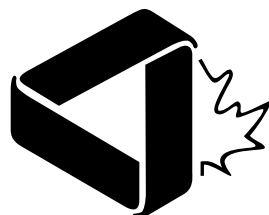




A TASTE OF MATHEMATICS



AIME-T-ON LES MATHÉMATIQUES

Volume / Tome XIII

QUADRATICS AND COMPLEX NUMBERS

Edward J. Barbeau

University of Toronto



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The booklets in the series, **A Taste of Mathematics**, are published by the Canadian Mathematical Society (CMS). They are designed as enrichment materials for high school students with an interest in and aptitude for mathematics. Some booklets in the series will also cover the materials useful for mathematical competitions at national and international levels.

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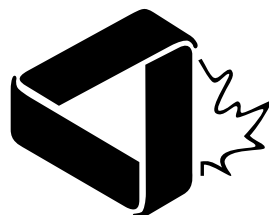
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His honours include the Fellowship of the Ontario Institute for Studies in Education, the David Hilbert Award from the World Federation of National Mathematics Competitions and the Adrien Pouliot Award from the Canadian Mathematical Society.

FOREWORD

While the quadratic equation is part of the standard syllabus in secondary school, the scope of this topic has been curtailed in many jurisdictions over the years. Treatment of any mathematical topic should be extensive enough to allow students to see some interesting problems and applications, as well as to get some inkling as to how it fits into the larger mathematical scheme of things. It is not enough for students to simply do basic factoring exercises and engage in rote application of the quadratic formula in solving equations.

This criticism has even more force when it comes to the topic of complex numbers. For many students, complex numbers arise only in the discussion of the roots of a quadratic equation with negative discriminant. Students have no idea of their theoretical and utilitarian importance in mathematics. Yet there is so much that can be done at the high school level, in particular in the solution of geometric and trigonometric problems.

This book is intended as a companion to the usual high school fare. The reader is assumed to have been introduced to polynomials and operations of addition, subtraction, multiplication and division, the remainder and factor theorems, simple factorizations, solution of quadratic equations by factoring, completing the square and the quadratic formula, and the relationship between the roots and coefficients of a quadratic equation. In addition, the reader should know the fundamental trigonometric functions and their values at standard angles as well as simple relationships among them. I hope that the exercises and problems will interest and entertain the reader, as well as deepen their experience and prepare them for advanced mathematical study.





FACTORING THE DIFFERENCE OF SQUARES

Many years ago, I was asked to review a textbook to be used in the first two high school years. I was dismayed by the many low-level and tedious exercises, which seemed to offer little in enlightenment or interest. What were the students to gain from doing them? I was challenged by the editors to come up with something better, and the questions of this chapter represent my attempt to do this with the factoring of differences of squares. They are included in this book because they are, well, quadratic, but also because completion of the square and factoring difference of squares are important foundational topics in studying quadratics.

1. Let $a > b > 0$. By means of a diagram, in which one slices and rearranges the gnomon (L-shaped region) when a square of side length b is removed from a square of side length a , illustrate the identity

$$a^2 - b^2 = (a + b)(a - b) .$$

2. The identity $a^2 - b^2 = (a + b)(a - b)$ gives an approach for mental multiplication. Suppose that we wish to multiply two positive integers u and v , particularly if they are of the same parity. Show how we can find suitable positive numbers a and b such that u and v can be written in the form $a + b$ and $a - b$. Then the product of u and v is the difference of squares $a^2 - b^2$. All that is needed is to know how to compute squares.

(a) Use the suggested method to mentally compute 37×43 , 22×28 , 46×54 .

(b) By making use of the identity $(10t + u)^2 = 100t^2 + 20tu + u^2$, devise a mental algorithm for quick computation of squares of two-digit numbers.

3. Let n be a positive integer. The number $n!$ is defined to be the product of the first n positive integers.

(a) Verify that $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$.

(b) By factoring the difference of squares in each case and then writing each factor as a product of smaller factors (no multiplications are necessary) and recomposing some products where needed, verify the following numerical results:

$$\begin{aligned} 4! &= 5^2 - 1^2 = 7^2 - 5^2 \\ 5! &= 11^2 - 1^2 = 13^2 - 7^2 \\ 6! &= 27^2 - 3^2 = 28^2 - 8^2 = 29^2 - 11^2 \\ 7! &= 71^2 - 1^2 = 72^2 - 12^2 = 73^2 - 17^2 . \end{aligned}$$

(It has been conjectured, but never proved, that $4!$, $5!$ and $7!$ are the only numbers of the form $n!$ that are 1 less than a perfect square.)

(c) Verify that $13! = 78912^2 - 288^2 = 112296^2 - 79896^2$.

(d) Write $17!$ as the difference of two integer squares.

(e) Prove that $n!$ cannot be written as the difference of two integer squares if and only if $n = 2$ or $n = 3$.



4. You are equipped with a pocket calculator that can display integers up to eight digits long. Let

$$a = 4565486027761$$

$$b = 1061652293520$$

$$c = 4687298610289 .$$

(a) Verify that (a, b, c) is a pythagorean triple, *i.e.*, $a^2 + b^2 = c^2$. (Hint: Try factoring either $c^2 - b^2$ or $c^2 - a^2$.)

(b) Verify that c is a square.

(c) Verify that $a + b$ is a square.

5. (a) Make a list of the numbers between 1 and 20 inclusive, expressing each, when possible, as the difference of the squares of two integers. Make a conjecture as to a criterion under which such a representation of a difference of squares is possible. To formulate or check your conjecture, you might want to extend your list.

(b) Suppose that an integer n can be written in the form $u^2 - v^2$. By factoring the difference of squares, show that it is necessary that n can be written as the product of two integers of the same parity (*i.e.*, both even or both odd).

(c) Suppose that n can be written in the form hk where h and k are integers of the same parity. Prove that there are integers u and v for which $h = u + v$ and $k = u - v$, and deduce that n can be written as the difference of two squares.

(d) Show that any odd number can be written as the product of two odd integers.

(e) Show that each integer divisible by 4 can be written as the product of two even integers.

(f) Show that each integer that leaves a remainder 2 upon division by 4 cannot be written as the product of two integers that are both odd or both even.

(g) Use the previous sections to formulate and prove a necessary and sufficient condition that an integer can be expressed as the difference of two squares.

6. Prove that 4 is the only perfect square that is 1 more than a prime number.

7. Prove that two nonzero perfect squares of integers cannot differ by 1.

8. (a) Make a table listing the numbers from 1 to 20, and beside each entry write the product of that number and the next higher number and four times this product. Formulate and prove a conjecture.

(b) Prove that the product of two consecutive positive integers is never a perfect square.



9. Prove that the product of two consecutive odd integers is not a perfect square.

10. Prove that the product of two consecutive even integers is not a perfect square.

11. (a) Look at the difference between the squares of two consecutive integers, working up from 1. What do you observe?

(b) What is the sum of the first million positive odd integers?

12. Prove that

$$1000^2 - 999^2 + 998^2 - 997^2 + \cdots + 2^2 - 1^2$$

is equal to the sum of the first thousand positive integers. Formulate and prove a generalization.

13. A straight metal rail is 2400 cm long and is firmly fixed at both ends. On a warm day, its length increases to 2402 cm and so it buckles. Assuming that its final shape is closely approximated by an isosceles triangle, determine how far from the ground its midpoint rises.

14. It is possible to arrive at the factorization of $x^2 - y^2$ by the technique of adding in an extra term and subtracting it out again:

$$x^2 - y^2 = x^2 - xy + xy - y^2 = x(x - y) + y(x - y) = (x + y)(x - y) .$$

(a) Apply this technique to determine a factorization of $x^3 - y^3$.

(b) Consider the factorization of $x^4 - y^4$. Use the technique just described to write this polynomial as the product of $x - y$ and another polynomial. Check your answer by factoring $x^4 - y^4$ as a difference of squares, and then factoring a second difference of squares.

(c) Factor $x^n - y^n$ where n is a positive integer. Check the results of your method for $n = 5, 6, 7$.

(d) Determine a method for factoring $x^3 + y^3$ and $x^5 + y^5$, and generalize it to a method for factoring $x^n + y^n$ where n is any odd positive integer.

15. (a) By adding to and subtracting from $x^4 + 4$ a term which is a square, factor $x^4 + 4$ as a product of two polynomials with integer coefficients.

(b) Write $x^4 + 1$ as the product of two quadratic factors (in this case, not all of the coefficients will be integers).

16. (a) Let p and q be two distinct odd primes. Prove that the number

$$(pq + 1)^4 - 1$$

has at least four distinct prime divisors.

(b) Suppose that $p = 2$ and q is an odd prime. Does the conclusion of (a) still hold?



17. An ordered set (a, b, c) of three positive integers a, b, c is called a *pythagorean triple* if it satisfies $a^2 + b^2 = c^2$. The name derives from the fact that pythagorean triples represent side lengths of right triangles (because of the Theorem of Pythagoras). A pythagorean triple is *primitive* if the greatest common divisor of its members is 1.

(a) Verify that $(3, 4, 5)$ is a primitive pythagorean triple, but $(24, 45, 51)$ is a pythagorean triple that is not primitive.

(b) There exist pythagorean triples (a, b, c) for which $c = b + 1$; an example is $(5, 12, 13)$. Prove that each such triple is primitive and that the smallest number of each such triple must be odd. Determine 7 such triples.

(c) Show that we can determine a pythagorean triple (a, b, c) for which a is any number we choose except for 1 and 2.

(d) Prove that, if (u, v, w) is a pythagorean triple, then there is an integer k and a primitive pythagorean triple (a, b, c) for which $u = ka$, $v = kb$ and $w = kc$.

(e) It is possible to give a formula which will churn out all pythagorean triples. Suppose that (a, b, c) is a primitive pythagorean triple. Prove that exactly one of a and b is odd. Without loss of generality, let us suppose that a is even. Verify that $a^2 = (c - b)(c + b)$ and that the greatest common divisor of $c - b$ and $c + b$ is 2. Deduce that $c + b = 2m^2$ and $c - b = 2n^2$ for some integers m and n .

(f) Prove that (u, v, w) is a pythagorean triple if and only if there are integers k, m, n for which

$$u = 2kmn, \quad v = k(m^2 - n^2), \quad w = k(m^2 + n^2) .$$

(You have to show, first of all, that if u, v, w have this form, then (u, v, w) is a pythagorean triple. Secondly, you have to argue, possibly using (d) and (e), that if (u, v, w) is a pythagorean triple, then k, m, n can be found as required.)

(g) Let (m, n) be a pair of integers. Then from (f) it can be seen that

$$\left(\frac{m^2 - n^2}{m^2 + n^2}, \frac{2mn}{m^2 + n^2} \right)$$

is a point on the circumference of the unit circle with centre at the origin. Show that the angle between the radius vector from the origin to this point and the x -axis is equal to twice the angle whose tangent is n/m .

On the other hand, if $\tan \theta = n/m$, determine $\sin 2\theta$ and $\cos 2\theta$. Deduce that there is a one-one correspondence between primitive pythagorean triples and angles whose tangent is rational.



18. (a) By factoring the left side as a difference of squares, show that

$$\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 = xy .$$

(b) Use (a) to show that, when x and y are nonnegative, then

$$\sqrt{xy} \leq \frac{x+y}{2} .$$

When does equality occur?

(c) The sum of two positive integers is 56. What is the largest possible value of their product?

19. Consider the following numerical equations:

$$\begin{aligned} 3^2 + 4^2 &= 5^2 \\ 10^2 + 11^2 + 12^2 &= 13^2 + 14^2 \\ 21^2 + 22^2 + 23^2 + 24^2 &= 25^2 + 26^2 + 27^2 \\ 36^2 + 37^2 + 38^2 + 39^2 + 40^2 &= 41^2 + 42^2 + 43^2 + 44^2 . \end{aligned}$$

Suggest a generalization and verify these numerical equations in a way that will convince you that the generalization also is valid.

20. There are many ways to see that the number of primes is infinite, which is to say, that no matter how many primes you can identify, there is always one more. One way to see this uses the sequence of Fermat numbers, defined for nonnegative integers n by

$$F_n = 2^{2^n} + 1 .$$

Thus $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$.

Observe that $2+1 = (2-1)(2+1) = 2^2 - 1 = (2^2+1) - 2$ and $(2+1)(2^2+1) = (2-1)(2+1)(2^2+1) = (2^2-1)(2^2+1) = 2^4 - 1 = (2^4+1) - 2$. Generalize this to obtain the following result:

$$F_0 F_1 F_2 \cdots F_n = F_{n+1} - 2 ,$$

for each nonnegative integer n . Deduce from this fact that any two terms in the sequence $\{F_n\}$ are coprime; that is, have greatest common divisor 1.

Prove that there are at least $n+1$ distinct primes that divide the product $F_0 F_2 \cdots F_n$, and that therefore there are infinitely many primes.

21. Determine that positive integer n for which $n+9$, $16n+9$ and $27n+9$ are all perfect squares.

22. A round-robin tournament is played by n sporting teams. This means that each team plays each other team exactly once. Every match between a pair of teams results in a win for one and a loss for the other; there are



no ties. Suppose, for $1 \leq k \leq n$, the k th team has w_k wins and d_k defeats. Prove that

$$w_1^2 + w_2^2 + \cdots + w_n^2 = d_1^2 + d_2^2 + \cdots + d_n^2.$$

Comments, Answers and Solutions

3. (c) Observe that

$$78912 + 288 = 79200 = 2 \times 3 \times 4 \times 5 \times 6 \times 10 \times 11$$

$$78912 - 288 = 78624 = 7 \times 8 \times 9 \times 12 \times 13$$

$$112296 - 79896 = 32400 = 3 \times 4 \times 5 \times 6 \times 9 \times 10$$

$$112296 + 79896 = 192192 = 192 \times 1001 = (2 \times 8 \times 12) \times (7 \times 11 \times 13).$$

3. (d) To express a number n as a difference of squares, write it as a product of two numbers of the same parity: $n = uv$, with $u > v$. Then solve the system $a + b = u$ and $a - b = v$ for a and b .

4. (a) Instead of multiplying out the squares, we can get the result by factoring a difference of squares and taking out prime factors. It will sometimes help in comparing two products to see if certain terms of a product have a common divisor. This can be done by the Euclidean algorithm: if we seek the common divisor of a and b , where $a > b$, divide b into a to get the relation $a = bq + r$ where q is the quotient and r is the remainder; note that $0 \leq r < b$. Then the greatest common divisor of a and b is equal to the greatest common divisor of b and r . We can then treat the pair b and r in the same way.

We find that $c - a = 121812582528 = 2^7 \times 3^2 \times 7^2 \times 13^2 \times 113^2$, $c + a = 9252784638050 = 2 \times 5^2 \times 430181^2$ and $b = 2^4 \times 3 \times 5 \times 7 \times 13 \times 113 \times 430181$. The check using the factorization of $c^2 - b^2$ can be handled similarly.

4. (b, c) $c = 2165017^2$ and $a + b = 2372159^2$.

5. See the comment for 3(d). It is necessary and sufficient that the number can be written as the product of two factors of the same parity.

7. This result seems obvious, but many students cannot manage more than an “arm-waving” proof. A more solid argument can be based on consideration of the equation $1 = x^2 - y^2 = (x - y)(x + y)$. If $x > y > 0$, then the equation is impossible since the only factorization of 1 as a product of positive integers is $1 = 1 \times 1$.

8. (b) $n(n + 1)$ is a perfect square if and only if $4n(n + 1) = (2n + 1)^2 - 1$ is so. Alternatively, you can base an argument on the fact that the greatest common divisor of n and $n + 1$ is 1.

13. The isosceles triangle is obtained by abutting two right triangles with arm 1200 and hypotenuse 1201. The length of the vertical side is $\sqrt{1201^2 - 1200^2} = \sqrt{2401} = 49$ cm, or almost half a metre.



14. (c) Note that $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

15. (a) Note that $x^4 + 4 = (x^2 + 2)^2 - 4x^2$.

15. (b) Note that $x^4 + 1 = (x^2 + 1)^2 - 2x^2 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$.

16. Notice that

$$\begin{aligned}(pq + 1)^4 - 1 &= [(pq + 1)^2 - 1][(pq + 1)^2 + 1] \\ &= pq(pq + 2)[pq(pq + 2) + 2] .\end{aligned}$$

When p and q are both odd, all the factors are odd and pairwise coprime.

Note that when $p = 2$ and $q = 3$, then we get

$$7^4 - 1 = 48 \times 50 = 2^5 \times 3 \times 5^2 .$$

19. Observe that, for example,

$$\begin{aligned}(25^2 + 26^2 + 27^2) - (22^2 + 23^2 + 24^2) \\ &= (25^2 - 24^2) + (26^2 - 23^2) + (27^2 - 22^2) \\ &= (1 + 3 + 5) \times 49 = 3^2 \times 7^2 = 21^2 .\end{aligned}$$

21. Let $n+9 = x^2$ and $16n+9 = y^2$. Then $135 = 16x^2 - y^2 = (4x+y)(4x-y)$. The number n is equal to 280.

22. The key to the solution is to note that (i) $w_k + d_k$ takes the same value, $n - 1$, for each value of k , and (ii) the total number of wins equals the total number of losses equals the number of matches played. Thus,

$$0 = (w_1 - d_1) + (w_2 - d_2) + \cdots + (w_n - d_n) .$$

Multiply the first term by $w_1 + d_1$, the second by $w_2 + d_2$, and so on; then add.



COMPLEX NUMBERS

§1. Definitions and Notation

The quadratic formula for solving a quadratic equation whose discriminant is negative involves the square root of a negative number. This induces us to extend the number domain to include such square roots if we are to have a theory of the quadratic that allows for solutions of quadratic equations that is like that when the discriminant is positive. While the idea of taking the square root of a negative number might seem strange, nevertheless our enlarged number system will turn out to be consistent with all the arithmetic laws that we have for real numbers remaining valid.

A complex number has the form $x + yi$ where x and y are real numbers and $i^2 = -1$. They can be added, subtracted, multiplied and divided following the rules of ordinary algebra with the simplification that i^2 can be replaced by -1 .

