\}_{n=1}^{\infty}$.

The questions we ask about a series are: "Is it convergent?" and, if the series is convergent, "What is its sum?" To make clear what we are talking about we begin with a definition.

Definition 8. The n-th *partial sum* S_n of a sequence $\{a_n\}_{n=1}^{\infty}$ is the sum of the first n terms of that sequence. That is,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

Note that the partial sums S_n form another sequence as n grows,

$$S_1 = a_1, \quad S_2 = a_1 + a_2, \quad S_3 = a_1 + a_2 + a_3, \quad \ldots \quad S_n = a_1 + a_2 + \cdots + a_n.$$

The limit as $n \to \infty$ of the sequence of partial sums, if it exists, gives the *sum* of the series:

$$\lim_{n\to\infty} S_n = a_1 + a_2 + a_3 + \cdots.$$

Definition 9. If the limit of the sequence of partial sums exists, the series is *convergent*. Otherwise it is *divergent*.

This definition is due to Augustin-Louis Cauchy and appeared in his book, *Cours d'analyse*, in 1821.

In 1820 Joseph Fourier introduced a shorthand way to write a series called *sigma notation*. The partial sum S_n, represented by sigma notation, is:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n \quad = \quad \sum_{k=1}^{n} a_k.$$

The series and its sum, if it exists, will both be denoted by

$$\sum_{k=1}^{\infty} a_k.$$

The following properties are used in algebraic manipulations

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k, \qquad \sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k, \qquad \sum_{k=1}^{n} c = cn. \qquad (10)$$

The first two of these properties are valid when n is replaced by ∞ provided the sequence(s) on the right of the equation converge.

§2.2 Finite arithmetic and geometric sums

Sometimes we might be asked questions like: What is the sum of the first 100 natural numbers? Or, what is the sum of the first 25 positive integer powers of 2? The direct summation may seem tedious, but the questions can be quickly answered

There are general formulae for the sum of a finite number of terms of an arithmetic or geometric sequence, and we will talk about them in this section.

§2.2.1 Finite arithmetic sums. Triangular numbers.

Suppose that circles of equal diameter are packed tightly in n rows inside an equilateral triangle (see Figure 3). How many circles are there? It is clear that we have $1+2+3+\cdots+n$ circles. This sum has a special name, and appears frequently in mathematics. It is called the n-th *triangular number* and can be calculated as

$$T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Carl Friedrich Gauss found this formula for triangular numbers when he was around the age of ten. He used a unique method for this. He listed the n terms, wrote this list backwards under the original list, and added vertically.

$$
\begin{array}{ccccccccc}
T_n & = & 1 & + & 2 & + & \cdots & + & n \\
+ & & + & & + & & & & + \\
T_n & = & n & + & n-1 & + & \cdots & + & 1 \\
\Downarrow & & \Downarrow & & \Downarrow & & & & \Downarrow \\
2T_n & = & n+1 & + & n+1 & + & \cdots & + & n+1
\end{array}
$$

$$2T_n = n(n+1)$$

$$T_n = \frac{n}{2}(n+1)$$

For a more general arithmetic sequence $a_k = a + (k-1)d$, $k = 1, 2, ...$, the sum S_n of its first n terms is

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n}(a + (k-1)d) \\
&= \sum_{k=1}^{n} a + d\left(\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\right) \\
&= na + \frac{dn(n+1)}{2} - dn = \frac{n}{2}(2a + (n-1)d).
\end{aligned}
$$

Here we have used the properties (10).

The natural numbers form an arithmetic sequence with $a = 1$ and $d = 1$. Thus expression for S_n simplifies to T_n in this case.

Patterns show up in many triangular numbers. One example is the triangular numbers whose indices are powers of 10. For those, we have

$$
\begin{aligned}
T_{10} &= 55 \\
T_{100} &= 5050 \\
T_{1000} &= 500500.
\end{aligned}
$$

Some more general formulas are

$$
\begin{aligned}
T_{n+1}^2 - T_n^2 &= (n+1)^3 \\
T_{n+1} + T_n &= (n+1)^2.
\end{aligned}
$$

The reader can check these properties as a simple algebraic exercise.

Now let us present a problem which involves the triangular numbers.

Problem 14. Reconsider the construction stated earlier with the circles inside the equilateral triangle (Figure 3). Let $\mathcal{A}$ be the area of the equilateral triangle $\triangle ABC$ and $\mathcal{A}_n$ be the total area occupied by the n rows of circles, show that

$$
\lim_{n \to \infty} \frac{\mathcal{A}_n}{\mathcal{A}} = \frac{\pi}{2\sqrt{3}}
$$

Solution: First, we know that the number of circles inside the triangle is the n-th triangular number. Thus, the number of circles is $n(n+1)/2$.

We will denote the side length of the triangle as L and we will denote the radius of the circles as r. Now, the length of the side is the sum of the diameters of n circles and two small portions on each end of the length. We will denote each of these pieces as b and write $L = 2nr + 2b$. First, we find b in terms of r. For that we draw a small right triangle $\triangle BOE$ in the corner of the equilateral triangle. Its vertices will be at the vertex B of the equilateral triangle, center O of the closest circle, and point E of tangency of the closest circle with the side of the equilateral triangle.
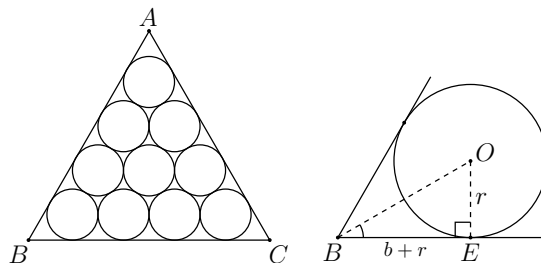
Figure 3. Circles inside an equilateral triangle.

Because the large triangle $\triangle ABC$ is equilateral, its angles are $\pi/3$ radians, and consequently the small angle $\angle OBE$ in the right triangle is half of it, $\pi/6$ radians. From this we can see that

$$
\tan(\pi/6) = r/(b+r) \Rightarrow b + r = \sqrt{3}r \Rightarrow b = (\sqrt{3} - 1)r.
$$

Now, $L = 2nr + 2(\sqrt{3} - 1)r = 2(n + \sqrt{3} - 1)r$. Thus, $r = \dfrac{L}{2(n + \sqrt{3} - 1)}$.

The total area covered by n rows of circles is

$$\mathcal{A}_n = \frac{n(n+1)}{2}\pi r^2 = \frac{\pi L^2 n(n+1)}{8(n + \sqrt{3} - 1)^2}.$$

Because the triangle is equilateral with side length L, its height is $(\sqrt{3}/2)L$ and consequently its area is $(1/2) \cdot L \cdot (\sqrt{3}/2)L = \sqrt{3}\,L^2/4$. Now,

$$
\begin{aligned}
\lim_{n \to \infty} \frac{\mathcal{A}_n}{\mathcal{A}} &= \lim_{n \to \infty} \left(\mathcal{A}_n \cdot \frac{1}{\mathcal{A}} \right) \\
&= \lim_{n \to \infty} \left(\frac{\pi L^2 n(n+1)}{8(n + \sqrt{3} - 1)^2} \cdot \left(\frac{4}{\sqrt{3}L^2} \right) \right) \\
&= \frac{\pi}{2\sqrt{3}} \lim_{n \to \infty} \left(\frac{n^2 + n}{n^2 + 2(\sqrt{3} - 1)n + (\sqrt{3} - 1)^2} \right) \\
&= \frac{\pi}{2\sqrt{3}} \cdot 1 = \frac{\pi}{2\sqrt{3}}.
\end{aligned}
$$

The limit is found using techniques learned in Section 1.4.

§2.2.2 Finite geometric sums

For a finite geometric series, there is a trick to find its sum as well. Let

$$\mathrm{S} = 1 + r + r^2 + \cdots + r^n;$$

then, we have

$$r \cdot \mathrm{S} = r + r^2 + r^3 + \cdots + r^n + r^{n+1}.$$

Consequently

$$
\begin{aligned}
\mathrm{S} - r\mathrm{S} &= 1 - r^{n+1}, \\
(1 - r)\mathrm{S} &= 1 - r^{n+1}, \\
\mathrm{S} &= \frac{1 - r^{n+1}}{1 - r}.
\end{aligned}
$$

In particular, the sum of the first 25 positive integer powers of 2 is 67,108,862.

§2.3 Infinite geometric sums

In this section we deal with an infinite geometric series of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots .$$

§2.3.1. The general formula

Geometric series are very nice due to the fact that there is a very simple way to determine their convergence, and, if convergent, find their sum. A geometric series diverges if $|r| \geq 1$; otherwise it converges, and its sum is the following:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad \text{for} \quad |r| < 1.$$

This formula has been known for a long time: it appeared in Euclid's *Elements*, written approximately 2300 years ago, and was used in the work of Archimedes.

There are several ways to prove this formula. One of them is motivated by the following example.

Consider a geometric series with $r = \frac{1}{2}$:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots .$$

Now, let us assume that the sum is a finite number and denote it by x. Then

$$2x = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 2 + x.$$

Thus, solving $2x = 2 + x$, we get $x = 2$. We have

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

Remember, we already had similar formula in the introduction, where its motivation was more geometrical than algebraic.

Now consider a geometric series with $r = 1/q$:

$$1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} + \cdots .$$

Again, let us assume that the sum is a finite number and denote it by x. Then

$$qx = q + 1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} + \cdots = q + x.$$

Thus $x = q/(q-1)$. Substitute $q = 1/r$ and do some algebra to obtain $x = 1/(1-r)$.

Until now, it seems as if there is no restriction on the value of r. The following example, where $r = 2$, shows that it is not so, and formal use of this approach leads to a contradiction. Let $x = 1 + 2 + 4 + 8 + 16 + \cdots$. The sum is positive since each term is positive. On the other hand, $2x = 2 + 4 + 8 + \cdots = x - 1$. Solving for x we have $x = -1$. Thus, we have just "proved" that the sum of an infinite number of positive terms is equal to a negative number. This is clearly false and thus, for $r = 2$ the series is divergent. Therefore, we conclude that this "proof" works only under the assumption that the series converges, but does not show when it does.

Another proof of the general formula, which gives a condition for convergence,

follows directly from the formula for a finite geometric sum through the use of Definition 9 and the properties of limits. Consider the n-th partial sum

$$S_n = 1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

Then the limit

$$\lim_{n \to \infty} S_n = \frac{1 - \lim\limits_{n \to \infty} r^{n+1}}{1 - r}$$

exists only if $|r| < 1$, when $\lim\limits_{n \to \infty} r^{n+1} = 0$ and thus by definition

$$1 + r + r^2 + \cdots = \lim_{n \to \infty} S_n = \frac{1}{1 - r}. \tag{11}$$

This result is illustrated for $0 < r < 1$ by Figure 4 below. It is a good exercise to convince yourself that from the similarity of the triangles ACD and AP_1B it follows that $r = (s - 1)/s$ and thus $s = (1 - r)^{-1}$.
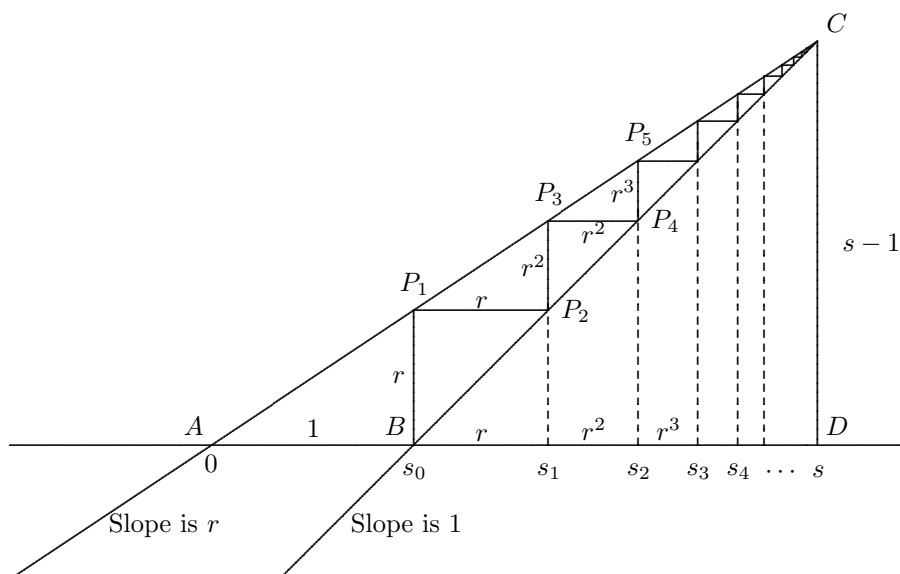
Figure 4. A sum of a geometric series.

§2.3.2 Problems involving geometric series

Let us consider the following situation. A fireman uses water from a tank at a rate three times as big as the rate at which water is supplied to the tank. If there were initially 600 litres of water in the tank, how many litres will be used before the tank is empty?

We can approach this question as follows. Observe that by the time a given amount of water is used by the fireman, a third of this amount will be supplied to

the tank. Thus after 600 litres are used, 600/3=200 litres will be supplied to the tank. Similarly, by the time 200 litres are used, 200/3=600/9 litres will be supplied to the tank, etc. If we continue in the same way, the total amount of water used will be represented by the infinite geometric series $600 \times (1+1/3+1/9+..)$. Using (11) with $r = 1/3$ we find $1 + 1/3 + 1/9 + .. = 3/2$. Thus, the answer is 900 litres.

The geometric sum (11) appears in many other problems involving an infinite sequence of similar geometric objects (segments, squares, triangles, circles). Some examples of this kind, including fractals, are collected in this section. Besides that, using geometric series, one can see why any repeating decimal number can be written as a fraction and thus is a rational number. Another example illustrates how formula (11) helps deal with an infinite product.

