Canadian Mathematical Olympiad Qualifying Repêchage 2023

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Official Solutions

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1 [10 points] There are two imposters and seven crewmates on Polus. How many ways are there for the nine people to split into three groups of three, such that each group has at least two crewmates? Assume that the two imposters and seven crewmates are all distinguishable from each other, but that the three groups are not distinguishable from each other.

Solution: There are $\binom{7}{2}$ ways to assign 2 crewmates to the first imposter, and $\binom{5}{2}$ ways to assign 2 crewmates to the second imposter, for a total of $\binom{7}{2}\binom{5}{2} = 210$ ways.

2 [10 points] How many ways are there to fill a 3×3 grid with the numbers $1, 2, 3, 4, 5, 6, 7, 8$, and 9, such that the set of three elements in every row and every column form an arithmetic progression in some order?

Solution: There are a few different ways to set up the casework. Since the ordering of the rows and columns do not matter, we may assume that 1 is in the top left corner. The two sequences containing 1 can be chosen from $\{1, 2, 3\}, \{1, 3, 5\}, \{1, 4, 7\}$, and $\{1, 5, 9\}$. Some of these sequences overlap and there are four cases we have to consider. However, in each case, there are 2 ways to choose the locations of each of the arithmetic progressions, and which one is a row or column, so we may choose one instantiation and multiply by 8.

Case 1:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & a & b \\ 7 & c & d \end{array}$$

Here, we must choose $\{b, d\} = \{6, 9\}, \{a, c\} = \{5, 8\}$. There are two arrangements, $(a, b, c, d) = (5, 6, 8, 9), (8, 6, 5, 9)$.

Case 2:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 5 & a & b \\ 9 & c & d \end{array}$$

Here, we must have $\{a, c\} = \{4, 6\}$, but there is then a contradiction as we cannot have $\{b, d\} = \{7, 8\}$.

Case 3:

$$\begin{array}{ccc} 1 & 3 & 5 \\ 4 & a & b \\ 7 & c & d \end{array}$$

Here, we must have $\{a, c\} = \{6, 9\}$ and $\{b, d\} = \{2, 8\}$, and there is only one arrangement: $(a, b, c, d) = (6, 2, 9, 8)$.

Case 4:

$$\begin{array}{ccc} 1 & 4 & 7 \\ 5 & a & b \\ 9 & c & d \end{array}$$

Here, we must have $\{b, d\} = \{6, 8\}$ and $\{a, c\} = \{2, 3\}$, and there is only one arrangement: $(a, b, c, d) = (2, 8, 3, 6)$.

In total, there are 4 arrangements, but we need to multiply by 72 since we chose the location of the 1 and the order of the elements in the same row/column as 1, for a final answer of 288.

3 [10 points] Let circles Γ_1 and Γ_2 have radii r_1 and r_2, respectively. Assume that $r_1 < r_2$. Let T be an intersection point of Γ_1 and Γ_2, and let S be the intersection of the common external tangents of Γ_1 and Γ_2. If it is given that the tangents to Γ_1 and Γ_2 at T are perpendicular, determine the length of ST in terms of r_1 and r_2.

Solution: Let A and B be the respective centers of Γ_1 and Γ_2, let H be the foot of the perpendicular from T to AB, and let U and V be the points of tangency of a tangent from S to Γ_1 and Γ_2, respectively. Note that S is on line AB by symmetry, so our goal is to compute TH and SH and use the Pythagorean Theorem to finish.

We have that $TH = \frac{r_1 r_2}{\sqrt{r_1^2+r_2^2}}$ by using similar triangles ATH and ABT, and the same pair of similar triangles allows us to conclude $AH = \frac{r_1^2}{\sqrt{r_1^2+r_2^2}}$. We use that triangles SUA and SVB are similar to conclude

$$\frac{BS}{AS} = \frac{r_2}{r_1},$$

and after subtracting 1 from both sides, we get

$$AS = \frac{r_1}{r_2 - r_1} AB = \frac{r_1\sqrt{r_1^2 + r_2^2}}{r_2 - r_1}.$$

Adding up AS and AH, we get

$$SH = \frac{r_1 r_2 (r_1 + r_2)}{\sqrt{r_1^2 + r_2^2}(r_2 - r_1)},$$

and so

$$ST = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}} \sqrt{1 + \left(\frac{r_1 + r_2}{r_2 - r_1}\right)^2} = \frac{\sqrt{2} r_1 r_2}{r_2 - r_1}.$$

4 [10 points] Let $a_1, a_2, \ldots$ be a sequence of numbers, each either 1 or -1. Show that if

$$\frac{a_1}{3} + \frac{a_2}{3^2} + \cdots = \frac{p}{q}$$

for integers p and q such that 3 does not divide q, then the sequence $a_1, a_2, \ldots$ is periodic; that is, there is some positive integer n such that $a_i = a_{n+i}$ for $i = 1, 2, \ldots$.