For real numbers represented on a number line, we can think of addition in terms of a translation along the line. For example, to add 2 and 5, the sum of 2 and 5 is represented by the point obtained by translating the point 5 by 2 units in the positive direction (or equivalently the point 2 by 5 units in the positive direction). Multiplication by a positive real corresponds on the line to a dilatation whose centre is at the origin. Thus, the product of 2 and 5 is the place where 2 ends up when the line has been expanded by a dilatation with factor 5. Multiplication by -1 corresponds to a reflection in 0, and multiplication by a negative number corresponds to the composite of this reflection and a dilatation whose factor is its absolute value.

This leads to a geometric representation of the complex numbers (Argand diagram). Since $1 \times -1 = -1$, we can think of the position of -1 as the result of applying a reflection about 0, or, equivalently on the line, a rotation of 180° about the origin to the line. Since -1 is the result of multiplying 1 by i twice, it is reasonable to represent i be the point on the plane which is the image of 1 after a counterclockwise rotation of 90° about the origin. Thus 1 corresponds to the point $(1, 0)$, i to the point $(0, 1)$ and -1 to the point $(-1, 0)$.

In general, we represent the complex number $x + yi$ with x and y real, by the point (x, y) in the plane. Addition of complex numbers corresponds to vector addition in the plane: $(x + yi) + (u + vi) = (x + u) + (y + v)i$. Also, $(x + yi)(u + vi) = (xu - yv) + (xv + yu)i$. The *absolute value* $|x + yi|$ is equal to $\sqrt{x^2 + y^2}$, which geometrically is the distance from 0 to $x + yi$ in the Argand diagram. The angle θ measured counterclockwise from the positive real axis to the segment joining 0 to $x + yi$ is called the *argument* of $x + yi$ and is denoted by $\arg(x + yi)$. It is given in radians and is determined up to a multiple of 2π .

For a complex number $x + yi$, x is called the *real part* and denoted by $\operatorname{Re}(x + yi)$, and y is called the *imaginary part* and is denoted by $\operatorname{Im}(x + yi)$. The number $x - yi$ is called the *complex conjugate* of $x + yi$ and is denoted by $\overline{x + yi}$.

If $r = \sqrt{x^2 + y^2}$ and $\theta = \arg(x + yi)$, then $x = r \cos \theta$ and $y = r \sin \theta$ and



we get the polar representation of the complex number:

$$r \cos \theta + ir \sin \theta .$$

§2. Exercises

In these exercises, we can use the standard notation

$$z = x + yi = r(\cos \theta + i \sin \theta)$$

and

$$w = u + vi = s(\cos \phi + i \sin \phi)$$

unless otherwise indicated.

2.1. Prove that $\overline{z + w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$ and $\bar{\bar{z}} = z$. (Otherwise stated, this says that the operation of complex conjugation is an isomorphism and an involution, *i.e.*, it preserves the arithmetic operations of complex numbers and is a transformation of period 2.)

2.2. (a) Prove that $|z|^2 = z\bar{z}$ and deduce that, for any nonzero complex number z ,

$$z^{-1} = \frac{\bar{z}}{|z|^2} .$$

(b) Deduce that, when $|z| = 1$, then $\bar{z} = z^{-1}$.

(c) Provide a geometric interpretation of the mapping $z \longrightarrow z^{-1}$.

2.3. Prove that

$$\operatorname{Re} z = \frac{1}{2}(z + \bar{z}) \leq |z|$$

and

$$\operatorname{Im} z = \frac{1}{2i}(z - \bar{z}) \leq |z| .$$

2.4. Prove that $|zw| = |z||w|$ and that $|z + w| \leq |z| + |w|$. The latter inequality can be obtained algebraically by expressing $|z + w|^2$ as $(z + w)(\bar{z} + \bar{w})$, multiplying out and observing that $z\bar{w}$ is the complex conjugate of $\bar{z}w$.

Give a geometric interpretation of the inequality

$$|z + w| \leq |z| + |w| .$$

2.5. Prove that

$$\frac{1}{\sqrt{2}}(|x| + |y|) \leq |z| \leq (|x| + |y|) .$$

2.6. Prove that

$$[r(\cos \theta + i \sin \theta)][s(\cos \phi + i \sin \phi)] = rs(\cos(\theta + \phi) + i \sin(\theta + \phi)) .$$

Deduce that $\arg zw \equiv \arg z + \arg w$ modulo 2π and give a geometric interpretation in the complex plane of the product of two complex numbers z and w .



2.7. Prove that the vectors from the origin to the points represented by z and w in the complex plane are perpendicular if and only if z/w is pure imaginary (*i.e.*, its real part is 0, so that it is a real multiple of i).

2.8. Prove, for integers n , de Moivre's theorem:

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n .$$

Use this result to obtain $\cos k\theta$ and $\sin k\theta$ as polynomials in $\cos \theta$ and $\sin \theta$ for $k = 2, 3, 4$.

2.9. Suppose that $z = \cos \theta + i \sin \theta$. Prove that

(a) $|z| = 1$ and $z^{-1} = \cos \theta - i \sin \theta$.

(b) $\cos \theta = \frac{1}{2}(z + \bar{z})$ and $\sin \theta = \frac{1}{2i}(z - \bar{z})$.

(c) For each integer n ,

$$\cos n\theta = \frac{1}{2}(z^n + \bar{z}^n) = \frac{1}{2}(z^n + z^{-n})$$

and

$$\sin n\theta = \frac{1}{2i}(z^n - \bar{z}^n) = \frac{1}{2i}(z^n - z^{-n}) .$$

2.10. Prove that, for each real value of θ , the value of

$$\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta}$$

is constant and determine the value of the constant. Find more than one argument for this result, including an argument that makes use of complex numbers.

2.11. Let $p(z)$ be a polynomial in the complex variable z with real coefficients. Prove that

$$\overline{p(z)} = p(\bar{z})$$

and deduce that if r is a root of $p(z)$ then so is its complex conjugate \bar{r} . Explain why every polynomial with real coefficients and odd degree must have at least one real root. Provide an example to show that these assertions are not necessarily true when a polynomial has at least one nonreal coefficient.

2.12. The function $f(\theta) = \cos \theta + i \sin \theta$ satisfies the equation $f(\theta + \phi) = f(\theta)f(\phi)$ and $f(0) = 1$, which makes it look like an exponential function. In fact, this is precisely what it is. Because this function satisfies the differential equation $f''(\theta) = f(\theta)$ as well as the "initial" conditions $f(0) = 1$ and $f'(0) = i$, we can write it in the form $e^{i\theta}$, where e is the base of the natural logarithms, a number that lies between 2 and 3. However, for our purposes, we can leave this at the formal level.



Sum the geometric progression

$$\sum_{k=0}^n e^{(i\theta)k}$$

and equate the real and imaginary parts of this sum to the real and imaginary parts of the closed form of the sum that you get to obtain an expression for the sums of the following trigonometric series:

$$\sin \theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta$$

and

$$1 + \cos \theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta .$$

Check that your expressions for these sums are correct when $n = 1, 2, 3$.

2.13. Establish the identity

$$\cos 7\theta = (\cos \theta + 1)(8 \cos^3 \theta - 4 \cos^2 \theta - 4 \cos \theta + 1)^2 - 1$$

and deduce that the three roots of the polynomial

$$8z^3 - 4z^2 - 4z + 1$$

are $\cos \frac{\pi}{7}$, $\cos \frac{3\pi}{7}$ and $\cos \frac{5\pi}{7}$. Deduce that

$$8z^3 - 4z^2 - 4z + 1 = 8(z - \cos \frac{\pi}{7})(z - \cos \frac{3\pi}{7})(z - \cos \frac{5\pi}{7}) .$$

Compare coefficients of the two sides of this equation to obtain three equations satisfied by these roots.

2.14. Let $p(z)$ be a polynomial of degree greater than 4 with complex coefficients. Prove that $p(z)$ must have a pair u, v of roots, not necessarily distinct, for which the real parts of both u/v and v/u are positive. Show that this does not necessarily hold for polynomials of degree 4.

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$$\begin{aligned} 2.4. \quad |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) = |z|^2 + z\bar{w} + w\bar{z} + |w|^2 \\ &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 \leq |z|^2 + 2|z||w| + |w|^2 \\ &= (|z| + |w|)^2 . \end{aligned}$$

2.10. Letting $z = \cos \theta + i \sin \theta$, we have that

$$\begin{aligned} \frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} &= \frac{z^3 - \bar{z}^3}{z - \bar{z}} - \frac{z^3 + \bar{z}^3}{z + \bar{z}} \\ &= (z^2 + 1 + \bar{z}^2) - (z^2 - 1 + \bar{z}^2) = 2 . \end{aligned}$$

Putting the difference over a common denominator yields

$$\frac{\sin 3\theta \cos \theta - \cos 3\theta \sin \theta}{\sin \theta \cos \theta} = \frac{\sin 2\theta}{\frac{1}{2} \sin 2\theta} = 2 .$$



Alternatively, we can expand $\sin 3\theta = \sin(2\theta + \theta)$ and $\cos 3\theta$ to obtain

$$\frac{\sin 3\theta}{\sin \theta} = \cos 2\theta + 2 \cos^2 \theta \quad \text{and} \quad \frac{\sin 3\theta}{\sin \theta} = \cos 2\theta + 2 \cos^2 \theta$$

from which the same result follows.

2.14. Solution. Since the degree of the polynomial exceeds 4, there must be two roots u, v in one of the four quadrants containing a ray from the origin along either the real or the imaginary axis along with all the points within the region bounded by this ray and the next such ray in the counterclockwise direction. The difference in the arguments between two such numbers must be strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\arg(u/v) = \arg u - \arg v$ and $\arg(v/u) = \arg v - \arg u$ both lie in this range, both u/v and v/u lie to the right of the imaginary axis, and so have positive real parts.

§3. Problems on Complex Numbers

These problems can be solved by complex techniques and you should do so. However, if you can solve them some other way, compare your solution with the complex one with respect to naturalness, ease of understanding and the insight it gives into the situation.

3.1. Using complex multiplication, show that the product of two integers that are equal to the sum of two squares is also equal to the sum of two squares. Use this to write 85 as the sum of two squares in two different ways.

3.2. Using complex numbers, prove that the angle subtended at the circumference of a circle by a diameter is right.

3.3. Some pirates wish to bury their treasure on an island. They find a tree T and two rocks U and V . Starting at T , they pace off the distance from T to U , then turn right and pace off an equal distance from U to a point P , which they mark. Returning to T , they pace off the distance from T to V , then turn left and pace off an equal distance from V (to TV) to a point Q which they mark. The treasure is buried at the midpoint of the line segment PQ .

Years later, they return to the island and discover to their dismay that the tree T is missing. One of them decides just to assume any position for the tree and carry out the procedure. Is this strategy likely to succeed?

3.4. Let ABC be a triangle and P any point in its plane. Let P_1 be the reflection of P in A , P_2 be the reflection of P_1 in B and P_3 be the reflection of P_2 in C . Suppose that I is the midpoint of the segment PP_3 .

(a) How does the position of I depend on P ?

(b) Is it possible for the points P and P_3 to coincide? Justify your answer.

3.5. Let a, b, c be three real numbers for which $0 \leq c \leq b \leq a \leq 1$ and let w be a complex root of the polynomial $z^3 + az^2 + bz + c$. Must $|w| \leq 1$?



3.6. For nonzero complex numbers z and w , show that

$$(|z| + |w|) \left| \frac{z}{|z|} + \frac{w}{|w|} \right| \leq 2(|z + w|) .$$

3.7. Determine the set of complex numbers z that satisfy each of the following equations:

(a) $\operatorname{Re}(wz) = c$, where w is a fixed nonzero complex number and c is a fixed real number.

(b) $|z| = k|z + 1|$ where k is a fixed positive real number.

(c) $|z - u| + |z - v| \leq k$ where u and v are fixed distinct complex numbers and k is a positive real number.

(d) $\operatorname{Im}(z^4) = (\operatorname{Re}(z^2))^2$.

3.8. Describe those triangles with vertices at the points z_1, z_2, z_3 in the complex plane for which

$$(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0 .$$

3.9. Evaluate

(a) $\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ$.

(b) $\sin 10^\circ \sin 50^\circ \sin 70^\circ$.

(c) $\sec 40^\circ + \sec 80^\circ + \sec 160^\circ$.

3.10. (a) A regular pentagon has side length a and diagonal length b . Prove that

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} = 3 .$$

(b) A regular heptagon (seven equal sides and equal angles) has diagonals of two different lengths. Let a be the length of a side, b the length of a shorter diagonal and c the length of a longer diagonal of a regular heptagon. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

and

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 .$$

(c) Can the results of (a) and (b) be generalized?

3.11. Suppose that z_1, z_2, z_3, z_4 are four distinct complex numbers for which there exists a real number t not equal to 1 such that

$$|tz_1 + z_2 + z_3 + z_4| = |z_1 + tz_2 + z_3 + z_4| = |z_1 + z_2 + tz_3 + z_4| .$$

Show that, in the complex plane, z_1, z_2, z_3, z_4 lie at the vertices of a rectangle.

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3.2. Let the circle be centred at the origin with radius 1. Suppose that the ends of the diameter are at -1 and $+1$ and that z is an arbitrary point on the circumference. Then it has to be established that $(z-1)$ and $(z+1)$ are perpendicular.

3.3. Suppose that the rocks U and V are at the respective points 0 and 1 in the complex plane and the tree is at the point z . Then P is at the point iz and Q is at the point $1-i(z-1)$. The midpoint of PQ is at $\frac{1}{2}(iz+1-i(z-1)) = \frac{1}{2}(1+i)$.

3.4. Place the triangle in the complex plane with A, B, C at the respective points a, b, c . Let the point P be at z . Then P_1 is at $2a-z$, P_2 is at $2b-(2a-z) = 2b-2a+z$ and P_3 is at $2c-(2b-2a+z) = 2(a+c-b)-z$. The midpoint of PP_3 is at $a+c-b$, which is independent of z . As an exercise, verify this when P is chosen to be any of the vertices of the triangle. The points P and P_3 will coincide when $z = a+c-b$ or $z-a = c-b$. Geometrically, this says that the two points will coincide when P is the fourth vertex of a parallelogram whose other vertices are A, B, C .

3.5. Let a, b, c be three real numbers for which $0 \leq c \leq b \leq a \leq 1$ and let w be a complex root of the polynomial $z^3 + az^2 + bz + c$. Must $|w| \leq 1$?

Solution 1. [L. Fei] Let $w = u + iv$, $\bar{w} = u - iv$ and r be the three roots. Then $a = -2u - r$, $b = |w|^2 + 2ur$ and $c = -|w|^2r$. Substituting for b, ac and c , we find that

$$|w|^6 - b|w|^4 + ac|w|^2 - c^2 = 0$$

so that $|w|^2$ is a nonnegative real root of the cubic polynomial $q(t) = t^3 - bt^2 + act - c^2 = (t-b)t^2 + c(at-c)$. Suppose that $t > 1$, then $t-b$ and $at-c$ are both positive, so that $q(t) > 0$. Hence $|w| \leq 1$.

Solution 2. [P. Shi; Y. Zhao]

$$\begin{aligned} 0 &= (1-w)(w^3 + aw^2 + bw + c) \\ &= -w^4 + (1-a)w^3 + (a-b)w^2 + (b-c)w + c \\ \Rightarrow w^4 &= (1-a)w^3 + (a-b)w^2 + (b-c)w + c \\ \Rightarrow |w|^4 &\leq (1-a)|w|^3 + (a-b)|w|^2 + (b-c)|w| + c. \end{aligned}$$

Suppose, if possible, that $|w| > 1$. Then

$$|w|^4 \leq |w|^3[(1-a) + (a-b) + (b-c) + c] = |w|^3$$

which implies that $|w| \leq 1$ and yields a contradiction. Hence $|w| \leq 1$.

Solution 3. There must be one real solution v to the equation $f(z) \equiv z^3 + az^2 + bz + c = 0$. If $v = 0$, then the remaining roots w and \bar{w} , the complex conjugate of w , must satisfy the quadratic equation $z^2 + az + b = 0$. Therefore $|w|^2 = w\bar{w} = b \leq 1$ and the result follows. Henceforth, let $v \neq 0$.



Observe that

$$f(-1) = -1 + a - b + c = -(1 - a) - (b - c) \leq 0$$

and that

$$f(-c) = -c^3 + ac^2 - bc + c \geq -c^3 + c^3 - bc + c = c(1 - b) \geq 0 ,$$

so that $-1 \leq v \leq -c$. The polynomial can be factored as

$$(z - v)(z^2 + pz + q)$$

where $c = -qv$ so that $q = c/(-v) \leq 1$. But $q = w\bar{w}$, and the result again follows.

3.6. Prove that, for any complex numbers z and w ,

$$(|z| + |w|) \left| \frac{z}{|z|} + \frac{w}{|w|} \right| \leq 2|z + w| .$$

Solution 1.

$$\begin{aligned} & (|z| + |w|) \left| \frac{z}{|z|} + \frac{w}{|w|} \right| \\ &= \left| z + w + \frac{|z|w}{|w|} + \frac{|w|z}{|z|} \right| \\ &\leq |z + w| + \frac{1}{|z||w|} |\bar{z}zw + \bar{w}zw| \\ &= |z + w| + \frac{|zw|}{|z||w|} |\bar{z} + \bar{w}| = 2|z + w| . \end{aligned}$$

Solution 2. Let $z = ae^{i\alpha}$ and $w = be^{i\beta}$, with a and b real and positive. Then the left side is equal to

$$\begin{aligned} |(a + b)(e^{i\alpha} + e^{i\beta})| &= |ae^{i\alpha} + ae^{i\beta} + be^{i\alpha} + be^{i\beta}| \\ &\leq |ae^{i\alpha} + be^{i\beta}| + |ae^{i\beta} + be^{i\alpha}| . \end{aligned}$$

Observe that

$$\begin{aligned} |z + w|^2 &= |(ae^{i\alpha} + be^{i\beta})(ae^{-i\alpha} + be^{-i\beta})| \\ &= a^2 + b^2 + ab[e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)}] \\ &= |(ae^{i\beta} + be^{i\alpha})(ae^{-i\beta} + be^{-i\alpha})| \end{aligned}$$

from which we find that the left side does not exceed

$$|ae^{i\alpha} + be^{i\beta}| + |ae^{i\beta} + be^{i\alpha}| = 2|ae^{i\alpha} + be^{i\beta}| = 2|z + w| .$$

Solution 3. Let $z = ae^{i\alpha}$ and $w = be^{i\beta}$, where a and b are positive reals. Then the inequality is equivalent to

$$\left| \frac{1}{2}(e^{i\alpha} + e^{i\beta}) \right| \leq |\lambda e^{i\alpha} + (1 - \lambda)e^{i\beta}|$$



where $\lambda = a/(a+b)$. But this simply says that the midpoint of the segment joining $e^{i\alpha}$ and $e^{i\beta}$ on the unit circle in the Argand diagram is at least as close to the origin as another point on the segment.