Problem 15. The path of a ball. A ball falls from a height of 1m and bounces to heights of 1/3 m, 1/9 m, 1/27 m, and so on. Find the total distance that the ball covers after an infinite number of bounces.

Solution: The total distance is

$$1 + 2 \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \right) = 1 + \frac{2}{3} \left(1 + \frac{1}{3} + \frac{1}{9} + \cdots \right) = 1 + \frac{2}{3} \cdot \frac{1}{1 - \frac{1}{3}} = 1 + 1 = 2.$$

Here formula (11) with $r = 1/3$ was used. The answer is 2 m.

Problem 16. An infinite sequence of squares.

i) Consider an infinite sequence of squares such that vertices of each square coincide with the midpoints of the sides of the previous square. Find the sum of the areas of the squares, if the first square has an area of 1 m^2.
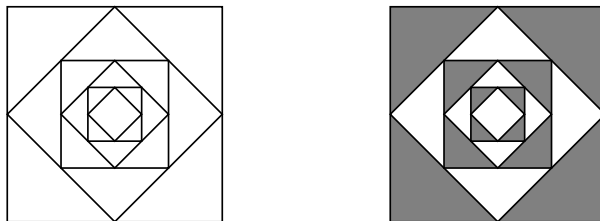
Figure 5. A sequence of squares.

ii) Let the first square be black, the second one be white, the third one be black again, and so on. The colouring is shown on the right. Find the total area covered by the black colour.

Solution: First note that the area of a square, whose vertices are the midpoints of the sides of a given square, is exactly half that of the given square. Since this pattern continues, we have an infinite sequence of squares in which each square area is one-half as large as the area of the previous one. Thus we have a geometric sequence of the areas of the squares: 1, 1/2, 1/4, 1/8, ...

In question (i) we are asked about the total area of the squares, which according to our previous discussion is the sum of the geometric series with $r = 1/2$:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2$$

The answer for question (i) is 2 m^2.

To answer question (ii) we have to add every black square area and subtract every white square area, which results in geometric series with $r = -1/2$:

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}.$$

The answer for question (ii) is $2/3$ m^2.

Problem 17. The length of a jagged line in an angle.

Consider an angle $0 < \theta < \pi/2$. Start with any point A_0 on the side of the angle and draw the perpendicular to the other side. Denote the point of intersection by A_1. Then draw the perpendicular to the opposite side of the angle and denote the intersection by A_2, and so on. By the jagged line we understand the resulting piece-wise linear curve with turning points at $A_0, A_1, A_2, \ldots A_k, \ldots$.

The problem is to find the length of the jagged line given the length of the first segment $|A_0 A_1| = b$.
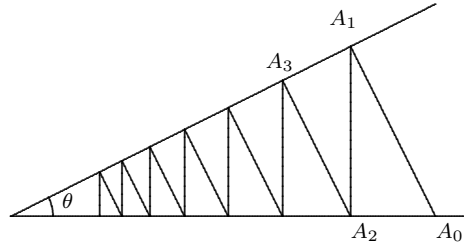
Figure 6. A jagged line.

Solution: We will start by denoting the segments' lengths by

$$|A_0 A_1| = a_1, \quad |A_1 A_2| = a_2, \quad |A_2 A_3| = a_3, \qquad \ldots$$

Consider a sequence of similar right triangles $A_0 A_1 A_2$, $A_1 A_2 A_3$, ... They all contain the angle θ with the side of length a_{n+1} adjacent the angle θ, and side of length a_n as the hypotenuse. Thus $\cos \theta = a_{n+1}/a_n$ where θ is a constant. That implies the recurrence $a_{n+1} = a_n \cos \theta$ for all $n \geq 1$. In particular, $a_2 = a_1 \cos \theta$, and $a_3 = a_2 \cos \theta = a_1 \cos^2 \theta$. Similarly, we have $a_{n+1} = a_1 \cos^n \theta$.

Recall that $a_1 = b$, and so the segments' lengths form a geometric sequence $a_{n+1} = b \cos^n \theta$ for all $n \geq 1$. The common ratio $a_{n+1}/a_n = \cos \theta$. Since $0 < \theta < \pi/2$, $0 < \cos \theta < 1$. This condition ensures us that the corresponding geometric

series, which gives the required length of the jagged line, converges, and we obtain:

$$|A_0 A_1| + |A_1 A_2| + |A_2 A_3| + \cdots \quad = \quad a_1 + a_2 + a_3 + \cdots =$$

$$b + b\cos\theta + b\cos^2\theta + \cdots \quad = \quad b(1 + \cos\theta + \cos^2\theta + \cdots) =$$

$$b\sum_{n=0}^{\infty}(\cos\theta)^n \quad = \quad \frac{b}{1 - \cos\theta}$$

Thus the length of the jagged line in an angle of size θ with first segment of length b is $b\,(1 - \cos\theta)^{-1}$.

We invite the reader to check directly some special cases (e.g. $\theta = \pi/4$ and $\theta = \pi/3$). What do you think will happen when θ is very close to 0 or $\pi/2$?

Problem 18. Two examples of a reduction to a geometric series.
18A. Find the infinite product

$$3^{1/3} \cdot 9^{1/9} \cdot 27^{1/27} \cdot 81^{1/81} \cdots$$

Solution: Denote the product by N. Then $N = 3^M$, where

$$M = \frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} \cdots.$$

Note that

$$\frac{M}{3} = \frac{1}{9} + \frac{2}{27} + \frac{3}{81} + \cdots.$$

Thus

$$M - \frac{M}{3} = \frac{2}{3}M = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \cdots = \frac{1}{3}\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right).$$

This geometric series in the parentheses with $r = 1/3$ has the sum $3/2$. Therefore, $2M/3 = 1/2$, and $M = 3/4$. Finally, the product is $N = 3^{3/4}$.

18B. A number is written in the form of an "infinite" nested root with the following periodic pattern

$$\sqrt{4\sqrt{4\sqrt{4\sqrt{4\sqrt{4...}}}}}$$

Find the value of this number.
Solution: You may be familiar with the following solution of this problem. Let the number be x. We have $\sqrt{4x} = x$. Squaring both sides we get $4x = x^2$. Since $\sqrt{4} > 0$ and the sequence of finite nested roots increases, we need $x > 0$. Thus $x = 4$.

However, one may also use geometric series to solve this question. Note that

$$\sqrt{4\sqrt{4\sqrt{4\sqrt{4...}}}} = 4^M, \text{ where } M = 1/2 + (1/2)^2 + (1/2)^3 + ... + (1/2)^k + ...$$

Using the formula for the sum of geometric series with first term $1/2$ and common

ratio $r = 1/2$, we obtain $M = \dfrac{1/2}{1 - 1/2} = 1$. Thus, the value is $4^1 = 4$.

Problem 19. Conversion of a repeating decimal to fractional form.
Convert $0.\overline{543} = 0.543543543\ldots$ into a fraction.
Solution:

$$
\begin{aligned}
0.543543543\ldots &= 543 \cdot (0.001001001\ldots) \\
&= 543 \cdot (0.001 + 0.000001 + 0.000000001 + \cdots) \\
&= 543 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{1000} \right)^n \\
&= 543 \cdot \left(\left(\sum_{n=0}^{\infty} \left(\frac{1}{1000} \right)^n \right) - 1 \right) \\
&= 543 \cdot \left(\frac{1}{1 - \frac{1}{1000}} - 1 \right) \\
&= 543 \cdot \left(\frac{1000}{999} - 1 \right) = 543 \cdot \left(\frac{1}{999} \right) = \frac{543}{999}.
\end{aligned}
$$

Problem 20. Consider an isoceles $\triangle A_0 B_0 C$ with angles $\widehat{A_0} = \widehat{B_0}$, $\widehat{C} = \alpha$, and side $|A_0 B_0| = 1$. Consider an infinite sequence of circles approaching angle C and touching each other and the sides of the triangle (refer to the figure). Find the total area occupied by the circles in terms of the angle $\alpha \in (0, \pi)$.
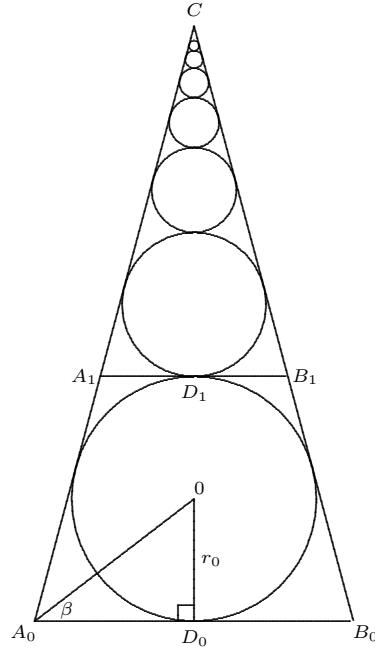

Figure 7. Circles in an isosceles triangle.

Solution: Note that the first circle of the sequence is inscribed in $\triangle A_0 B_0 C$, and the kth circle of the sequence is inscribed in a triangle which is similar to $\triangle A_0 B_0 C$. We claim that the *coefficient of similarity* (see Appendix B.4) is $(\tan \beta)^{2k}$, where $\beta = (\pi - \alpha)/4$, and will show that now.

Let CD_0 be the altitude, and let O be the incentre of $\triangle A_0 B_0 C$. Angle $\widehat{CA_0 B_0} = (\pi - \alpha)/2 = 2\beta$ and from $\triangle A_0 C D_0$ which is right angled, we obtain the length $|CD_0| = (1/2)\tan(2\beta)$. From $\triangle A_0 O D_0$ which is also right angled, we get the inradius $r_0 = |OD_0| = (1/2)\tan \beta$. Thus, the diameter of the first circle is $\tan \beta$.

Denote the point of tangency of the first and the second circles by D_1. Draw a line parallel to the side $A_0 B_0$ through D_1. Denote the points of intersection of the line with the sides $A_0 C$ and $B_0 C$ by A_1 and B_1 respectively. The second circle is inscribed in $\triangle A_1 B_1 C$. $\triangle A_1 B_1 C$ is similar to $\triangle A_0 B_0 C$ with coefficient $(\tan \beta)^2$ which follows from calculating the ratio of their heights (see Appendix B for double angle formulas):

$$\frac{|CD_1|}{|CD_0|} = \frac{|CD_0| - 2r_0}{|CD_0|} = 1 - \frac{2\tan \beta}{\tan(2\beta)} = (\tan \beta)^2.$$

By the same process it can be shown that the third circle of the sequence is inscribed in $\triangle A_2 B_2 C$ which is similar to $\triangle A_1 B_1 C$ with coefficient $(\tan \beta)^2$, and thus similar to $\triangle A_0 B_0 C$ with coefficient $(\tan \beta)^4$. In fact, for each circle, the corresponding circumscribed triangle $A_k B_k C$ is similar to the previous one $A_{k-1} B_{k-1} C$ with coefficient of similarity $(\tan \beta)^2$.

From the coefficients of similarity, we know the ratio of the radii of inscribed circles. Let r_k be the inradius of $A_k B_k C$. Then

$$\frac{r_1}{r_0} = (\tan \beta)^2, \quad \frac{r_2}{r_0} = (\tan \beta)^4, \quad \frac{r_k}{r_0} = (\tan \beta)^{2k}, \text{ for all } k \geq 1.$$

From this we immediately obtain the total area of the circles

$$\pi r_0^2 + \sum_{k=1}^{\infty} \pi r_k^2 = \pi r_0^2 \left(1 + \sum_{k=1}^{\infty} \frac{r_k^2}{r_0^2} \right) = \pi r_0^2 \sum_{k=0}^{\infty} \frac{r_k^2}{r_0^2} = \pi r_0^2 \sum_{k=0}^{\infty} (\tan \beta)^{4k} = \frac{\pi r_0^2}{1 - (\tan \beta)^4}.$$

Recall that $r_0 = \dfrac{\tan \beta}{2}$, so the total area occupied by the circles is $\dfrac{\pi}{4} \dfrac{\tan^2 \beta}{(1 - \tan^4 \beta)}$, where $\beta = (\pi - \alpha)/4$. Note that, if the triangle is equilateral, then $\alpha = \dfrac{\pi}{3}$, $\beta = \dfrac{\pi}{6}$, and the total area of the circles is $\dfrac{3\pi}{32}$.

Problem 21a. The Cantor Set. The Cantor set, named after Georg Cantor is constructed as follows. We start with the closed interval $[0,1]$ which clearly has length 1. We divide it into 3 equal intervals, with the middle interval being an open interval (it does not contain its endpoints). We remove the middle interval $\left(\frac{1}{3}, \frac{2}{3}\right)$, and this leaves the two intervals $\left[0, \frac{1}{3}\right]$, and $\left[\frac{2}{3}, 1\right]$. We take each of these intervals, divide them into 3 identical sized intervals, and remove the middle open interval again. We repeat this process forever. The Cantor set is the set of all real numbers which remain after all the intervals have been removed.

The problem is to show that the total length of all the intervals removed is 1, which is equal to the length of the original interval. (Despite this fact, the Cantor set contains infinitely many numbers!)

Solution: The first interval removed has length $\frac{1}{3}$. There are now 2 other intervals remaining, each with length $\frac{1}{3}$. We remove an interval from each of the remaining ones, each interval removed has length $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. There are now 4 intervals remaining. Each of them has length $\frac{1}{9}$. We remove an interval from each of them. Each interval removed has length $\frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$. Therefore, if we repeat this process, the total length of intervals removed is:

$$
\frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \cdots = 2^0 \cdot \frac{1}{3^1} + 2^1 \cdot \frac{1}{3^2} + 2^2 \cdot \frac{1}{3^3} + \cdots
$$

$$
= \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1.
$$

Thus, as desired, the total length of the intervals removed is 1.