Solution: First, we note that

$$\frac{1}{3} + \frac{1}{3^2} + \cdots = \frac{1}{2},$$

so

$$\frac{(1+a_1)/2}{3} + \frac{(1+a_2)/2}{3} + \cdots = \frac{2p+q}{4q}.$$

Now, the left hand side is the unique base-3 representation of the right hand side (since each $(1+a_i)/2$ is either 0 or 1), so it must be eventually periodic. To show that it is periodic, note that if we have some periodic sequence

$$\frac{b_1}{3} + \frac{b_2}{3^2} + \ldots,$$

with $b_i = b_{n+i}$, then this sum is equal to

$$\frac{b_1 3^{n-1} + b_2 3^{n-2} + \cdots + b_n}{3^n - 1},$$

which has denominator not divisible by 3. Thus, given some eventually periodic sequence, we compare it to the periodic sequence; neither of them has denominator divisible by 3 (since $4q$ has no factor of 3), and their difference must have denominator divisible by 3 if non-zero, which means that the difference is exactly 0, and so the sequence is actually periodic.

5 [10 points] Six decks of n cards, numbered from 1 to n, are given. Melanie arranges each of the decks in some order, such that for any distinct numbers x, y, and z in $\{1, 2, \ldots, n\}$, there is exactly one deck where card x is above card y and card y is above card z. Show that there is some n for which Melanie cannot arrange these six decks of cards with this property.

Solution: Fix a card k. Then, assign to every other card c an ordered sextuplet $(c_1, \ldots, c_6)$, where $c_i = 0$ if c is above k in deck i, and $c_i = 1$ otherwise. If $n > 65$, there are at least two cards c and c' with $c_i = c'_i$ for all i; this means that the relative positions of c_i and c'_i are the same in all decks, and in particular, we never have k between c and c' in any deck.

6 [10 points] Given triangle ABC with circumcircle Γ, let D, E, and F be the midpoints of sides BC, CA, and AB, respectively, and let the lines AD, BE, and CF intersect Γ again at points J, K, and L, respectively. Show that the area of triangle JKL is at least that of triangle ABC.

Solution: Let G be the centroid of triangle ABC, and let a, b, c denote the side lengths of BC, CA, AB, respectively. Then, we can compute $AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$, and by Power of a Point, $DJ = \frac{a^2}{2\sqrt{2b^2+2c^2-a^2}}$. We thus have

$$GJ = \frac{1}{3}AD + DJ = \frac{a^2 + b^2 + c^2}{3\sqrt{2b^2 + 2c^2 - a^2}},$$

and

$$\frac{GJ}{GA} = \frac{a^2 + b^2 + c^2}{2b^2 + 2c^2 - a^2}.$$

We get similar formulas for GJ/JB and GJ/JC, and so we have

$$\frac{[JKL]}{[ABC]} = \frac{1}{3}\left(\frac{[JGK]}{[AGB]} + \frac{[KGL]}{[BGC]} + \frac{[LGJ]}{[CGA]}\right) = \frac{1}{3}\left(\frac{GJ \cdot GK}{GA \cdot GB} + \frac{GK \cdot GL}{GB \cdot GC} + \frac{GL \cdot GJ}{GC \cdot GA}\right).$$

Plugging in our formulas, we thus have

$$\frac{[JKL]}{[ABC]} = \frac{(a^2 + b^2 + c^2)^2}{3}\frac{(2b^2 + 2c^2 - a^2) + (2c^2 + 2a^2 - b^2) + (2a^2 + 2b^2 - c^2)}{(2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)}$$

$$= \frac{(a^2 + b^2 + c^2)^3}{(2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)}.$$

To finish, note that the three terms in the denominator average to $a^2 + b^2 + c^2$, so the whole quantity is at least 1 by AM-GM.

7 [20 points]

1. Let u, v, and w be the real solutions to the equation $x^3 - 7x + 7 = 0$. Show that there exists a quadratic polynomial f with rational coefficients such that $u = f(v)$, $v = f(w)$, and $w = f(u)$.