Solution 4. [G. Goldstein] Observe that, for each $\mu \in \mathbf{C}$,

$$\left| \frac{\mu z}{|\mu z|} + \frac{\mu w}{|\mu w|} \right| = \left| \frac{z}{|z|} + \frac{w}{|w|} \right|,$$

$$|\mu|(|z| + |w|) = |\mu z + \mu w|,$$

and

$$|\mu||z + w| = |\mu z + \mu w|.$$

So the inequality is equivalent to

$$(|t| + 1) \left| \frac{t}{|t|} + 1 \right| \leq 2|t + 1|$$

for $t \in \mathbf{C}$. (Take $\mu = 1/w$ and $t = z/w$.)

Let $t = r(\cos \theta + i \sin \theta)$. Then the inequality becomes

$$\begin{aligned} (r+1)\sqrt{(\cos \theta + 1)^2 + \sin^2 \theta} &\leq 2\sqrt{(r \cos \theta + 1)^2 + r^2 \sin^2 \theta} \\ &= 2\sqrt{r^2 + 2r \cos \theta + 1}. \end{aligned}$$

Now,

$$\begin{aligned} &4(r^2 + 2r \cos \theta + 1) - (r+1)^2(2 + 2 \cos \theta) \\ &= 2r^2(1 - \cos \theta) + 4r(\cos \theta - 1) + 2(1 - \cos \theta) \\ &= 2(r-1)^2(1 - \cos \theta) \geq 0, \end{aligned}$$

from which the inequality follows.

Solution 5. [R. Mong] Consider complex numbers as vectors in the plane. $q = (|z|/|w|)w$ is a vector of magnitude z in the direction w and $p = (|w|/|z|)z$ is a vector of magnitude w in the direction z . A reflection about the angle bisector of vectors z and w interchanges p and w , q and z . Hence $|p+q| = |w+z|$. Therefore

$$\begin{aligned} &(|z| + |w|) \left| \frac{z}{|z|} + \frac{w}{|w|} \right| \\ &= |z + q + p + w| \leq |z + w| + |p + q| \\ &= 2|z + w|. \end{aligned}$$

3.7. (a) Determine the set of complex numbers for which $\operatorname{Re}(wz) = c$.

Solution. Let $v = c/w = c\bar{w}/|w|^2$. The point v is on the locus and furthermore, $\operatorname{Re}(w(z-v)) = 0$ for any point z on the locus. Therefore



$w(z - v) = ti$ for some real number t . Thus, for every point on the locus, we have that

$$z = v + (t/w)i = (1/w)(c + ti) .$$

Conversely, any point z of this form satisfies the equation of the locus. Geometrically, the equation of the locus is the line $\operatorname{Re} z = c$ rotated through the angle $-\arg w$.

(b) Determine the set of complex numbers z which satisfy $|z| = k|z + 1|$ for $k > 0$.

Solution. With $z = x + yi$, the equation can be written as

$$(k^2 - 1)x^2 + 2k^2x + (k^2 - 1)y^2 + 1 = 0 .$$

When $k = 1$, the locus is the straight line $\operatorname{Re} z = -\frac{1}{2}$. Otherwise the locus is a circle (of Apollonius).

(c) Determine the set of complex numbers z which satisfy

$$|z - u| + |z - v| \leq k$$

where u and v are complex and k is positive.

Solution. The locus includes all curves

$$|z - u| + |z - v| = c$$

where $c \leq k$. The equation signifies that the sum of the distances from z to u and v is a constant, and so is void when $c < |u - v|$, is the segment joining u and v when $c = |u - v|$ and an ellipse with foci at u and v when $c > |u - v|$. Thus the locus of the inequality is void when $k < |u - v|$, a line segment when $k = |u - v|$ and the interior and boundary of an ellipse when $k > |u - v|$.

(d) Determine the set of complex numbers z which satisfy

$$\operatorname{Im}(z^4) = (\operatorname{Re}(z^2))^2 ,$$

and sketch this set in the complex plane.

Solution 1. Let $z = x + yi$ and $z^2 = u + vi$. Then $u = x^2 - y^2$, $v = 2xy$ and $z^4 = (u^2 - v^2) + 2uvi$. $\operatorname{Im}(z^4) = (\operatorname{Re}(z^2))^2$ implies that $2uv = u^2$. Thus, $u = 0$ or $u = 2v$. These reduce to $x^2 = y^2$ or $(x - 2y)^2 = 5y^2$, so that the locus consists of the points z on the lines determined by the equations $y = x$, $y = -x$, $y = (\sqrt{5} - 2)x$, $y = (-\sqrt{5} - 2)x$.

Solution 2. Let $z = r(\cos \theta + i \sin \theta)$; then $z^2 = r^2(\cos 2\theta + i \sin 2\theta)$ and $z^4 = r^4(\cos 4\theta + i \sin 4\theta)$. The condition is equivalent to

$$r^4 \sin 4\theta = (r^2 \cos 2\theta)^2 \Leftrightarrow 2 \sin 2\theta \cos 2\theta = \cos^2 2\theta .$$

Hence $\cos 2\theta = 0$ or $\tan 2\theta = \frac{1}{2}$. The latter possibility leads to $\tan^2 \theta + 4 \tan \theta - 1 = 0$ or $\tan \theta = -2 \pm \sqrt{5}$. This yields the same result as



Solution 3. Let $z = x + yi$. Then $z^2 = x^2 - y^2 + 2xyi$ and $z^4 = (x^4 - 6x^2y^2 + y^4) + 4xy(x^2 - y^2)i$. Then the condition in the problem is equivalent to

$$4xy(x^2 - y^2) = (x^2 - y^2)^2,$$

which in turn is equivalent to $y = \pm x$ or $y^2 + 4xy - x^2 = 0$, i.e., $y = (-2 \pm \sqrt{5})x$.

3.8. The condition can be rewritten as

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

Without loss of generality, we may translate the points in the complex plane so that $z_1 + z_2 + z_3 = 0$. In this case, we can show that $z_1 + z_2 + z_3 = z_1z_2 + z_2z_3 + z_3z_1 = 0$, so that z_1, z_2, z_3 are roots of an equation of the form $z^3 = c$. Therefore, we can argue that they are at the vertices of an equilateral triangle.

3.9. (a) Determine the value of

$$\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ.$$

Solution 1. Note that the expression is equal to the real part of

$$\begin{aligned} &(\cos 5^\circ + i \sin 5^\circ)[1 + (\cos 72^\circ + i \sin 72^\circ) + (\cos 144^\circ + i \sin 144^\circ) \\ &+ (\cos 216^\circ + i \sin 216^\circ) + (\cos 288^\circ + i \sin 288^\circ)]. \end{aligned}$$

Let $\zeta = \cos 72^\circ + i \sin 72^\circ$ so that ζ is a nonreal root of

$$0 = z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1).$$

Hence $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$; using de Moivre's theorem, and taking the real part of this equation, we find that

$$1 + \cos 72^\circ + \cos 144^\circ + \cos 216^\circ + \cos 288^\circ = 0.$$

(Note that taking the imaginary part yields a triviality.)

Solution 2. A vectorial way of seeing that $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$ is to note that the vectors represented by the five roots of unity sum bound a closed regular pentagon and so sum to zero.

Non-complex arguments. Before getting into the next solution, we will discuss how to obtain the trigonometric ratios of certain angles related to 36° . It is useful for you to know some of these techniques, as these angles tend to come up in problems, and to be on the safe side in a contest, you should try to include a justification for assertions that you make about these angles. Here is one way to evaluate $t = \cos 36^\circ$. Observe that

$$\begin{aligned} t &= -\cos 144^\circ = 1 - 2\cos^2 72^\circ \\ &= 1 - 2(2t^2 - 1)^2 = -8t^4 + 8t^2 - 1 \end{aligned}$$



from which we see that

$$0 = 8t^4 - 8t^2 + t + 1 = (2t - 1)(t + 1)(4t^2 - 2t - 1) .$$

Since t is equal to neither 1 nor $\frac{1}{2}$, we must have that $4t^2 = 2t + 1$. Solving this equation will give you an actual numerical value (can you justify your choice of root?).

A very useful relation is $4 \cos 36^\circ \cos 72^\circ = 1$. This can be checked geometrically. Let PQS be a triangle for which $\angle P = \angle S = 36^\circ$ and $\angle PQS = 108^\circ$. Let R be a point on the side PS for which $\angle PQR = 72^\circ$ and $\angle SQR = 36^\circ$. Then $PQ = PR$, $PQ = QS$ and $QR = RS$; let r be the common length of PQ , PR , QS and let s be the common length of QR and RS . Then $\cos 72^\circ = s/2r$ and $\cos 36^\circ = r/2s$ and the desired result follows. An algebraic derivation of this result can also be given.

$$\begin{aligned} 4 \cos 36^\circ \cos 72^\circ &= \frac{4 \sin 36^\circ \cos 36^\circ \cos 72^\circ}{\sin 36^\circ} \\ &= \frac{2 \sin 72^\circ \cos 72^\circ}{\sin 36^\circ} \\ &= \frac{\sin 144^\circ}{\sin 36^\circ} = 1 . \end{aligned}$$

We also have that

$$1 + 2 \cos 72^\circ - 2 \cos 36^\circ = 1 + \cos 72^\circ + \cos 144^\circ + \cos 216^\circ + \cos 288^\circ = 0 .$$

Solution 3. Let $\cos 36^\circ = t$. Then

$$\begin{aligned} &\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ \\ &= [\cos 5^\circ + \cos 293^\circ] + [\cos 77^\circ + \cos 221^\circ] + \cos 149^\circ \\ &= \cos 149^\circ [2 \cos 144^\circ + 2 \cos 72^\circ + 1] \\ &= \cos 149^\circ [-2 \cos 36^\circ + 2 \cos 72^\circ + 1] = 0 . \end{aligned}$$

Alternatively, this is seen to be equal to

$$\cos 149^\circ [-2t + 2(2t^2 - 1) + 1] = \cos 149^\circ [-2t + 4t^2 - 1] = 0 .$$

Solution 4. [C. Huang]

$$\begin{aligned} &\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ \\ &= \cos 5^\circ + 2 \cos 185^\circ \cos 108^\circ + 2 \cos 185^\circ \cos 36^\circ \\ &= \cos 5^\circ [1 + 2(\cos 72^\circ - \cos 36^\circ)] \\ &= \cos 5^\circ [1 - 4 \sin 18^\circ \sin 54^\circ] \\ &= \cos 5^\circ [1 - 4 \cos 72^\circ \cos 36^\circ] = 0 . \end{aligned}$$



3.9. (b) Using complex numbers, or otherwise, evaluate

$$\sin 10^\circ \sin 50^\circ \sin 70^\circ .$$

Solution 1. Let $z = \cos 20^\circ + i \sin 20^\circ$, so that $1/z = \cos 20^\circ - i \sin 20^\circ$. Then, by De Moivre's Theorem, $z^9 = -1$. Now,

$$\sin 70^\circ = \cos 20^\circ = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z} ,$$

$$\sin 50^\circ = \cos 40^\circ = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right) = \frac{z^4 + 1}{2z^2} ,$$

and

$$\sin 10^\circ = \cos 80^\circ = \frac{1}{2}\left(z^4 + \frac{1}{z^4}\right) = \frac{z^8 + 1}{2z^4} .$$

Hence

$$\begin{aligned} \sin 10^\circ \sin 50^\circ \sin 70^\circ &= \frac{z^2 + 1}{2z} \cdot \frac{z^4 + 1}{2z^2} \cdot \frac{z^8 + 1}{2z^4} \\ &= \frac{1 + z^2 + z^4 + z^6 + z^8 + z^{10} + z^{12} + z^{14}}{8z^7} \\ &= \frac{1 - z^{16}}{8z^7(1 - z^2)} \\ &= \frac{1 - z^7 z^9}{8(z^7 - z^9)} \\ &= \frac{1 + z^7}{8(z^7 + 1)} = \frac{1}{8} . \end{aligned}$$

Solution 2. We have that

$$\begin{aligned} \sin 10^\circ \sin 50^\circ \sin 70^\circ &= \frac{1}{2}[\cos 40^\circ - \cos 60^\circ] \sin 70^\circ \\ &= \frac{1}{2}\left[\cos 40^\circ \sin 70^\circ - \frac{1}{4}\right] \sin 70^\circ \\ &= \frac{1}{4}[\sin 110^\circ + \sin 30^\circ] - \frac{1}{4} \sin 70^\circ \\ &= \frac{1}{4}[\sin 110^\circ - \sin 70^\circ] + \frac{1}{8} = \frac{1}{8} . \end{aligned}$$



Solution 3. Observe that

$$\begin{aligned}
 \sin 20^\circ \sin 70^\circ \sin 50^\circ \sin 10^\circ &= \sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ \\
 &= \frac{1}{2} \sin 40^\circ \cos 40^\circ \cos 80^\circ \\
 &= \frac{1}{4} \sin 80^\circ \cos 80^\circ \\
 &= \frac{1}{8} \sin 160^\circ = \frac{1}{8} \sin 20^\circ .
 \end{aligned}$$

Since $\sin 20^\circ \neq 0$, we can cancel this factor from both sides to obtain that $\sin 10^\circ \sin 50^\circ \sin 70^\circ = 1/8$.

Solution 4. [O. Ivrii]

$$\begin{aligned}
 \sin 10^\circ \sin 50^\circ \sin 70^\circ &= \frac{1}{2} \sin 10^\circ [\cos 20^\circ - \cos 120^\circ] \\
 &= \frac{1}{2} \sin 10^\circ \cos 20^\circ + \frac{1}{4} \sin 10^\circ \\
 &= \frac{1}{4} [\sin 30^\circ - \sin 10^\circ] + \frac{1}{4} \sin 10^\circ \\
 &= \frac{1}{4} \sin 30^\circ = \frac{1}{8} .
 \end{aligned}$$

Solution 5. [L. Chindelevitch] Observe that $\sin 10^\circ \sin 50^\circ \sin 70^\circ = \cos 20^\circ \cos 40^\circ \cos 80^\circ$ and that $\cos 6\theta = -1/2$ is satisfied by $\theta = \pm 20^\circ, \pm 40^\circ, \pm 80^\circ, \pm 100^\circ, \pm 140^\circ$ and $\pm 160^\circ$. Now, $\cos 6\theta$ is equal to the real part of $(\cos \theta + i \sin \theta)^6$, namely $32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$.

Thus, the sextic equation $32x^6 - 48x^4 + 18x^2 - 1 = -1/2$ is satisfied by $x = \pm \cos 20^\circ, \pm \cos 40^\circ, \pm \cos 60^\circ$. As the equation can be rewritten

$$x^6 - \frac{3}{2}x^4 + \frac{9}{16}x^2 - \frac{1}{64} = 0 ,$$

the product of its six roots is $-1/64$. Thus

$$-(\cos 20^\circ \cos 40^\circ \cos 80^\circ)^2 = -1/64$$

and the result follows.

Solution 6. [A. Critch] Observe that $\sin 10^\circ \sin 50^\circ \sin 70^\circ = -\cos 40^\circ \cos 80^\circ \cos 160^\circ$, and that $\theta = 40^\circ, 80^\circ, 160^\circ$ satisfy $\cos 3\theta = -1/2$. Thus the cosines of $40^\circ, 80^\circ$ and 160° are the roots of the cubic equation $4x^3 - 3x + \frac{1}{2} = 0$. The result follows, since the product of the roots of this equation is $-1/8$.

3.9. (c) Determine $\sec 40^\circ + \sec 80^\circ + \sec 160^\circ$.