What are the numbers that are left in the Cantor set? Some examples are:

$$
0, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}.
$$

If we look carefully at the procedure which constructs the set we find that a number is in the Cantor set if and only if it is the sum of a series

$$
\frac{a_1}{3^1} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \quad \text{where } a_k \text{ is either } 0 \text{ or } 2.
$$

The series is infinite in some cases and in others not. For example, $\frac{8}{9} = \frac{2}{3} + \frac{2}{3^2}$ corresponds to $a_1 = a_2 = 2$, $a_k = 0$ for $k \geq 3$. However, $\frac{1}{3} = \sum_{k=2}^{\infty} \frac{2}{3^k}$ is represented by an infinite geometric series and corresponds to $a_1 = 0$, $a_k = 2$ for $k \geq 2$. The removal of the middle third of each interval corresponds to the condition $a_k \neq 1$ in the representation above. The Cantor set includes the endpoints of each removed open interval, but not only these points. Even if we had removed the middle closed interval in each step instead of removing the open interval, the set of points remaining would have been infinite.

Problem 21b. The Sierpinski Carpet. The Sierpinski carpet is very similar to the Cantor set. It can be viewed as being a 2-dimensional counterpart of the Cantor set. It is constructed as follows. We start with a square of side length

1. We partition this square into 9 smaller squares of equal area, and remove the interior of the middle square. Now, we take each of the remaining 8 squares and partition them into 9 smaller squares. We again remove the interior of the middle square in each one. We take each of the $8 \cdot 8 = 64$ remaining squares and partition them, and again remove the interior of the middle square. We continue the process forever. Show that the total area of the removed squares is 1, which implies that the Sierpinski carpet has an area 0.
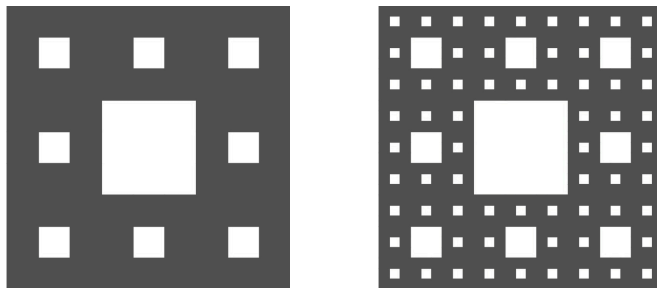

Figure 8. The Sierpinski Carpet.

Solution: The first square removed has area $\dfrac{1}{3^2}$. Then we remove 8 more squares, each with area $\left(\dfrac{1}{3^2}\right)^2 = \dfrac{1}{3^4}$. Now we remove 64 more squares, each with area $\left(\dfrac{1}{3^3}\right)^2 = \dfrac{1}{3^6}$. We continue this process forever. The total area of the removed squares is:

$$
\begin{aligned}
\frac{1}{3^2} + 8 \cdot \frac{1}{3^4} + 64 \cdot \frac{1}{3^6} + \cdots &= 8^0 \cdot \frac{1}{9} + 8^1 \cdot \frac{1}{9^2} + 8^2 \cdot \frac{1}{9^3} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{8^n}{9^{n+1}} = \frac{1}{9} \sum_{n=0}^{\infty} \left(\frac{8}{9}\right)^n = \frac{1}{9} \cdot \frac{1}{1 - \frac{8}{9}} = 1.
\end{aligned}
$$

Thus, as desired, the total area of the removed squares is 1. Just as in the Cantor set, even though it may appear that there is nothing remaining, there are infinitely many points remaining. In particular, if $\mathcal{C}$ is the Cantor set, then the sets[4] $\mathcal{C} \times [0, 1]$ and $[0, 1] \times \mathcal{C}$ are both subsets of the Sierpinski carpet.

Problem 22. The Koch Snowflake. The Koch snowflake is constructed as follows. Start with an equilateral triangle of side length 1. Then, partition each side into 3 equal portions. On the middle portion of each side, place another equilateral triangle, with side length equal to the length of that portion. Now, do the same thing on each side of the resulting figure and continue indefinitely. The figure that appears as the limit of these constructions is the Koch Snowflake.

---

[4]Here $A \times B$ denotes the *Cartesian product* of two sets A and B, that is, the set of all pairs (a, b), where $a \in A$ and $b \in B$.
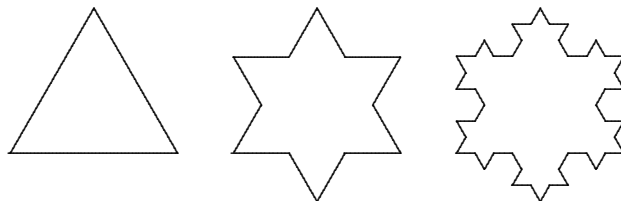
Figure 9. The Koch Snowflake.

If you were to calculate the area and perimeter for the first few steps, you will notice that they are both increasing with each step. There is one major difference though. The sequence of areas is bounded because each figure can actually be inscribed inside the same circle as the original triangle. Thus, the sequence of areas is bounded from above and monotonically increasing and is therefore convergent, the resulting area is $\dfrac{2\sqrt{3}}{5}$. The same is not true for the sequence representing the perimeter. That sequence increases without bound, and thus, the perimeter is infinite. You should try to show both of these facts using geometric series.

Problem 23. The Four Mice Problem.

Four mice are placed initially at the corners A, B, C, and D of a square room. Each mouse runs directly towards the mouse on its right, continuously changing direction as the mouse on the right moves on its path. All four mice run at the same constant speed, and they continue running, spiralling inwards, until they all meet at the centre of the square. The side length of the room is 1 unit. What is the length of the path run by each mouse?

This problem can be found under various names such as the turtle, dog, beetle, or mice problem. Some Internet sites also provide animations and variations (see e.g. http://mathworld.wolfram.com/MiceProblem.html). There are many approaches to solving it. Here we give a solution which relies on the geometric sum.
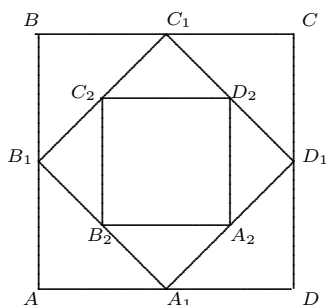

Figure 10. An auxiliary Four Mice Problem.

First, we consider an auxiliary problem. In this problem the difference from the previous one is that the mice change direction at discrete intervals instead of

continuously. The mouse starting from A initially runs half of the length of side AD ending at A_1, and similarly for the other three mice. So at the end of the first interval the four mice have reached A_1, B_1, C_1, D_1 respectively. Now the mice change direction. The mouse at A_1 runs half the length of the side A_1D_1 ending at A_2, and similarly for the other three mice. At the end of the second interval the four mice have reached A_2, B_2, C_2, D_2 respectively (see Figure 10). The mice continue moving in this fashion. After infinitely many iterations they meet at the centre of the square. The question is the same as before: what is the total length of the path run by each mouse?

Solution: Observe that if the side length of $ABCD$ is 1 unit then the side length of A_1, B_1, C_1, D_1 is $1/\sqrt{2}$, the side length of the next square A_2, B_2, C_2, D_2 is $(1/\sqrt{2})^2$, and the side length of the n-th square A_n, B_n, C_n, D_n is $(1/\sqrt{2})^n$. Since each mouse runs a half of the side of each square in this infinite sequence of squares, the path length is

$$\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}} + \cdots + \left(\frac{1}{\sqrt{2}}\right)^n + \cdots\right) = \frac{1}{2}\left(\frac{1}{1 - 1/\sqrt{2}}\right) = \frac{2}{2 - \sqrt{2}}.$$

Next, we modify the auxilary problem as follows. Let $k \geq 2$. As before each of the four mice is pursuing the mouse on its right, each starting from one of the four corners A, B, C, D of the room. Again, the mice change direction at discrete intervals. This time, at the first stage, the mouse which starts from A runs $1/k$ along the side towards D stopping at A_1 which is $1/k$ from A. Similarly, for the other three mice. At the second stage the mouse at A_1 runs $1/k$ of the side A_1D_1 towards D_1 stopping at A_2 which is $(1/k) \cdot A_1D_1$ from A_1. Similarly, for the other three mice. The question is the same as before: what is the total length of the path run by each mouse?

Solution: Note that for $k = 2$ we have the problem that we already solved. In general, by the Pythagorean theorem each square $A_{n+1}B_{n+1}C_{n+1}D_{n+1}$ in the sequence is

$$\sqrt{(1/k)^2 + [1 - (1/k)]^2} = \frac{\sqrt{k^2 - 2k + 2}}{k}$$

times the size of the previous square $A_nB_nC_nD_n$. Since each mouse runs $1/k$ of the side of each square in the sequence, the total path length is:

$$\frac{1}{k}\sum_{n=0}^{\infty}\left(\frac{\sqrt{k^2 - 2k + 2}}{k}\right)^n = \frac{1}{k - \sqrt{k^2 - 2k + 2}}$$

In the limit as $k \to \infty$, the paths run by the mice become the paths run when each mouse is always running directly towards the mouse on its right — as in the original Four Mice Problem. Thus the solution to the Four Mice Problem is

$$L = \lim_{k \to \infty} \frac{1}{k - \sqrt{k^2 - 2k + 2}}.$$

To evaluate the limit we multiply numerator and denominator by $k + \sqrt{k^2 - 2k + 2}$:

$$\frac{1}{k - \sqrt{k^2 - 2k + 2}} = \frac{k + \sqrt{k^2 - 2k + 2}}{k^2 - (k^2 - 2k + 2)} = \frac{k + \sqrt{k^2 - 2k + 2}}{2k - 2}$$

Now we see that

$$\lim_{k\to\infty} \frac{k + \sqrt{k^2 - 2k + 2}}{2k - 2} \ = \ \lim_{k\to\infty} \frac{1 + \sqrt{1 - (2/k) + (2/k^2)}}{2 - (2/k)} \quad \text{dividing top and} \\ \text{bottom by } k$$

$$= \ \frac{\lim_{k\to\infty} 1 + \sqrt{1 - (2/k) + (2/k^2)}}{\lim_{k\to\infty} 2 - (2/k)} \ = \ \frac{1 + \sqrt{1}}{2} \ = \ 1.$$

Therefore $L = 1$ and the path of each mouse has length 1.

§2.4 Telescoping

In this section we will introduce a method which can help to find the sum of a given series. This method only works for a certain class of problems, and it requires more creativity than the use of the geometric formula. Nevertheless, it is a very important method because it is one of the very few ways to find the sum of infinite as well as finite summations.

§2.4.1. Finding telescoping sums

The main idea of the method is to change the appearance of the sum and look for similar terms to cancel with each other. We will first illustrate the idea with a pair of finite summations.

Problem 24. Give an explicit formula for the following sum.

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!$$

Solution. Use $k \cdot k! = (k + 1) \cdot k! - k! = (k + 1)! - k!$. Then

$$1\cdot1! + 2\cdot2! + 3\cdot3! + \cdots + n\cdot n! = (2! - 1!) + (3! - 2!) + (4! - 3!) + \cdots + ((n+1)! - n!).$$

After cancellations we obtain $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n + 1)! - 1$.

Problem 25. Find the sum

$$\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{1000 \cdot 1001}.$$

Solution: Of course the sum could be found by adding all the terms manually, but it can be done faster and easier by telescoping.

Note that

$$\frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3}, \qquad \frac{1}{3 \cdot 4} = \frac{1}{3} - \frac{1}{4}, \quad \cdots, \quad \frac{1}{1000 \cdot 1001} = \frac{1}{1000} - \frac{1}{1001}.$$

Then we rewrite the sum as

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{1000} - \frac{1}{1001}. \tag{12}$$

Many terms cancel each other. In fact, all terms other than the first and the last

cancel. Thus, the sum is $\dfrac{1}{2} - \dfrac{1}{1001} = \dfrac{999}{2002}$.

Problem 26. Consider a sequence of triangles the n-th having vertices:

$$(1, 2), \quad \left(\frac{1}{n}, 0\right), \quad \left(\frac{1}{n+1}, 0\right), \qquad n = 2, 3, 4, \ldots, 1000.$$
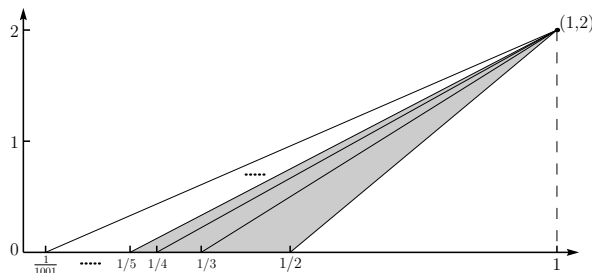
Find their total area.

Figure 11. A sequence of triangles.

Solution: Each of the triangles has $(1, 2)$ as one vertex and two other vertices on the x-axis. Thus, each of the triangles has height 2. The length of base of the n-th triangle on the x-axis is $n^{-1} - (n+1)^{-1}$. Since the area of a triangle is a half of the height times base, the area of the n-th triangle is the same as the length of its base. To find the total area we need to sum the areas of all 999 triangles. This, in fact, leads to the same telescoping procedure (12) as in the previous problem. After cancellations we again obtain $1/2 - 1/1001 = 999/2002$.

However, the problem has a geometric solution. The segment joining $(1, 2)$ and $\left((n+1)^{-1}, 0\right)$ is a common side of the n-th and $(n+1)$-th triangles, for $2 \leq n \leq 999$. Thus the total area is the area of the big triangle with vertices $(1, 2)$, $(1/2, 0)$, $(1/1001, 0)$ which has height 2 and base length $1/2 - 1/1001 = 999/2002$. So the area of the big triangle is $999/2002$.

This problem is useful for us in two ways. First it provides a geometric interpretation of the telescoping procedure, and second, we can generalize this result to the case of arbitrary $N > 2$ (not only $N = 1000$ as in the two problems above):

$$\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{N \cdot (N+1)} = \frac{1}{2} - \frac{1}{N+1}. \tag{13}$$

Both algebraic and geometric (using triangles' area) solutions will work.

The advantage of having a general formula is that we can now ask ourselves: What will happen if we let N go to infinity?

