Canadian Mathematical Olympiad Qualifying Repêchage 2023

A competition of the Canadian Mathematical Society.

Official Problem Set

1. There are two imposters and seven crewmates on Polus. How many ways are there for the nine people to split into three groups of three, such that each group has at least two crewmates? Assume that the two imposters and seven crewmates are all distinguishable from each other, but that the three groups are not distinguishable from each other.

2. How many ways are there to fill a 3×3 grid with the numbers $1, 2, 3, 4, 5, 6, 7, 8$, and 9, such that the set of three elements in every row and every column form an arithmetic progression in some order? (Each number must be used exactly once)

3. Let circles Γ_1 and Γ_2 have radii r_1 and r_2, respectively. Assume that $r_1 < r_2$. Let T be an intersection point of Γ_1 and Γ_2, and let S be the intersection of the common external tangents of Γ_1 and Γ_2. If it is given that the tangents to Γ_1 and Γ_2 at T are perpendicular, determine the length of ST in terms of r_1 and r_2.

4. Let $a_1, a_2, \ldots$ be a sequence of numbers, each either 1 or -1. Show that if

$$\frac{a_1}{3} + \frac{a_2}{3^2} + \cdots = \frac{p}{q}$$

for integers p and q such that 3 does not divide q, then the sequence $a_1, a_2, \ldots$ is periodic; that is, there is some positive integer n such that $a_i = a_{n+i}$ for $i = 1, 2, \ldots$.

5. Six decks of n cards, numbered from 1 to n, are given. Melanie arranges each of the decks in some order, such that for any distinct numbers x, y, and z in $\{1, 2, \ldots, n\}$, there is exactly one deck where card x is above card y and card y is above card z. Show that there is some n for which Melanie cannot arrange these six decks of cards with this property.

6. Given triangle ABC with circumcircle Γ, let D, E, and F be the midpoints of sides BC, CA, and AB, respectively, and let the lines AD, BE, and CF intersect Γ again at points J, K, and L, respectively. Show that the area of triangle JKL is at least that of triangle ABC.

7. (a) Let u, v, and w be the real solutions to the equation $x^3 - 7x + 7 = 0$. Show that there exists a quadratic polynomial f with rational coefficients such that $u = f(v)$, $v = f(w)$, and $w = f(u)$.

(b) Let u, v, and w be the real solutions to the equation $x^3 - 7x + 4 = 0$. Show that there does not exist a quadratic polynomial f with rational coefficients such that $u = f(v)$, $v = f(w)$, and $w = f(u)$.

8. A point starts at the origin of the coordinate plane. Every minute, it either moves one unit in the x-direction or is rotated θ degrees counterclockwise about the origin.

   (a) If $\theta = 90^\circ$, determine all locations where the point could end up.

   (b) If $\theta = 45^\circ$, prove that for every location L in the coordinate plane and every positive number ε, there is a sequence of moves after which the point has distance less than ε from L.

   (c) Determine all rational numbers θ such that for every location L in the coordinate plane and every positive number ε, there is a sequence of moves after which the point has distance less than ε from L.

   (d) Prove that when θ is irrational, for every location L in the coordinate plane and every positive number ε, there is a sequence of moves after which the point has distance less than ε from L.