Solution 1. Let $z = \cos 40^\circ + i \sin 40^\circ$. Then $z^9 = 1$. In fact, since $z^9 - 1 = (z - 1)(z^2 + z + 1)(z^6 + z^3 + 1)$ and the first two factors fail to vanish,

$z^6 + z^3 + 1 = 0$. Also $1 + z + z^2 + \cdots + z^8 = (1 + z + z^2)(1 + z^3 + z^6) = 0$. Observe that $\cos 40^\circ = \frac{1}{2}(z + \frac{1}{z})$, $\cos 80^\circ = \frac{1}{2}(z^2 + \frac{1}{z^2})$ and $\cos 160^\circ = \frac{1}{2}(z^4 + \frac{1}{z^4})$, so that the given sum is equal to

$$\begin{aligned}
 & 2 \left[\frac{z}{1+z^2} + \frac{z^2}{1+z^4} + \frac{z^4}{1+z^8} \right] \\
 &= 2 \left[\frac{z}{1+z^2} + \frac{z^2}{1+z^4} + \frac{z^5}{1+z} \right] \\
 &= 2 \left[\frac{z(1+z+z^4+z^5) + z^2(1+z+z^2+z^3) + z^5(1+z^2+z^4+z^6)}{(1+z)(1+z^2)(1+z^4)} \right] \\
 &= 2 \left[\frac{z^7 + z^6 + 3z^5 + z^4 + z^3 + 3z^2 + z + 1}{(1+z)(1+z^2)(1+z^4)} \right] \\
 &= 2 \left[\frac{(z+1)(z^6 + z^3 + 1) + 3z^2(z^3 + 1)}{(1+z)(1+z^2)(1+z^4)} \right] \\
 &= 2 \left[\frac{0 - 3z^8}{1+z+z^2+z^3+z^4+z^5+z^6+z^7} \right] = 2 \left[\frac{-3z^8}{-z^8} \right] = 6.
 \end{aligned}$$

Solution 2. The values 40° , 80° and 160° all satisfy $\cos 3\theta = -1/2$, or $8\cos^3\theta - 6\cos\theta + 1 = 0$. Thus, $\cos 40^\circ$, $\cos 80^\circ$ and $\cos 160^\circ$ are the roots of the cubic equation $8x^3 - 6x + 1 = 0$, so that their reciprocals $\sec 40^\circ$, $\sec 80^\circ$ and $\sec 160^\circ$ are the roots of the cubic equation $x^3 - 6x^2 + 8 = 0$. The sum of the roots of this cubic is

$$\sec 40^\circ + \sec 80^\circ + \sec 160^\circ = 6.$$

Solution 3. [T. Liu]

$$\begin{aligned}
 & \sec 40^\circ + \sec 80^\circ + \sec 160^\circ \\
 &= \frac{\cos 40^\circ + \cos 80^\circ}{\cos 40^\circ \cos 80^\circ} + \frac{1}{\cos 160^\circ} = \frac{2 \cos 60^\circ \cos 20^\circ}{\cos 40^\circ \cos 80^\circ} + \frac{1}{\cos 160^\circ} \\
 &= \frac{\cos 20^\circ \cos 160^\circ + \cos 40^\circ \cos 80^\circ}{\cos 40^\circ \cos 80^\circ \cos 160^\circ} \\
 &= \frac{\cos 180^\circ + \cos 140^\circ + \cos 120^\circ + \cos 40^\circ}{\cos 40^\circ (\cos 240^\circ + \cos 80^\circ)} \\
 &= \frac{-1 - 1/2}{(1/2)(-\cos 40^\circ + \cos 120^\circ + \cos 40^\circ)} = \frac{-3/2}{-1/4} = 6.
 \end{aligned}$$

Solution 4. Let $x = \cos 40^\circ$, $y = \cos 80^\circ$ and $z = \cos 160^\circ$. Then

$$x + y + z = 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ = 0$$



and

$$\begin{aligned}
 & xy + yz + zx \\
 &= \frac{1}{2} [\cos 120^\circ + \cos 140^\circ + \cos 240^\circ + \cos 80^\circ + \cos 200^\circ + \cos 120^\circ] \\
 &= \frac{1}{2} \left[-\frac{3}{2} + x + y + z \right] \\
 &= -\frac{3}{4}.
 \end{aligned}$$

Now

$$\begin{aligned}
 \sin 40^\circ \cos 40^\circ \cos 80^\circ \cos 160^\circ &= 4 \sin 80^\circ \cos 80^\circ \cos 160^\circ \\
 &= 2 \sin 160^\circ \cos 160^\circ = -\sin 40^\circ
 \end{aligned}$$

so that $xyz = -1/8$. Then the sum of the problem is equal to $(xy + yz + zx)/(xyz) = 6$.

3.10. (a) A regular pentagon has side length a and diagonal length b . Prove that

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} = 3.$$

Solution 1. Let the pentagon be placed in the complex plane with its vertices at the fifth roots of unity. If ζ is a primitive fifth root of unity, then the expression to be evaluated is

$$\begin{aligned}
 \frac{|\zeta^2 - 1|^2}{|\zeta - 1|^2} + \frac{|\zeta^4 - 1|^2}{|\zeta^2 - 1|^2} &= |\zeta + 1|^2 + |\zeta^2 + 1|^2 \\
 &= (\zeta + 1)(\zeta^4 + 1) + (\zeta^2 + 1)(\zeta^3 + 1) \\
 &= (2 + \zeta + \zeta^4) + (2 + \zeta^2 + \zeta^3) = 3.
 \end{aligned}$$

Solution 2. Let $ABCDE$ be the regular pentagon, and let triangle ABC be rotated about C so that B falls on D and A falls on E . Then ADE is a straight angle and triangle CAE is similar to triangle BAC . Therefore

$$\frac{a+b}{b} = \frac{b}{a} \implies \frac{b}{a} - \frac{a}{b} = 1 \implies \frac{b^2}{a^2} + \frac{a^2}{b^2} - 2 = 1$$

so that $b^2/a^2 + a^2/b^2 = 3$, as desired.

3.10. (b) A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let a be the length of a side, b be the length of a shorter diagonal and c be the length of a longer diagonal of a regular heptagon (so that $a < b < c$). Prove that:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6 \quad \text{and} \quad \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5.$$



Solution 1. There is no loss of generality in assuming that the vertices of the heptagon are placed at the seventh roots of unity on the unit circle in the complex plane. Then $\zeta = \cos(2\pi/7) + i\sin(2\pi/7)$ be the fundamental seventh root of unity. Then $\zeta^7 = 1$, $1 + \zeta + \zeta^2 + \cdots + \zeta^6 = 0$ and (ζ, ζ^6) , (ζ^2, ζ^5) , (ζ^3, ζ^4) are pairs of complex conjugates. We have that

$$\begin{aligned}a &= |\zeta - 1| = |\zeta^6 - 1| \\b &= |\zeta^2 - 1| = |\zeta^5 - 1| \\c &= |\zeta^3 - 1| = |\zeta^4 - 1| .\end{aligned}$$

It follows from this that

$$\frac{b}{a} = |\zeta + 1| \quad \frac{c}{b} = |\zeta^2 + 1| \quad \frac{a}{c} = |\zeta^3 + 1| ,$$

whence

$$\begin{aligned}\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} &= (\zeta + 1)(\zeta^6 + 1) + (\zeta^2 + 1)(\zeta^5 + 1) + (\zeta^3 + 1)(\zeta^4 + 1) \\&= 2 + \zeta + \zeta^6 + 2 + \zeta^2 + \zeta^5 + 2 + \zeta^3 + \zeta^4 \\&= 6 + (\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) = 6 - 1 = 5 .\end{aligned}$$

Also

$$\frac{a}{b} = |\zeta^4 + \zeta^2 + 1| \quad \frac{b}{c} = |\zeta^6 + \zeta^3 + 1| \quad \frac{c}{a} = |\zeta^2 + \zeta + 1| ,$$

whence

$$\begin{aligned}\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} &= (\zeta^4 + \zeta^2 + 1)(\zeta^3 + \zeta^5 + 1) + (\zeta^6 + \zeta^3 + 1)(\zeta + \zeta^4 + 1) \\&\quad + (\zeta^2 + \zeta + 1)(\zeta^5 + \zeta^6 + 1) \\&= (3 + 2\zeta^2 + \zeta^3 + \zeta^4 + 2\zeta^5) + (3 + \zeta + 2\zeta^3 + 2\zeta^4 + \zeta^6) \\&\quad + (3 + 2\zeta + \zeta^2 + \zeta^5 + 2\zeta^6) \\&= 9 + 3(\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) \\&= 9 - 3 = 6 .\end{aligned}$$

Solution 2. Suppose that the circumradius of the heptagon is 1. By considering isosceles triangles with base equal to the sides or diagonals of the heptagon and apex at the centre of the circumcircle, we see that

$$\begin{aligned}a &= 2 \sin \theta = 2 \sin 6\theta = -2 \sin 8\theta \\b &= 2 \sin 2\theta = -2 \sin 9\theta \\c &= 2 \sin 3\theta = 2 \sin 4\theta\end{aligned}$$



where $\theta = \pi/7$ is half the angle subtended at the circumcentre by each side of the heptagon. Observe that

$$\cos 2\theta = \frac{1}{2}(\zeta + \zeta^6) \quad \cos 4\theta = \frac{1}{2}(\zeta^2 + \zeta^5) \quad \cos 6\theta = \frac{1}{2}(\zeta^3 + \zeta^4)$$

where ζ is the fundamental primitive root of unity. We have that

$$\frac{b}{a} = 2 \cos \theta = 2 \cos 6\theta \quad \frac{c}{b} = 2 \cos 2\theta \quad \frac{a}{c} = -2 \cos 4\theta$$

whence

$$\begin{aligned} \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} &= 4 \cos^2 6\theta + 4 \cos^2 2\theta + 4 \cos^2 4\theta \\ &= (\zeta^3 + \zeta^4)^2 + (\zeta + \zeta^6)^2 + (\zeta^2 + \zeta^5)^2 \\ &= \zeta^6 + 2 + \zeta + \zeta^2 + 2 + \zeta^5 + \zeta^4 + 2 + \zeta = 6 - 1 = 5. \end{aligned}$$

Also

$$\begin{aligned} \frac{a}{b} &= \frac{\sin 6\theta}{\sin 2\theta} = 4 \cos^2 2\theta - 1 = (\zeta + \zeta^6)^2 - 1 \\ &= 1 + \zeta^2 + \zeta^5 \\ -\frac{b}{c} &= \frac{\sin 9\theta}{\sin 3\theta} = 4 \cos^2 3\theta - 1 = 4 \cos^2 4\theta - 1 \\ &= (\zeta^2 + \zeta^5)^2 - 1 = 1 + \zeta^4 + \zeta^3 \\ \frac{c}{a} &= \frac{\sin 3\theta}{\sin \theta} = 4 \cos^2 6\theta - 1 \\ &= (\zeta^3 + \zeta^4)^2 - 1 \\ &= 1 + \zeta^6 + \zeta, \end{aligned}$$

whence

$$\begin{aligned} \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} &= (3 + 2\zeta^2 + \zeta^3 + \zeta^4 + 2\zeta^5) + (3 + \zeta + 2\zeta^3 + 2\zeta^4 + \zeta^6) \\ &\quad + (3 + 2\zeta + \zeta^2 + \zeta^5 + 2\zeta^6) \\ &= 9 + 3(\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) \\ &= 9 - 3 = 6. \end{aligned}$$

Solution 3. Let A, B, C, D, E be consecutive vertices of the regular heptagon. Let AB, AC and AD have respective lengths a, b, c , and let $\angle BAC = \theta$. Then $\theta = \pi/7$, the length of BC , of CD and of DE is a , the length of AE is c , $\angle CAD = \angle DAE = \theta$, since the angles are subtended by equal chords of the circumcircle of the heptagon, $\angle ADC = 2\theta$, $\angle ADE = \angle AED = 3\theta$ and $\angle ACD = 4\theta$. Triangles ABC and ACD can be glued together along BC and DC (with C on C) to form a triangle similar to $\triangle ABC$, whence

$$\frac{a+c}{b} = \frac{b}{a}. \quad (1)$$



Triangles ACD and ADE can be glued together along CD and ED (with D on D) to form a triangle similar to $\triangle ABC$, whence

$$\frac{b+c}{c} = \frac{b}{a} . \quad (2)$$

Equation (2) can be rewritten as $\frac{1}{b} = \frac{1}{a} - \frac{1}{c}$. Whence $b = \frac{ac}{c-a}$.

Substituting this into (1) yields

$$\frac{(c+a)(c-a)}{ac} = \frac{c}{c-a}$$

which simplifies to

$$a^3 - a^2c - 2ac^2 + c^3 = 0 . \quad (3)$$

Note also from (1) that $b^2 = a^2 + ac$.

$$\begin{aligned} & \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} - 6 \\ &= \frac{a^4c^2 + b^4a^2 + c^4b^2 - 6a^2b^2c^2}{a^2b^2c^2} \\ &= \frac{a^4c^2 + (a^4 + 2a^3c + a^2c^2)a^2 + c^4(a^2 + ac) - 6a^2c^2(a^2 + ac)}{a^2b^2c^2} \\ &= \frac{a^6 + 2a^5c - 4a^4c^2 - 6a^3c^3 + a^2c^4 + ac^5}{a^2b^2c^2} \\ &= \frac{a(a^2 + 3ac + c^2)(a^3 - a^2c - 2ac^2 + c^3)}{a^2b^2c^2} = 0 . \end{aligned}$$

$$\begin{aligned} & \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} - 5 \\ &= \frac{(a^4 + 2a^3c + a^2c^2)c^2 + a^2c^4 + a^4(a^2 + ac) - 5a^2c^2(a^2 + ac)}{a^2b^2c^2} \\ &= \frac{a^6 + a^5c - 4a^4c^2 - 3a^3c^3 + 2a^2c^4}{a^2b^2c^2} \\ &= \frac{a^2(a + 2c)(a^3 - a^2c - 2ac^2 + c^3)}{a^2b^2c^2} = 0 . \end{aligned}$$

Solution 4. [R. Barrington Leigh] Let the heptagon be $ABCDEFGH$; let AD and BH intersect at P , and BE and CH intersect at Q . Observe that $|PD| = |GE| = b$, $|AP| = c - b$, $|GP| = |DE| = a$, $|BP| = b - a$, $|GQ| = |AB| = a$, $|CQ| = c - a$. From similarity of triangles, we obtain the following:

$$\begin{aligned} \frac{a}{c} = \frac{c-b}{a} &\implies \frac{a}{c} - \frac{c}{a} + \frac{b}{a} = 0 \quad (\triangle APG \sim \triangle ADE) \\ \frac{c-a}{a} = \frac{c}{b} &\implies \frac{c}{a} - \frac{c}{b} = 1 \quad (\triangle QBC \sim \triangle CEG) \end{aligned}$$

$$\frac{c-b}{a} = \frac{b-a}{b} \implies \frac{c}{a} - \frac{b}{a} + \frac{a}{b} = 1 \quad (\triangle APG \sim \triangle DPB)$$

$$\frac{b-a}{a} = \frac{b}{c} \implies \frac{b}{a} - \frac{b}{c} = 1 \quad (\triangle ABP \sim \triangle ADB).$$

Adding these equations in pairs yields

$$\frac{b}{a} + \frac{a}{c} - \frac{c}{b} = 1 \implies \frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{b^2} + 2\left(\frac{b}{c} - \frac{c}{a} - \frac{a}{b}\right) = 1$$

and

$$\frac{c}{a} + \frac{a}{b} - \frac{b}{c} = 2 \implies \frac{c^2}{a^2} + \frac{a^2}{b^2} + \frac{b^2}{c^2} + 2\left(\frac{c}{b} - \frac{b}{a} - \frac{a}{c}\right) = 4.$$

The desired result follows from these equations.

Solution 5. [of the second result by J. Chui] Let the heptagon be $ABCDEFGH$ and $\theta = \pi/7$. Using the Law of Cosines in the indicated triangles ACD and ABC , we obtain the following:

$$\cos 2\theta = \frac{a^2 + c^2 - b^2}{2ac} = \frac{1}{2}\left(\frac{a}{c} + \frac{c}{a} - \frac{b^2}{ac}\right)$$

$$\cos 5\theta = \frac{2a^2 - b^2}{2a^2} = 1 - \frac{1}{2}\left(\frac{b}{a}\right)^2$$

from which, since $\cos 2\theta = -\cos 5\theta$,

$$-1 + \frac{1}{2}\left(\frac{b}{a}\right)^2 = \frac{1}{2}\left(\frac{a}{c} + \frac{c}{a} - \frac{b^2}{ac}\right)$$

or

$$\frac{b^2}{a^2} = 2 + \frac{a}{c} + \frac{c}{a} - \frac{b^2}{ac}. \quad (1)$$

Examining triangles ABC and ADE , we find that $\cos \theta = b/2a$ and $\cos \theta = (2c^2 - a^2)/(2c^2) = 1 - (a^2/2c^2)$, so that

$$\frac{a^2}{c^2} = 2 - \frac{b}{a}. \quad (2)$$

Examining triangles ADE and ACF , we find that $\cos 3\theta = a/2c$ and $\cos 3\theta = (2b^2 - c^2)/(2b^2)$, so that

$$\frac{c^2}{b^2} = 2 - \frac{a}{c}. \quad (3)$$

Adding equations (1), (2), (3) yields

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 6 + \frac{c^2 - bc - b^2}{ac}.$$

By Ptolemy's Theorem, the sum of the products of pairs of opposite sides of a concyclic quadrilateral is equal to the product of the diagonals. Applying this to the quadrilaterals $ABDE$ and $ABCD$, respectively, yields $c^2 = a^2 + bc$



and $b^2 = ac + a^2$, whence $c^2 - bc - b^2 = a^2 + bc - bc - ac - a^2 = -ac$ and we find that

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 6 - 1 = 5.$$

Solution 6. [of the second result by X. Jin] By considering isosceles triangles with side-base pairs (a, b) , (c, a) and (b, c) , we find that $b^2 = 2a^2(1 - \cos 5\theta)$, $a^2 = 2c^2(1 - \cos \theta)$, $c^2 = 2b^2(1 - \cos 3\theta)$, where $\theta = \pi/7$. Then

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 2[3 - (\cos \theta + \cos 3\theta + \cos 5\theta)].$$

Now,

$$\begin{aligned} & \sin \theta (\cos \theta + \cos 3\theta + \cos 5\theta) \\ &= \frac{1}{2} [\sin 2\theta + (\sin 4\theta - \sin 2\theta) + (\sin 6\theta - \sin 4\theta)] \\ &= \frac{1}{2} \sin 6\theta = \frac{1}{2} \sin \theta, \end{aligned}$$

so that $\cos \theta + \cos 3\theta + \cos 5\theta = 1/2$. Hence $b^2/a^2 + c^2/b^2 + a^2/c^2 = 2(5/2) = 5$.

3.11. Let z_1, z_2, z_3, z_4 be distinct complex numbers for which $|z_1| = |z_2| = |z_3| = |z_4|$. Suppose that there is a real number $t \neq 1$ for which

$$|tz_1 + z_2 + z_3 + z_4| = |z_1 + tz_2 + z_3 + z_4| = |z_1 + z_2 + tz_3 + z_4|.$$

Show that, in the complex plane, z_1, z_2, z_3, z_4 lie at the vertices of a rectangle.

Solution. Let $s = z_1 + z_2 + z_3 + z_4$. Then

$$|s - (1 - t)z_1| = |s - (1 - t)z_2| = |s - (1 - t)z_3|.$$

Therefore, s is equidistant from the three distinct points $(1 - t)z_1, (1 - t)z_2$ and $(1 - t)z_3$; but these three points are on the circle with centre 0 and radius $(1 - t)|z_1|$. Therefore $s = 0$.

Since $z_1 - (-z_2) = z_1 + z_2 = -z_3 - z_4 = (-z_4) - z_3$ and $z_2 - (-z_3) = z_2 + z_3 = -z_4 - z_1 = (-z_4) - z_1$, $z_1, -z_2, z_3$ and $-z_4$ are the vertices of a parallelogram inscribed in a circle centered at 0, and hence of a rectangle whose diagonals intersect at 0. Therefore, $-z_2$ is the opposite of one of z_1, z_3 and $-z_4$. Since z_2 is unequal to z_1 and z_3 , we must have that $-z_2 = z_4$. Also $z_1 = -z_3$. Hence z_1, z_2, z_3 and z_4 are the vertices of a rectangle.