Following the geometric interpretation, the areas of the $N - 1$ triangles add up to the area of the triangle with vertices $(1, 2)$, $(1/2, 0)$, $(1/(N+1), 0)$. In the limit $N \to \infty$, the vertices of the resulting triangle become $(1, 2)$, $(1/2, 0)$, $(0, 0)$; the area of the triangle is $1/2$, and this is the sum of the infinite series:

$$\sum_{n=2}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2}. \tag{14}$$

When we find the area (13) of the triangle with vertices $(1, 2)$, $(1/2, 0)$, $(1/(N+1), 0)$, we, in fact, find a partial sum of the series. Thus, finding the total sum this way agrees with our previous concept of convergence of a series in terms of the limit of the sequence of its partial sums (see Definition 3):

$$S = \lim_{N \to \infty} \left(\frac{1}{2} - \frac{1}{N+1} \right) = \frac{1}{2}.$$

There is one more geometrical interpretation of this formula which is provided by the following example.

Problem 27. Figure 11 shows two circles α and β of radius 1 that touch at X. τ is a common tangent line and γ_1 is the unique circle touching α, β, and τ; γ_2 is the unique circle touching α, β, and γ_1. For each $n \geq 1$, let γ_{n+1} be the unique circle touching α, β, and γ_n. Show that the diameter d_n of γ_n is equal to

$$d_n = \frac{1}{n(n+1)}, \qquad n \geq 1.$$
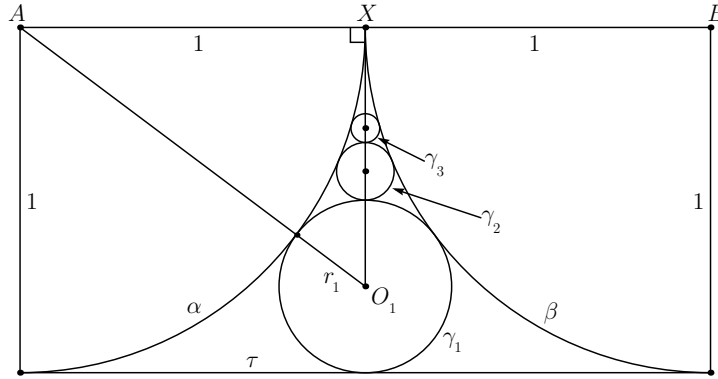
Figure 12. A sequence of circles.

Solution: We will start by finding the radius r_1 of γ_1. Let A and O_1 be centres of cercles α and γ_1 respectively.

Draw a right triangle with vertices O_1, A, and X. The legs of this right triangle have lengths $XO_1 = 1 - r_1$ and $AX = 1$. The hypotenuse of this triangle has length $AO_1 = 1 + r_1$. We can use the Pythagorean theorem to obtain:

$$
\begin{aligned}
(1 - r_1)^2 + 1 &= (1 + r_1)^2 \\
1 - 2r_1 + r_1^2 + 1 &= 1 + 2r_1 + r_1^2 \\
1 &= 4r_1 \\
r_1 &= \frac{1}{4}
\end{aligned}
$$

Thus, the diameted of γ_1 is $d_1 = 2r_1 = 1/2$.

The following statement establishes a relationship between diameters of two touching circles in a slightly more general situation.

Lemma. Let α, β be circles of radius 1, with centres A, B respectively, which touch at X. Let γ be a circle which touches both α and β externally and δ be the unique circle which touches α, β, and γ externally as in Figure 12. If the diameter of γ is $1/[c(c+1)]$, where $c > 0$, then the diameter of δ is $1/[(c+1)(c+2)]$.
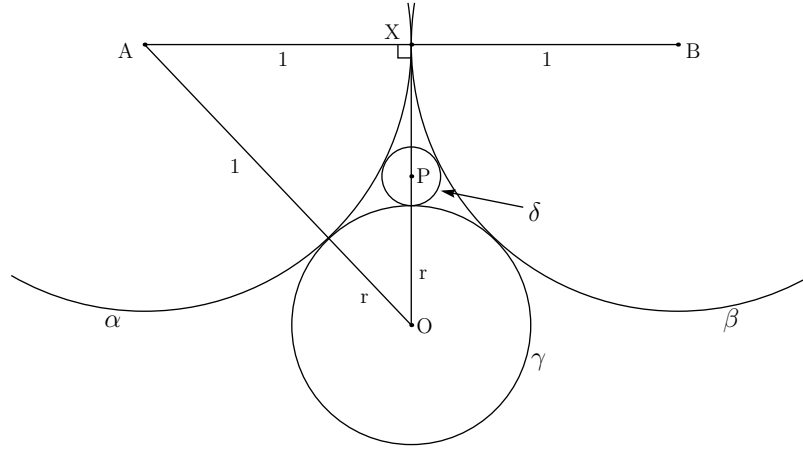

Figure 13. Touching circles

Note. Since every $d > 0$ can be written uniquely in the form $1/[c(c + 1)]$ with $c > 0$ the lemma applies to every circle γ touching α and β externally. It is convenient to write the diameter of γ in the form $1/[c(c + 1)]$ because it gives us a neat way of expressing the relationship between the two diameters.

Proof: Let O, P be the centres of γ, δ respectively, and r, s their radii. Applying the Pythagorean theorem to $\triangle AXP$ gives

$$(r + 1)^2 = XO^2 + 1^2.$$

Therefore $XO = \sqrt{(r + 1)^2 - 1}$ and similarly $XP = \sqrt{(s + 1)^2 - 1}$. Since $OP = XO - XP$ and γ and δ touch, we have

$$\sqrt{(r + 1)^2 - 1} - \sqrt{(s + 1)^2 - 1} = r + s$$

which is the same as

$$\sqrt{(r + 1)^2 - 1} - r = \sqrt{(s + 1)^2 - 1} + s \tag{15}$$

Substituting $r = 1/[2c(c + 1)]$, the left hand side of (15) becomes:

$$\sqrt{\left(\frac{1}{2c(c + 1)} + 1\right)^2 - 1} - \frac{1}{2c(c + 1)}$$

$$= \frac{1}{2c(c + 1)}\left(\sqrt{1 + 4c(c + 1)} - 1\right) = \frac{1}{c + 1}$$

Substituting in (15) gives the equation $1/(c+1) = \sqrt{(s+1)^2 - 1} + s$ for s. Subtracting s from both sides and then squaring both sides we get

$$\frac{1}{(c+1)^2} - \frac{2s}{c+1} + s^2 = s^2 + 2s$$

The unique solution is $s = 1/[2(c+1)(c+2)]$ and so the diameter of δ is $1/[(c+1)(c+2)]$ as claimed.

The result stated in Problem 27 follows by induction on n. We already know that $d_1 = 1/2$. The induction step is provided by the lemma because, for each n, γ_{n+1} is related to γ_n in exactly the same way as δ to γ in the lemma. Consequently we have

$$d_1 = \frac{1}{1 \cdot 2}, \quad d_2 = \frac{1}{2 \cdot 3}, \quad d_3 = \frac{1}{3 \cdot 4}, \cdots, d_n = \frac{1}{n(n+1)}, \cdots$$

As one can see from the Figure 13, the infinite sum of the diameters d_n is equal to the radius of the two largest circles, which is 1. Thus we obtain

$$\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Note that this formula easily follows from (14) by adding $1/2$ to each side.

Problem 28. Find the sum

$$\sum_{k=0}^{\infty} \frac{k^2 + 3k + 1}{(k+2)!}.$$

Solution: Note that $k^2 + 3k + 1 = (k^2 + 3k + 2) - 1 = (k+1)(k+2) - 1$. Also, recall that $k! = 1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k$. Thus, $(k+2)! = k!(k+1)(k+2)$. With this in hand we can rewrite the general term as

$$\frac{k^2 + 3k + 1}{(k+2)!} = \frac{(k^2 + 3k + 2) - 1}{(k+2)!} = \frac{(k+1)(k+2)}{(k+2)!} - \frac{1}{(k+2)!} = \frac{1}{k!} - \frac{1}{(k+2)!}.$$

Using the convention that $0! = 1$ the N-th partial sum is now

$$S_N = \frac{1}{0!} - \frac{1}{2!} + \frac{1}{1!} - \frac{1}{3!} + \frac{1}{2!} - \frac{1}{4!} \cdots + \frac{1}{N!} - \frac{1}{(N+2)!}.$$

After cancellations we obtain

$$S_N = 1 + 1 - \frac{1}{(N+1)!} - \frac{1}{(N+2)!}.$$

Finally, we take the limit as $N \to \infty$ and see that $S_N \to 2$.

Thus, the answer is

$$\sum_{k=0}^{\infty} \frac{k^2 + 3k + 1}{(k+2)!} = 2.$$

The next two examples refer to the Fibonacci numbers F_n. Telescoping rearrangements of the series are possible due to the identities which we derived in Section 1.9.1.

Problem 29. Show that:

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1} \cdot F_{n+1}} = 1$$

Solution: Use the identity $\dfrac{1}{F_{n-1} \cdot F_n} - \dfrac{1}{F_n \cdot F_{n+1}} = \dfrac{1}{F_{n-1} \cdot F_{n+1}}$ from Problem 13 to write the general term as a difference of two others

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1} \cdot F_{n+1}} = \sum_{n=2}^{\infty} \left(\frac{1}{F_{n-1} \cdot F_n} - \frac{1}{F_n \cdot F_{n+1}} \right).$$

Denoting $\dfrac{1}{F_{n-1} \cdot F_n}$ by a_n for $n \geq 2$ observe that the series has the form $\sum_{n=2}^{\infty} (a_n - a_{n+1})$.

Therefore the partial sum is

$$S_N = \sum_{n=2}^{N} \frac{1}{F_{n-1} \cdot F_{n+1}} = 1 - a_{N+1} = 1 - \frac{1}{F_N \cdot F_{N+1}}.$$

We now can take the limit of the partial sum to get the desired result:

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1} \cdot F_{n+1}} = \lim_{N \to \infty} \left(1 - \frac{1}{F_N \cdot F_{N+1}} \right) = 1.$$

Problem 30. Show that:

$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1} \cdot F_{n+1}} = 2.$$

Solution: We start by using the definition of the Fibonacci sequence to change the form of F_n in the numerator, and then write the expression as a difference of two terms:

$$a_n = \frac{F_n}{F_{n-1} \cdot F_{n+1}} = \frac{F_{n+1} - F_{n-1}}{F_{n-1} \cdot F_{n+1}} = \frac{1}{F_{n-1}} - \frac{1}{F_{n+1}}$$

Writing out the sum of the first N terms, we get:

$$S_N = \left(1 - \frac{1}{2} \right) + \left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{8} \right) + \cdots + \left(\frac{1}{F_{N-1}} - \frac{1}{F_{N+1}} \right).$$

For each $n \geq 1$, the negative portion of a_n will always cancel with the positive portion of a_{n+2}. Thus, the N-th partial sum is:

$$S_N = 1 + 1 - \frac{1}{F_N} - \frac{1}{F_{N+1}} = 2 - \frac{1}{F_N} - \frac{1}{F_{N+1}}$$

Taking the limit, we get the desired result:

$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1} \cdot F_{n+1}} = \lim_{N \to \infty} \left(2 - \frac{1}{F_N} - \frac{1}{F_{N+1}} \right) = 2.$$

Note that even if you manage to find a rearrangement which allows you to cancel terms, an infinite series may still be divergent. Remember, to prove that a series converges you need to show that the sequence of partial sums converges. Otherwise, the series is divergent. A few examples where the telescoping technique does not lead to a convergent result are presented below.

Problem 31. Explain why the series

$$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 - \cdots$$

is divergent.
Solution: It looks as though the cancellations take place again and the sum must be 0. We have to work out this example by the definition, and thus find the limit of the sequence of partial sums. We obtain

$$S_1 = -1, \quad S_2 = 0, \quad S_3 = -1, \quad S_4 = 0, \quad \text{etc.}$$

We observe that all the terms of the sequence of partial sums with even indices are equal to 0 while those with odd indices are equal to -1. Thus, the sequence alternates between two numbers and therefore does not have a limit. The series is therefore divergent.

Problem 32. Is the following series convergent or divergent?

$$\sum_{n=2}^{\infty} \ln \left(\frac{n}{n+1} \right)$$

Solution: Since $\ln \left(\frac{n}{n+1} \right) = \ln n - \ln(n+1)$, the N-th partial sum is

$$S_N = (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + (\ln 4 - \ln 5) + \cdots + (\ln N - \ln(N+1)).$$

After cancellations we have $S_N = \ln 2 - \ln(N+1)$.

Notice that so far everything looks similar to Example 2, but that does not mean that we will have a similar result. The calculation of the limit gives the final answer. We have

$$\lim_{N \to \infty} S_N = \ln 2 - \lim_{N \to \infty} \ln(N+1) = -\infty.$$

Thus, the series is divergent.

Problem 33. Is the following series convergent or divergent?

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Solution: First we use $(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n}) = n+1-n = 1$ to write

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} = \sqrt{n+1} - \sqrt{n}.$$

Now the N-th partial sum is

$$S_N = \sqrt{1} - \sqrt{0} + \sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \cdots + \sqrt{N+1} - \sqrt{N}.$$

After cancellations we have $S_N = \sqrt{N+1}$. Here again, the calculation of the limit gives the final answer. We have $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sqrt{N+1} = \infty$. Thus, the series is divergent.

From these examples we see that there are many ways to make the telescoping method work. In the next subsection we show that the method of *partial fractions* is a very valuable tool when finding the sum of a telescoping series whose n-th term is a rational function of n.