2. Let u, v, and w be the real solutions to the equation $x^3 - 7x + 4 = 0$. Show that there does not exist a quadratic polynomial f with rational coefficients such that $u = f(v)$, $v = f(w)$, and $w = f(u)$.

Solution: For now, we take the general case where the cubic equation is $x^3 - mx + n = 0$. By Vieta's formulas, we know that $u + v + w = 0$, $uv + vw + wu = -m$, and $uvw = -n$. Let $u = av^2 + bv + c, v = aw^2 + bw + c, w = au^2 + bu + c$ for some rational a, b, c.

Multiplying the first quadratic equation above by v, we get

$$uv = av^3 + bv^2 + cv = a(mv - n) + bv^2 + cv.$$

Summing this and the two similar expressions $vw = a(mw - n) + bw^2 + cw$ and $wu = a(mu - n) + bu^2 + cu$, we get

$$-m = uv + vw + wu = am(u + v + w) - 3an + b(u^2 + v^2 + w^2) + c(u + v + w) = 2bm - 3an.$$

Separately, subtracting the first two quadratic equations, we get

$$u - v = (v - w)(a(v + w) + b) = (v - w)(b - au).$$

Multiplying this and the two similar expressions, we get

$$(u - v)(v - w)(w - u) = (u - v)(v - w)(w - u)(b - au)(b - av)(b - aw),$$

so after canceling (why can we cancel?), we get

$$1 = (b - au)(b - av)(b - aw) = b^3 - mba^2 + na^3.$$

Combining this and the previous equation with a and b, we get

$$1 = b^3 + \left(\frac{(2b + 1)m}{3n}\right)^2 \frac{(1 - b)m}{3}. \qquad (\star)$$

$b = 1$ is always a solution. But when $b = 1$, we have $a = m/n$. Adding the three original equations, we get

$$0 = u + v + w = a(u^2 + v^2 + w^2) + b(u + v + w) + 3c = 2m^2/n + 3c,$$

so $c = -2m^2/3n$.

Finally, plugging in $m = u^2 + uv + v^2$ and $n = uv(u + v)$ into $u = (m/n)v^2 + v - (2m^2/3n)$, we get

$$3(u - v)n = m(3v^2 - 2m) \iff 3(u - v)(u + v)(uv) = -(u^2 + uv + v^2)(u - v)(u + 2v)$$
$$\iff u^3 + 6u^2v + 6uv^2 + 2v^3 = 0.$$

The same equation must hold upon cyclically permuting the indices, so we get that $u/v, v/w, w/u$ are roots of the polynomial $x^3 + 6x^2 + 6x + 2 = 0$, which is impossible since they multiply to 1. Thus, we have eliminated the case $b = 1$.

Our equation $(\star)$ now becomes

$$27n^2(b^2 + b + 1) = m^3(4b^2 + 4b + 1),$$

which factors as

$$(4m^3 - 27n^2)(2b + 1)^2 = 81n^2.$$

This has rational roots if and only if $4m^3 - 27n^2$ is a perfect square.

(a) When $m = 7, n = 4$, $4m^3 - 27n^2$ is not a perfect square, so we cannot find a, b, c.

(b) When $m = n = 7$, we have

$$2b + 1 = \pm 9,$$

so $b = 4, -5$. We choose $b = 4$, and from the above discussion we get $a = 3$ and $c = -2am/3 = -14$. We now need to verify that we can indeed choose the roots u, v, w such that $u = 3v^2 + 4v - 14$; that is, the roots are $v, u = 3v^2 + 4v - 14, w = -3v^2 - 5v + 14$. But this holds since we can verify that the symmetric sums of the roots are $uv + vw + wu = -7$ and $uvw = -7$.

8 [20 points] A point starts at the origin of the coordinate plane. Every minute, it either moves one unit in the x-direction or is rotated θ degrees counterclockwise about the origin.

(a) If $\theta = 90^\circ$, determine all locations where the point could end up.

(b) If $\theta = 45^\circ$, prove that for every location L in the coordinate plane and every positive number ε, there is a sequence of moves after which the point has distance less than ε from L.

(c) Determine all rational numbers θ such that for every location L in the coordinate plane and every positive number ε, there is a sequence of moves after which the point has distance less than ε from L.

(d) Prove that when θ is irrational, for every location L in the coordinate plane and every positive number ε, there is a sequence of moves after which the point has distance less than ε from L.