THE QUADRATIC

§1. Exercises on Basic Properties

1.1. Let $f(x)$ be a quadratic polynomial and suppose that we divide it by $x - u$ and obtain a remainder r , where u and r are constants, giving: $f(x) = (x - u)g(x) + r$.

(a) Explain why the degree of $g(x)$ is 1.

(b) Prove that $r = f(u)$.

(c) Prove that u is a root of the equation $f(x) = 0$ if and only if there is a linear polynomial $g(x)$ for which $f(x) = (x - u)g(x)$.

1.2. (a) How many different quadratic polynomials $f(x)$ can you find for which $f(0) = 5$, $f(1) = 3$ and $f(2) = -7$? Determine all of them.

(b) Determine all of the polynomials $g(x)$ of degree not exceeding 2 for which $g(-3) = 2$, $g(-1) = -1$ and $g(4) = 0$.

1.3. (a) Suppose that $f(x)$ and $g(x)$ are two polynomials of degree not exceeding 2 for which $f(u) = g(u)$, $f(v) = g(v)$ and $f(w) = g(w)$ for three distinct numbers u , v and w . Prove that $f(x)$ and $g(x)$ must be the same polynomial.

(b) Suppose that $h(x)$ is a quadratic polynomial that vanishes at the two distinct numbers u and v , i.e., $h(u) = h(v) = 0$. Prove that $h(x)$ must be a constant multiple of $(x - u)(x - v)$.

(c) Let a , b , c be three distinct numbers. Determine a quadratic polynomial $h(x)$ for which $h(a) = h(b) = 0$ and $h(c) = 1$.

1.4. (a) Suppose that a , b and c are three distinct numbers and that $f(x)$, $g(x)$ and $h(x)$ are quadratic polynomials for which

$$h(a) = h(b) = g(a) = g(c) = f(b) = f(c) = 0$$

and

$$f(a) = g(b) = h(c) = 1.$$

Let $p(x) = uf(x) + vg(x) + wh(x)$ for some constants u , v and w . Determine the values of $p(a)$, $p(b)$ and $p(c)$.

(b) Suppose that $p(x)$ is a polynomial of degree less than three for which $p(a)$, $p(b)$ and $p(c)$ are specified. Prove that, for every x ,

$$p(x) = p(a) \frac{(x-b)(x-c)}{(a-b)(a-c)} + p(b) \frac{(x-a)(x-c)}{(b-a)(b-c)} + p(c) \frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

(c) Use the format of (b) to determine the polynomials $f(x)$ and $g(x)$ asked for in Exercise 1.2. Check that you get the same answer as you did before.



(d) Use (b) to give a necessary and sufficient condition involving an arbitrary set $\{a, b, c\}$ of numbers that the polynomial p has degree strictly less than 2. [Hint: Look at the coefficient of x^2 in (b).]

1.5. (a) For each of the quadratic polynomials

$$x^2, \quad 1 - x^2, \quad \frac{1}{2}x(x+1), \quad x^2 + 3x + 3$$

construct a table listing in order nonnegative integral values of x , the corresponding values of the polynomial and the difference between the values of the polynomials at consecutive integers. What do you notice about the sequence of differences? If you take differences of consecutive differences, what happens?

(b) Let $p(x) = ax^2 + bx + c$ be a general quadratic polynomial. Verify that, if $q(x) = p(x+1) - p(x)$ and $r(x) = q(x+1) - q(x)$, then $q(x)$ is a linear polynomial and $r(x)$ is a constant polynomial.

(c) It is given that $f(x)$ is a quadratic polynomial. Fill in the missing entries in the following table:

x	$f(x)$	$g(x) = f(x+1) - f(x)$	$h(x) = g(x+1) - g(x)$
0	5	4	?
1	?	-1	?
2	?	?	?
3	?	?	?
4	?	?	?

What do you think $f(x)$ is?

1.6. Exercise 1.4 provided an “all-at-once” method for constructing a quadratic polynomial $p(x)$ with assigned values $p(a)$, $p(b)$ and $p(c)$ at the respective distinct values a , b , c . An alternative approach is to build it up by degree. The polynomial $f(x)$ of lowest degree for which $f(a) = p(a)$ is clearly the constant polynomial $f(x) \equiv p(a)$.

(a) The next step is to “correct” $f(x)$ to a new polynomial $g(x)$ which will take the desired values at both a and b . Argue that $g(x)$ must have the form $f(x) + r(x-a)$ for some constant r , and verify that

$$r = \frac{p(b) - p(a)}{b - a}.$$

(b) Finally, argue that we can get the quadratic polynomial $p(x)$ we want by trying

$$p(x) = p(a) + \left(\frac{p(b) - p(a)}{b - a} \right)(x - a) + s(x - a)(x - b)$$



where s is selected to make the right side equal to $p(c)$ when $x = c$. Determine the value of s that will achieve this.

(c) Use this method to obtain the polynomials asked for in Exercise 1.2. Try it with the three values at which the polynomial is to be computed in different orders.

1.7. Let $p(x)$, $q(x)$, $r(x)$ be as given in Exercise 1.5.(b). Suppose that $p(0)$, $q(0)$ and $r(0)$ are given.

(a) Prove that, for each positive integer n ,

$$p(n) = p(0) + q(0) + q(1) + \cdots + q(n-1)$$

$$q(n) = q(0) + r(0) + r(1) + \cdots + r(n-1) = q(0) + (n-1)r(0) .$$

(b) Determine a formula for $p(n)$ in terms of $p(0)$, $q(0)$ and $r(0)$.

1.8. (a) Write down some values of the polynomial $x^2 + x + 1$ for $x = 0, 1, 2, 3, \dots$. Observe that the product of two consecutive values in the list occur elsewhere in the list. Formulate and prove a general result.

(b) Answer (a) for the polynomial $x^2 + x = x(x+1)$.

(c) Any integer that is the product of two consecutive integers is called *oblong*. Part (b) can be used to show that there are infinitely many triples (a, b, c) of oblong numbers for which $c = ab$. Investigate the existence of triples of oblong numbers no two of which are consecutive but for which the product of two of them is equal to the third.

1.9. Let $p(x)$ be a *monic* quadratic polynomial. (This means that the *leading coefficient* is 1, so that it has the form $p(x) = x^2 + bx + c$.) Suppose also that its coefficients are integers.

(a) Prove that there exists an integer k such that $p(0)p(1) = p(k)$. How many possible such values of k are there?

(b) More generally, prove that for each integer n , there is at least one integer m for which $p(n)p(n+1) = p(m)$.

(c) Are there any values of n for which the value of m determined in (b) is unique?

1.10. Suppose that $p(x)$ is a quadratic polynomial for which $p(a) = p(b)$ for distinct values of a and b . Prove that $p(x) = r(x-c)^2 + s$ for some constants r and s , where $c = -\frac{1}{2}(a+b)$. Deduce that if $p(u) = p(v)$ for some other pair u, v of distinct numbers, then $a+b = u+v$.

1.11. Determine all quadratic polynomials f that satisfy

$$f(f(1)) = f(f(2)) = f(f(3)).$$



1.12. Does there exist a quadratic polynomial $f(x)$ with integer coefficients and the unusual property that, whenever x is a positive integer which consists only of 1's, then $f(x)$ is also a positive integer consisting only of 1's (where the representation is to base 10)?

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1.2. (a) $f(x) = -4x^2 + 2x + 5$.

1.2. (b) Note that $g(x)$ has the form $(x-4)(ax-b)$,

1.3. (a) The polynomial $f(x) - g(x)$ vanishes when $x = 0, 1, 2$ and so is divisible by the cubic $a(x-1)(x-2)$, This is possible if and only if $f(x) - g(x)$ is the zeropolynomial.

1.3. (c) $h(x) = (x-a)(x-b)/(c-a)(c-b)$.

1.6. (b) Let

$$p(x) = p(a) + r(x-a) + s(x-a)(x-b)$$

where $r = (p(b) - p(a))/(b-a)$. Then

$$\begin{aligned} s &= \frac{1}{(c-a)(c-b)} \left[(p(c) - p(a)) - \frac{p(b) - p(a)}{b-a} (c-a) \right] \\ &= \frac{(p(c) - p(a))(b-a) - (p(b) - p(a))(c-a)}{(c-a)(c-b)(b-a)} \\ &= \frac{(p(c) - p(b))(b-a) - (p(b) - p(a))(c-b)}{(c-a)(c-b)(b-a)} \\ &= \frac{1}{c-a} \left[\frac{p(c) - p(b)}{c-b} - \frac{p(b) - p(a)}{b-a} \right]. \end{aligned}$$

This method can be generalized for polynomials f of higher degree. Suppose that we have numbers a_1, a_2, \dots and that we want a polynomial f such that $f(a-1) = b_1, f(a_2) = b_2, \dots$. We can systemize matters by calculating what are known as *divided differences*. With $f(a_1) = b_1$ and $f(a_2) = b_2$, then the divided difference of f for these two points is given by

$$\Delta(a_1, a_2; b_1, b_2) = \frac{b_2 - b_1}{a_2 - a_1}.$$

For brevity, we may write this as $\Delta(a_1, a_2)$ when the values of b_1 and b_2 are understood. Observe that $\Delta(a_1, a_2) = \Delta(a_2, a_1)$.

The second order divided difference is defined with reference to three points in the domain of f :

$$\Delta^2(a_1, a_2, a_3; b_1, b_2, b_3) = \frac{1}{a_3 - a_1} [\Delta(a_2, a_3; b_2, b_3) - \Delta(a_1, a_2; b_1, b_2)].$$

There are many interesting equivalences for divided differences. For example, we see that

$$\Delta^2(a_1, a_2, a_3) = \frac{1}{a_3 - a_2} [\Delta(a_1, a_3) - \Delta(a_1, a_2)].$$



To check that this is useful, consider the polynomial:

$$f(x) = b_1 + \Delta(a_1, a_2)(x - a_1) + \Delta^2(a_1, a_2, a_3)(x - a_1)(x - a_2) .$$

We have already noted that $f(a_1) = b_1$ and $f(a_2) = b_2$. Also,

$$\begin{aligned} f(a_3) &= b_1 + \Delta(a_1, a_2)(a_3 - a_1) + \Delta^2(a_1, a_2, a_3)(a_3 - a_1)(a_3 - a_2) \\ &= b_1 + (a_3 - a_1) \left[\Delta(a_1, a_2) + \frac{(\Delta(a_2, a_3) - \Delta(a_1, a_2))(a_3 - a_2)}{a_3 - a_1} \right] \\ &= b_1 + [\Delta(a_1, a_2)(a_3 - a_1) + \Delta(a_2, a_3)(a_3 - a_2) - \Delta(a_1, a_2)(a_3 - a_2)] \\ &= b_1 + [\Delta(a_1, a_2)(a_2 - a_1) + \Delta(a_2, a_3)(a_3 - a_2)] \\ &= b_1 + (b_2 - b_1) + (b_3 - b_2) = b_3 . \end{aligned}$$

If we want to additionally make $f(a_4) = b_4$, we can define

$$\begin{aligned} &\Delta^3(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4) \\ &= \frac{1}{a_4 - a_1} [\Delta^2(a_2, a_3, a_4; b_2, b_3, b_4) - \Delta^2(a_1, a_2, a_3; b_1, b_2, b_3)] . \end{aligned}$$

Let

$$\begin{aligned} f(x) &= b_1 + \Delta(a_1, a_2)(x - a_1) + \Delta^2(a_1, a_2, a_3)(x - a_1)(x - a_2) \\ &\quad + \Delta^3(a_1, a_2, a_3, a_4)(x - a_1)(x - a_2)(x - a_3) . \end{aligned}$$

Then $f(a_4)$ is equal to

$$\begin{aligned} &b_1 + \Delta(a_1, a_2)(a_4 - a_1) + \Delta^2(a_1, a_2, a_3)(a_4 - a_1)(a_4 - a_2) \\ &\quad + \Delta^3(a_1, a_2, a_3)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) \\ &= b_1 + \Delta(a_1, a_2)(a_4 - a_1) + (a_4 - a_1)(a_4 - a_2) \left[\Delta^2(a_1, a_2, a_3) + \right. \\ &\quad \left. \left(\frac{\Delta^2(a_2, a_3, a_4) - \Delta^2(a_1, a_2, a_3)}{a_4 - a_1} \right) (a_4 - a_3) \right] \\ &= b_1 + \Delta(a_1, a_2)(a_4 - a_1) + (a_4 - a_2) [\Delta^2(a_2, a_3, a_4)(a_4 - a_3) + \\ &\quad \Delta^2(a_1, a_2, a_3)(a_3 - a_1)] \\ &= b_1 + \Delta(a_1, a_2)(a_4 - a_1) + \\ &\quad (a_4 - a_2) \left[\frac{\Delta(a_3, a_4) - \Delta(a_2, a_3)}{a_4 - a_2} (a_4 - a_3) + \Delta(a_2, a_3) - \Delta(a_1, a_2) \right] \\ &= b_1 + \Delta(a_1, a_2)(a_4 - a_1) + \Delta(a_3, a_4)(a_4 - a_3) - \Delta(a_2, a_3)(a_4 - a_3) + \\ &\quad \Delta(a_2, a_3)(a_4 - a_2) - \Delta(a_1, a_2)(a_4 - a_2) \\ &= b_1 + \Delta(a_1, a_2)(a_2 - a_1) + \Delta(a_2, a_3)(a_3 - a_2) + \Delta(a_3, a_4)(a_4 - a_3) \\ &= b_1 + (b_2 - b_1) + (b_3 - b_2) + (b_4 - b_3) = b_4 . \end{aligned}$$

Thus, the function satisfies $f(a_i) = b_i$ for $1 \leq i \leq 4$.



In an analogous way, we can define divided differences of higher order and extend the interpolating polynomial indefinitely to accommodate as many interpolation points as we require.

$$1.8. (a) (x^2 + x + 1)[(x + 1)^2 + (x + 1) + 1] = (x + 1)^4 + (x + 2)^2 + 1.$$

$$1.8. (c) x(x + 1) \cdot (x + 1)(x + 2) = (x^2 + 2x)(x^2 + 2x + 1).$$

1.9. (a) $p(0)p(1) = c + bc + c^2 = p(c) = p(p(0))$. The equation $p(x) = p(c)$ has two solutions $x = c$ and $x = -b - c$.

1.9. (b) Let $q(x) = p(n + x)$. Then

$$p(n)p(n + 1) = q(0)q(1) = q(q(0)) = p(n + q(0)) = p(n + p(n)) .$$

1.10. Let $p(x) = rx^2 + tx + w$. Then $p(a) - p(b) = 0$ leads to $r(a + b) + t = 0$, so that $p(x) = rx^2 - r(a + b)x + w$.

1.11. Because $f(x)$ is quadratic, it cannot be the case either that all of $f(1)$, $f(2)$, $f(3)$ are equal or that all of $f(1)$, $f(2)$, $f(3)$ are distinct. There are three possibilities: (i) $f(1) = f(2) = u$, $f(3) = v$; (ii) $f(1) = f(3) = u$, $f(2) = v$; (iii) $f(1) = u$, $f(2) = f(3) = v$, for distinct values of u and v . In the solution, we make use of the result of Exercise 1.10.

(i) We must have that $f(x) = 4a(x - \frac{3}{2})^2 + b$ for some constants a and b , from which $f(1) = f(2) = a + b$ and $f(3) = 9a + b$. Therefore, from Exercise 1.10, $3 = f(1) + f(3) = 10a + 2b$. This is a necessary and sufficient condition that f takes the same values at $f(1)$ and $f(3)$. Thus, $f(x)$ must have the form

$$f(x) = 4a\left(x - \frac{3}{2}\right)^2 + \left(\frac{3}{2} - 5a\right) = 4ax^2 - 12ax + 4a + \frac{3}{2} .$$

(ii) We must have that $f(x) = a(x - 2)^2 + b$ for some constants a and b . Then $u = a + b$ and $v = b$, so that $a + 2b = 4$. Thus $f(x) = (4 - 2b)(x - 2)^2 + b$. A particular example is $f(x) = 2(x - 2)^2 + 1 = 2x^2 - 8x + 9$.

1.12. Let x be a number all of whose digits are 1. Then x must have the form $(1/9)(x^n - 1)$ for some positive integer n . We calculate the square of this in an attempt to get something that involves x . It can be checked that $x^2 = (10^{2n} - 1)/81 - 2x/9$. The desired polynomial is $9x^2 + 2x$.

§2. Exercises on polynomials of higher degree

The properties outlined for quadratics in Section 1 can be generalized to polynomials of higher degree. This section will briefly introduce them.

2.1. Formulate and prove a generalization to Exercise 1.4 to give an expression for a polynomial p of degree less than n that assumes the values $p(a_i)$ at the distinct numbers a_i ($1 \leq i \leq n$).



2.2. The Remainder Theorem. Let $f(x)$ be a polynomial of degree $n \geq 2$ and let a be a real number. Suppose that $f(x)$ be divided by $x - a$ with remainder r :

$$f(x) = g(x)(x - a) + r ,$$

where $g(x)$ is a polynomial of degree $n - 1$. Prove that $r = f(a)$.

2.3. The Factor Theorem. Prove that a is a root of the polynomial $f(x)$ (i.e., $f(a) = 0$) if and only if $f(x)$ is divisible by the polynomial $x - a$, i.e.,

$$f(x) = (x - a)g(x)$$

for some polynomial $g(x)$.

2.4. Suppose that $f(x)$ is a polynomial of degree exceeding 2, and that $f(x)$ be divided by $(x - a)(x - b)$ where $a \neq b$. Prove that the remainder is

$$\frac{f(b)}{b - a}(x - a) + \frac{f(a)}{a - b}(x - b) .$$

2.5. Formulate and prove a generalization of Exercise 2.4 when the polynomial $f(x)$ of degree at least n is divisible by $(x - a_1)(x - a_2) \cdots (x - a_n)$.