§2.4.2 Partial fractions

In this subsection we deal with sequences whose n-th term is a rational expression of n. That is, $a_n = P(n)/Q(n)$, where P and Q are polynomials with the degree of P less than that of Q. We know that any polynomial in n can be factored as a product of linear terms $n - d$ and irreducible quadratic terms $an^2 + bn + c$ with $b^2 < 4ac$ (recall that if $b^2 \geq 4ac$ then such a quadratic term can be factored into two linear terms, and thus it is called *reducible*). The method of partial fractions aims to rewrite the ratio $a_n = P(n)/Q(n)$ as a sum of fractions whose denominators are factors of $Q(n)$. The trick from Problems 25 and 26 (section 2.4.1) uses such a representation:

$$\frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)} - \frac{1}{(n+2)}.$$

The goal of this subsection is to convert this "trick" into a rigorous method. The following example shows how the method works.

Problem 34. Find the partial fraction representation for the rational expression $\dfrac{1}{n^2 + 3n + 2}$.

Solution: We know that $n^2 + 3n + 2 = (n+1)(n+2)$; let us asume that there exist real numbers A and B such that

$$\frac{1}{n^2 + 3n + 2} = \frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}.$$

Now we need to find the values of A and B. This task is called *equating coefficients*. Multiplying both sides of the last equation by $(n + 1)(n + 2)$ we obtain

$$1 = A(n + 2) + B(n + 1) = (A + B)n + (2A + B).$$

As you can see, the coefficient of n on the left side of this equation is zero, so $A + B = 0$. Also, the constant on the left is 1 while the constant on the right is $2A + B$. So $2A + B = 1$. We now have two equations and two unknowns so we can solve the equations for A and B. The first equation yields $A = -B$. Then the second equation gives $A = 1$ and $B = -1$. Thus,

$$\frac{1}{n^2 + 3n + 2} = \frac{1}{n + 1} - \frac{1}{n + 2}.$$

The numerators in the partial fractions will not always be constants. They were only constants in the previous example because the denominators were linear functions of n.

However, for a denominator of degree $d > 1$ the corresponding numerator is a polynomial in n of degree $d - 1$. For example, if the denominator has factors $(n + 1)(n^2 + 1)$ then the corresponding numerators are A and $Bn + C$.

Problem 35. Find the partial fraction decomposition for

$$\frac{n^2 + 2n + 2}{(n + 2)(n^2 + 3n + 3)}.$$

Solution: Since $n^2 + 3n + 3$ is irreducible (because $b^2 - 4ac = 9 - 12 = -3 < 0$), it will be one of the denominators of the partial fractions, while $n + 2$ is the other one. Thus, we have

$$\frac{n^2 + 2n + 2}{(n + 2)(n^2 + 3n + 3)} = \frac{A}{n + 2} + \frac{Bn + C}{n^2 + 3n + 3}.$$

Multiplying both sides by $(n + 2)(n^2 + 3n + 3)$, we get

$$\begin{aligned}
n^2 + 2n + 2 &= A(n^2 + 3n + 3) + (Bn + C)(n + 2) \\
&= An^2 + 3An + 3A + Bn^2 + 2Bn + Cn + 2C \\
&= (A + B)n^2 + (3A + 2B + C)n + (3A + 2C).
\end{aligned}$$

Now, by equating coefficients, we just have to solve the system

$$A + B = 1, \qquad 3A + 2B + C = 2, \qquad 3A + 2C = 2.$$

This can be easily solved to find $A = 2$, $B = -1$, and $C = -2$. Therefore

$$\frac{n^2 + 2n + 2}{(n + 2)(n^2 + 3n + 3)} = \frac{2}{n + 2} - \frac{n + 2}{n^2 + 3n + 3}.$$

Sometimes partial fractions and telescoping work well together. Here are two such examples.

Problem 36. Find

$$\sum_{n=1}^{\infty} \frac{k}{k^4 + k^2 + 1}.$$

Solution: It is easy to see that the denominator can be factored using a quick trick

$$k^4 + k^2 + 1 = (k^4 + 2k^2 + 1) - k^2 = (k^2 + 1)^2 - k^2$$

which is a difference of squares, so $k^4 + k^2 + 1 = (k^2 + k + 1)(k^2 - k + 1)$. Now we can find the partial fractions for the general term. Let for some A, B, C, D

$$\frac{k}{k^4 + k^2 + 1} = \frac{k}{(k^2 + k + 1)(k^2 - k + 1)} = \frac{Ak + B}{k^2 + k + 1} + \frac{Ck + D}{k^2 - k + 1}.$$

Multiplying both sides by $k^4 + k^2 + 1$ we obtain

$$\begin{aligned} k &= (Ak + B)(k^2 - k + 1) + (Ck + D)(k^2 + k + 1) \\ &= (A + C)k^3 + (-A + B + C + D)k^2 + (A - B + C + D)k + (B + D). \end{aligned}$$

Solving corresponding linear system yields: $A = 0$, $B = -1/2$, $C = 0$, and $D = 1/2$. Thus, the general term is now equal to:

$$\begin{aligned} a_k = \frac{k}{k^4 + k^2 + 1} &= \frac{1}{2}\left(\frac{1}{k^2 - k + 1} - \frac{1}{k^2 + k + 1}\right) \\ &= \frac{1}{2}\left(\frac{1}{k^2 - k + 1} - \frac{1}{(k + 1)^2 - (k + 1) + 1}\right) \end{aligned}$$

When we write each of the first n terms in this fashion and add them together the negative portions of a_k cancels with the positive portion of a_{k+1} for all $1 \leq k \leq n$ leaving

$$S_n = \frac{1}{2(1^1 - 1 + 1)} - \frac{1}{2((n + 1)^2 - (n + 1) + 1)} = \frac{1}{2}\left(1 - \frac{1}{n^2 + n + 1}\right).$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{k^4 + k^2 + 1} = \lim_{n \to \infty} S_n = \frac{1}{2}.$$

Comment: The critical point in a telescoping sum is to express a_k in the form $f(k) - f(k + 1)$.

Problem 37. Find the sum:
$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$$

Solution: It is easy to see that for $k \geq 1$ the k-th term
$$a_k = \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2(k+1)-1}\right).$$

When we write each of the first n terms in this fashion and add them together the negative portions of a_k cancels with the positive portion of a_{k+1} for all $1 \leq k \leq n$ leaving
$$S_n = \frac{1}{2(2\cdot 1 - 1)} - \frac{1}{2(2(n+1)-1)} = \frac{1}{2} - \frac{1}{4n+2}.$$

Taking the limit as $n \to \infty$ the desired sum is

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{4n+2}\right) = \frac{1}{2}.$$

§2.5 Harmonic series and paradox of shifted cards

Recall that the *harmonic mean* of two numbers a and b is given by $\dfrac{2}{a^{-1}+b^{-1}}$.
The series for which each term is equal to the harmonic mean of its neighbors is called the *harmonic series*, and it has the very simple form

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots .$$

In 1673, Gottfried Wilhelm Leibniz knew that this series was divergent. This is not an obvious observation, because it diverges very slowly, and if one calculates its partial sums it is hard to say whether they are bounded or not. Leonhard Euler's book *Differential Calculus* written in 1755, gives the following examples: the sum of the first 1000 terms of the Harmonic series is approximately 7.485, and the sum of first $1,000,000$ terms of the series is approximately 14.393. Euler himself found the numbers by a certain trick, not by direct summation. To have the sum close to 100, we would have to take approximately 2^{143} terms.

Another interesting fact is that none of the partial sums for the harmonic series is an integer.

To prove that the harmonic series is divergent we group the terms of the series as follows

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \left(\frac{1}{17} + \cdots + \frac{1}{32}\right) + \cdots$$

Note that in the n-th set of parentheses, the last (and smallest) term is $1/2^n$, and the number of terms is 2^{n-1}. Thus, the sum in each set of parentheses is greater than $2^{n-1}/2^n = 1/2$. For instance,

$$\frac{1}{3} + \frac{1}{4} > 2\cdot\frac{1}{4} = \frac{1}{2}, \qquad \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 4\cdot\frac{1}{8} = \frac{1}{2},$$

and so on. The total sum therefore is greater than an infinite sum of halves and thus, the series is divergent.

This divergence helps explain an interesting effect — an experiment which you can perform.

Place cards, say, from a standard deck, at the left edge of a desk so that the card on top extends half its length beyond the card underneath it; the second top card extends a quarter of its length beyond the card underneath it; the third, fourth,..., n-th from the top, respectively, extends one-sixth, one-eighth,...,$\frac{1}{2n}$ of its length beyond the card immediately underneath it. The bottom card rests on the desk with its left edge lying along the left edge of the desk. This pile of cards will not fall to the floor even though the top card extends far beyond the edge of the desk.
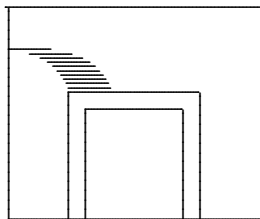

Figure 14. Paradox of the shifted cards.

Now we will explain this effect. Let N be the number of cards. We suppose that the length of each card is 1, and that the origin of measurement is the left edge of the top card.
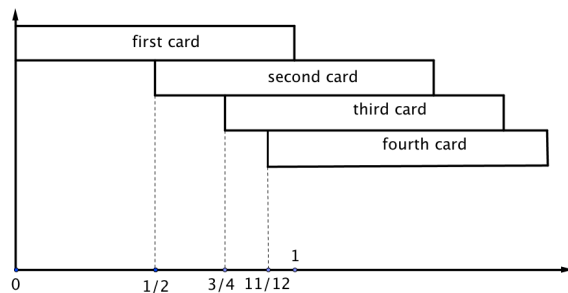
Figure 15. Shifted cards in a coordinate plane.

Let x_n be the coordinate of the left edge of the n-th card from the top. Thus $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, and generally for $n \geq 2$,

$$x_n = x_{n-1} + \frac{1}{2(n-1)} = \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) = \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k}.$$

The position of the centre of mass of the n-th card is at $\bar{x}_n = x_n + \frac{1}{2}$.

Since all the cards are assumed to be identical, the centre of mass $\bar{g}_n$ of the top n cards is located at the mean of the centres of mass of the individual cards, so that

$$\bar{g}_n = \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_n}{n}.$$

In the table below we give the values of x_n, $\bar{x}_n$, and $\bar{g}_n$ for small values of n.

n	$x_n = x_{n-1} + \frac{1}{2(n-1)}$	$\bar{x}_n = x_n + \frac{1}{2}$	$\bar{g}_n = (\bar{x}_1 + \cdots + \bar{x}_n)/n$
1	0	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2}$	1	$\frac{3}{4}$
3	$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$	$\frac{5}{4}$	$\frac{11}{12}$
4	$\frac{3}{4} + \frac{1}{6} = \frac{11}{12}$	$\frac{17}{12}$	$\frac{25}{24}$

We make some observations. Since the edge of the desk is located at x_N and since the harmonic series diverges, the top card in the pile can be arbitrarily far to the left of the edge of the desk when sufficiently many cards are laid.

The second observation is that from the table $\bar{g}_n = x_{n+1}$ for $1 \leq n \leq 4$. We prove by induction that it holds for all positive integers n. Suppose that it holds up to the index k, that is $\bar{g}_n = x_{n+1}$ for $1 \leq n \leq k$. Then

$$(k+1)\bar{g}_{k+1} = \bar{x}_1 + \cdots + \bar{x}_{k+1} = (\bar{x}_1 + \cdots \bar{x}_k) + \bar{x}_{k+1} = k\bar{g}_k + \bar{x}_{k+1}.$$

Now, use the assumption of the induction step and the relation $\bar{x}_n = x_n + \frac{1}{2}$ to get

$$(k+1)\bar{g}_{k+1} = kx_{k+1} + \left(x_{k+1} + \frac{1}{2} \right) = (k+1)x_{k+1} + \frac{1}{2}.$$

Finally, use $x_{k+2} = x_{k+1} + \frac{1}{2(k+1)}$ to obtain

$$(k+1)\bar{g}_{k+1} = (k+1)\left(x_{k+2} - \frac{1}{2(k+1)} \right) + \frac{1}{2} = (k+1)x_{k+2}.$$

This implies that $\bar{g}_{k+1} = x_{k+2}$, and thus by induction $\bar{g}_n = x_{n+1}$ for $1 \leq n < N$.

What is the significance of this result? It says that the centre of mass of the top n cards lie directly on top of the left edge of $(n+1)$-th card from the top, and that the centre of mass of all the cards above the bottom card lying on the desk is directly above the edge of the desk. Thus, even though the upper cards in the stack extend far to the left of the desk, the stack will not fall to the floor.

The harmonic series is a member of a large class of series which are called hyperharmonic. We will talk about them in a later section.

§2.6 Leibniz series and an unexpected effect of permutation

The series named after Gottfried Wilhelm Leibniz is a convergent one:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2. \tag{16}$$

Note that it differs from the divergent harmonic series only by the alternation of the sign of its terms, and hence it is classified as an *alternating series*.

It is not hard to show that the series is convergent. In fact, any alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n, \qquad a_n > 0$$

whose general term a_n is both decreasing and tends to zero (i.e. $a_{n+1} \leq a_n$ and $a_n \to 0$) is convergent. The partial sums of such a series jump above and below its limit, getting closer as n increases.

It is a little more tricky to show that the series converges to $\ln 2$. To see this you will need to read a later section about power series representations of functions and use formula (23) for the logarithmic function $\ln(x+1)$ at $x = 1$.

The series is convergent, but it has an amazing property: by changing the order of the terms, one can get different results for its sum! Series with this property were called *conditionally convergent* by Georg Bernhard Riemann around 1854.

Let us show for this example that after a certain permutation of the terms, the sum is equal to $(3/2)\ln 2$.

First write our original statement

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

Now divide both sides by 2 and fill in zeros for further convenience

$$\frac{\ln 2}{2} = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \frac{1}{10} + \cdots$$

Now add the two formulas to obtain

$$\frac{3\ln 2}{2} = 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \cdots$$

Note that on the right side, you have the same terms as in the Leibniz series, but written in a different order. On the left side, you have a different result!