Solution:

(a) The answer is all locations with integer coordinates. To get to such a location (x, y) in the first quadrant, we move right y times, rotate once, and move right x more times. To get to such a location in any other quadrant, we move to the corresponding location in the first quadrant and rotate. After every minute, the point's coordinates are integers, so these are all the possible final positions for the point.

(b) For this and the next part, we first prove a lemma: we can get to every location of the form

$$\sum_{j=0}^{k} a_j(cos(j\theta), \sin(j\theta)),$$

where the a_j are non-negative integers. We achieve this by moving right a_k times, rotating, moving right a_{k-1} times, rotating, and so on.

For this problem, we note that j and $j+4$ give the same vector but in opposite directions for $(\cos(j\theta), \sin(j\theta))$, so we may without loss of generality assume that the a_k are any integers. Since $\sqrt{2}$ is irrational, there are some integers m, n such that $|m + n\sqrt{2} - c| < \epsilon/2$. Given coordinates (c_1, c_2), we may thus find vectors adding to $(m_1 + n_1\sqrt{2}, m_2 + n_2\sqrt{2})$ where $|m_i + n_i\sqrt{2} - c_i| < \epsilon/2$ for $i = 1, 2$. This point satisfies the properties.

(c) The answers are θ a multiple of 60° or 90°. Recall from the previous part that we can get to any location of the form

$$\sum_{j=0}^{k} a_j(cos(j\theta), \sin(j\theta)),$$

where the a_j are non-negative integers.

When $\theta = 60^\circ$, we can only get to locations of the form $(m, n\sqrt{3})$ and $(m+\frac{1}{2}, (n+\frac{1}{2})\sqrt{3})$ for integers m and n. The $\theta = 90^\circ$ case is dealt with in part a).

Now assume θ is irrational. We may find j, k such that $0 < j\theta - 180 - 360k < \epsilon$. Let $j\theta - 180 - 360k = \delta_1$. Thus, $(1,0) + (\cos(j\theta), \sin(j\theta)) = (O(\delta_1^2), -\delta_1 + O(\delta_1^2))$. Similarly, by choosing $-\epsilon < j\theta - 180 - 360k < 0$, and letting $180 + 360k - j\theta = \delta_2$, we can get a vector of $(O(\delta_2^2), \delta_2 + O(\delta_2^2))$. Choosing j_1, k_1, j_2, k_2 such that $-\epsilon/2 < j_1\theta - 90 - 360k_1 < 0 < j_2\theta - 270 - 360k_2 < \epsilon/2$, and letting $j_2\theta - j_1\theta - 360(k_2 - k_1) - 180 = \delta_3$, we can get a vector of $(\delta_3 + O(\delta_3^2), O(\delta_3^2))$. and choosing j_1, k_1, j_2, k_2 such that $-\epsilon/2 < j_1\theta - 270 - 360k_1 < 0 < j_2\theta - 90 - 360k_2 < \epsilon/2$, and letting $j_2\theta - j_1\theta - 360(k_2 - k_1) + 180 = \delta_4$, we can get a vector of $(-\delta_4 + O(\delta_4^2), O(\delta_4^2))$. Combining these for sufficiently small ϵ allow us to get near any location (c_1, c_2).

Finally, assume θ is rational and replace θ with $\gcd(\theta, 360)$. We assume that $\theta \neq 60, 90, 120, 180, 360$, which necessarily means $\theta \leq 72$. We note that $\cos\theta$ must be irrational. Assume otherwise, and let $\theta = p/q$. Note that if $q = 2$, the only possibilities are $p = 1, 2$, which lead to $\theta = 60, 90$, respectively. Now assume $q > 2$. Assume $k\theta = 360$. We note that $\cos(k\theta)$ is a degree k integer polynomial of $\cos\theta$ with leading coefficient 2^k. Thus, multiplying both sides of $\cos(k\theta) = 1$ by q^{k-1} and expanding the left hand side into a polynomial of $\cos\theta$, we see that the first term in the left hand side is not an integer, but the other terms on both sides are, which is a contradiction.

(d) We first draw a series of n unit vectors that go from 0 to L (this is always possible for some n since our unit vectors can be in any direction). Then, we approximate these unit vectors with unit vectors that are positive multiples of θ, each of which has distance at most ε/n; since θ is irrational, positive multiples of θ can approximate any angle with arbitrary precision. We then add up these n vectors and get something within ϵ of L.