2.6. In this exercise, we will generalize the method of Exercise 1.6 in the special case that we construct the polynomial of smallest degree that assumes specified values $f(0), f(1), f(2), \dots$. In this case, we define the ordinary (undivided) differences by

$$\Delta^0 f(n) \equiv If(n) = f(n)$$

$$\Delta^1 f(n) = \Delta f(n) = f(n + 1) - f(n)$$

$$\Delta^2 f(n) = \Delta f(n + 1) - \Delta f(n) = f(n + 2) - 2f(n + 1) + f(n)$$

and generally

$$\Delta^k f(n) = \Delta^{k-1} f(n + 1) - \Delta^{k-1} f(n)$$

for $k \geq 2$. There is a useful operational calculus that we can bring into play. Define $Ef(n) = f(n + 1)$. Then $E^k f(n) = f(n + k)$ and $\Delta = (E - I)$, so that $\Delta^k = (E - I)^k$. From the above, we see that

$$\begin{aligned} \Delta^2 f(n) &= f(n + 2) - 2f(n + 1) + f(n) = E^2 f(n) - 2Ef(n) + If(n) \\ &= (E - I)^2 f(n) . \end{aligned}$$

By means of this *operational calculus*, we have the formula

$$\Delta^k f(n) = \sum_{j=0}^k (-1)^j \binom{k}{j} f(n + k - j) .$$



We can treat polynomials in I , E and Δ as satisfying the regular rules of algebra. In particular

$$f(n) = E^n f(0) = (I + \Delta)^n f(0) = \sum_{i=0}^n \binom{n}{i} \Delta^i f(0),$$

for each positive integer k , facts that can be verified directly by unpacking the operational notation.

We define the *factorial power* of x by

$$x^{(r)} = x(x-1)(x-2) \cdots (x-r+1)$$

for each integer $r \geq 1$, with $x^{(0)} = 1$.

- (a) Verify that that $\Delta x^{(r)} = r x^{(r-1)}$ for each nonnegative integer r .
- (b) Deduce that f is a polynomial of degree n if and only if $\Delta^n f(x)$ is a nonzero constant and $\Delta^{n+1} f(x)$ is identically 0.
- (c) Prove the identity

$$\sum_{k=0}^n (-1)^k \binom{n}{k} f(x+k) = 0$$

for all x whenever $f(x)$ is a polynomial of degree strictly less than n .

Let $f(x)$ be a polynomial. Establish that

$$f(x) = (I + \Delta)^x f(0) = \sum_{j=0}^{\infty} \binom{x}{j} \Delta^j f(0),$$

where $\binom{x}{j} = x(x-1) \cdots (x-j+1)/j! = x^{(j)}/j!$, and the series terminates after a finite number of terms. When the differences at some point are known, this gives us another way of representing a polynomial. Check this formula on various polynomials of your choice. For example, use this formula and a difference table to find the formula for the sum of the fourth powers of integers from 1 to x inclusive.

2.7. Prove that, if a polynomial of degree n takes integer values at $n+1$ consecutive integers, it takes integer values at every integer and is a sum of polynomials of the form $r \binom{x}{k}$, for constant r and nonnegative integer k . Thus, every polynomial in x taking integer values when x is an integer has the form $\sum_{k=0}^n a_k \binom{x}{k}$.

Comment

2.4, 2.5. Observe that, if $f(x) = (x-a_1)(x-a_2) \cdots (x-a_k)g(x) + h(x)$, where the degree of $h(x)$ is less than k , then $h(x)$ is a uniquely determine polynomial whose values agree with those of $f(x)$ when $x = a_i$ for $(1 \leq i \leq k)$.



§3. Exercises on Completing the Square and Transformations

3.1. Let $f(x)$ be a real-valued function.

- (a) Compare the graphs of the equations $y = f(x)$ and $y = f(2x)$.
- (b) Compare the graphs of the equation $y = f(x)$ and $y = 4f(x)$.
- (c) Corroborate your answers to parts (a) and (b) with the example $f(x) = x^2$.

3.2. Let $ax^2 + bx + c$ be a quadratic polynomial with real coefficients.

- (a) Verify that it equals

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{1}{4a}(b^2 - 4ac)$$

- (b) From (a), argue that, when $a > 0$, the quadratic assumes its minimum value when $x = -b/2a$, while if $a < 0$, it assumes its maximum value when $x = -b/2a$.

- (c) Use this to describe a transformation in the plane that takes the graph of the equation

$$y = ax^2 + bx + c$$

to the graph of the equation $y = x^2$.

3.3. One geometric definition of a *parabola* is that it is the locus of points whose distance from a fixed point (the focus) is equal to its distance from a fixed line (the directrix). It is asserted that the graph of the equation $y = ax^2 + bx + c$ is a parabola. Is this consistent with the geometric definition? If so, what is the focus? What is the directrix? Look first at some special cases, such as $y = x^2$, $y = x^2 + c$, $y = ax^2$, $y = x^2 - 3x + 2$.

3.4. Is it true that all parabolas are the same shape? Explain.

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3.2. (c) The required transformation is the composite of a vertical translation of $(b^2 - 4ac)/4a$ upwards, a horizontal translation to the right of $b/2a$ and a vertical dilatation (that leaves points on the x -axis fixed) with factor $1/a$.

3.3. For the parabola with equation $y = x^2$, the focus is at $(0, \frac{1}{4})$ and the directrix is the line $y = -\frac{1}{4}$.

3.4. Yes. Any pair consisting of a line and a point not on the line can be carried to any other such pair by a similarity transformation, that preserves angles and relative distances.



§4. Exercises on Solutions of Quadratics

4.1. Suppose that $a \neq c$ and that $x = (b - d)/(a - c)$ satisfies one of the equations $x^2 - ax + b = 0$ and $x^2 - cx + d = 0$. Prove that this value of x satisfies the other.

4.2. Let $p(x)$ and $q(x)$ be two quadratic polynomials with integer coefficients. Suppose that there is an *irrational* number c for which $p(c) = q(c) = 0$. Prove that one of the polynomials $p(x)$ and $q(x)$ is a constant multiple of the other.

4.3. Let $p(x) = x^2 + bx + c$. Suppose that $p(0)$ and $p(1)$ are solutions of the quadratic equation $p(x) = 0$. What are the possible values of the pair (b, c) ?

4.4. (a) Show that, for every quadratic equation $(x - p)(x - q) = 0$, there exist constants a, b, c with $c \neq 0$ such that $(x - b)(b - x) = c$ is equivalent to the original equation and the following reasoning “either $x - a$ or $b - x$ must equal to c ” yields the correct answers “ $x = p$ or $x = q$ ”.

(b) Determine constants a, b, c with $c \neq 0$ so that the equation

$$(x - 19)(97 - x) = 0$$

can be “solved” in this manner.

[Round 25 of the International Mathematical Talent Search.]

4.5. Suppose x and y are integers. Solve the equation

$$x^2y^2 - 7x^2y + 12x^2 - 21xy - 4y^2 + 63x + 70y - 174 = 0.$$

[Problem 2332 from *CruX Mathematicorum*.]

4.6. Two nested concentric rectangles, are given, with corresponding sides parallel and each side of the inner rectangle the same distance from the corresponding side of the outer.

(a) Prove that, if the area of the inner rectangle is exactly half that of the outer rectangle, then the perimeter of the inner rectangle is equal to the sum of the lengths of the diagonals of the outer rectangle.

(b) Verify the result in (a) when the outer rectangle has dimensions 3×4 , 8×15 and more generally $(m^2 - n^2) \times 2mn$ where m and n are positive integers.

4.7. The illegal moves method for quadratics.

One method for solving quadratics is to factor them by what is known as the “illegal moves method”. This is described below and you are invited to analysis it and see whether or not it is legitimate and how it could be justified.

Suppose that $ax^2 + bx + c$ has to be factored, where a, b, c are integers with no common factors save 1, a is neither 0 nor 1, and the discriminant $b^2 - 4ac$ is the square of an integer, so that the roots of the quadratic are rational.



Follow these steps:

1. Remove the coefficient of the x^2 term by multiplying it into the constant term. This produces $x^2 + bx + ac$.
2. Now factor $x^2 + bx + ac$. Since this factors over the integers, there exist integers b_1 and b_2 for which $x^2 + bx + ac = (x + b_1)(x + b_2)$.
3. Next, divide each of the constants by a : $(x + b_1/a)(x + b_2/a)$.
4. Reduce the fractions to lowest terms:

$$(x + b_1/a)(x + b_2/a) = (x + u_1/v_1)(x + u_2/v_2) .$$

5. "Squeeze" the denominator of each fraction in front of the binomials to get the desired factorization: $(v_1x + u_1)(v_2x + u_2)$.

Here is how the method goes on an example: factor $6x^2 - 7x - 3$.

1. Remove the 6 and multiply it into the constant term: $x^2 - 7x - 18$.
2. Factor the new trinomial: $(x - 9)(x + 2)$.
3. Divide each constant by 6: $(x - \frac{9}{6})(x + \frac{2}{6})$.
4. Reduce fractions to lowest terms: $(x - \frac{3}{2})(x + \frac{1}{3})$.
5. Finally, move each denominator to the leading term: $(2x - 3)(3x + 1)$, which is the factored form of $6x^2 - 7x - 3$.

Comments, Answers and Solutions

- 4.1. For a quick argument, note that the stated value of x satisfies the difference of the two equations.
- 4.2. Let a and c be the roots of $p(x) = 0$ and let b and c be the roots of $q(x) = 0$. Observe that $a - b = (a + c) - (b + c)$ and $(a - b)0c = ac - bc$ must both be rational.
- 4.3. Observe that $p(p(0)) = 0$ if and only if $c(c + b + 1) = 0$. Let $c = 0$. Then $0 = p(p(1))$ if and only if $(1 + b)^2 + b(1 + b) = 0$ or $b = 0$ and $b = -\frac{1}{2}$. Suppose that $b + c = -1$, so that

$$p(x) = x^2 + bx - (b + 1) = (x - 1)(x + b + 1) = (x - 1)(x - c) .$$

Then $p(1) = 0$ is a solution of the equation $p(x) = 0$ if and only if $b = -1$ or $c = 0$.

- 4.4. (a) The two equivalent equations are $x^2 - (p + q)x + pq = 0$ and $x^2 - (a - b)x + (ab + c) = 0$, so that $p + q = a + b$ and $pq = ab + c$. Since $x = a + c$ and $x = b - c$ satisfy the equations, we may take $p = a + c$ and $q = b - c$. Thus $c = pq - ab = (b - a)c - c^2$, whence $c = b - a - 1$. Therefore $p = b - 1$ and $q = a + 1$. We may take $(a, b, c) = (q - 1, p + 1, p - q + 1)$.



4.4. (b) (a, b, c) can be either $(96, 20, -77)$ or $(18, 09, 79)$.

4.5. The equation can be rewritten

$$\begin{aligned}
 0 &= x^2(y^2 - 7y + 12) - 21x(y - 3) - 2(2y^2 - 35y + 57) \\
 &= (y - 3)[(y - 4)x^2 - 21x - 2(2y - 29)] \\
 &= (y - 3)[y(x^2 - 4) - (4x^2 + 21x - 58)] \\
 &= (y - 3)(x - 2)[y(x + 2) - (4x + 29)] \\
 &= (y - 3)(x - 2)(xy + 2y - 4x - 29) \\
 &= (y - 3)(x - 2)[(x + 2)(y - 4) - 21] .
 \end{aligned}$$

The solutions are

$$\begin{aligned}
 (x, y) &= (s, 3), (2, t), (19, 5), (5, 7), (1, 11), \\
 &\quad (-1, 25), (-3, -17), (-5, -3), (-9, 1), (-23, 3) ,
 \end{aligned}$$

where s and t are arbitrary.

4.6. (a) Let $x \times y$ be the dimensions of the outer rectangle and z the difference between corresponding sides of the two rectangles. Then $2(x - z)(y - z) = xy$ implies that

$$9 = xy - 2(x + y)z + 2z^2 = \frac{1}{2}[(x + y - 2z)^2 - (x^2 + y^2)]$$

from which the result follows.

4.6.(b) The inner rectangle has dimensions $(m^2 - mn) \times (n^2 + mn)$.

4.7. Both the equations $ax^2 + bx + c = 0$ and $x^2 + bx + ac = 0$ yield the same numerator in the quadratic formula for their roots so that the roots of the former are equal to $1/a$ times the roots of the latter; from this, it is easy to see how a factorization of the second quadratic induces a factorization of the first. It may be that this comparison of the roots of the two quadratic equations is what led to the promulgation of the method in the first place.

§5. Exercises on Inequalities

5.1. Let a and b be positive real numbers. Using the fact that the quadratic equation $0 = (x - a)(x - b) = x^2 - (a + b)x + ab$ has real roots and the discriminant condition, verify the Arithmetic-Geometric Means Inequality

$$\sqrt{ab} \leq \frac{a + b}{2} .$$

When does equality occur?

5.2. (a) Suppose that a, b, c, u, v, w are real numbers. Using the fact that the quadratic polynomial

$$\begin{aligned}
 &(ax + u)^2 + (bx + v)^2 + (cx + w)^2 \\
 &= (a^2 + b^2 + c^2)x^2 + 2(au + bv + cw)x + (u^2 + v^2 + w^2)
 \end{aligned}$$



is always nonnegative, argue that it has either coincident real roots or nonreal roots. Use the discriminant condition for this to obtain the Cauchy-Schwarz Inequality

$$au + bv + cw \leq (a^2 + b^2 + c^2)^{\frac{1}{2}}(u^2 + v^2 + w^2)^{\frac{1}{2}} .$$

When does equality hold?

(b) Generalize (a) to obtain an inequality for a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n .

5.3. Consider the function

$$f(x) = x^2(1 - x) .$$

(a) Verify that $f(x) \geq 0$ when $x \leq 1$ and $f(x) < 0$ when $x > 1$. What do you think the graph of the equation $y = f(x)$ looks like?

(b) Argue that, if $0 \leq u \leq 1$, then the equation $f(x) = f(u)$ has three real solutions, one of which is $x = u$.

(c) On the interval $\{x : 0 \leq x \leq 1\}$, the function $f(x)$ assumes a positive maximum value m . Argue that the equation $f(x) = k$ has exactly one real solution when $k < 0$ or $k > m$, and exactly three real solutions when $0 < k < m$. You can see this graphically by examining how lines of equation $y = k$ intersect the graph of $y = f(x)$. Observe that the lines $y = 0$ and $y = m$ are tangent to the graph of $y = f(x)$.

(d) Let u be any real number. By the factor theorem, $f(x) - f(u) = (x - u)g(x)$ for some polynomial $g(x)$. Determine this polynomial $g(x)$. Show that $g(u) = 0$ if and only if $u = 0$ and $u = \frac{2}{3}$.

(e) Suppose that $f(u) = m$. Then the cubic equation $f(x) - f(u) = 0$ has u as a double root in the sense that as k approaches m from below, two of the intersection points of the line $y = k$ with the graph of $y = f(x)$ coalesce into a single point, the point of tangency. Explain why

$$f(x) \leq f(2/3) = 4/27$$

for $0 \leq x \leq 1$. Thus, $4/27$ is the maximum value assumed by $f(x)$ when $0 \leq x \leq 1$.

(f) As a check on the conclusion of (e), factor the polynomial

$$\frac{4}{27} - f(x)$$

and show that it is non-negative when $x \geq -1/3$.

§6. Exercises on Sum and Product of Roots

6.1. The roots of the quadratic equation $x^2 + bx + c = 0$ are m and n . Verify that b and c satisfy the quadratic equation

$$x^2 + (m + n - mn)x - mn(m + n) = 0 .$$

6.2. (a) Determine any solution in positive integers to the diophantine equation

$$x^2 + y^2 + z^2 + w^2 = xyzw .$$



(b) It is possible to show that the diophantine equation in (a) has infinitely many solutions in positive integers by the following argument. Suppose that we have found a solution $(x, y, z, w) = (a, b, c, d)$. Consider the quadratic equation

$$x^2 - bcdx + (b^2 + c^2 + d^2) = 0 \quad .$$

One root of this equation is the integer a . Argue that there is a second root a' which is also an integer. Show that $(x, y, z, w) = (a', b, c, d)$ is another solution of the equation in (a). Use this strategy, starting with the solution you found in (a), to obtain a sequence of different solutions to the equation.

Comments, Answers and Solutions

6.2. You can start with the solution $(x, y, z, w) = (2, 2, 2, 2)$.

§7. More Exercises on Polynomials of Higher Degree

7.1. Consider the equation $x^4 - 2x^3 - x^2 - 2x + 1 = 0$. Since the coefficients $(1, -1, -2, -1, 1)$ are symmetric about the middle one, it turns out that there is a special method for solving such an equation which reduces to the solution of quadratic equations.

(a) Prove that, if the equation has a nonzero solution $x = u$, then $x = 1/u$ also satisfies the equation.

(b) Verify that $x = 0$ does not satisfy the equation. Deduce that the equation is equivalent to

$$\left(x^2 + \frac{1}{x^2}\right) - 2\left(x + \frac{1}{x}\right) - 1 = 0 \quad .$$

(c) Set $t = x + \frac{1}{x}$ and verify that $x^2 + \frac{1}{x^2} = t^2 - 2$. Verify that the equation, with this substitution, becomes $t^2 - 2t - 3 = 0$. Solve for t .

(d) Solve the equations $x + \frac{1}{x} = -1$ and $x + \frac{1}{x} = 3$, and argue that the solutions to these two equations satisfy the original equation.

(e) Use the result in (c) to obtain a factorization of $x^4 - 2x^3 - x^2 - 2x + 1$ as a product of two quadratic polynomials.

7.2. (a) Write down several examples of products of four consecutive integers, such as $3 \times 4 \times 5 \times 6 = 360$.

(b) Observe that in each case the result is not a square. Why do you think this is?

(c) Extending the observation in (b), it appears on the basis of numerical evidence that the product of four consecutive integers is 1 less than a perfect square. This suggests that we might introduce variables to check the truth of this in general. What is the general form for the product of consecutive integers?

(d) Consider $f(x) = x(x+1)(x+2)(x+3)$. Rewriting the terms (think why one might want to do this) thus, $f(x) = [x(x+3)][(x+1)(x+2)]$, verify that



$(x+1)(x+2) = x(x+3) + 2$ and so

$$f(x) = [x(x+3)]^2 + 2[x(x+3)]$$

and use this to show that $f(x) + 1$ is the square of a quadratic polynomial. What is this quadratic polynomial?