An even more surprising result, known as Riemann's Theorem, is as follows: *one can get any number by rearranging the terms of a conditionally convergent series*. Niels Henrik Abel was so upset by this statement that he once said that conditionally convergent series are the devil's production. Some more examples of the devil's production are series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^s}, \quad 0 < s < 1.$$

In particular, for $s = 1/2$,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$$

is conditionally convergent.

§2.7 Hyperharmonic series and the Riemann zeta function

The Riemann zeta function is a well known object in analysis, in particular due to a problem that has remained open for more than a century and which is known as the 8th Hilbert problem or the Riemann hypothesis about the zeroes of the zeta function. Who knows, maybe you will witness or even participate in its solving!

The most characteristic property of the zeta function is its representation by either the infinite sum with respect to all natural numbers n, also known as hyperharmonic series,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

or an infinite product with respect to all prime numbers p,

$$\zeta(s) = \prod_{p \, \text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Either of these formulas may serve as a definition of the zeta function when the corresponding expression converges.

Recall that a prime number is a natural number which has only two divisors: one and itself. The fundamental significance of prime numbers is manifested in the uniqueness (up to the order of the factors) of the representation of any natural number as a product of powers of primes, like $72 = 3^2 \cdot 2^3$. This fact is the basis of the equivalence of the two representations of the zeta function given above. It also explains why the study of the zeta function can have applications in number theory, in particular for questions about the distribution of the prime numbers. The equivalence of the two representations for the zeta function was first established by Euler, who considered only real values of the argument s. Later, Riemann introduced the function $\zeta(s)$ by means of another equation, valid for any value of the argument s, including complex numbers. He showed that his definition agrees with the one provided by the hyperharmonic series for all s, both real and complex, for which the series converges.

Our goal in this section is to investigate the convergence of hyperharmonic series for real values of s. It is clear that for $s = 0$ and any $s < 0$, e.g. $s = -1/2$, $s = -2$, the hyperharmonic series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is divergent. The answer is less obvious for positive s, but we know that for $s = 1$ it is the harmonic series which diverges. In fact, the value $s = 1$ is the *delimiter* between the convergent and divergent hyperharmonic series. That is, $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent for $s > 1$, and divergent otherwise. For instance, it converges for $s = 1.1$ and diverges for $s = 1/2$. (However, a hyperharmonic series with $0 < s \leq 1$ becomes conditionally convergent with the insertion of an alternating sign $(-1)^n$ because $a_n = n^{-s}$ is decreasing and tends to 0 as $n \to \infty$.)

Let us first demonstrate the convergence of the series for $\zeta(2)$ given by the hyperharmonic series for $s = 2$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Consider the sequence of partial sums

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

It is obvious that the sequence grows; that is, $S_n < S_{n+1}$.

Let us show that the sequence is bounded above by 2.

Using the technique of telescoping sum from Section 4.2, for every n we have

$$\begin{aligned}
S_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\
&\leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} \\
&= 1 + \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n} \right) \\
&= 2 - \frac{1}{n} < 2.
\end{aligned}$$

This confirms that the partial sums S_n are indeed bounded above by 2. The series representing $\zeta(2)$ does converge.

To prove convergence of $\sum_{n=1}^{\infty} \frac{1}{n^s}$ for all $s > 1$ one may observe that such series have positive terms and are *bounded from above* by a convergent geometric series, and therefore converge. In detail, we write

$$\zeta(s) = 1 + \left(\frac{1}{2^s} + \frac{1}{3^s} \right) + \left(\frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} \right) + \left(\frac{1}{8^s} + \cdots + \frac{1}{15^s} \right) + \cdots$$

Here we group together all terms from the 2^n-th to the $(2^{n+1} - 1)$-th position, for $n = 1, 2, 3...$ (just like we did to to show that the harmonic series diverges). For example, in the first group we have the 2nd and 3rd terms, then the 4th to 7th terms in the second group, the 8th to 15th terms in the third group, and so on. Note that there are 2^n terms in nth group. For instance, two in the first group, four in the second, eight in the third and so on. Obviously, since $s > 1$ the larget term in each group in the first one, $1/(2^n)^s$. Thus, the nth group's sum is less than

$$2^n \times \frac{1}{(2^n)^s} = \frac{1}{2^{n(s-1)}} = \left(\frac{1}{2^{s-1}} \right)^n.$$

Thus the whole series is less then the geometric series $\sum_{n=1}^{\infty} r^n$, where $r = 1/2^{s-1}$. Now for $s > 1$ we have $0 < r < 1$, and so the geometric series converges. But the sum of our hyperharmonic series is even less than the the sum of the geometric series, therefore $\sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s > 1$ converges as well.

Note that for $s = 1$ we get $r = 1$, and the above approach no longer works. This by itself does not show the divergence for $s = 1$. But we have proved the divergence earlier, showing that the harmonic series can be compared to an infinite sum of constant terms $(1/2)$, and thus its sum is infinite.

To find the actual value of the sum of a convergent hyperharmonic series is a harder problem. For $s = 2$, it was solved by Euler in 1734. We show its value here without a proof:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{17}$$

It turns out that when the argument is an even number, the value of the zeta function can be expressed using π. For example

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

The same is not true for odd values of the argument though. For instance, it took many researchers quite an effort to prove that $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is an irrational number. This problem was considered hopeless until Roger Apéry finally solved it. This number is now called the Apéry constant. An approximate value is

$$\zeta(3) \approx 1.20205690315959428539973816151145$$

It was first obtained with such precision by Andrey Markov in 1889. This improved a result of Thomas Stieltjes from 2 years earlier when he published a table of the values of the Riemann zeta function with 32 decimals places for integral values of p from 2 to 70.

To appreciate these results you may try to obtain $\zeta(3)$ by definition using your pocket calculator. See how many terms you must add to get just a few correct digits. Again, mathematicians devised various tricks to get the result.

§2.8 Power series and approximate evaluation of functions

Not so long ago people didn't have calculators, but mathematicians knew how to calculate an approximate value for some expressions like $\sqrt{3}$ or $\sin(\pi/12)$. Even now in the computer era, have you ever wondered how your calculator or computer performs such operations? Here is a hint.

Everyone can evaluate a polynomial at any number.

For example, if $f(x) = x^2 + 2x + 1$ and $x = 3$ then $f(3) = 16$.

It is interesting that many other functions can be approximated arbitrarily closely by polynomials and, if we extend these polynomials into infinite sums (such sums are called power series), they represent the corresponding function. A good example of how this works is given by the geometric series. Rewriting the formula for the geometric series backwards, and replacing r with x we obtain

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1. \tag{18}$$

This means that for any $|x| < 1$, the function $f(x) = (1-x)^{-1}$ is the same as the infinite sum on the right hand side. This example gives you an idea of such representations, but it looks like the function itself is much simpler than the power series. Thus, in this case there is no profit in the replacement of the former by the latter. Nevertheless, one can use the right hand side to evaluate the function

approximately by taking just the first few terms of the infinite series. For example, the formula

$$\frac{1}{1-x} \approx 1 + x, \quad |x| < 1 \tag{19}$$

allows you to find such results as

$$\frac{1}{0.9} \approx 1.1 \quad \text{or} \quad \frac{1}{1.002} \approx 0.998.$$

quickly in your head.

In the first case we took $x = 0.1$ and in the second case we took $x = -0.002$. When we say *approximately equal* we normally need to estimate the accuracy of our approximated result, that is, the difference between the exact and approximate values. Formula (19) gives better accuracy for smaller x. For larger x, like $x = 0.9$, using just two terms is not as accurate. To improve the accuracy, one needs to take more terms from the right hand side of (18), like $(1 - x)^{-1} \approx 1 + x + x^2$.

Before discussing the approximation of functions by power series we state some important properties of power series whose justification lies beyond the scope of this work.

The first point is that for every power series

$$S(x) : \ a_0 + a_1 x + a_2 x^2 + \dots$$

there exists $r \geq 0$, which is either a real number or ∞, called the *radius of convergence of* S such that $S(x)$ converges for every x such that $|x| < r$, and $S(x)$ diverges for every x such that $|x| > r$. When $0 < r < \infty$, $S(x)$ may converge when x takes one of the values r, $-r$, both, or neither. The open interval $(-r, r)$ is called the *interval of convergence*.

If $r = 0$, the series is not interesting to us. So below assume that $r > 0$. With $S(x)$ we associate a function f by:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots \qquad (\text{ for all } x \text{ such that } S(x) \text{ converges}) \tag{20}$$

A crucial property of power series is that, in the interval of convergence of $S(x)$, $f(x)$ is differentiable and

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \qquad (-r < x < r).$$

That is, $f'(x)$ exists and is represented by the power series obtained by differentiating $S(x)$ term by term. Moreover, the resulting series has the same interval of convergence as $S(x)$.

Thus differentiating (18) we get the formula

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots \qquad (-1 < x < 1) \tag{21}$$

Since the interval of convergence does not change we can differentiate (20) term by term as many times as we like to obtain series representing the successive derivatives of $f(x)$. Differentiating n times we get

$$f^{(n)}(x) = (n \cdot (n-1) \cdot \dots 3 \cdot 2 \cdot 1)a_n + (\text{a series of powers of } x) \qquad (-r < x < r).$$

Substituting $x = 0$, we see that $f^{(n)}(0) = n!\, a_n$, which implies that $a_n = f^{(n)}(0)/n!$.

This is important. It shows that, just as we obtain a function from a power series by (21), so from a function f we obtain a power series:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots \tag{22}$$

This is called the *Maclaurin series* of f. From the properties stated above, if there is any series which represents f in a non-empty interval $(-r, r)$, then it is the Maclaurin series of f.

Approximating functions by power series

To approximate f by a power series in some interval $(-r, r)$ we can proceed as follows:

Step 1. By computing the derivatives of f at 0 compute the Maclaurin series of f:

$$a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n, \qquad a_n = \frac{f^{(n)}(0)}{n!}, \ n \geq 0$$

Notice that (18) is the Maclaurin series of $1/(1-x)$ and (21) is the Maclaurin series of $1/(1-x)^2$.

Step 2. Find $r > 0$ such that

$$\sum_{k=0}^{\infty} a_k x^k = f(x) \qquad (-r < x < r).$$

Of course, the larger r is the better.

Step 3. Let $S_n(x)$ denote the partial sum $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ of the Maclarin series of f. Find an expression $B(x, n)$ in x and n which bounds $|f(x) - S_n(x)|$ for $x \in (-r, r)$ such that

$$\lim_{n \to \infty} B(x, n) = 0 \qquad (-r < x < r).$$

In the case of (18),

$$\left| \frac{1}{1-x} - \sum_{k=0}^{n} x^k \right| = \left| \sum_{k=n+1}^{\infty} x^k \right| = \frac{|x|^{n+1}}{1-x} \qquad (-1 < x < 1).$$

So we can take $B(n, x) = |x|^{n+1}/(1-x)$ in this case.

In the case of (21), one can use the *Cauchy remainder formula* to see that

$$\left| \frac{1}{(1-x)^2} - \sum_{k=1}^{n} k x^{k-1} \right| < (n+1)n \frac{|x|^n}{(1-x)^3} \qquad (-1 < x < 1).$$

Step 4. Given $\epsilon > 0$ and a value of x in the interval found in step 2 for which we want the value of $f(x)$, use the expression $B(x, n)$ in step 3 to find out how many terms of the series are needed to compute $f(x)$ with an error of less than ϵ.

In example (17), letting $x = 1/4$ and $\epsilon = 0.01$, the bound for the error is $B(1/4, n) = 1/(3 \cdot 4^n)$. Since for $n = 3$, $(1/3 \cdot 4^3) < 0.01$, the first four terms $1 + x + x^2 + x^3$ are sufficient in this case.

In example (21), letting $x = 1/4$ and $\epsilon = 0.01$, the bound for the error from step 3 is

$$B(1/4, n) = \frac{(n+1)n}{3^3 \cdot 4^{n-3}}$$

Since for $n = 7$, $(8 \cdot 7)/(3^3 \cdot 4^4) < 0.01$, the first seven terms $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6$ are sufficient in this case.

The technique for approximating functions by series described above is most effective for approximating $f(x)$ when x is close to 0. The same technique can be used to approximate $f(x)$ for x near a. Let $g(x) = f(x + a)$. Assuming that $g(x)$ is represented by its Maclaurin series in some interval, we have

$$f(x + a) = g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k \qquad (-r < x < r)$$

for some $r > 0$. Note that $g^{(k)}(0) = f^{(k)}(a)$. Thus, substituting $x - a$ for x in the line above we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k \qquad (-r < x - a < r).$$

This is called the *Taylor series for $f(x)$ at $x = a$*.

The following Maclaurin series for some other well-known functions are very easily computed:

$$\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad (-1 < x \leq 1) \tag{23}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \qquad (|x| < \infty) \tag{24}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \qquad (|x| < \infty) \tag{25}$$

In general, the tasks mentioned in steps 2 and 3 above, finding the interval on which the Maclaurin series represents the function and finding an error bound $B(n, x)$ are beyond what we can explain in this work. However, there are a lot of cases in which suitable error bounds can be found in a simple way. For example, in (23), for $0 \leq x \leq 1$, the terms are decreasing in absolute value and alternating in sign. So we can take $B(n, x) = x^{n+1}/(n + 1)$ for $0 \leq x \leq 1$.

Note that power series yield interesting examples for summations. Fixing x, we obtain a number series and its value. For example, let us find the value of

$$1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \cdots$$

It is easy to see that the value is -1, by noticing that the sum is of the form $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$ with $x = \pi$. Thus, by (25) we immediately see that the sum is $\cos \pi = -1$.

Another example is the use of (21) with $x = \frac{1}{3}$ to show that

$$\sum_{n=1}^{\infty} \frac{n}{3^{n-1}} = \frac{9}{4}. \tag{26}$$

Finally, using formula (23) with $x = 1$ gives (16) from our discussion of Leibniz series.