(e) Some might prefer to represent the product of four consecutive integers are $g(x) = (x-1)x(x+1)(x+2)$. Is this equally valid? Why might one choose this form? Prove that $g(x) + 1$ is equal to the square of a quadratic polynomial.

7.3. In his paper, *Recherches sur les racines imaginaires des équations*, published in *Mem. de l'academie des sciences de Berlin* (5) (1749), 1751, 222-288 = *Opera omnia* (1) 6, 78-141, Leonard Euler (1707-1783) presents what turns out to be a subtly incorrect proof of a version of the Fundamental Theorem of Algebra, that each polynomial with real coefficients can be written as a product of linear and quadratic polynomials with real coefficients. However, his argument works in the case of a quartic polynomial $h(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$ with $A \neq 0$.

(a) Prove that $h(x)$ can be factored as a product of quadratic polynomials if and only if $h(kx)$ and $h(x-k)$ can be so factored for any nonzero constant k . [Hint: If $h(x) = f(x)g(x)$ is an identity in x , what happens if you replace x by kx and $x-k$?

(b) Let $k = -B/4A$. Verify that the coefficient of x^3 in $h(x+k)$ is 0.

(c) From (a) and (b), argue that, without loss of generality, it is enough to prove that any polynomial of the form

$$h(x) = x^4 + ax^2 + bx + c$$

can be factored as a product of real quadratics.

Henceforth, we will suppose that $h(x)$ has this form.

(d) Suppose that $b = 0$ so that $h(x) = x^4 + ax^2 + c$. Let $a^2 \geq 4c$. Use the theory of the quadratic to prove that $h(x)$ can be written as a product of the form $(x^2 - r)(x^2 - s)$ for real values of r and s .

(e) Suppose that $b = 0$ and that $a^2 < 4c$. Verify that $c > 0$ and that

$$x^4 + ax^2 + c = (x^2 + \sqrt{c})^2 - (2\sqrt{c} - a)x^2$$

so that $h(x)$ can be factored as a difference of squares.

(f) We now turn to the case

$$h(x) = x^4 + ax^2 + bx + c,$$

where $b \neq 0$. The polynomial $h(x)$ can be factored as a product of quadratics if and only if real numbers u, v, w can be found for which

$$x^4 + ax^2 + bx + c = (x^2 + ux + v)(x^2 - ux + w).$$



By expanding the right side and comparing coefficients on the two sides of the equation, obtain the set of conditions

$$\begin{aligned}a &= v + w - u^2 \\b &= u(w - v) \\c &= vw ,\end{aligned}$$

which, in turn, are equivalent to

$$\begin{aligned}w + v &= a + u^2 \\w - v &= \frac{b}{u} \\4vw &= 4c .\end{aligned}$$

Thus, if we can find a suitable real value of u , then the real values of v and w can be obtained from the first two of these equations and we can write out the desired factorization. Verify that

$$\begin{aligned}2w &= u^2 + a + \frac{b}{u} \\2v &= u^2 + a - \frac{b}{u}\end{aligned}$$

and thus show that u must satisfy

$$u^6 + 2au^4 + (a^2 - 4c)u^2 - b^2 = 0 .$$

(g) In (f), it suffices to show that the sextic equation is satisfied by some real value of u . One way to do this is through a result called the *Intermediate Value Theorem* for continuous functions, which applies in particular to polynomials. Let

$$f(x) = x^6 + 2ax^4 + (a^2 - 4c)x^2 - b^2 = x^6(1 + 2ax^{-2} + (a^2 - 4c)x^{-4} - b^2x^{-6}) .$$

Verify that $f(0) < 0$ and that $f(x)$ is positive for very large values of x . The graph of $f(x)$ is a continuous curve which lies below the x -axis when $x = 0$ but lies above the axis when x is large. Deduce that it must cross the axis somewhere, and so there must be a real number u such that $f(u) = 0$.

(h) Here is an erroneous argument to show that $f(0) < 0$. (It was this approach that got Euler into trouble with polynomials of higher degree.) Can you spot the difficulty? As above, try

$$x^4 + ax^2 + bx + c = (x^2 + ux + v)(x^2 - ux + w) .$$

Each of the quadratics can be factored as a product of linear polynomials, so that

$$x^4 + ax^2 + bx + c = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta) .$$

By comparing coefficients, verify that $\alpha + \beta + \gamma + \delta = 0$, and that u is the sum of two of the roots. There are six ways of pairing the roots and the correspond to six possible values of u :

$$\alpha + \beta , \quad \alpha + \gamma , \quad \alpha + \delta ,$$



$$\beta + \gamma, \quad \beta + \delta, \quad \gamma + \delta.$$

Observe that, for any possible value of u , its negative is also a possible value of u , so that the sextic equation $f(x) = 0$ satisfied by u has the form

$$f(u) \equiv (u^2 - \lambda^2)(u^2 - \mu^2)(u^2 - \nu^2) = 0.$$

The left side has an odd number ($\frac{1}{2}\binom{4}{2} = 3$) of terms, and so its constant coefficient, $f(0)$, being the product of three squares, must be negative.

Comments, Answers and Solutions

7.1 (c) $t = -1, 3$.

§8. Exercises on Rational Functions

8.1. Let $f(x) = (x^2 + 2x + 2)/(x + 1)$ be defined for all real values of x not equal to -1 .

(a) By considering the solvability of the quadratic equation

$$x^2 + 2x + 2 = k(x + 1),$$

prove that $f(x)$ cannot assume any value strictly between -2 and 2 but that it can assume all other real values.

(b) By considering the signs of the expressions $f(x) - 2$ and $f(x) + 2$, corroborate the result of (a).

(c) Use a calculator to obtain the graph of $y = f(x)$. Does this validate (a) and (b)?

(d) Verify that $f(x) = x + 1 + \frac{1}{x+1}$. Use this representation to obtain a rough sketch of the graph of $y = f(x)$. Does this agree with (c)? Describe the asymptotes of the graph.

8.2. Use the techniques of Exercise 8.1 to analyze the range of values and the graphs of the following rational functions:

(a)

$$\frac{x^2 + x + 4}{x + 1}$$

(b)

$$\frac{x^2 + 4x - 4}{x + 2}.$$

8.3. Let a, b, c be parameters and let

$$f(x) = \frac{x^2 + bx + c}{x + a}.$$

Assume that $x + a$ is not a factor of $x^2 + bx + c$. We are concerned with conditions on a, b, c under which each real number can be written in the form $f(x)$ for some real x , *i.e.*, for each real k , $f(x) = k$ is solvable. Three approaches will be followed in this and the next two problems.



(a) Verify that $f(x) = k$ is equivalent to

$$x^2 + (b - k)x + (c - ak) = 0 .$$

(b) Verify that the discriminant of the quadratic equation in (a) is

$$k^2 - 2(b - 2a)k + (b^2 - 4c) = [k - (b - 2a)]^2 - 4(a^2 - ab + c) .$$

(c) Using (b), prove that $f(x) = k$ is solvable for *each* real value of k if and only if $a^2 - ab + c < 0$.

(d) Prove that, if $f(x) = k$ is solvable for each real value of k , then $x^2 + bx + c = 0$ must have real roots. Is the converse of this result true?

8.4. Let $f(x)$ be the function of Exercise 8.3.

(a) Verify that

$$\begin{aligned} f(x) &= \frac{x^2 + bx + c}{x + a} \\ &= x + (b - a) + \frac{c + a^2 - ab}{x + a} \\ &= (x + a) + \frac{c + a^2 - ab}{x + a} + (b - 2a) . \end{aligned}$$

(b) Suppose that $c + a^2 - ab > 0$ and that $x > -a$. Verify that

$$f(x) \geq 2\sqrt{c + a^2 - ab} + (b - 2a)$$

with equality if and only if $x + a = \sqrt{c + a^2 - ab}$.

(c) Suppose that $c + a^2 - ab > 0$ and that $x < -a$. Verify that

$$f(x) \leq -2\sqrt{c + a^2 - ab} + (b - 2a)$$

with equality if and only if $x + a = -\sqrt{c + a^2 - ab}$.

(d) Deduce from (f) and (g) that $f(x) = k$ is not solvable when $c + a^2 - ab > 0$ and

$$b - 2a - 2\sqrt{c + a^2 - ab} < k < b - 2a + 2\sqrt{c + a^2 - ab} .$$

(e) Suppose that $c + a^2 - ab < 0$. Argue that, as x increases from $-a$, then $f(x)$ passes through all real values. Similarly argue that as x decreases from $-a$, then $f(x)$ passes through all real values. Observe that this result, along with (d), corroborates the result of Exercise 8.3(c).

8.5. Let $f(x)$ be the function of Exercise 8.3.

(a) Suppose the roots r, s of $x^2 + bx + c = 0$ are both real with $r \leq s$. Observe that $x^2 + bx + c < 0$ if and only if $r < x < s$.



(b) Suppose that r and s lie on the same side of $-a$. Without loss of generality, let $-a < r \leq s$. Argue that, when $x > -a$, $f(x)$ must assume a minimum value, say m_2 when $x = x_2$, while if $x < -a$, then $f(x)$ must assume a maximum value, say m_1 when $x = x_1$, where $m_1 < 0$. We will argue that $m_1 < m_2$ so that $f(x)$ cannot assume any value between m_1 and m_2 . Consider $f(x) - m_2$; this is a quadratic polynomial that vanishes when $x = x_2$ and is nonnegative when $x > -a$. Deduce that

$$f(x) - m_2 = \frac{(x - x_2)^2}{x + a}$$

so that $f(x)$ assumes the value m_2 only when $x = x_2$, and so never when $x < -a$. Conclude that $f(x)$ cannot assume any value between m_1 and m_2 .

(c) Suppose that r and s lie on opposite sides of $-a$, so that $r < -a < s$. Prove that $f(x) > 0$ for $r < x < -a$ and that $f(x) < 0$ for $-a < x < s$. Indeed, show that $f(x)$ passes through all real values as x increases from or decreases from $-a$.

(d) Deduce from (a) and (b) that $f(x) = k$ is solvable for all real k if and only if $r < -a < s$. Using the fact that $r = \frac{1}{2}(-b - \sqrt{b^2 - 4c})$ and $s = \frac{1}{2}(-b + \sqrt{b^2 - 4c})$, show that this is equivalent to $a^2 - ab + c < 0$.

(e) Suppose that the equation $x^2 + bx + c = 0$ has nonreal roots. Observe that this has two consequences: $4c > b^2$, and $f(x)$ never assumes the value 0. Deduce that

$$a^2 - ba + c > (a - \frac{b}{2})^2 \geq 0.$$

(f) Using the results of parts (a) to (e), prove that $f(x) = k$ is solvable for each real value of k if and only if $a^2 - ba + c < 0$.

Comments, Answers and Solutions

8.1. (a) The discriminant of the quadratic equation is

$$(2 - k)^2 - 4(2 - k) = k^2 - 4,$$

which is negative when $|k| < 2$. Hence the rational function does not assume these values of k .

8.1. (b) $f(x) - 2 = x^2(x + 1)^{-1} \geq 0$ for $x > -1$
and $f(x) + 2 = (x + 2)^2(x + 1)^{-1} \leq 0$ for $x < -1$.

8.1. (d) The asymptotes are the lines $x = -1$ and $y = x + 1$. There is a relative minimum at $(0, 2)$ and a relative maximum at $(-2, -2)$.

8.2. (a) The range of values of $(-\infty, -5] \cup [3, \infty)$.

8.5. (d) $r < -a$ if and only if $-\sqrt{b^2 - 4c} < b - 2a$; $s > -a$ if and only if $\sqrt{b^2 - 4c} > b - 2a$.

These two conditions together are equivalent to $|b - 2a| < \sqrt{b^2 - 4c}$, or $b^2 - 4ab + 4a^2 < b^2 - 4c$.



§9. Exercises on Second Order Recursions

9.1. Suppose that $x = u$ and $x = v$ satisfy the quadratic equation $x^2 = px + q$. Define

$$w_0 = 2, \quad w_1 = u + v, \quad w_2 = u^2 + v^2, \quad w_3 = u^3 + v^3, \quad \dots, \quad w_n = u^n + v^n, \quad \dots$$

(a) Prove that, when $n \geq 2$, then $w_n = pw_{n-1} + qw_{n-2}$.

(b) Check (a) when u and v are the solutions of the equations (i) $x^2 = 3x - 2$ and (ii) $x^2 = 3x + 2$.

(c) Suppose that $x_n = 7u^n - 5v^n$, where u and v are as defined above. Is it true that $x_n = px_{n-1} + qx_{n-2}$ for $n \geq 2$ for some numbers p and q ?

9.2. Let x_n be a sequence satisfying a *second order recursion*. This means that two consecutive terms, say x_0 and x_1 can be chosen arbitrarily, and that there are fixed multipliers p and q such that for all values of n , $x_n = px_{n-1} + qx_{n-2}$.

(a) Write out the first few terms of the following sequences satisfying a second order recursion in each of the following cases:

(i) $x_0 = 0, x_1 = 1, p = q = 1$ (Fibonacci sequence);

(ii) $x_0 = 0, x_1 = 1, p = 2, q = -1$;

(iii) $x_0 = 1, x_1 = 1, p = 2, q = 1$;

(iv) $x_0 = 3, x_1 = -2, p = 1, q = -2$;

(v) $x_0 = 3, x_1 = 2, p = 1, q = -1$.

(b) Verify that a geometric progression $\{a, ar, ar^2, ar^3, \dots\}$ satisfies the recursion $x_n = px_{n-1} + qx_{n-2}$ if and only if r is a solution of the quadratic equation $x^2 = px + q$.

(c) Suppose that the equation $x^2 = px + q$ has two distinct solutions $x = r$ and $x = s$. Let x_0 and x_1 be any two numbers. Solve the system of equations

$$\begin{aligned} y + z &= x_0 \\ ry + sz &= x_1 \end{aligned}$$

for y and z . Prove that, if $\{x_n\}$ satisfies the second order recursion $x_n = px_{n-1} + qx_{n-2}$, then $x_n = yr^n + zs^n$ for each value of n .

(d) Use (c) to obtain the general term of the sequences in (a). Are there any situations in which the method does not work? Why?

9.3. In this exercise, we examine the situation of a second order recursion as described in Exercise 9.2 in which the associated quadratic equation $x^2 = px + q$ has a double solution.

(a) Prove that $x^2 = px + q$ has a double solution if and only if $q = -p^2/4$.



In the notation introduced above, we thus are interested in investigating sequences satisfying a recursion of the type

$$x_n = px_{n-1} - \frac{p^2}{4}x_{n-2} . \quad (*)$$

We cannot proceed as in Exercise 14 to get the general solution of the recursion as we have only a single solution of the related quadratic to manipulate. Suppose that r is this root.

(b) Verify that $r = p/2$.

(c) Define y_n by $x_n = r^n y_n$. By substituting into (*), show that $y_n - y_{n-1} = y_{n-1} - y_{n-2}$ for each value of n . What does this tell us about the nature of the sequence $\{y_n\}$?

(d) Verify that $x_n = (\alpha n + \beta)r^n$ satisfies (*).

(e) Imitate the strategy of Exercise 9.2 to show that every solution to the recursion (*) is in the form given by (c).

(f) Solve the recursion $x_n = 6x_{n-1} - 9x_{n-2}$ ($n \geq 2$) where $x_0 = -2$ and $x_1 = -3$.

9.4. DeMoivre's Formula

(a) Verify that the sequence defined by $x_n = \cos n\theta$ satisfies the recursion

$$x_{n+2} = (2 \cos \theta)x_{n+1} - x_n$$

for $n \geq 0$ with initial conditions $x_0 = 1$ and $x_1 = \cos \theta$.

(b) Verify that the sequence defined by $x_n = \sin n\theta$ satisfies the recursion

$$x_{n+2} = (2 \cos \theta)x_{n+1} - x_n$$

for $n \geq 0$ with initial conditions $x_0 = 0$ and $x_1 = \sin \theta$.

(c) Verify that the solutions of the quadratic equation $t^2 - (2 \cos \theta)t + 1 = 0$ are $\cos \theta + i \sin \theta$ and $\cos \theta - i \sin \theta$.

(d) Solve the recursions in (a) and (b) to obtain

$$\cos n\theta = \frac{1}{2}(\cos \theta + i \sin \theta)^n + \frac{1}{2}(\cos \theta - i \sin \theta)^n$$

and

$$\sin n\theta = \frac{1}{2i}(\cos \theta + i \sin \theta)^n - \frac{1}{2i}(\cos \theta - i \sin \theta)^n .$$

Deduce from this, De Moivre's Rule:

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta .$$



This idea is from the note

V.J. Matsko, *De Moivre's Rule, Recursion Relations, and Number Theory* Mathematics and Informatics Quarterly 8 (1998), 12-14.

9.5. Consider the problem of determining the value of $ax^4 + by^4$ when it is given that

$$\begin{aligned}a + b &= 23 \\ax + by &= 79 \\ax^2 + by^2 &= 217 \\ax^3 + by^3 &= 691 .\end{aligned}$$

One way to approach the solution is to derive from each pair of successive equations the relations $b(y-x) = 79-23x$, $by(y-x) = 217-79x$, $by^2(y-x) = 691-217x$. Therefore

$$\begin{aligned}y &= \frac{217-79x}{79-23x} = \frac{651-237x}{237-69x} \\&= \frac{691-217x}{217-79x} = \frac{40+20x}{-20-10x} = -2 .\end{aligned}$$

Doing the same manoeuvre to isolate the value of x leads to $x = -2$. However, it is straightforward to see that $(x, y) = (-2, -2)$ cannot satisfy the equation (otherwise the numbers on the right side would be in geometric progression).

What goes wrong with the foregoing argument? Find a correct solution to the problem.

Comments, Answers and Solutions

9.4. Observe that $\cos(n+2)\theta + \cos n\theta = \cos[(n+1)\theta + \theta] + \cos[(n+1)\theta - \theta]$ and that a similar treatment obtains for $\sin(n+2)\theta + \sin \theta$.

9.5. Note that the numbers x and y are the roots of a quadratic equation of the form $t^2 = ut + v$, so that

$$217 = 79u + 23v$$

and

$$691 = 217u + 79v .$$

This system has the solution $(u, v) = (1, 6)$ and we find that $ax^4 + by^4 = 691 + 6 \times 217 = 1993$. We can go ahead to obtain the values of all the variables:

$$(a, b; x, y) = (-2, 25; -2, 3), (25, -2; 3, -2) .$$

Why did the first method not turn up these solutions?