Caution. We can only use this if we know that the power series is truly representative of the function at a given point x. We already saw that otherwise, this may lead to nonsense like the sum of positive terms being equal to a negative number ($\sum_{n=0}^{\infty} 2^k = -1$) if we use $x = 2$ in formula (18) while it is only valid for $|x| < 1$.

Let us conclude this section with a problem that incorporates one of the examples.

Problem 38. The double sum. (Putnam, 1999).

Evaluate

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)}$$

Solution: First note that if we interchange the summation variables, m and n, the result will be the same. That is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n^2 m}{3^n(m3^n + n3^m)}.$$

With this, we can make the expression look more symmetric:

$$\frac{1}{2}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n^2 m}{3^n(m3^n + n3^m)}\right).$$

Rewriting the summands with a common denominator, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)} = \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{3^n m^2 n + 3^m n^2 m}{3^m 3^n(n3^m + m3^n)}$$

$$= \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{mn(n3^m + m3^n)}{3^m 3^n(n3^m + m3^n)}$$

$$= \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{mn}{3^m 3^n}$$

$$= \frac{1}{2}\sum_{m=1}^{\infty}\left(\frac{m}{3^m}\left(\frac{1}{3}\sum_{n=1}^{\infty} \frac{n}{3^{n-1}}\right)\right)$$

Now we use (26) and continue to get

$$
\begin{aligned}
&= \frac{1}{2} \sum_{m=1}^{\infty} \left(\frac{m}{3^m} \frac{1}{3} \cdot \frac{9}{4} \right) \\
&= \frac{1}{2} \cdot \frac{9}{12} \sum_{m=1}^{\infty} \frac{m}{3^m} \\
&= \frac{9}{24} \cdot \frac{1}{3} \sum_{m=1}^{\infty} \frac{m}{3^{m-1}} \\
&= \frac{9}{72} \cdot \frac{9}{4} = \frac{9}{32}.
\end{aligned}
$$

Please note that a double sum is simply a sum whose summands are themselves sums. Also, we found the common denominator by switching the order of the sums. We were able to do this because each term is positive and the series is convergent. In general, we have to be careful with switching the order of summation, as seen in a past example with a conditionally convergent series. Finally, we were able to bring $m/(3^m)$ outside of the sum whose index variable is n because, with respect to n, $m/(3^m)$ is a constant.

§2.9 L'Hospital's Rule and power series

Let

$$
f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n
$$

be two functions represented by the power series shown in an interval $(-r, r)$, where $r > 0$. Then we have:

An instance of L'Hospital's rule:

If $f(0) = g(0) = 0$ and one of the limits

$$
\lim_{x \to 0} \frac{f(x)}{g(x)}, \qquad \lim_{x \to 0} \frac{f'(x)}{g'(x)}
$$

exists, then so does the other and the limits are equal.

Here is the explanation. We start with an example. Suppose that

$$
g(x) = b_5 x^5 + b_6 x^6 + \cdots, \qquad f(x) = a_{5+k} x^{5+k} + a_{6+k} x^{6+k} + \cdots,
$$

where $k \geq -4$ is a fixed integer number, and $b_5 \neq 0$, $a_{5+k} \neq 0$. Then

$$
\frac{f(x)}{g(x)} = \frac{a_{5+k} x^{5+k} + a_{6+k} x^{6+k} + \cdots}{b_5 x^5 + b_6 x^6 + \cdots} = \frac{a_{5+k} x^k + a_{6+k} x^{1+k} + \cdots}{b_5 + b_6 x + \cdots}
$$

and

$$\frac{f'(x)}{g'(x)} = \frac{a_{5+k}(5+k)x^{4+k} + a_{6+k}(6+k)x^{5+k} + \cdots}{5b_5 x^4 + 6b_6 x^5 + \cdots}$$

$$= \frac{a_{5+k}(5+k)x^k + a_{6+k}(6+k)x^{1+k} + \cdots}{5b_5 + 6b_6 x + \cdots}$$

Now there are three cases to consider. First, for $k = 0$

$$\frac{f(x)}{g(x)} = \frac{a_5 + a_6 x + \cdots}{b_5 + b_6 x + \cdots}, \qquad \frac{f'(x)}{g'(x)} = \frac{5a_5 + 6a_6 x + \cdots}{5b_5 + 6b_6 x + \cdots}$$

so

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{a_5}{b_5}.$$

Second, for $k > 0$ (e.g. $k = 1$)

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0.$$

Third, for $-4 \le k < 0$ (e.g. $k = -2$) in both ratios f/g and f'/g' in the reduced form the limit of the denominator is a constant (b_5 and $5b_5$ respectively), but the numerator is unbounded as $x \to 0$. Thus, limits of both ratios do not exist.

We can proceed similarly in the general case. Let $m(f)$ be the least n such that $a_n \neq 0$, and $m(g)$ be the least n such that $b_n \neq 0$. Since $f(0) = g(0) = 0$, $a_0 = b_0 = 0$ and so $m(f)$, $m(g) \ge 1$. It follows that $m(f') = m(f) - 1$, and $m(g') = m(g) - 1$. In the above example $m(f) = 5 + k \ge 1$ (since $k \ge -4$), $m(g) = 5$, $m(f') = 4 + k$, and $m(g') = 4$.

Case 1. $m(f) < m(g)$. This corresponds to $-4 \le k < 0$ in the above example. For ease of notation let p denote $m(f)$. Then

$$\frac{f(x)}{g(x)} = \frac{a_p x^p + a_{p+1} x^{p+1} + a_{p+2} x^{p+2} + \ldots}{b_{p+1} x^{p+1} + b_{p+2} x^{p+2} + \ldots}$$

$$= \frac{a_p + a_{p+1} x + a_{p+2} x^2 + \ldots}{b_{p+1} x + b_{p+2} x^2 + \ldots}$$

Note that here we require that the leading coefficient in the numerator $a_p \neq 0$, but there is no assumption about the leading coefficient of the denominator. Since power series functions are continuous in the interval of convergence and, in particular at 0, as $x \to 0$ the limit of the numerator is $a_p \neq 0$ and the the limit of the denominator is 0. Therefore the last fraction is unbounded as $x \to 0$. So $\lim_{x \to 0} f(x)/g(x)$ does not exist. By the same token, since $m(f') < m(g')$, $\lim_{x \to 0} f'(x)/g'(x)$ also does not exist.

Case 2. $m(f) \ge m(g)$. This corresponds to $k \ge 0$ in the above example. For ease

of notation let q denote $m(g)$. Then

$$\frac{f(x)}{g(x)} = \frac{a_q x^q + a_{q+1} x^{q+1} + a_{q+2} x^{q+2} + \ldots}{b_q x^q + b_{q+1} x^{q+1} + b_{q+2} x^{q+2} + \ldots}$$

$$= \frac{a_q + a_{q+1} x + a_{q+2} x^2 + \ldots}{b_q + b_{q+1} x + b_{q+2} x^2 + \ldots}$$

Note that here we require that the leading coefficient in the denominator $b_q \neq 0$, but there is no assumption about the leading coefficient of the numerator. In the last fraction the limit as $x \to 0$ of the numerator is a_q and the limit of the denominator is $b_q \neq 0$. So $\lim_{x \to 0} f(x)/g(x) = a_q/b_q$ exists. Also, $m(f') \geq m(g')$. For f' and g' the corresponding coefficients are qa_q, qb_q respectively. Thus $\lim_{x \to 0} f'(x)/g'(x) = (qa_q)/(qb_q) = a_q/b_q$ also exists.

We can prove a corresponding version of L'Hospital's rule at $x = a$ for functions f, g represented by power series in $x - a$ in an interval $(a - r, a + r)$.

Let us give two examples of the instance of L'Hospital's rule shown above.

Example 12. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

In this example we verify the result from 2.6.3 using two new methods. First, substituting $x = 0$ gives $\left[\frac{0}{0}\right]$. So the hypothesis of L'Hospital's rule is satisfied and we have

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1$$

Alternatively, using (23), the Maclaurin series for $\sin x$, we get

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}{x}$$

$$= \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) = 1$$

because the function represented by $1 - \frac{x^2}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ is continuous at $x = 0$.

Example 13. $\lim\limits_{x \to 0} \dfrac{\ln(1 + x)}{x} = 1$.

Applying the rule, the given limit is equal to

$$\lim_{x \to 0} \frac{d/dx\,[\ln(1 + x)]}{d/dx[x]} = \lim_{x \to 0} \frac{(1 + x)^{-1}}{1} = \frac{\lim_{x \to 0}(1 + x)^{-1}}{\lim_{x \to 0} 1} = \frac{1}{1} = 1.$$

This result can be used to calculate the limit of the Euler sequence, see 2.6.2. The first observation we need is that

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

provided that the limit on the right exists. This is clear from the definitions of the limits on the two sides. So it is sufficient to evaluate the limit on the right.

Let $\exp(x) = e^x$ denote the exponential function which is the inverse function of $\ln(x)$. The rest of the calculation is:

$$
\begin{aligned}
\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x &= \lim_{y \to 0^+} (1 + y)^{1/y} && \text{by substituting } x = 1/y \\
&= \lim_{y \to 0} (1 + y)^{1/y} && \text{provided this limit exists} \\
&= \lim_{y \to 0} \exp\left(\ln\left((1 + y)^{1/y}\right)\right) && \text{since exp is the inverse of ln} \\
&= \lim_{y \to 0} \exp\left(\frac{\ln(1 + y)}{y}\right) && \text{by a property of ln} \\
&= \exp\left(\lim_{y \to 0} \frac{\ln(1 + y)}{y}\right) && \begin{array}{l}\text{since exp is continuous} \\ \text{assuming inner limit exists}\end{array} \\
&= \exp(1) \;=\; e && \text{from Example 13}
\end{aligned}
$$

One should be careful to apply L'Hospital's rule only when the hypotheses are satisfied. For example, the application

$$
\lim_{x \to 0} \frac{\cos x}{1 + x} \;=\; \lim_{x \to 0} \frac{d/dx\,[\cos x]}{d/dx\,[1 + x]} \;=\; \lim_{x \to 0} \frac{-\sin x}{1} \;=\; 0
$$

gives nonsense. The function $(\cos x)/(1 + x)$ is continuous at $x = 0$ because it is a quotient of continuous function with denominator nonzero at $x = 0$. Thus the true value of the limit is $(\cos 0)/(1 + 0) = 1$.

§2.10 Fourier series

In this section we touch on the subject of Fourier series. We begin with a problem about a finite trigonometric sum called the *Dirichlet kernel*.

Problem 39. Prove that for $N \geq 0$ and $x \neq 2\pi k$, $k \in \mathbb{Z}$,

$$
1 + 2\cos x + 2\cos(2x) + 2\cos(3x) + \cdots + 2\cos(Nx) = \frac{\sin[(N + 1/2)x]}{\sin[x/2]}.
$$

Solution: Multiply both sided by $\sin[x/2]$ to get rid of the fraction on the righthand side. To simplify the new lefthand side use the trigonometric identity

$$
2\cos A \sin B = \sin(A + B) - \sin(A - B),
$$

with $A = nx$, $n = 1, 2, \ldots N$, and $B = x/2$. Then we have

$$
\sin\left(\frac{x}{2}\right) + \left[\sin\left(\frac{3x}{2}\right) - \sin\left(\frac{x}{2}\right)\right] + \left[\sin\left(\frac{5x}{2}\right) - \sin\left(\frac{3x}{2}\right)\right] + \cdots
$$

$$
+ \left[\sin\left(N + \frac{1}{2}\right)x - \sin\left(N - \frac{1}{2}\right)x\right].
$$

After cancelling $\sin(x/2)$ with $-\sin(x/2)$, $\sin(3x/2)$ with $-\sin(3x/2)$, and so on, we are left with only one term $\sin\left[(N + \frac{1}{2})x\right]$ as required. Notice that we have used the method of telescoping which we studied in §2.4.

The Dirichlet kernel is related to a very important kind of series called a *Fourier series* which takes the form:

$$\frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos(2x) + b_1 \sin(2x) + a_3 \cos(3x) + b_3 \sin(3x) + \ldots$$

and may also be written

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right). \tag{27}$$

Here the coefficients a_n and b_n are fixed real numbers and x is a free variable. As with a power series in x, when x is given a particular value, we get a series of numbers which may or may not converge. Thus, just as a power series in x determines a function $f(x)$ defined for those x for which the series converges, the same is true of a Fourier series.

For each $n \geq 1$, the functions $\cos(nx)$ and $\sin(nx)$ are *periodic with period* $(2\pi)/n$ in the sense that, for all x,

$$\cos\left[n\left(x + \frac{2\pi}{n}\right)\right] = \cos(nx), \quad \sin\left[n\left(x + \frac{2\pi}{n}\right)\right] = \sin(nx).$$

Since 2π is the least common multiple of 2π, $(2\pi)/2$, $(2\pi)/3$, $(2\pi)/4$, ..., if $f(x)$ is represented by the Fourier series shown above, then $f(x)$ is periodic with period 2π. This is the big difference between representing a function by a power series and representing a function by a Fourier series.

We saw in §2.8 that there is a way of choosing the coefficients in $\sum_{n=0}^{\infty} a_n x^n$ to obtain a series which represents a given function f. We take $a_n = f^{(n)}(0)/n!$. As we saw above this "recipe" works in many important cases.

The situation for Fourier series is analogous. Let f be a function with domain $(-\pi, \pi)$. If a_n and b_n are given by the formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \quad (n \geq 0), \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \quad (n \geq 1),$$

then (27) is called *the Fourier series of* $f(x)$.

In many cases the Fourier series of $f(x)$ represents $f(x)$ on $(-\pi, \pi)$. We give two examples.

Example 1. $x = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx) \qquad (-\pi < x < \pi).$

Notice that all the coefficients a_n are 0 in this case because x is an odd function. Similarly, in the power series expansion of an odd function such as $\sin x$ only odd

powers of x appear.

Example 2. $\quad x^2 = \dfrac{\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty} (-1)^n \dfrac{4}{n^2} \cos{(nx)} \qquad (-\pi \le x \le \pi).$

Notice that all the coefficients b_n are 0 in this case because x^2 is an even function. Similarly, in the power series expansion of an even function such as $\cos x$ only even powers of x appear.

Proving that the Fourier series of $f(x)$ converges to the function $f(x)$ at a given point x requires mathematical techniques which can not be discussed in this text. However, once such a correspondence is established, Fourier series expansions give rise to some interesting results regarding sums of series. For example, letting $x = \pi/2$ in the above Example 1 we obtain

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots = \frac{\pi}{4}.$$

As well, letting $x = \pi$ in Example 2 we obtain

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2},$$

which immediately gives the value of $\zeta(2)$ shown in (17).

The Dirichlet kernel with which we began this section plays a key role in the proof that, under certain conditions, the Fourier series of $f(x)$ converges to $f(x)$ for all x in $(-\pi, \pi)$.

Fourier series were invented by Joseph Fourier in his study of the diffusion of heat in a metal plate. He published his work on the problem in *Théorie Analytique de la Chaleur* in 1822. Fourier series have found many applications in physics and engineering.

Exercises

1. Find an appropriate formula or method from this chapter to show that

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}$

(ii) $\displaystyle\sum_{n=1}^{1000} 2n + 3 = 1,004,000$

(iii) $1.\overline{76} = \dfrac{175}{99}$

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n} = \frac{4}{9}$

(vi) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n}{4^{n-1}} = 48$

(vii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{24}$

(viii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

(ix) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

(x) $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n}{n!} = e^{\pi}$

(xi) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{6^n}{n7^{n+1}} = \frac{\ln 13 - \ln 7}{7}$

(xii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} = \ln 2$

(xiii) $\displaystyle\lim_{x \to 0} \frac{\ln(1+kx)}{x} = k$

(xiv) $\displaystyle\lim_{n \to \infty} \left(1 + \frac{k}{n}\right)^n = e^k$

(xv) $\displaystyle\lim_{n \to \infty} n\sin(k/n) = k$

(xvi) $\displaystyle\lim_{x \to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}$

(xvii) $1 + 2\displaystyle\sum_{n=1}^{N} \cos(nx) = \frac{\sin(Nx) + \sin[(N+1)x]}{\sin x}, \quad N \geq 1, \quad x \neq \pi k,\ k \in \mathbb{Z}$

2. Recall where in the text the each of the following names were mentioned and what problem or theory is related to them.

Abel, Apéry, Archimedes, Cantor, Cauchy, Euclid, Euler, Fibonacci, Fourier, Gauss, L'Hospital, Koch, Leibniz, Markov, Maclaurin, Riemann, Sierpinsky, Stieltjes, Taylor, Zeno.

APPENDIX 1: DERIVATIVES

In this book we use derivatives as an auxiliary tool in several places. First, they are useful to check whether a function or sequence is monotonic. Secondly, they are needed for finding Maclaurin series representations of functions. And finally, they are required for L'Hopital's Rule for finding limits. In this appendix we will introduce the definition of the derivative, along with the derivatives of some useful functions, and also a few rules which can be used to simplify our task of finding a derivative.

A 1.1 Definition of the derivative

The derivative of a function $y = f(x)$ at a point x_0 is a number. It can be denoted in many ways including $y'(x_0), f'(x_0)$, and $\frac{dy}{dx}(x_0)$. Geometrically, this number represents the slope of the line tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. Algebraically, it is the instantaneous rate of change of the function at the point, so it shows how the function changes if the argument rises from x to $x + h$ as $h \to 0$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
\end{aligned}
$$

Let us use the definition of the derivative to find the derivative of an elementary function.

Problem 40. Let $f(x) = ax$, where a is a constant. Then

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{a(x+h) - ax}{h} \\
&= \lim_{h \to 0} \frac{ah}{h} \\
&= \lim_{h \to 0} a \\
&= a
\end{aligned}
$$

which makes sense because $f(x) = ax$ is a straight line with slope a, and the rate of change of this function is the slope. Thus, the derivative is also equal to a at any point x_0.

Here's another example.

Problem 41. Let $f(x) = ax^2$, where a is a constant. Then,

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{a(x+h)^2 - ax^2}{h} \\
&= \lim_{h \to 0} \frac{ax^2 + 2ahx + ah^2 - ax^2}{h} \\
&= \lim_{h \to 0} \frac{2ahx + ah^2}{h} \\
&= \lim_{h \to 0} (2ax + ah) \\
&= 2ax.
\end{aligned}
$$

This tells you that the derivative, and the slope of the tangent line, of the curve $f(x) = ax^2$ at any point $(x_0, f(x_0))$, is equal to $2ax_0$. For instance, the curve has a horizontal tangent line at the origin since the derivative at $x_0 = 0$ is $2 \cdot 0 = 0$. The curve has a positive slope for $x > 0$, meaning that the rate of change of the function is positive and therefore, the function is increasing for $x > 0$. Likewise, for all $x < 0$ the slope of the tangent line is negative and the function is decreasing.

The relationship between the slope of the tangent line and the rate of change of a function, expressed numerically using the derivative, becomes a valuable tool for the monotonicity check.

i) If we have a positive derivative for all x in an interval, then the function is increasing on that interval.

ii) If we have a negative derivative for all x in an interval, then the function is decreasing on that interval.

iii) If we have a derivative equal to zero for all x in an interval, then the function is constant on that interval.

Now, we will present several rules and techniques for finding derivatives. Each of them can be derived from the definition of the derivative.

A 1.2. Important derivatives

It would be inappropriate for someone to be expected to use limits whenever they need to find a derivative because this can sometimes be a long and tedious process. The following are important derivatives that are presented here without proof:

$f(x)$	$f'(x)$
c	0
x^n	nx^{n-1}
e^x	e^x
$a^x,\ a > 0$	$(\ln a)a^x$

$f(x)$	$f'(x)$
$\ln x$	$1/x$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

In these tables a, c, and n are real constants with $a > 0$.

These derivatives can be combined with the following rules to easily find the derivatives of many functions. We will refer to a function which has a derivative as *differentiable*.

A 1.3. Rules and properties

1. The Constant Multiple Rule

The constant multiple rule states that if f is a differentiable function and c is a real number, then cf is also differentiable and:

$$\frac{d}{dx}[cf(x)] \quad = \quad cf'(x),$$

where $\frac{d}{dx}$ means "the derivative with respect to x".

Problem 42. What is $\frac{d}{dx}[3x^2]$?
Solution: With $c = 3$ and $f(x) = x^2$ we can use the constant multiple rule. We have, $f'(x) = 2x$, and therefore $\frac{d}{dx}[3x^2] = 3(2x) = 6x$.

2. The Sum and Difference Rule

The sum (and difference) rule states that the sum (or difference) of two differentiable functions is differentiable and is equal to the sum (or difference) of their derivatives.

$$\frac{d}{dx}[f(x) \pm g(x)] \quad = \quad f'(x) \pm g'(x)$$

Problem 43. What is $\frac{d}{dx}[e^x + 3x^2]$?
Solution: We will use the sum rule, with $f(x) = e^x$ and $g(x) = 3x^2$. $f'(x) = e^x$ and $g'(x) = 6x$ which we found in the last example. Thus, $\frac{d}{dx}[e^x + 3x^2] = e^x + 6x$.

3. The Product Rule

The product rule states that the product of two differentiable functions is also differentiable, and

$$\frac{d}{dx}[f(x)g(x)] \quad = \quad f'(x)g(x) + f(x)g'(x)$$

We can see from this that the constant multiple rule is just a special case of the product rule, with $g(x) = c$.

Problem 44. What is $\frac{d}{dx}[3x \sin x]$?
Solution: We will use the product rule with $f(x) = x$ and $g(x) = \sin x$. $f'(x) = 1$ and $g'(x) = \cos x$, so $\frac{d}{dx}[x \sin x] = \sin x + x \cos x$ and using the constant multiple rule we get $\frac{d}{dx}[3x \sin x] = 3 \sin x + 3x \cos x$.

4. The Quotient Rule

The quotient rule states that the quotient f/g of two differentiable functions (with $g(x) \neq 0$) is itself differentiable and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \quad = \quad \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Problem 45. What is $\frac{d}{dx}\left[\frac{x}{e^x}\right]$?

Solution: We will use the quotient rule with $f(x) = x$ and $g(x) = e^x$. $f'(x) = 1$ and $g'(x) = e^x$. Therefore $\frac{d}{dx}\left[\frac{x}{e^x}\right] = \frac{e^x - xe^x}{e^{2x}} = \frac{1-x}{e^x}$.

5. The Chain Rule

The chain rule involves the composition of functions. The composition read "f of g of x" can be written as $(f \circ g)(x)$ or $f(g(x))$. Finding the composition of the two functions involves substituting $g(x)$ for x in the expression for $f(x)$.

Problem 46. Find the composition $f(g(x))$ for $f(x) = \sin x$ and $g(x) = e^x$.

Solution: Replacing x by $g(x)$ in the expression for $f(x)$ we quickly obtain

$$f(g(x)) = f(e^x) = \sin(e^x).$$

The chain rule states, if f and g are differentiable, then their composition of f and g is also differentiable, and

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x).$$

Thus for the composite function $\sin(e^x)$ we have

$$\frac{d}{dx}[\sin(e^x)] = \cos(e^x) \cdot e^x.$$

Problem 47. What is $\frac{d}{dx}[\ln e^x]$?

First Solution: This is $f(g(x))$ with $f(x) = \ln x$ and $g(x) = e^x$. $f'(x) = 1/x$ and $g'(x) = e^x$ so $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) = \frac{1}{e^x} \cdot e^x = 1$

Second Solution: $\ln x$ and e^x are inverse functions of each other, so $\ln e^x = x$. Therefore, $\frac{d}{dx}[\ln e^x] = \frac{d}{dx}[x] = 1$.

As you can see, both of the previous solutions had the exact same answer even though they were approached using totally different methods.

Exercises

Find the derivatives of the following functions:

1. i) $17x^{100}$ ii) $3\cos x$ iii) $-4\ln x$
   iv) $5x$ v) 4^{x-1} vi) $\pi \sin x$

2. i) $e^x + e^{2x}$ ii) $x^3 + x^2$ iii) $\cos x + \sin x$
   iv) $5 - x$ v) $3^x - x^3$ vi) $\ln x - x$

3. i) $\sin x \cdot \cos x$ ii) $1/2 \sin 2x$ iii) $e^x \cdot e^x$
   iv) $x \ln x - x$ v) $3^x \cdot e^x$ vi) $e^x \sin x$

4. i) $(\sin x)/(\cos x)$ ii) $(e^x)/(x^2)$ iii) $(3^x)/(4^x)$
   iv) $(x^3)/(x^2)$ v) $(\sin x)/(e^x)$ vi) $(e^x)/(\ln x)$

5. i) $\ln(\ln x)$ ii) $e^{\sin x}$ iii) $e^{\ln x}$
   iv) $(\sin x)^4$ v) $\ln(\sin x)$ vi) $(\ln x)^{12}$

APPENDIX B: TRIANGLES

Here we have collected a few facts which are useful for problems dealing with triangles.

I. **Similar triangles**

Similar figures are re-scaled images of each other. If either of the following is true, then the two triangles ABC and $A'B'C'$ are similar:

1. $\hat{A} = \hat{A}'$, $\hat{B} = \hat{B}'$.

2. $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'} = \dfrac{BC}{B'C'} = k$. The number k is called coefficient of similarity.

3. $\hat{A} = \hat{A}'$, $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$.
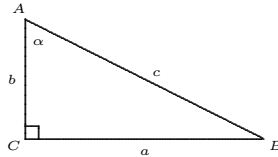
II. **Right triangle**

Figure 16. Right triangle.

1. $a^2 + b^2 = c^2$
2. $\cos\alpha = \dfrac{b}{c}$, $\sin\alpha = \dfrac{a}{c}$, $\tan\alpha = \dfrac{a}{b}$.

α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	undefined

3. $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$, $\sin(2\alpha) = 2\sin\alpha\cos\alpha$, $\tan(2\alpha) = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$.
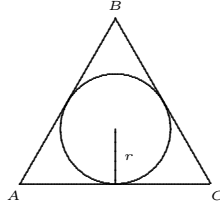
III. Equilateral triangle

Figure 17. Equilateral triangle.

1. $\hat{A} = \hat{B} = \hat{C} = \pi/3$, $|AB| = |BC| = |CA| = a$.
2. Area is $\dfrac{\sqrt{3}a^2}{4}$.
3. Any altitude is an angle bisector and a median as well. Altitudes intersect at a point, which is also the incenter and circumcenter.
4. Radius of the incircle is $\dfrac{\sqrt{3}a}{6}$.
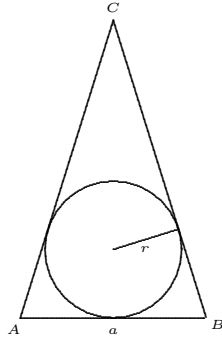
IV. Isoceles triangle

Figure 18. Isoceles triangle.

1. $\hat{A} = \hat{B} = (\pi - \hat{C})/2$, $|BC| = |CA|$, $|BA| = a$.
2. Let CD be the altitude. Then $|CD| = \dfrac{a}{2}\tan\hat{A}$.
3. The center of incircle lies on CD. The radius r of incircle is $\dfrac{a}{2}\tan\left(\dfrac{\hat{A}}{2}\right)$.

ATOM

A Taste Of Mathematics / Aime-T-On les Mathématiques

1. Edward J. Barbeau   *Mathematical Olympiads' Correspondence Program* (*1995-1996*)

2. Bruce L.R. Shawyer   *Algebra — Intermediate Methods*

3. Peter I. Booth, John Grant McLoughlin and Bruce L.R. Shawyer   *Problems for Mathematics Leagues*

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