The problem, as is often the case, is dealing with a fraction for which the denominator can possibly vanish. The rational function $(40+20x)/(-20-10x)$ is indeterminate when $x = -2$. However, the other rational functions



appearing in the expression for y are well-defined and take the value 3 when $x = -2$. And indeed we find that $(x, y) = (-2, 3)$ appears as part of the solution of the system. Observe that when $x = 3$, then all of the rational functions yield the value -2 and we get $(x, y) = (3, -2)$. When x assumes some other value, then the rational functions assume different values and the equations for y do not hold. A similar phenomenon occurs when we solve for x in terms of y .

§10. Exercises on Geometry and Trigonometry

10.1. (a) Sketch the parabola with equation $y^2 = 4x$. Consider the family of parallel chords with equation $y = mx + b$, where m is a fixed parameter and b is allowed to vary. Argue that the midpoint of the chord of equation $y = mx + b$ is given by (X, Y) where $X = \frac{1}{2}(x_1 + x_2)$ and $Y = mX + b$, with x_1 and x_2 the two solutions of the quadratic equation

$$(mx + b)^2 = 4x \quad \text{or} \quad m^2x^2 + (2bm - 4)x + b^2 = 0 \quad .$$

(b) Without solving the quadratic equation in (a), use the relationship between the coefficients and roots to obtain an expression for X . Show that Y does not depend on b . What does this tell you about the locus of (X, Y) ?

(c) Redo parts (a) and (b) by setting up an equation in y rather than x and computing $Y = \frac{1}{2}(y_1 + y_2)$ directly.

10.2. A *diameter* of a conic section is the locus of the midpoints of a family of parallel chords.

(a) Sketch the ellipse with equation $(x^2/9) + (y^2/4) = 1$ along with some chords in the family $y = x + k$ where k is a parameter. (This could be done with a calculator or with some geometric computer software. In the latter case, try to trace the midpoints of the chords.)

(b) Follow the strategy used in Exercise 10.1 to show that the locus of the midpoints of the chords is a straight line.

(c) Generalize to the general conic section of equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad .$$

Corroborate your findings by taking particular choices of coefficients and chord slopes and graphing them with a computer or calculator.

10.3. Suppose that, in a triangle ABC , one angle B and two sidelengths $a = |BC|$ and $b = |AC|$ are known. What is the length of the remaining side? One way to obtain this is to use the Law of Cosines $b^2 = a^2 + c^2 - 2ac \cos B$ to obtain $c = |AB|$. Let us rewrite this third sidelength as a variable x and arrange the equation to

$$x^2 - (2a \cos B)x + (a^2 - b^2) = 0 \quad . \quad (*)$$

This is a quadratic equation, and so will have two solutions, which could be real or nonreal, positive or negative, or coincident. In this exercise, we will see how this relates to the geometry of the situation.

(a) Verify that the discriminant D of the quadratic in (*) is $4(b^2 - a^2 \sin^2 B)$. Explain why D is nonnegative if and only if a , b and B correspond to data for a feasible triangle. What happens if $D = 0$? Explain how the geometry supports the fact that (*) has a single solution in this case.

(b) Suppose that a , b and B are data for a feasible triangle. By considering the sum of the roots, explain why (*) has at least one positive solution.

(c) Determine conditions on a and b that (*) has (i) exactly one, (ii) exactly two, positive solutions. Relate this to the geometric possibilities for the triangle. In the case where there is a negative solution, explain how it might be interpreted.

10.4. Let a , b , c be real numbers. We consider solutions of the quadratic equation $az^2 + bz + c = 0$ where $z = x + yi$ is a complex number.

(a) Show that the complex equation $az^2 + bz + c = 0$ is equivalent to the system of real equations:

$$a(x^2 - y^2) + bx + c = 0 \quad (1)$$

$$axy + by = 0 \quad (2)$$

(b) Considering (2) in the form $y(ax + b) = 0$, describe its locus.

(c) Show that (1) can be written in the form

$$\left(x + \frac{b}{2a}\right)^2 - y^2 = \left(\frac{1}{2a}\right)^2 (b^2 - 4ac) .$$

Describe the locus of this equation in the three cases: (i) $b^2 = 4ac$; (ii) $b^2 > 4ac$; (iii) $b^2 < 4ac$.

(d) The solutions of the system (1) and (2) are represented in the plane by points (x, y) that lie on the intersection of the loci of (1) and (2). When $b^2 = 4ac$, show that there is a single such point and that it lies on the real axis. When $b^2 > 4ac$, show that there are two points on the real axis, each a reflection of the other in the line $\operatorname{Re} z = -b/2a$. When $b^2 < 4ac$, show that there are two points not on the real axis that are mirror images of each other with respect to the real axis. Explain how this is consistent with what you already now about real and imaginary roots of a quadratic.

Comments, Answers and Solutions

10.1. (b) $(X, Y) = (-(bm - 2)/m^2, 2/m)$. The locus is parallel to the axis of the parabola.

10.3.(a) $D \geq 0 \iff b \geq a \sin B$. Note that $a \sin B$ is the length of the perpendicular dropped from C to the line AB ; this length cannot exceed the



distance from C to any other point on this line. If $D = 0$, the triangle has a right angle at A .

10.3. (b) The sum of the roots of $2a \cos B$. When $B < 90^\circ$, this is positive and so at least one root is positive. When $B = 90^\circ$, then $0 \leq D = 4(b^2 - a^2)$, and there is one real root $\sqrt{b^2 - a^2}$. When $B > 90^\circ$, then $b > a$ and both the sum and product of the roots are negative, so exactly one root is positive.

10.3. (c) When $B < 90^\circ$, there is exactly one positive root when $a \leq b$. Sketch diagrams to illustrate the situations according as a is less than, equal to or greater than b .

§11. Exercises on Approximation

11.1. Let c be a positive real number. A standard way to approximate the square root of c is to begin with a positive guess u and then proceed to a new guess $v = \frac{1}{2}(u + c/u)$. (Note that c/u is another approximation to \sqrt{c} that lies on the other side of \sqrt{c} to u .) This is repeated over and over until the desired degree of approximation is reached.

(a) Verify that if $c = 2$ and the first guess is 1, then this process yields the sequence of approximants: 1 , $\frac{3}{2} = 1.5$, $\frac{17}{12} = 1.416667$, $\frac{577}{408} = 1.414216$ (where the decimals forms are not exact).

(b) Use the process to approximate $\sqrt{3}$.

(c) Show that $u < \sqrt{c}$ if and only if $c/u > \sqrt{c}$ and that $u > \sqrt{c}$ if and only if $c/u < \sqrt{c}$. Noting that v is the average of u and c/u , explain why it is reasonable to expect that v might be a better approximation than u .

(d) Verify that

$$v - \sqrt{c} = \frac{1}{2u}(u - \sqrt{c})^2.$$

Deduce that every approximation beyond the first exceeds \sqrt{c} , and prove that from this point on the sequences decreases. Why does the sequence tend towards \sqrt{c} ?

11.2. We look at the geometry of the situation of Exercise 10.1. As before, we have that $c > 0$.

(a) Let $x > 0$. Use the Arithmetic-Geometric Means Inequality (Exercise 4.1) to prove that $\frac{1}{2}(x + c/x) \geq \sqrt{c}$. with equality if and only if $x = \sqrt{c}$.

(b) Verify that

$$\frac{1}{2}\left(x_1 + \frac{c}{x_1}\right) - \frac{1}{2}\left(x_2 + \frac{c}{x_2}\right) = \frac{1}{2}(x_1 - x_2)\left(1 - \frac{c}{x_1x_2}\right).$$

Use this to argue that $\frac{1}{2}(x + c/x)$ is a decreasing function of x for $0 < x < \sqrt{c}$ and an increasing function of x for $\sqrt{c} < x$.



(c) With the same axes, sketch the graphs of both of the curves $y = x$ and $y = \frac{1}{2}(x + c/x)$ for $x > 0$. Where do these curves intersect? What are the asymptotes of the second curve?

(d) Using the graphs in (c), we can illustrate the behaviour of the approximating sequence for \sqrt{c} described in Exercise 10.1. Let $u_1 > 0$ be the first approximant. Locate on your sketch a possible position of $(u_1, 0)$. Let $u_2 = \frac{1}{2}(u_1 + c/u_1)$. Locate (u_1, u_2) , (u_2, u_2) and $(u_2, 0)$. These three points will be on the respective curves $y = \frac{1}{2}(x + c/x)$, $y = x$ and $y = 0$. We continue on in this way. Suppose that u_n has been found. Let

$$u_{n+1} = \frac{1}{2} \left(u_n + \frac{c}{u_n} \right).$$

Locate $(u_n, 0)$, (u_n, u_{n+1}) , (u_{n+1}, u_{n+1}) and $(u_{n+1}, 0)$. Describe from your diagram what eventually happens to the terms of the sequence $\{u_n\}$.

11.3. The recursion of Exercise 11.1 can be defined when c is negative, even though c does not have a real square root in this case. What will happen? To focus the discussion, consider the case $c = -1$.

(a) Sketch the curve

$$y = \frac{1}{2} \left(x - \frac{1}{x} \right)$$

for real nonzero x , and attempt an analysis as in Exercise 10.2.(d), using various starting points. In this case, you may find it helpful to use a calculator or computer to generate the terms of the sequence of “approximants”, or even to use the computer to draw the whole situation for you.

(b) To get a handle on the situation, we note that any real number can be written in the form $x = \cot \theta$ for some number θ lying strictly between 0 and π . Consider the transformation

$$T : x \longrightarrow \frac{1}{2} \left(x - \frac{1}{x} \right).$$

If $x = \cot \theta$, show that the image of x under this transformation is $\cot 2\theta$. Thus, in terms of θ the mapping is *conjugate* (essentially the same in its mathematical structure) to U : $\theta \rightarrow 2\theta$ (modulo π) (this simply means that if you add, subtract two angles or multiply by a constant, you add an integral multiple of π to put the result of the operation in the interval $(0, \pi)$ using a kind of “clock arithmetic”).

(c) Does the transformation T have any *fixed points*? (These are points x for which $T(x) = x$. You can answer this question directly, but also by looking at the mapping U and reinterpreting what you find in terms of T .)

(d) Let $U^2(\theta) = U(U(\theta))$ and for $n \geq 3$, let $U^n(\theta) = U(U^{n-1}(\theta))$. Determine a simple expression for $U^n(\theta)$.



(e) Does the transformation T have any points of *period 2*? (This asks whether there are any numbers u for which $T(u) = v$ for some number v and $T(v) = u$, so that two applications of the mapping T take the point back to itself.) Answer this question directly by looking at the equation

$$T(T(x)) = x \ .$$

Now answer it by working through the operator U . For what values of θ does $U(U(\theta)) = 4\theta$ differ from θ by a multiple of π . Are your results consistent?

(f) A point p is a point of *period k* for T if and only if $T^k(p) = p$, where $T^1(x) = T(x)$ and $T^k(x) = T(T^{k-1}(x))$ for $k \geq 2$. Either directly or working through the operator U , determine if T has points of period k for k is a positive integer exceeding 1. Use a calculator to work out the approximate values of such points and check the result by applying the operator T .

§12. Exercises on the Logistic Dynamical System

We suppose that k is a positive parameter and define the function $p_k(x) = kx(1 - x)$ for $0 \leq x \leq 1$. We can use p_k to define a *dynamical system* as follows:

Begin with any point x_0 in the closed interval $[0, 1] \equiv x : 0 \leq x \leq 1$. For each nonnegative integer n , define $x_{n+1} = p_k(x_n)$.

12.1. One can use graphical methods in helping us visualize how the sequence defined for the dynamical system behaves. Suppose that we have a sketch of the curves with equations

$$y = p_k(x)$$

and

$$y = x \ .$$

For each nonnegative integer n , plot the points $(x_n, 0)$ and $(x_n, p_k(x_n)) = (x_n, x_{n+1})$. By drawing lines parallel to the axes and making use of the line $y = x$, indicate geometrically how the point $(x_{n+1}, 0)$ can be found. Thus, we can indicate on the x -axis the progress of the sequence $\{x_n\}$.

12.2. Consider the case $0 < k < 1$. Sketch the curves as indicated in (a) and use your diagram to argue that $\lim_{n \rightarrow \infty} x_n = 0$. Verify this analytically, by first verifying that $0 < x_{n+1} < kx_n$ whenever $0 < x_0 < 1$.

12.3. Suppose that $k > 1$. Determine a number u for which $0 < u < 1$ and $p_k(u) = u$.

12.4. Consider the case $1 < k < 2$. Sketch the curves as in (a), being careful to indicate on which side of the line $x = \frac{1}{2}$ the curves intersect. Analyze the types of behaviour of the sequence for values of x_0 in the closed interval $[0, 1]$.

12.5. Consider the case $2 < k < 3$. Sketch the curves as in (a) and analyze the behaviour or sequences $\{x_n\}$. Verify that

$$x_{n+1} - u = k(x_n - u)(1 - u - x_n)$$



and use this to check that, when $x_n > 1 - u$, $x_{n+1} - u$ and $x_n - u$ have opposite signs and $|x_{n+1} - u| < |x_n - u|$. Analyze the behaviour of the sequence $\{x_n\}$ for various cases of x_0 in $[0, 1]$.

12.6. Let $k > 1$ and let u be as defined in part (c). Determine $p'_k(u)$ in terms of k , where p'_k denotes the derivative of p_k . Prove that $|p'_k(u)| < 1$ if and only if $1 < k < 2$. What effect do you think that the value of the derivative of p_k at u has on the behaviour of sequences $\{x_n\}$ that start off with a value x_0 close to u ?

12.7. We study the possibility of sequences $\{x_n\}$ of *period 2*, i.e., there are two distinct values v and w for which $x_n = v$ when n is even and $x_n = w$ when n is odd, so that the sequence proceeds $\{u, v, u, v, \dots\}$. To do this, we define the *second iterate* of p_k :

$$q_k(x) = p_k(p_k(x)) = kp_k(x)(1 - p_k(x)) \quad .$$

Determine the polynomial q_k and specify its degree. Prove that if $p_k(v) = w$ and $p_k(w) = v$, then $q_k(v) = v$ and $q_k(w) = w$.

12.8. To solve the equation $x = q_k(x)$, we can write it in the form

$$x - q_k(x) = 0 \quad .$$

Explain why $x - p_k(x)$ is a factor of the left side, and use this fact to write the left side as a product of quadratics. Thus determine v and w .

12.9. For the cases $1 < k < 2$, $2 < k < 3$, $k = 3$ and $k < 3$, show on a graph the location of v and w .

12.10. Investigate the behaviour of the sequence $\{x_n\}$ when $k > 3$. You may find a pocket calculator of some use in this enterprise.

§13. Exercises on Composition of Quadratics

The function $f(x)$ is said to be the *composite* of functions g and h if $f(x) = g(h(x))$. The binary operation that takes the ordered pair (g, h) to f is called *composition*. The functions g and h are said to *commute* under composition if $g(h(x)) = h(g(x))$. A polynomial is *monic* if its leading coefficient (i.e. the coefficient of the highest power of the variable) is equal to 1.

13.1. Suppose that g and h are polynomials. How is the degree of the composite $f(x) = g(h(x))$ related to the degrees of g and h ?

13.2. Prove that two linear polynomials commute under composition if and only if one of the following conditions hold:

- (a) one of them is the identity polynomial x ;
- (b) both of them have the form $x + k$ for some values of the constant k ;
- (c) both of them have a common fixed point, i.e., a number c that gets mapped to itself by both polynomials.



13.3. (a) Suppose that $f(x)$ is a monic quadratic polynomial with integer coefficients, then for each integer m , there is an integer n for which $f(m)f(m+1) = f(n)$. Determine a formula that indicates n as a function of m .

(b) The result of (a) has an interesting generalization found by James Rickards: a monic quartic polynomial (of degree 4) is the composite of two monic quadratic polynomials if and only if the sum of two of the roots of the quartic is equal to the sum of the other two.

(c) Generalize the result of (b) so that the polynomials need not to be monic.

Comments

13.3. (a) First, we note that if $f(x) = x^2 + bx + c$, then

$$f(0)f(1) = c + bc + c^2 = f(c) = f(f(0)) .$$

Let m be any integer and define $g(x) = f(m+x)$. Then

$$f(m)f(m+1) = g(0)g(1) = g(g(0)) = g(f(m)) = f(m+f(m))$$

so that $n = m + f(m)$ is the desired integer.

Observe that, if the roots of $f(x) = 0$ are r and s , then the roots of $f(x+1) = 0$ are $r-1$ and $s-1$. Thus, $f(x)f(x+1)$ is a quartic polynomial whose four roots satisfy the relationship $r + (s-1) = (r-1) + s$ and which has the form $g(h(x))$ where $g(x) = f(x)$ and $h(x) = x + f(x)$.

(b) Suppose that the quartic polynomial $f(x) = u(x)v(x)$ where $u(x) = (x-a)(x-b)$, $v(x) = (x-c)(x-d)$ and $a+b = c+d$. Then $v(x) = u(x)+k$ for some constant k . Then $f(x) = g(h(x))$ where $h(x) = u(x)$ and $g(x) = x(x+k)$.

On the other hand, let $f(x) = g(h(x))$, where g and h are monic quadratic polynomials with $g(x) = (x-r)(x-s)$.

Then $f(x) = g(h(x)) = (h(x)-r)(h(x)-s)$. The sum of the roots of $h(x)-r$, being the negative of the linear coefficient of this polynomial, is equal to the sum of the roots of $h(x) - s$.

There is another criterion which determines when a quartic polynomial $ax^4 + bx^3 + cx^2 + dx + e$ is the composite of two quadratics: $4abc - 8a^2d = b^3$. (*Polynomials*, by E.J. Barbeau, Springer, 2003, p. 266) Problem 931 in the *College Mathematics Journal* 47:4 (September, 2010), 329, requires the solver to show that for any polynomial $f(x)$, $f(f(x)+x) = f(x)g(x)$ for some polynomial $g(x)$ and that the remainder when $g(x)$ is divided by $f(x)$ is $f'(x) + 1$, where f' is the derivative of f .





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